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Connected Search for a Lazy Robber

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Abstract

The node search game against a lazy (or, respectively, agile) invisible robber has been introduced as a search-game analogue of the treewidth parameter (and, respectively, pathwidth). In the *connected* variants of the above games, we additionally demand that, at each moment of the search, the *clean* territories are *connected*. The connected search game against an agile and invisible robber has been extensively examined. The monotone variant (where we demand that the clean territories are progressively increasing) of this game, corresponds to the graph parameter of *connected pathwidth*. It is known that *the price of connectivtiy* to search for an agile robber is bounded by 2, that is the connected pathwidth of a graph is at most twice (plus some constant) its pathwidth. We investigate the study of the connected search game against a *lazy* robber. A lazy robber moves only when the cops' strategy threatens the vertex that he currently occupies. We introduce two alternative graph-theoretical formulations of this game, one in terms of connected tree decompositions and one in terms of (connected) layouts, leading to the graph parameter of *connected treewidth*. We observe that connected treewidth parameter is closed under contractions and prove that for every $k \geq 2$, the set of contraction obstructions of the class of graphs with connected treewidth at most k is infinite. Our main result is a complete characterization of the obstruction set for $k = 2$. We also show that, in contrast to the agile robber game, the price of connectivity is unbounded.

Keywords: Graph Searching, Cops and Robbers Game, Connected Treewidth, Contraction Obstructions, Price of Connectivity.

1 Introduction

In a *graph search game* the competing parts are a group of cops and a robber that move in a graph while having opposite goals. The goal of the cops is to capture the robber, while the robber is trying to avoid capture. Typically, the robber is assumed to be *omniscient*, implying that he/she always makes the best possible move towards avoiding capture. Numerous and quite diverse variants of this game can be defined, depending on the rules that determine how the cops and the robber can move and what the definition of "capture" is. A search strategy represents a series of moves that eventually yields the capture of the robber. Given the rules of the game, the cost of a search strategy is the maximum number of cops simultaneously present on the graph during the search. Then the

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corresponding *search number* of the graph is defined as the minimum cost of a search strategy. The study of graph searching parameters is an active field of graph theory as several important graph parameters have their search-game analogues that provides useful insights. For related surveys, see [2,3,10,25,29,46].

Node-search against an invisible searcher. One of the most classic graph-search games is the one of *node-search* introduced by Kirousis and Papadimitriou [38,39]. In this version, both the cops and the robber occupy vertices of the graph and they play in turn. At the first round the robber selects a vertex to occupy, and then, at his turn, he moves on any vertex that can be reached from his position by a path whose vertices are not occupied by cops. The capture of the robber happens when some cop and the robber simultaneously appear at the same vertex, meaning that the robber cannot escape along a free path. The search number of a graph is the minimum number of cops required to guarantee the capture of the robber.

In this paper we consider the variant of node-search where the robber is *invisible*, that is, while a search strategy is being deployed, the cops do not have any knowledge of robber's actual position. This implies that the search strategy should provide, before the beginning of the game, the full sequence of cops' moves that will result to the capture of the robber. Thereby a search strategy is independent of the reaction of the robber. As in this paper we exclusively deal with variants of the node-search game against an *invisible* robber we simply use the term "search game" to refer to this setting.

Several variants of the search game arise, depending on the rules imposed to the search strategy and to the moves of the robber. We outline below the most classic ones. The formal definitions are given in Section 2.

Monotonicity. A common feature of a search strategy is *monotonicity*: we say that a search strategy is *monotone* if it does not permit the robber to move to vertices that have already been occupied by cops. In other words, a monotone strategy should guarantee that, the "clean" territory (i.e,. the set of vertices from which the robber has been expelled), is gradually increasing (and, certainly, finally occupy the whole graph). The *monotone search number* is defined as the minimum number of cops required to capture the robber by using a monotone search strategy. We say that a search game is *monotone* when the corresponding search number is equal to the non-monotone one, i.e,. monotone strategies are as good as the non-monotone ones.

Agility and laziness. Different variants of the game arise depending on the mobility rules of the robber, e.g. a robber can be lazy or agile. A lazy robber residing on vertex v may move only if the next move of the search strategy is a placement of a cop on v. In other words, the lazy robber stays put, unless his position is threatened by the cop's strategy. On the other hand, an agile robber may always move no matter what the next move of the search strategy is. The distinction between a lazy and an agile robber has been introduced for the first time in [16]. Parameterizations of the game that oscillate between the lazy and agile variants have been studied in [48] (see also [27]).

There is an extensive amount of research on the four versions of the search game that are generated by the above variations. The reason for this is that they correspond to well-studied graph theoretical parameters. The monotone search number of a graph G against an agile (resp. lazy) robber is equal to the pathwidth (resp. treewidth) of G plus one [16,36,38,39,44,51]. Also, it was proven that the non-monotone versions of the above variants are equal to their monotone counterparts [8,9,28,41,51]. This induced a clear landscape on how to connect treewidth and pathwidth to graph searching and established the intuition that the qualitative difference between the two parameters is expressed by the agility or the laziness of the robber. Moreover, the monotonicity proofs were based on min-max theorems and alternative definitions of the corresponding parameters [9,28,51].

The connectivity issue. All the above variants are assuming that there is some place "somewhere outside" the graph where the cops go when they are removed from the graph and they stay there in order to readily be placed to some vertex, as required by the search strategy. For this reason, such search games have also been called "helicopter search games" (as suggested in [51]). This is not always realistic, as the search may take place in a building or in a system of caves where the cops start their persecution from some particular vertex (the entry point) and they do not have the ability of "teleporting themselves" to non-neighboring vertices. This natural restriction was considered and studied for the first time in [5]. A search strategy is *connected* when at each moment of the search the clean territories induce a connected subgraph[1]. This inspired the question on the "price of connectivity", asking whether there is some universal constant c such that the connected search number is no more than c times its non-connected counterpart. In its original form, this question was made in [5] for the agile variant and, in the same paper, it was answered affirmatively for the case of trees (see also [6,22–24,26,30,45] for related results). Later, it was proved for all graphs by Dariusz Dereniowski in [17]. In particular, in [18], a connected counterpart of pathwidth was suggested, called connected pathwidth, that is equivalent to the monotone connected agile search number. Then it was proved that this parameter is always upper bounded by twice the pathwidth plus one.

Much less is known about the non-monotone variants of the connected search game. The monotonicity question for the connected search against an agile robber was resolved negatively in [52]. Analogous negative results have been derived in the case where the fugitive is *visible* and agile [31] (which is equivalent to the *invisible* and lazy case).

Connected treewidth. This paper initiates the study of the monotone connected search against a *lazy* robber. Our first step is to provide two alternative definitions of this parameter: one in terms of tree decompositions and one in terms of layouts. Before we proceed, we need to give the formal definition of tree decompositions and treewidth.

A *tree-decomposition* of a graph $G = (V, E)$ is a pair $(T, \mathcal{F})$ where $T = (V_T, E_T)$ is a tree and $\mathcal{F} = \{X_t \subseteq V \mid t \in V_T\}$ such that

1. $\bigcup_{t \in V_T} X_t = V$,
2. for every edge $e \in E$, there exists a node $t \in T$ such that $e \subseteq X_t$, and
3. for every vertex $x \in V$, the set $\{t \in V_T \mid x \in X_t\}$ induces a connected subgraph of T.

We refer to V_T as the set of *nodes of* T and the sets of $\mathcal{F}$ as the *bags* of $(T, \mathcal{F})$. The *width* of a tree-decomposition $(T, \mathcal{F})$ is $\mathsf{width}(T, \mathcal{F}) = \max\{|X| - 1 \mid X \in \mathcal{F}\}$ and the *tree-width* of a graph

[1]Interestingly, the motivating story of one of the foundational articles on graph searching, authored by Torrence Parsons [47] in 1976, was inspired by an earlier article of Breisch in *Southwestern Cavers Journal* [13] proposing a "speleotopological" approach for the problem of finding an explorer lost in a system of dark caves. It is worth to stress that that setting neglected the natural connectivity requirement.

G is

$$\mathsf{tw}(G) = \min\left\{\mathsf{width}(T,\mathcal{F}) \mid (T,\mathcal{F}) \text{ is a tree-decomposition of} G\right\}. \quad (1)$$

A tree-decomposition $(T,\mathcal{F})$ is *connected* if the following fourth condition is satisfied:

4. there is a node $r \in V_T$ such that for each subtree $T' = (V_{T'}, E_{T'})$ of T that contains r, the set $\bigcup_{t\in V_{T'}} X_t$ is inducing a connected subgraph of G.

Figure 1 illustrates the difference between connected and non-connected tree-decomposition. If, in the above definitions, T is a path instead of a tree, then we define the notion of a *path-decomposition* and the parameter of *pathwidth* which we denote by $\mathsf{pw}(G)$ accordingly.

Observe that none of the tree-decompositions of a disconnected graph is connected. So for now on, we only consider connected graphs. The *connected treewidth* of a (connected) graph is defined over the set of connected tree-decompositions as follows:

$$\mathsf{ctw}(G) = \min\left\{\mathsf{width}(T,\mathcal{F}) \mid (T,\mathcal{F}) \text{ is a connected tree-decomposition of} G\right\}. \quad (2)$$

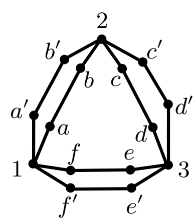

Figure 1: A series-parallel graph G with $\mathsf{tw}(G) = 2$ and $\mathsf{ctw}(G) = 3$. A connected tree-decomposition of minimum width is given the path-decomposition $(P,\mathcal{F})$ where $V(P) = \{x_1, \ldots x_8\}$ and $\mathcal{F} = \{X_1 = \{1,a,b,2\}, X_2 = \{1,a',b',2\}, X_3 = \{1,2,c,d\}, X_4 = \{1,2,d,3\}, X_5 = \{1,2,3,c'\}, X_6 = \{1,3,c',d'\}, X_7 = \{1,3,e,f\}, X_8 = \{1,3,e',f'\}\}$, the root node being x_1. The fact that $\mathsf{ctw}(G) \geq 3$ follows as G can be contracted to the graph W_1 in Figure 8 that is one of the contraction obstructions that we find in this paper.

In the case where T is a path and r is one of its endpoints, we obtain the corresponding notion of a *connected path-decomposition* and the parameter *connected pathwidth* which we denote by $\mathsf{cpw}(G)$ (this is the same as the connected pathwidth given in [18]).

Our first result is that connected treewidth is equal to the monotone, connected, and lazy search number minus one. Our proof (see Section 3) comes together with a second equivalent definition, given in terms of layouts. A layout is *connected* if every prefix induces a connected subgraph. If we apply the standard layout-based definition of treewidth, given in [16] (see also [14]) to *connected layouts*, then we again obtain the monotone, connected, and lazy search number minus one. We stress that both equivalences constitute natural adaptations of known definitions of treewidth to the connected setting. They also provide a useful combinatorial background for further investigations.

Alternative notions of connected treewidth. We now make a short deviation as this is not the first time a "connected" counterpart of treewidth is proposed. We give two alternative definitions below.

- In [30], Fraigniaud and Nisse consider tree-decompositions with the following additional connectivity property: for every edge e of the tree-decomposition $(T, \mathcal{F})$, if T_1 and T_2 are two connected components created after removing e from T, then, for every $i \in \{1, 2\}$, the union of the bags of T_i induce a connected subgraph of G. The main result of [30] was that if we define connected treewidth under this condition, then the resulting parameter is again equal to treewidth. Interestingly, this equality breaks if we transfer this definition to pathwidth. To see this, take the graph $K_{1,3}^{(2)}$ in Figure 2 and observe that $\mathsf{pw}(\mathsf{K}_{1,3}^{(2)}) = 2$ while, under the connectivity restriction of [30], its connected pathwidth is 3.

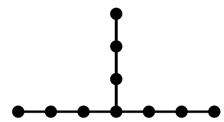

Figure 2: The graph $K_{1,3}^{(2)}$.

- Another way to define connected treewidth is to consider tree decompositions where for every $t \in V_T$, the bag X_t induces a connected subgraph of G. Let us call the variant *bag-connected treewidth*. This definition was introduced independently, in different contexts, by Jégou and Terrioux in [33] and by Diestel and Müller in [20]. The definition of bag-connected treewidth is quite natural and defines a different parameter that has been the subject of several investigations both in theory and in applications.

  The results in [33] used bag-connected tree-decompositions in the context of solving Constraint Satisfaction Problems (CSPs) and they show experimentally that this leads to significant improvements in the resolution of CSPs by decomposition methods. On the other hand, [20] initiated a combinatorial study of bag-connected treewidth and revealed interesting relations with graph-geometric parameters such as the geodesic cycle number, graph hyperbolicity (see also [32]).

  It follows from the definitions that, for connected graphs, the connected treewidth is sandwiched between treewidth and bag-connected treewidth. Moreover the three parameters can be different. Observe that the bag-connected treewidth of the graph G of Figure 1 is 4 while as noticed before $\mathsf{tw}(G) = 2$, $\mathsf{ctw}(G) = 3$.

Contraction obstructions. We say that a graph H is a *contraction* of G, denoted by $H \preceq G$, if a graph isomorphic to H can be obtained from G by a series of *edge contractions* (see definition in Subsection 2.1). We also say that H is a *minor* of G, denoted by $H \leq G$, if H is a contraction of some subgraph of G. We say that a graph class $\mathcal{G}$ is *closed under minors* (*contractions*, respectively) if every minor (contraction, respectively) of a graph in $\mathcal{G}$ belongs to $\mathcal{G}$. We also define the *minor obstructions* (*contraction obstructions*, respectively) of $\mathcal{G}$, denoted by $\mathbf{obs}_{\leq}(\mathcal{G})$ ($\mathbf{obs}_{\preceq}(\mathcal{G})$, respectively), as the set of all minor (contraction, respectively) minimal graphs that do not belong to $\mathcal{G}$. It is easy to see that when $\mathcal{G}$ is minor (contraction, respectively) closed, then $\mathbf{obs}_{\leq}(\mathcal{G})$ ($\mathbf{obs}_{\preceq}(\mathcal{G})$, respectively) provides a *complete* characterization of a minor closed class $\mathcal{G}$: a graph belongs to $\mathcal{G}$ if and only if it excludes all graphs in $\mathbf{obs}_{\leq}(\mathcal{G})$ (respectively $\mathbf{obs}_{\preceq}(\mathcal{G})$) as minors (contractions, respectively). Moreover, in the case of the minor relation, we know from the theorem of Roberston and Seymour [49] (that was the main outcome of the Graph Minors series) that the set $\mathbf{obs}_{\leq}(\mathcal{G})$ is always finite and therefore the aforementioned characterization provides a *finite characterization* of any minor closed class in terms or forbidden minors. To identify (or even to compute) $\mathbf{obs}_{\leq}(\mathcal{G})$ for different instantiations of minor closed graph classes is an interesting topic in graph theory (see [1,42]). For instance, if $\mathcal{T}_k$ is the class of graphs with treewidth at most k, then $\mathbf{obs}_{\leq}(\mathcal{T}_k)$ is known for every $k \leq 3$ [4] and remains unknown for $k > 3$ (see [50] for some partial results for the case where $k = 4$). Similarly, if $\mathcal{P}_k$ is the class of graphs with pathwidth at most k, then $\mathbf{obs}_{\leq}(\mathcal{P}_k)$ is known for $k \leq 2$ [37] and remains unknown for $k > 2$. Bounds for the size of the graphs in $\mathbf{obs}_{\leq}(\mathcal{T}_k)$ and $\mathbf{obs}_{\leq}(\mathcal{P}_k)$ have been proved in [40].

Unfortunately, the landscape is more obscure for the contraction relation as contraction obstruction sets are not finite in general. Contraction obstruction sets are only known for a few contraction-closed classes. For instance, the contraction obstruction set for planar graphs is described in [15]. A more elaborate example of a finite contraction obstruction set was identified in [7], containing 177 graphs, for the class of graphs whose connected mixed search number (for an agile and invisible robber) is at most 2. A class characterized by an infinite set of contraction obstructions is discussed in [34].

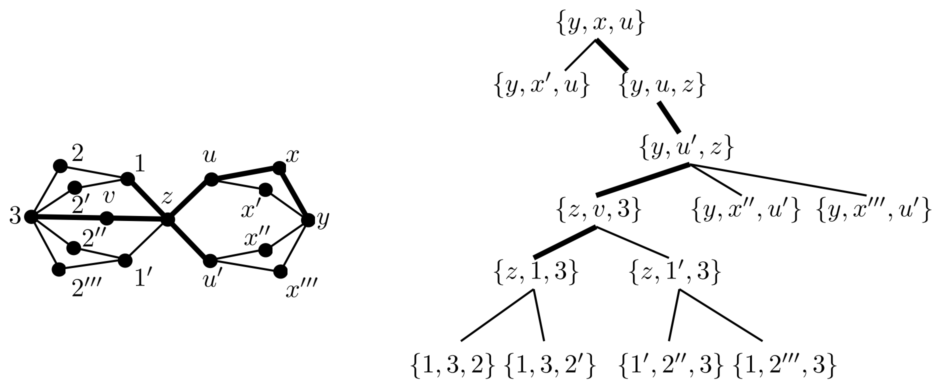

Figure 3: A graph $G \in \mathcal{T}_2^c$ such $G - uv \notin \mathcal{T}_2^c$ and $G - v \notin \mathcal{T}_2^c$. The fact that $G \in \mathcal{T}_2^c$ is certified by the above connected tree decomposition where the bag corresponding to the root is $\{y, x, u\}$. We depict by bold edges in G the connected subgraph induced by the vertices in the bags of the path (also depicted in bold) between the root bag and the bag $\{z, 1, 3\}$. The fact that $G - uv \notin \mathcal{T}_2^c$ and $G - v \notin \mathcal{T}_2^c$ follows from the fact that $G - v \notin \mathcal{T}_2^c$ is a contraction of $G - uv \notin \mathcal{T}_2^c$ and the fact that $G - v \notin \mathcal{T}_2^c$ is one of the contraction obstructions that we find in this paper (in particular, the graph $2 \times \mathbf{Y}_x^{(2)}$ – see Figure 7 for the definition of $\mathbf{Y}_x^{(2)}$ and Subsection 2.1 for the definition of $\times$).

Let $k \in \mathbb{N}$. By $\mathcal{T}_k^c$, we denote the class of all (connected) graphs with connected treewidth at most k. We observe that $\mathcal{T}_2^c$ is not minor closed. Indeed, removing a vertex or an edge (as illustrated by the graph G of Figure 3) may increase the connected treewidth. Therefore, no characterization via minor obstruction exists. However, $\mathcal{T}_k^c$ is contraction-closed, for every k, and it is a challenging problem to identify $\mathbf{obs}_{\preceq}(\mathcal{T}_k^c)$ for distinct values of k, especially since we have no guarantee that this set is finite. Moreover, in case $\mathbf{obs}_{\preceq}(\mathcal{T}_k^c)$ is infinite, we are essentially looking for a finite description of this set.

To warm up, it is easy to observe that $\mathbf{obs}_{\preceq}(\mathcal{T}_1^c) = \{K_3\}$ where K_3 is the complete graph on 3 vertices. As $\mathsf{ctw}(K_3) = 2$, every graph that can be contracted to K_3 has connected treewidth at least 2, while if G is a tree, then it has an obvious tree decomposition of width 2, that is also a connected one. The identification of $\mathbf{obs}_{\preceq}(\mathcal{T}_2^c)$ is a much harder problem. The main result of this paper is a proof that $\mathbf{obs}_{\preceq}(\mathcal{T}_2^c)$ is an infinite set that we completely describe in Section 5. As a preliminary part of our results we prove several properties for the contraction obstructions of every $\mathcal{T}_k^c$ that are later used to identify $\mathbf{obs}_{\preceq}(\mathcal{T}_2^c)$. Additionally to our results we give for every $k \geq 2$ an *infinite* subset of $\mathbf{obs}_{\preceq}(\mathcal{T}_k^c)$ consisting of graphs of treewidth 2 (Section 6). Consequently, the price of connectivity on treewidth is unbounded and this makes a sharp contrast with the corresponding result on pathwidth. To conclude, for monotone search, the price of connectivity is bounded when we are searching for an agile robber while this price goes to infinity when the robber is lazy.

2 Formal definitions of the search game and its variants

2.1 Preliminaries

Numbers and sequences. All numbers in this paper are integers. Given a number n, we define $[n] = \{1, \ldots, n\}$ and, given two numbers n, m where $n \leq m$, we define $[n, m] = \{n, \ldots, m\}$. Given a set U and a collection $\mathcal{U}$ of subsets of U, we set $\bigcup \mathcal{U} = \cup_{U' \in \mathcal{U}} U'$. We also denote by 2^U the set of all subsets of U.

Given a finite set U, a *sequence* σ over U is a bijection $\sigma : U \to [|U|]$. For $x \in U$, $\sigma(x)$ is the position of x in σ. For two elements x and $y \in U$, we write $x <_\sigma y$ if $\sigma(x) < \sigma(y)$. To simplify the notation we define $\sigma_i = \sigma^{-1}(i)$. We also define the sets $\sigma_{<i} = \{x \in U \mid \sigma(x) < i\}$. The subsets $\sigma_{\leqslant i}$, $\sigma_{>i}$ and $\sigma_{\geqslant i}$ are defined similarly and, for $i, j \in [|U|]$, we denote $\sigma_{i,j} = \sigma_{\geq i} \cap \sigma_{\leq j}$. Alternatively, we also denote a sequence by $\sigma = \langle \sigma_1, \ldots, \sigma_n \rangle$. We define the *concatenation* of two sequences σ and σ' over the sets U and U' respectively, as the sequence $\sigma \odot \sigma'$ over $U \cup U'$ such that for $x \in U$, $(\sigma \odot \sigma')(x) = \sigma(x)$ and for $y, z \in U' \setminus U$, $y <_{\sigma \odot \sigma'} z$ if and only if $y <_{\sigma'} z$.

Let $v \in U$ be such that $\sigma(v) = i$. Then we denote by $\sigma \setminus v$ the sequence $\sigma_{<i} \odot \sigma_{>i}$. If $S \subseteq U$, we define $\sigma \setminus S$ as the sequence σ' that is recursively defined as follows: If $|S| = 1$ and $S = \{v\}$, then $\sigma' = \sigma \setminus v$. If $|S| > 2$ and $v \in S$, then $\sigma' = (\sigma \setminus (S \setminus \{v\})) \setminus v$. We finally define $\sigma \cap S$ as the sequence $\sigma \setminus (U \setminus S)$.

Graphs. We consider undirected, simple graphs. Given a graph G, we let $V(G)$ and $E(G)$ denote its vertex and edge set, respectively. We set $|G| = |V(G)|$. We use the shortcut xy for an edge $\{x, y\}$ of G, agreeing that xy and yx denote the same edge. Let $S \subseteq V(G)$ be a vertex subset of G. Then $N_G(S)$ is the set of vertices of G that do not belong to S and are adjacent to some vertex in S. The subgraph induced by S, denoted $G[S]$, has vertex set S and edge set $\{yx \in E \mid x, y \in S\}$. We say that S is *connected in* G if $G[S]$ is connected.

A vertex subset $S \subseteq V(G)$ is a *separator* if $G \setminus S = G[V \setminus S]$ contains more connected components than G. A connected component H of $G \setminus S$ is a *full S-component* of G if $N_G(V(H)) = S$. We denote by $\mathcal{C}(G, S)$ the set of all full S-components of G. We denote by $\mathcal{F}(G, S)$ the set containing every induced subgraph $G[S \cup C]$ with $C \in \mathcal{C}(G, S)$. A separator S is a *minimal separator* if $|\mathcal{F}(G, S)| \geq 2$. A minimal separator S is a *minimal $\langle x, y \rangle$-separator* if x and y belong to different full S-components. A vertex $x \in V(G)$ is a *cut-vertex* if $\{x\}$ is a separator. The set of cut-vertices of a graph G is denoted $C(G)$. A graph G is *biconnected* if it is connected and $C(G) = \emptyset$. A *biconnected component* of a graph is any biconnected subgraph of G that is vertex-maximal. A *bridge* in a graph G is an edge whose removal increases the number of connected components. A *separating edge* is an edge

$xy \in E(G)$ such that $\{x, y\}$ is a minimal separator of G. Observe that a bridge is not a separating edge.

Given a tree T and $x, y \in V(T)$, we denote by xTy the unique path in T that has endpoints x and y.

Rooted graphs. Let q be a non-negative integer. A *q-rooted* graph is a pair $\mathbf{G} = (G, \mathbb{R})$ where G is a graph and $\mathbb{R}$ is a sequence of q pairwise distinct vertices of G. We say that $\mathbf{G}$ is *rooted* on $\mathbb{R}$ and we call G the *underlying graph* of $\mathbf{G}$. We let $V(\mathbb{R})$ denote the set of *roots* of $\mathbf{G}$. We also set $|\mathbb{R}| = |V(\mathbb{R})|$. A *rooted graph* is any q-rooted graph, where $q \geq 0$. The rooted graph $(G, \mathbb{R})$ is *connected* if either G is connected or if every connected component of G contains at least one vertex from $V(\mathbb{R})$. It is *biconnected* if the graph obtained from G by adding an edge between every pair of root vertices (if such an edge does not already exist) is biconnected. The *gluing* of two q-rooted graphs $(G_1, \mathbb{R}_1)$ and $(G_2, \mathbb{R}_2)$ results in the graph $(G_1, \mathbb{R}_1) \oplus (G_2, \mathbb{R}_2)$ obtained by identifying the vertex $\mathbb{R}_1(i)$ with $\mathbb{R}_2(i)$ for every $i \in [q]$. Given a collection $\mathcal{K} = \{\mathbf{K}_1, \ldots, \mathbf{K}_q\}$ of rooted graphs, all rooted on the same tuple $\mathbb{R}$, we recursively define their *union*, denoted as $\oplus\mathcal{K}$ as the rooted graph $(\mathbf{K}_1 \oplus \mathbf{K}', \mathbb{R})$ where $\mathbf{K}'$ is the union of the rooted graphs in $\mathcal{K} \setminus \{\mathbf{K}_1\}$. Observe that the gluing operation yields a graph while the union operation yield a rooted graph. If $\mathbf{K}$ is a rooted graph and $k \geq 2$ is an integer, then we denote $k \times \mathbf{K} = \oplus\mathcal{K}$ where $\mathcal{K}$ is a set of k disjoint copies of $\mathbf{K}$. Given a rooted graph $\mathbf{G} = (G, \mathbb{R})$ and some $x \in V(\mathbb{R})$, we define $\mathbf{G}[x] = (G, \langle x \rangle)$, i.e,. we reroot G to some singleton of its root. Also, we define $\mathbf{G}^+ = (G^+, \mathbb{R})$ where G^+ is the graph obtained from G if we add all possible edges between the vertices in $V(\mathbb{R})$.

We treat every graph G as the 0-rooted graph $(G, \langle\rangle)$. For simplicity we denote each 0-rooted graph $(G, \langle\rangle)$ by its underlying graph G. We say that two q-rooted graphs $(G_1, \mathbb{R}_1)$ and $(G_2, \mathbb{R}_2)$ are *isomorphic* if there is an isomorphism ψ from G_1 to G_2 that sends each vertex of $V(\mathbb{R}_1)$ to the equally indexed vertex of $V(\mathbb{R}_2)$.

Layouts of rooted graphs. A *layout* σ of a rooted graph $\mathbf{G} = (G, \mathbb{R})$ is a sequence over $V(G)$ such that for every $1 \leq j \leq |\mathbb{R}|$, $\sigma^{-1}(j) \in \mathbb{R}$. We say that σ is *connected* if, for every $i \in [|\mathbb{R}|, |G|]$, every connected component of $(G[\sigma_{\leq i}], \mathbb{R})$ contains a root vertex of $\mathbb{R}$. We denote by $\mathcal{L}(\mathbf{G})$ the set of all layouts of $\mathbf{G}$ and by $\mathcal{L}^c(\mathbf{G})$ the set of all connected layouts of $\mathbf{G}$.

2.2 Node search

We now give the formal definition of the node search game and its variants.

Search strategies. Given a graph G, a *search strategy on* G is a sequence $\mathcal{S} = \langle S_1, \ldots, S_r \rangle$, with $r \in \mathbb{N}$, over the set of subsets of vertices of $V(G)$ where

- $|S_1| = 1$.
- For all $i \in [r-1]$, the symmetric difference of S_i and S_{i+1} has cardinality one.

Notice that a search strategy $\mathcal{S}$ indicates a sequence of *moves* of cops on G. Such a move may be either a placement of a cop to a vertex or the removal of a cop from a vertex. To see this, consider consecutive elements S_{i-1} and S_i of $\mathcal{S}$ for some $i \in [r]$. If $S_i \setminus S_{i-1} = \{v\}$, then the corresponding move is the *placement of a cop on vertex v*. If $S_{i-1} \setminus S_i = \{v\}$, then the corresponding move is the *removal of a cop from vertex v*. The *cost* of a search strategy $\mathcal{S}$ is $\max\{|S_i| \mid i \in [r]\}$ and is denoted by $\mathsf{costcost}(\mathcal{S})$.

Let $\mathcal{S} = \langle S_1, \ldots, S_r \rangle$ be a search strategy on G. We define the *sequence of robber spaces* of an *agile* robber with respect to $\mathcal{S}$ as the sequence $\mathcal{F}^{(\mathsf{a})}_{\mathcal{S}} = \langle F_1, \ldots, F_r \rangle$ where

- $F_1 = V(G) \setminus S_1$.
- For $i \in [2, r]$, let $F_i = (F_{i-1} - S_i) \cup \{v \in V - S_i :$ there is a path from a vertex $u \in F_{i-1}$ to v whose vertices except u belong to $V - S_i\}$.

Similarly, we define the *sequence of robber spaces* of a *lazy* robber with respect to $\mathcal{S}$ as the sequence $\mathcal{F}^{(\mathsf{l})}_{\mathcal{S}} = \langle F_1, \ldots, F_r \rangle$ where

- $F_1 = V(G) \backslash S_1$.
- For $i \in [2, r]$, let $F_i = (F_{i-1} - S_i) \cup \{v \in V - S_i :$ there is a path from a vertex $u \in F_{i-1} \cap (S_i - S_{i-1})$ to v whose vertices except u belong to $V - S_i\}$.

Properties of search strategies. Given the type $\mathsf{t} \in \{\mathsf{a}, \mathsf{l}\}$ of the robber (agile or lazy) and a search strategy $\mathcal{S}$ on G, where $\mathcal{F}^{(\mathsf{t})}_{\mathcal{S}} = \langle F_1, \ldots, F_r \rangle$, we define by $\bar{\mathcal{F}}^{(\mathsf{t})}_{\mathcal{S}} = \langle \bar{F}_1, \ldots, \bar{F}_r \rangle$ as the set of *clean territories*, where $\bar{F}_i = V(G) \setminus F_i, i \in [1, r]$. Notice that for every $i \in [2, r]$, $|\bar{F}_i \setminus \bar{F}_{i-1}| \leq 1$. We say that the search strategy $\mathcal{S}$ is

- *complete*, if $F_r = \emptyset$ (or, alternatively $\bar{F}_r = V(G)$),
- *connected*, if for each $i \in [r]$, $\bar{F}_i$ is connected in G, and
- *monotone*, if for each $i \in [r-1]$, $F_{i+1} \subseteq F_i$.

Observe that connected strategies only exist if the graph is connected. We define

$$\begin{aligned} \mathsf{ans}(G) &= \min\{\mathsf{cost}(\mathcal{S}) \mid \mathcal{S} \text{ is a complete search strategy on } G \text{ against an agile robber}\} \text{ and} \\ \mathsf{lns}(G) &= \min\{\mathsf{cost}(\mathcal{S}) \mid \mathcal{S} \text{ is a complete search strategy on } G \text{ against a lazy robber}\}. \end{aligned}$$

If in the above definitions we consider only monotone strategies, then we obtain the parameters mans and mlns. In the context of connected graphs, restricting to connected searches yields to the definition of the cans, clns, mcans and mclns parameters.

Search strategies of rooted graphs. A *search strategy* on a rooted graph $\mathbf{G} = (G, \mathbb{R})$ is defined as a search strategy on G with the difference that $S_1 = V(\mathbb{R})$ (i.e,. the cops first occupy the root vertices). The set of robber space and clean territories ($\mathcal{F}^{(\mathsf{t})}_{\mathcal{S}}$ and $\bar{\mathcal{F}}^{(\mathsf{t})}_{\mathcal{S}}$, for $\mathsf{t} \in \{\mathsf{a}, \mathsf{l}\}$) is defined as in the case of unrooted graphs with the difference that now $F_1 = V(\mathbb{R})$. The notions of monotonicity, connectivity, and completeness of a search strategy are defined as in the case of unrooted graphs (where connectivity is interpreted as connectivity of rooted graph).

2.3 Vertex separations.

Supporting sets. Let σ be a layout of G on n vertices. For every $i \in [n]$, we define the *tree-supporting set* of position i as

$$S_\sigma^{(\mathsf{t})}(i) = \{x \in V(G) \mid \sigma(x) < i \text{ and there exists a } (x,\sigma_i)\text{-path whose internal vertices belong to } \sigma_{>i}\}$$

We also define the *path-supporting set* of position i as

$$S_\sigma^{(\mathsf{p})}(i) = N_G(\sigma_{\geq i}).$$

Notice that $S_\sigma^{(\mathsf{p})}(i)$ contains all vertices in $\sigma_{<i}$ with a neighbour in $\sigma_{\geqslant i}$ and that $S_\sigma^{(\mathsf{t})}(i) \subseteq S_\sigma^{(\mathsf{p})}(i)$.

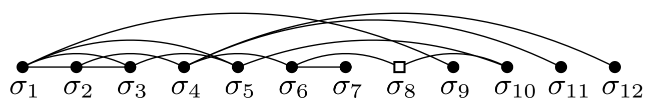

Figure 4: A connected layout σ of a graph G. Notice that $S_\sigma^{\mathsf{p}}(8) = \{\sigma_1, \sigma_4, \sigma_5, \sigma_6\}$ while $S_\sigma^{\mathsf{t}}(8) = \{\sigma_6, \sigma_5\}$. Also, $\mathsf{pcost}(G,\sigma) = 4$ and $\mathsf{tcost}(G,\sigma) = 3$.

We define $\mathsf{tcost}(G,\sigma) = \max\{|S_\sigma^{(\mathsf{t})}(i)| \mid i \in [n]\}$ and $\mathsf{pcost}(G,\sigma) = \max\{|S_\sigma^{(\mathsf{p})}(i)| \mid i \in [n]\}$ (see Figure 4 for an example). We also define the *tree vertex separation number* of G and the *path vertex separation number* of G as

$$\mathsf{tvs}(G) = \min\{\mathsf{tcost}(G,\sigma) \mid \sigma \in \mathcal{L}(G)\} \text{ and} \tag{3}$$

$$\mathsf{pvs}(G) = \min\{\mathsf{pcost}(G,\sigma) \mid \sigma \in \mathcal{L}(G)\} \tag{4}$$

respectively[2]. It is known that, for every graph G, $\mathsf{tvs}(G) = \mathsf{mlns}(G) = \mathsf{lns}(G) = \mathsf{tw}(G)$ [16,51] and $\mathsf{pvs}(G) = \mathsf{mans}(G) = \mathsf{ans}(G) = \mathsf{pw}(G)$ [9,36,38,39,41,44].

In the context of connected graphs, if we restrict in (3) and (4) the set of layouts to be connected (i.e,. we replace $\mathcal{L}(G)$ by $\mathcal{L}^c(G)$), then the defined parameters are the *connected tree vertex separation number* of G and the *connected path vertex separation number* of G denoted $\mathsf{ctvs}(G)$ and $\mathsf{cpvs}(G)$ respectively.

All the definitions of this subsection are extended to rooted graphs accordingly to the definition of rooted layout and connectivity of rooted graphs. For this, we replace G by $(G,\mathbb{R})$ and we define $\mathsf{tvs}(G,\mathbb{R})$, $\mathsf{pvs}(G,\mathbb{R})$, $\mathsf{ctvs}(G,\mathbb{R})$, and $\mathsf{cpvs}(G,\mathbb{R})$.

Contractions. Let G be a graph. *Contracting* an edge $e = xy \in E(G)$ yields the graph G/e obtained by removing x and y from G, introduce a new vertex and make it adjacent with all vertices in $N_G(\{x,y\}) \setminus \{x,y\}$. Notice that contraction of an edge does not create multiple edges and the resulting graph remains simple. If e is incident to a degree-2 vertex x, then contracting e is equivalent to *dissolving* x, that is removing x from the vertex set and adding an edge between

[2] In the literature [36] the path vertex separation number is known as the «vertex separation number». This alternative term is adopted in this paper in order to make clear the distinction with the concept of «tree separation number».

its two neighbours. If F is a subset of edges of G, then G/F is the graph obtained by contracting the edges of F. Observe that the order of the contraction does not matter. We present below an extension of contraction to rooted graphs.

Given two q-rooted graphs $(H, \mathbb{T})$ and $(G, \mathbb{R})$, where $\mathbb{T} = \langle t_1, \ldots, t_q \rangle$ and $\mathbb{R} = \langle r_1, \ldots, r_q \rangle$, and a surjection $\rho : V(G) \to V(H)$, we say that $(H, \mathbb{T})$ is a *ρ-contraction* of $(G, \mathbb{R})$, denoted by $(H, \mathbb{T}) \preceq_\rho (G, \mathbb{R})$, if

- $\forall x \in V(H)$, the subgraph $G[\rho^{-1}(x)]$ is connected.
- $\forall x, y \in V(H)$, we have $xy \in E(H)$ iff the subgraph $G[\rho^{-1}(x) \cup \rho^{-1}(y)]$ is connected.
- $\forall i \in [q]$, $r_i \in \rho^{-1}(t_i)$.

We say that $(H, \mathbb{T})$ is a *contraction* of $(G, \mathbb{R})$, denoted by $(H, \mathbb{T}) \preceq (G, \mathbb{R})$, if $(H, \mathbb{T}) \preceq_\rho (G, \mathbb{R})$ for some surjection $\rho : V(G) \to V(H)$. We say that a graph G_1 is a contraction of a graph G_2 if $(G_1, \varnothing) \preceq (G_2, \varnothing)$. Notice that $(H, \mathbb{T}) \preceq (G, \mathbb{R})$ holds if a rooted graph isomorphic to $(H, \mathbb{T})$ can be obtained after a series of edge contractions on G, under the constraint that no path between two vertices of $V(\mathbb{R})$ can be contracted to a single vertex. We say that $(H, \mathbb{T})$ is a *proper* contraction of $(G, \mathbb{R})$ if $(H, \mathbb{T}) \preceq (G, \mathbb{R})$ and $(H, \mathbb{T})$ is not isomorphic to $(G, \mathbb{R})$. For an edge e incident to at least one non-root vertex, we let $\mathbf{G}/e$ denote the rooted graph obtained by contracting the edge e. Similarly, if F is a set of edges not containing the edge set of a path between to root vertices, then $\mathbf{G}/F$ is the rooted graph resulting from the contraction of the edges in F. Observe that if the vertex $v \in V(H)$ results from the contraction of an edge incident to a root vertex of $\mathbb{R}$, then v is a root vertex of $\mathbb{T}$.

Let $k \geq 0$. Given a set $\mathcal{C}$ of k-rooted graphs and a k-rooted graph $\mathbf{G}$, we say that $\mathbf{G}$ is $\mathcal{C}$-free if for every $\mathbf{H} \in \mathcal{C}$, $\mathbf{H} \not\preceq \mathbf{G}$. If $\mathcal{C} = \{\mathbf{H}\}$ we say that $\mathbf{G}$ is $\mathbf{H}$-*free* to denote that G is $\{\mathbf{H}\}$-free.

Lemma 1. *Let $(G_1, \mathbb{R}_1)$ and $(G_2, \mathbb{R}_2)$ be two q-rooted graphs such that $(G_1, \mathbb{R}_1) \preceq (G_2, \mathbb{R}_2)$. Then* $\mathsf{ctvs}(G_1, \mathbb{R}_1) \leq \mathsf{ctvs}(G_2, \mathbb{R}_2)$.

Proof. Let n be the number of vertices of G_2. Let $e = xy$ be an edge of G_2 such that $x \notin V(\mathbb{R}_2)$ or $y \notin V(\mathbb{R}_2)$. We denote $(G', \mathbb{R}) = \mathbf{G_2}/e$. Let us consider an arbitrary connected layout $\sigma \in \mathcal{L}^c(G_2, \mathbb{R}_2)$. Let v_e be the vertex resulting from the contraction of e and let $i = \sigma(x)$ and $j = \sigma(y)$, where $i < j$. We construct the following layout σ' of G': for $h < i$, set $\sigma'_h = \sigma_h$; set $\sigma'_i = v_e$; for $i < h < j$, set $\sigma'_h = \sigma_h$; for $j < h$, set $\sigma'_{h-1} = \sigma_h$. As contracting an edge does not disconnect a graph, the fact that σ is connected implies that every subgraph induced by a prefix of σ' is connected and henceforth, $\sigma' \in \mathcal{L}^c(G', \mathbb{R})$

We claim that $\mathsf{tcost}(G', \mathbb{R}, \sigma') \leqslant \mathsf{tcost}(G, \mathbb{R}, \sigma)$. To see this, observe that for $0 < h \leqslant i$, $S^{(t)}_\sigma(h) = S^{(t)}_{\sigma'}(h)$. For $i < h < j$, if $v_e \in S^{(t)}_{\sigma'}(h)$, then $x \in S^{(t)}_\sigma(h)$. As $S^{(t)}_{\sigma'}(h) \setminus \{v_e\} \subseteq S^{(t)}_\sigma(h) \setminus \{x\}$, it follows that $|S^{(t)}_{\sigma'}(h)| \leq |S^{(t)}_\sigma(h)|$. Similarly, for $j \leq h \leq n-1$, if $v_e \in S^{(t)}_{\sigma'}(h)$, then either $x \in S^{(t)}_\sigma(h+1)$ or $y \in S^{(t)}_\sigma(h+1)$. As $S^{(t)}_\sigma(h+1) \setminus \{x, y\} = S^{(t)}_{\sigma'}(h) \setminus \{v_e\}$, we have that $|S^{(t)}_{\sigma'}(h)| \leqslant |S^{(t)}_\sigma(h+1)|$. $\square$

3 Equivalence of parameters

In this section, we prove the following theorem.

Theorem 1. *For every connected graph G, it holds that* $\mathsf{ctw}(G) = \mathsf{ctvs}(G) = \mathsf{mclns}(G) - 1$.

The proof is a consequence of the next three lemmata.

Lemma 2. *Let G be a connected graph. Then* $\mathsf{ctvs}(G) + 1 \leq \mathsf{mclns}(G)$.

Proof. Let n be the number of vertices of G. Let $\mathcal{S} = \langle S_1, \ldots, S_r \rangle$ be a connected, monotone, and complete strategy against a lazy robber certifying that $\mathsf{mclns}(G) \leq k+1$. We associate with $\mathcal{S}$ the layout σ of G defined as the ordering in which the vertices are first occupied by a cop. More formally, consider the sequence $\mathcal{F}^{(\mathsf{l})}_{\mathcal{S}} = \langle F_1, \ldots, F_r \rangle$ of rober spaces. We let j_i for $i \in [n]$ be the sequence of indices such that $j_1 = 1$ and for $i > 1$, $j_i \in [r]$ is the i-th index such that $|F_{j_i - 1} \setminus F_{j_i}| = 1$. As $\mathcal{S}$ is complete and monotone, the sequence of indexes j_i for $i \in [n]$, is well-defined. Then σ_i, for $i \in [n]$, is the unique vertex in $F_{j_i - 1} \setminus F_{j_i} = S_{j_i} \setminus S_{j_i - 1}$. Observe that, for every $i \in [n]$, as $\mathcal{S}$ is monotone, $\bar{F}_{j_i} = \sigma_{\leqslant i}$, and since $\mathcal{S}$ is connected, $\bar{F}_{j_i} = \sigma_{\leq i}$ is connected in G. This implies that every prefix of σ induces a connected subgraph of G, and thereby σ is a connected layout of G.

We now prove that $\mathsf{tcost}(G, \sigma) \leq k$ or, equivalently, that, for every $i \in [n]$, the tree-supporting set $S^{(\mathsf{t})}_{\sigma}(i)$ has size at most k. By definition of a tree-supporting set, $S^{(\mathsf{t})}_{\sigma}(i) \subseteq \sigma_{<i}$ and for every $y \in S^{(\mathsf{t})}_{\sigma}(i)$, G contains a (σ_i, y)-path whose internal vertices belong to $\sigma_{>i} = F_{j_i}$. It follows that vertices of $S^{(\mathsf{t})}_{\sigma}(i)$ have to be occupied by a cop before a new cop is placed on σ_i. This implies that $S^{(\mathsf{t})}_{\sigma}(i) \subseteq S_{j_i - 1}$. As $|S_{j_i}| = |S_{j_i - 1}| + 1$ and $|S_{j_i}| \leq k+1$, we have that $|S_{j_i - 1}| \leq k$, therefore $|S^{(\mathsf{t})}_{\sigma}(i)| \leq k$, as required. □

Lemma 3. *For every connected graph G,* $\mathsf{ctw}(G) \leq \mathsf{ctvs}(G)$.

Proof. Let n be the number of vertices of G. Suppose without loss of generality that $n > 1$. Let $\sigma \in \mathcal{L}^c(G)$ be a connected layout such that $\mathsf{tcost}(G, \sigma) \leq k$. For $i \in [n]$, we set $B_i = S^{(\mathsf{t})}_{\sigma}(i) \cup \{\sigma_i\}$. Consider the graph $\hat{G}$ obtained from G by completing, for every $i \in [n]$, B_i into a clique. Observe that by construction of $\hat{G}$, σ is a connected layout of $\hat{G}$ such that $\mathsf{tcost}(\hat{G}, \sigma) \leq k$. For $i \in [n]$, we set $\hat{G}_i = \hat{G}[\sigma_{\leq i}]$ and $G_i = G[\sigma_{\leq i}]$.

To prove the statement, we show by induction on $i \in [n]$ that $\hat{G}_i$ has a tree decomposition $(T_i, \mathcal{F}_i)$ of width at most k that is also a connected tree-decomposition of G_i. We assume that for some $i \in [n-1]$, $(T_i, \mathcal{F}_i)$ is a tree-decomposition of $\hat{G}_i$ such that:

(1) $V(T_i) = [i]$,

(2) $\mathcal{F}_i = \{B_j \mid j \in [i]\}$,

(3) $\forall j \in [i]$, $\sigma_j \in B_j$

(4) for every subtree T' of T_i containing node 1, $\cup_{t \in V_{T'}} X_t$ induces a connected subgraph of G.

Clearly by construction, if Conditions (1)–(4) hold, then $\mathsf{width}(T_i, \mathcal{F}_i) \leq k$. For the induction base, the four conditions trivially hold for $i = 1$, by setting $T_1 = (\{1\}, \emptyset)$, and $\mathcal{F}_1 = \{\{\sigma_1\}\}$.

Notice that $B_{i+1} \setminus \{\sigma_{i+1}\} \subseteq \sigma_{\leq i}$ and that B_{i+1} induces a clique in $\hat{G}_{i+1}$. This implies that $S^{(\mathsf{t})}_{\sigma}(i) = B_{i+1} \setminus \{\sigma_{i+1}\}$ induces a clique in $\hat{G}_i$. So T_i contains a node $h \in [i]$ such that $S^{(\mathsf{t})}_{\sigma}(i) \subseteq B_h$ (see e.g,. Lemma 4 in [12]). We construct a tree-decomposition $(T_{i+1}, \mathcal{F}_{i+1})$ of $\hat{G}_{i+1}$ as follows: we set $V(T_{i+1}) = V(T_i) \cup \{i+1\} = [i+1]$, $E(T_{i+1}) = E(T_i) \cup \{\{h, i+1\}\}$ and $\mathcal{F}_i = \mathcal{F}_{i-1} \cup \{B_{i+1}\}$. It is easy to verify that $(T_{i+1}, \mathcal{F}_{i+1})$ is a tree-decomposition of $\hat{G}_{i+1}$ that satisfies Conditions (1)–(3) when we replace i by $i+1$.

Let us now prove that $(T_{i+1}, \mathcal{F}_{i+1})$ satisfies (4). For every $j \in [j+1]$, let P_j denote the unique path in T_{i+1} between nodes 1 and j and set $V^j = \cup_{t \in V_{P_j}} X_t$. Observe that by the induction hypothesis, if V^h induces a connected subgraph of G, then for every subtree T' of T_{i+1} containing node 1, $\cup_{t \in V_{T'}} X_t$ induces a connected subgraph of G. To see this, recall that σ is a connected layout of G and therefore σ_{i+1} has some neighbour, say x, that belongs to $\sigma_{\leq i}$. Clearly $x \in S_\sigma^{(\mathsf{t})}(i) \subseteq B_h$. As P_h is a subtree of T_i, by induction hypothesis, V^h induces a connected subgraph of G. Since $B_{j+1} = B_h \cup \{\sigma_{j+1}\}$, it follows that V^{i+1} induces a connected subgraph of G. □

Lemma 4. *For every connected graph G, $\mathsf{mclns}(G) \leq \mathsf{ctw}(G) + 1$.*

Proof. Let $(T, \mathcal{F})$ with $\mathcal{F} = \{X_t \mid T \in V(T)\}$ be a connected tree-decomposition of G such that $\mathsf{width}(T, \mathcal{F}) \leq k$. We set $q = |V(T)|$. As a connected-tree decomposition T has a root node r such that

$$\forall \text{ subtree } T' \text{ of } T \text{ containing } r, G[\cup\{X_j \mid j \in V(T')\}] \text{ is connected.} \tag{5}$$

Let us consider a connected layout σ of the tree T where $\sigma_1 = r$. For every $i \in [q]$, we set $l_i = |X_i|$, $T_i = T[\sigma_{\leq i}]$, $V_i = \bigcup_{j \in [i]} X_{\sigma_i}$. Observe that because of (5), for every $i \in [q]$, the subgraph $G_i = G[V_i]$ is connected.

We define the following three operations that generate a sequence of sets:

- $\mathsf{remove}(Y, Y')$, defined for two subsets Y and Y' of vertices such that $Y' \subseteq Y$ with $Y \setminus Y' = \{y_1, \dots y_q\}$, returns the sequence of subsets $\langle Y \setminus \{y_1\}, Y \setminus \{y_1, y_2\}, \dots, Y' \rangle$;
- $\mathsf{re\text{-}position}(Y', Y)$, defined for two subsets Y and Y' of vertices such that $Y' \subseteq Y$ with $Y \setminus Y' = \{y_1, \dots y_q\}$, returns the sequence of subsets $\langle Y' \cup \{y_1\}, Y' \cup \{y_1, y_2\}, \dots, Y \rangle$;
- $\mathsf{place}(G, \mathbb{R})$, with $(G, \mathbb{R})$ a n vertex connected rooted graph with $|\mathbb{R}| = q$, returns the sequence of subsets $\langle \delta_{\leq q+1}, \delta_{\leq q+2}, \dots \delta_{\leq n} \rangle$ where δ is an arbitrary connected layout of $(G, \mathbb{R})$.

The proof goes by induction on $i \in [q]$, proving that G_i has a search strategy $\mathcal{S}_i = \langle S_{1_1}, \dots, S_{i_{l_i}} \rangle$ against a lazy robber such that $\mathsf{tcost}(\mathcal{S}_i) \leq k + 1$ and

(1) $\mathcal{S}_i$ is monotone and connected,

(2) $S_{i_{\ell_i}} = X_{\sigma_i}$, and

(3) $\bar{F}_{i_{\ell_i}} = V_i$ (it is complete for G_i).

For, $i = 1$, consider an arbitrary vertex $x \in X_r = V_1$. Observe that $(G_1, \langle x \rangle)$ is a connected rooted graph and that $|V_1| \leq k + 1$. Then $\mathcal{S}_1 = \langle \{x\} \rangle \odot \mathsf{place}(G_r, \langle x \rangle)$ is a search strategy of G_1 satisfying the above conditions.

Assume now that for some $i \in [q-1]$, $\mathcal{S}_i = \langle S_{1_1}, \dots, S_{i_{\ell_i}} \rangle$ is a search strategy of G_i against a lazy robber satisfying Conditions (1)–(3) such that $\mathsf{tcost}(\mathcal{S}_i) \leq k + 1$. We extend $\mathcal{S}_i$ to a search strategy $\mathcal{S}_{i+1} = \langle S_1, \dots, S_{i+1_{\ell_{i+1}}} \rangle$ of G_{i+1} against a lazy robber such that $\mathsf{tcost}(\mathcal{S}_{i+1}) \leq k + 1$ and Conditions (1)–(3) hold if we replace i by $i + 1$.

For this, notice first that σ_{i+1} is a leaf of T_{i+1} and let σ_h be its unique neighbor in T_{i+1}. Let $\mathbf{B}_{i+1}$ be the rooted graph $(G[X_{\sigma_{i+1}}], \mathbb{R}_{i+1})$ where $\mathbb{R}_{i+1}$ is any ordering of $X_{\sigma_h} \cap X_{\sigma_{i+1}}$. Observe that as G_{i+1} is connected, $\mathbf{B}_{i+1}$ is connected as well. We construct the following search strategy of G_{i+1}:

$$\mathcal{S}_{i+1} = \mathcal{S}_i \odot \mathsf{remove}(X_{\sigma_i}, X_{\sigma_i} \cap X_{\sigma_{i+1}}) \odot \mathsf{re\text{-}position}(X_{\sigma_i} \cap X_{\sigma_{i+1}}, X_{\sigma_h} \cap X_{\sigma_{i+1}}) \odot \mathsf{place}(G[X_{\sigma_{i+1}}], \mathbb{R}_{i+1}).$$

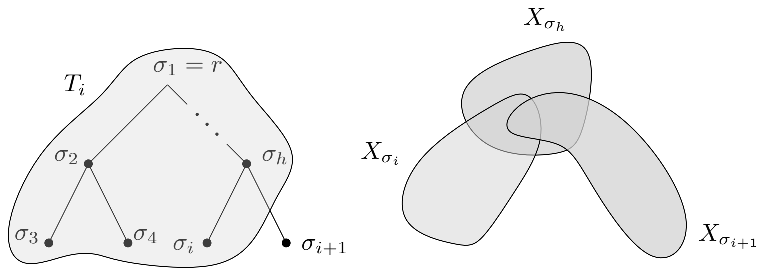

Figure 5: The sets $X_{\sigma_i}, X_{\sigma_h}$, and $X_{\sigma_{i+1}}$ in the proof of Lemma 4.

We set $j_1 = |X_{\sigma_i} \setminus X_{\sigma_{i+1}}|$, $j_2 = |(X_{\sigma_h} \cap X_{\sigma_{i+1}}) \setminus X_{\sigma_i}|$, $j_3 = |X_{\sigma_{i+1}} \setminus X_{\sigma_h}|$ and $\ell_{i+1} = j_1 + j_2 + j_3$ (see Figure 5 for a visualization of the sets $X_{\sigma_i}, X_{\sigma_h}$, and $X_{\sigma_{i+1}}$). Observe that the last subset of vertices of $\mathcal{S}_i$ is $S_{i_{\ell_i}} = X_{\sigma_i}$ and the clean territory is $\bar{F}_{i_{\ell_i}} = V_i$. The next j_1 moves consists in removing cops from $X_{\sigma_i} \setminus X_{\sigma_{i+1}}$ one by one in an arbitrary order. Observe then that the current set of cops is $X_{\sigma_i} \cap X_{\sigma_{i+1}} \subseteq X_{\sigma_h} \cap X_{\sigma_{i+1}}$. If some vertices of $X_{\sigma_h} \cap X_{\sigma_{i+1}}$ are not currently occupied by a cop, then the next j_2 moves consists in placing them back one by one in an arbitrary order. Observe that, by the laziness of the robber, after each of theses $j_1 + j_2$ moves, the clean territory remains unchanged, that is for every $j \in [i_{\ell_i}, i_{\ell_i} + j_1 + j_2]$, $\bar{F}_j = V_i$. Now that $S_{i_{\ell_i}+j_1+j_2} = X_{\sigma_h} \cap X_{\sigma_{i+1}}$, we can complete the search strategy $\mathcal{S}_{i+1}$ by adding the cops of $X_{\sigma_{i+1}} \setminus X_{\sigma_h}$ one by one following a connected search strategy of the connected rooted graph $\mathbf{B}_{i+1}$.

By construction $\mathcal{S}_{i+1}$ is a complete search of G_{i+1} such that $S_{i+1_{\ell_{i+1}}} = X_{i+1}$. To prove the monotonicity of $\mathcal{S}_{i+1}$, it suffices to observe that $X_{\sigma_h} \cap X_{\sigma_{i+1}}$ is a separator between every vertex of $X_{\sigma_{i+1}} \setminus V_i$ and every vertex of $V_i \setminus X_{\sigma_{i+1}}$. As $S_{i_{\ell_i}+j_1+j_2} = X_{\sigma_h} \cap X_{\sigma_{i+1}}$ and $F_{i_{\ell_i}+j_1+j_2} = X_{\sigma_{i+1}} \setminus V_i$, cops can be safely placed on vertices of $X_{\sigma_{i+1}} \setminus V_i$. The fact that $\mathcal{S}_{i+1}$ is connected follows from the fact that $\mathcal{S}_i$ is connected and that vertices of $X_{\sigma_{i+1}} \setminus V_i$ are searched with respect to a connected layout δ of $(G[X_{\sigma_{i+1}}], \mathbb{R}_{i+1})$. Finally as $\mathsf{width}(T, \mathcal{F}) \leq k$, we have that $|X_{\sigma_{i+1}}| \leqslant k+1$. Moreover for every $S_j \in \mathcal{S}_{i+1}$ with $j \in [i_{\ell_i}, i+1_{\ell_{i+1}}]$, we have that $|S_j| \leq \max\{|X_{\sigma_i}|, |X_{\sigma_{i+1}}|\}$. As $\mathsf{tcost}(\mathcal{S}_i) \leq k+1$, we also have that $\mathsf{tcost}(\mathcal{S}_{i+1}) \leq k+1$. □

We stress that if in the proofs of the above three lemmata, we use connected path decompositions instead of connected tree decompositions, we obtain the following counterpart of Theorem 1.

Theorem 2. *For every connected graph G, it holds that* $\mathsf{cpw}(G) = \mathsf{cpvs}(G) = \mathsf{mcans}(G) - 1$.

4 General properties of obstructions

We let denote $\mathcal{O}_k = \mathbf{obs}_{\preceq}(\mathcal{T}_k^c)$. Recall that $\mathcal{O}_k$ contains every graph G where $\mathsf{ctvs}(G) > k$ and where for every *proper* contraction H of G it holds that $\mathsf{ctvs}(H) \leq k$. A graph belonging to $\mathcal{O}_k$ is called an *obstruction* for ctvs at most k. We extend this definition of obstruction to rooted graphs: for every $q \geq 1$, we define $\mathcal{O}_k^{(q)}$ as the set containing every q-rooted graph $\mathbf{G} = (G, \mathbb{R})$, where $\mathsf{ctvs}(\mathbf{G}) > k$ and for every proper contraction G' of G, $\mathsf{ctvs}(G', \mathbb{R}') \leq k$.

4.1 General structure of $\mathcal{O}_k$

Let $x \in C(G)$ be a cut-vertex of G. The pair (G, x) is called an *s-pair*. If $Z \in \mathcal{F}(G, \{x\})$, then the 1-rooted graph $(Z, \langle x \rangle)$ is a 1-*component* of the s-pair (G, x). Given a vertex subset $U \subseteq V(G) \setminus \{x\}$, the 1-component $(Z, \langle x \rangle)$ of the s-pair (G, x) is U-*avoiding* if $U \cap Z = \emptyset$. We make the convention that when $U = \{u\}$, then we say u-avoiding instead of U-avoiding.

Similarly, if $\{x, y\}$ is a minimal separator of G, then the triple (G, x, y) is called an *s-triple*. A 2-rooted graph $(H, \langle x, y \rangle)$ is a 2-*component* of the s-triple (G, x, y) if $\{x, y\}$ is a minimal separator of G and $H \in \mathcal{F}(G, \{x, y\})$ (notice that H is not necessarily 2-connected.). Given a vertex set $U \subseteq V(G) \setminus \{x, y\}$, we say that a 2-component $(H, \langle x, y \rangle)$ of (G, x, y) is U-*avoiding* if $U \cap V(H) = \emptyset$.

In the rest of this paper, all profs are written using the layout definition of connected treewidth. However, some times the reader may find it more intuitive (while less formal) to translate layouts to searching strategies (an equivalence that is formally proved in Section 3). In this sense, it is helpful to see the root vertices of s-pairs and s-triples as «departure points» of connected search strategies restricted to the corresponding subgraphs.

A vertex v of a graph G is called k-*simplicial* if it has degree at most k and its neighborhood induces a complete subgraph.

Lemma 5. *If a connected graph G contains a k-simplicial vertex, then $G \notin \mathcal{O}_k$.*

Proof. Suppose to the contrary that $G \in \mathcal{O}_k$ contains a k-simplicial vertex v. Observe that $G^- = G \setminus v$ is a proper contraction of G and thereby $\mathsf{ctvs}(G') \leqslant k$. Let $\sigma' \in \mathcal{L}^c(G^-)$ such that $\mathsf{tcost}(G^-, \sigma') \leq k$. We define $\sigma = \sigma' \odot \langle v \rangle$ and we observe that $\sigma \in \mathcal{L}^c(G)$ and $\mathsf{tcost}(G, \sigma) \leq k$, a contradiction to the fact that $G \in \mathcal{O}_k$. □

Lemma 6. *Let v be a cut-vertex of a connected graph G. If $\mathsf{ctvs}(G) \leq k$ for some $k \in \mathbb{N}$, then there exists $C^* \in \mathcal{F}(G, \{v\})$ such that for every $C \in \mathcal{F}(G, \{v\}) \setminus \{C^*\}$, $\mathsf{ctvs}(C, \{v\}) \leq k$.*

Proof. Let $\sigma \in \mathcal{L}^c(G)$ such that $\mathsf{tcost}(G, \sigma) \leq k$. We choose $C^* \in \mathcal{F}(G, \{v\})$ such that $\sigma(1) \in V(C^*)$ and we set $\{C_1, \ldots, C_r\} := \mathcal{F}(G, \{v\}) \setminus \{C^*\}$ (clearly, $r \geq 1$). We also set $\sigma^{(i)} := \sigma \cap V(C_i), i \in [r]$. Observe that, as σ is connected, each $\sigma^{(i)}$ is a layout of $\mathcal{L}^c(C_i)$ which first vertex is v. Therefore $\mathsf{ctvs}(C_i, \{v\}) \leq k$, for each $i \in [r]$. □

Lemma 7. *Let G be a connected graph. If $G \in \mathcal{O}_k$ and contains a cut-vertex v, then the s-pair (G, v) contains exactly two 1-components and v is the unique cut-vertex of G.*

Proof. Let v be a cut-vertex of G. Suppose to the contrary that $|\mathcal{F}(G, \{v\})| \geq 3$ and let us consider three subgraphs $G_0, G_1, G_2 \in \mathcal{F}(G, \{v\})$ with $G_i = G[C_i \cup \{v\}]$ for distinct $C_i \in \mathcal{C}(G, \{v\})$ $(0 \leq i \leq 2)$. From Lemma 1 and the fact that $G \in \mathcal{O}_k$,

$$\forall i \in \{0, 1, 2\},\ \mathsf{ctvs}(G[C_i \cup C_{(i+1) \bmod 3} \cup \{v\}]) \leqslant k.$$

Lemma 6 along with above inequality imply that, for every $i \in \{0, 1, 2\}$, $\mathsf{ctvs}(G_i, \{v\}) \leq k$ or $\mathsf{ctvs}(G_{(i+1) \bmod 3}, \{v\}) \leq k$. Combining these disjonctions implies that for some $0 \leq i \leq 2$, $\mathsf{ctvs}(G_i, \{v\}) \leq k$ and $\mathsf{ctvs}(G_{(i+1) \bmod 3}, \{v\}) \leq k$. W.l.o.g,. we assume that this holds when $i = 1$. So for $j = 1, 2$, there exists $\sigma^{(j)} \in \mathcal{L}^c(G_j)$ such that $\sigma^{(j)}(v) = 1$, $\mathsf{tcost}(G_j, \sigma^{(j)}) \leq k$. As the induced subgraph $G' = G[V(G) \setminus (C_1 \cup C_2)]$, is a proper contraction of G, by Lemma 1 there is a layout $\sigma' \in \mathcal{L}^c(G')$ such that $\mathsf{tcost}(G', \sigma') \leq k$. We define the layout $\sigma = \sigma' \odot \sigma^{(1)}_{>1} \odot \sigma^{(2)}_{>1}$. It is now easy to observe that $\sigma \in \mathcal{L}^c(G)$ and that $\mathsf{tcost}(G,\sigma) \leqslant k$. It follows that $\mathsf{ctvs}(G) \leqslant k$, a contradiction to the fact that $G \in \mathcal{O}_k$.

Suppose that G has two distinct cut-vertices, say x and y. Let G_x (resp. G_y) be the graph in $\mathcal{F}(G,\{x\})$ (resp. in $\mathcal{F}(G,\{y\})$) that does not contain y (resp. x) and let C_x (resp. C_y) the corresponding components of $\mathcal{C}(G,x)$ (resp. of $\mathcal{C}(G,y)$). Notice that, by the discussion above, the choice for G_x and G_y is unique. Let $G_{xy} = G[V(G) \setminus (C_x \cup C_y)]$. Notice that, in case $\{x,y\}$ is a bridge, G_{xy} is the one-edge graph.

As $G \in \mathcal{O}_k$, observe that the graph $\hat{G}_x$, resp. $\hat{G}_y$ and $\hat{G}_{xy}$, obtained after contracting in G all the edges in G_x, resp. G_y and G_{xy}, satisfies $\mathsf{ctvs}(\hat{G}_x) \leqslant k$, resp. $\mathsf{ctvs}(\hat{G}_y) \leqslant k$, $\mathsf{ctvs}(\hat{G}_{xy}) \leqslant k$. As before, from Lemma 6 applied to $\hat{G}_{xy}$, $\hat{G}_y$, $\hat{G}_x$, we deduce that the following holds :

- $\mathsf{ctvs}(G_x,x) \leq k$ or $\mathsf{ctvs}(G_y,y) \leq k$,
- $\mathsf{ctvs}(G_x,x) \leq k$ or $\mathsf{ctvs}(G_{xy},x) \leq k$, and
- $\mathsf{ctvs}(G_y,y) \leq k$ or $\mathsf{ctvs}(G_{xy},y) \leq k$.

We extract the following cases (see Figure 6):

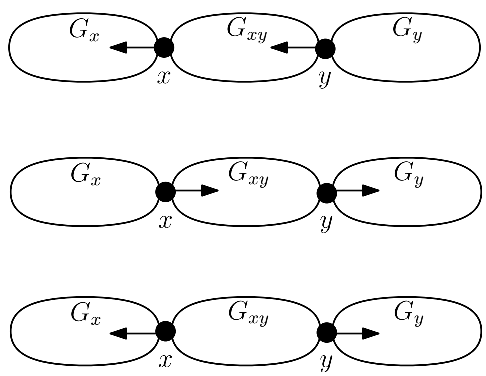

Figure 6: The three cases of the proof of Lemma 7.

(1) $\mathsf{ctvs}(G_x,x) \leq k$ and $\mathsf{ctvs}(G_{xy},y) \leq k$: Let us consider $\sigma^{(x)} \in \mathcal{L}^c(G_x)$ and $\sigma^{(xy)} \in \mathcal{L}^c(G_{xy})$ such that the following holds: $\mathsf{tcost}(G_x,\sigma^{(x)}) \leq k$ and $\sigma^{(x)}(x) = 1$, $\mathsf{tcost}(G_{xy},\sigma^{(xy)}) \leq k$ and $\sigma^{(xy)}(y) = 1$. Observe that as G_y is a contraction of G, $\mathsf{ctvs}(G_y) \leq k$. Consider $\sigma^1 \in \mathcal{L}^c(G_y)$ such that $\mathsf{tcost}(G_y,\sigma^1) \leq k$. We define $\tau^1 = \sigma^1 \odot \sigma^{(xy)}_{>1} \odot \sigma^{(x)}_{>1}$.

(2) If $\mathsf{ctvs}(G_{xy},x) \leq k$ and $\mathsf{ctvs}(G_y,y) \leq k$: Let us consider $\sigma^{(xy)} \in \mathcal{L}^c(G_{xy})$ and $\sigma^{(y)} \in \mathcal{L}^c(G_y)$ such that the following holds: $\mathsf{tcost}(G_{xy},\sigma^{(xy)}) \leq k$ and $\sigma^{(xy)}(x) = 1$, $\mathsf{tcost}(G_y,\sigma^{(y)}) \leq k$ and $\sigma^{(y)}(y) = 1$. Observe that as G_x is a contraction of G, $\mathsf{ctvs}(G_x) \leq k$. Consider $\sigma^2 \in \mathcal{L}^c(G_x)$ such that $\mathsf{tcost}(G_x,\sigma^2) \leq k$. We define $\tau^2 = \sigma^2 \odot \sigma^{(x)}_{>1} \odot \sigma^{(y)}_{>1}$.

(3) If $\mathsf{ctvs}(G_x,x) \leq k$ and $\mathsf{ctvs}(G_y,y) \leq k$: Let us consider $\sigma^{(x)} \in \mathcal{L}^c(G_x)$ and $\sigma^{(y)} \in \mathcal{L}^c(G_y)$ such that the following holds: $\mathsf{tcost}(G_x,\sigma^{(x)}) \leq k$ and $\sigma^{(x)}(x) = 1$, $\mathsf{tcost}(G_y,\sigma^{(y)}) \leq k$ and $\sigma^{(y)}(y) = 1$. Observe that as G_{xy} is a contraction of G, $\mathsf{ctvs}(G_{xy}) \leq k$. Consider $\sigma^3 \in \mathcal{L}^c(G_{xy})$ such that $\mathsf{tcost}(G_{xy},\sigma^3) \leq k$. We define $\tau^3 = \sigma^3 \odot \sigma^{(x)}_{>1} \odot \sigma^{(y)}_{>1}$.

It is easy to notice that, for every $1 \leq i \leq 3$, $\tau^i \in \mathcal{L}^c(G)$ and that $\mathsf{tcost}(G, \tau^i) \leq k$, a contradiction to the fact that $G \in \mathcal{O}_k$. □

Lemma 8. *For every $k \geq 1$ and every connected graph G, $G \in \mathcal{O}_k$ is not biconnected iff $G \in \{\mathbf{A} \oplus \mathbf{B} \mid \mathbf{A}, \mathbf{B} \in \mathcal{O}_k^{(1)}\}$.*

Proof. ($\Rightarrow$) Let G be a graph in $\mathcal{O}_k$ that has a cut-vertex v. From Lemma 7, there exists two rooted graphs $\mathbf{H}_1 = (H_1, \langle v \rangle)$ and $\mathbf{H}_2 = (H_2, \langle v \rangle)$ such that $H_1, H_2 \in \mathcal{F}(G, \{v\})$ and $G = \mathbf{H}_1 \oplus \mathbf{H}_2$. It remains to prove that both $\mathbf{H}_1$ and $\mathbf{H}_2$ belong to $\mathcal{O}_k^{(1)}$. We present the proof only for $\mathbf{H}_1$ as the proof for $\mathbf{H}_2$ is symmetric.

We first prove that $\mathsf{ctvs}(\mathbf{H_1}) > k$. Suppose to the contrary, that there is a $\sigma^1 \in \mathcal{L}^c(\mathbf{H_1})$ such that $\mathsf{tcost}(H_1, \sigma_1) \leq k$. As H_2 is a proper contraction of G and $G \in \mathcal{O}_k$, there is a layout $\sigma^2 \in \mathcal{L}^c(H_2)$ such that $\mathsf{tcost}(H_2, \sigma^2) \leq k$. As v is the root of $\mathbf{H_1}$, $\sigma^1(v) = 1$. This impies that $\sigma = \sigma^2 \odot (\sigma^1 \setminus v) \in \mathcal{L}^c(G)$ and $\mathsf{tcost}(G, \sigma) \leq k$, a contradiction to the fact that $G \in \mathcal{O}_k$.

It remains to prove that if $\mathbf{J} = (J, \langle v \rangle)$ is a proper contraction of $\mathbf{H}_1$, $\mathsf{ctvs}(\mathbf{J'}) \leq k$. To that aim, we prove that for any edge e of H_1, the rooted graph $\mathbf{J} = (J, \langle v \rangle)$, where $J = H_1/e$, satisfies $\mathsf{ctvs}(\mathbf{J}) \leq k$. Then the conclusion follows from Lemma 1.

Claim 1. *If $\mathbf{J} = (J, \langle v \rangle)$ is the result of the contraction of some edge in $\mathbf{H}_1$, then $\mathsf{ctvs}(\mathbf{J}) \leq k$.*

Proof of claim: From Lemma 7, H_1 is biconnected and from Lemma 5 H_1 contains at least 3 vertices, as otherwise it would contain a pendant vertex which is k-simplicial. This means that J has at least two vertices, therefore v is a cut vertex also in $G' = \mathbf{J} \oplus \mathbf{H}_2$. As G' is a proper contraction of G, it follows that $\mathsf{ctvs}(G') \leq k$. We can now apply Lemma 6 on G' and obtain that $\mathsf{ctvs}(\mathbf{J}) \leq k$ or $\mathsf{ctvs}(\mathbf{H}_2) \leq k$. Applying the same arguments as those we used above for proving that $\mathsf{ctvs}(\mathbf{H}_1) > k$, we deduce that $\mathsf{ctvs}(\mathbf{H}_2) > k$, therefore $\mathsf{ctvs}(\mathbf{J}) \leq k$, as required. ◇

($\Leftarrow$) Consider a graph G such that $G = \mathbf{H}_1 \oplus \mathbf{H}_2$ where $\mathbf{H}_1 = (H_1, \langle v \rangle) \in \mathcal{O}_k^{(1)}$ and $\mathbf{H}_2 = (H_2, \langle v \rangle) \in \mathcal{O}_k^{(1)}$. Observe that from Lemma 6, $\mathsf{ctvs}(\mathbf{H_1}) > k$ and $\mathsf{ctvs}(\mathbf{H_2}) > k$ implies that $\mathsf{ctvs}(G) > k$. It remains to prove that for any proper contraction G' of G, $\mathsf{ctvs}(G') \leq k$. To that aim, it suffices to prove the following claim and conclude with Lemma 1.

Claim 2. *If G' is the result of the contraction of some edge e in G then* $\mathsf{ctvs}(G') \leq k$.

Proof of claim 2: Assume without loss of generality that e is an edge of H_1, that is $G' = \mathbf{J} \oplus \mathbf{H}_2$, where $\mathbf{J} = (H_1/e, \langle v \rangle)$. As $\mathbf{J}$ is a proper contraction of $\mathbf{H}_1$ and $\mathbf{H}_1 \in \mathcal{O}_k^{(1)}$ we have that $\mathsf{ctvs}(\mathbf{J}) \leq k$. Therefore there exists $\sigma \in \mathcal{L}^c(\mathbf{J})$ such that $\mathsf{tcost}(\mathbf{J}, \sigma) \leq k$. As H_2 is a proper contraction of G, $\mathsf{ctvs}(H_2) \leq k$ and there exists $\sigma^2 \in \mathcal{L}^{(c)}(H_2)$ such that $\mathsf{ctvs}(H_2, \sigma^2) \leq k$. As v is the root of $\mathbf{J}$, $\sigma(v) = 1$. Observe that $\sigma' = \sigma^2 \odot \sigma \in \mathcal{L}^c(G')$ and $\mathsf{tcost}(G', \sigma') \leq k$ implying that $\mathsf{ctvs}(G') \leq k$, as required. ◇

□

Lemma 8 says that if G is a non-biconnected graph in $\mathcal{O}_k$ then it should have only one cut-vertex and two biconnected components. Moreover, each biconnected component will be the underlying graph of an 1-rooted obstruction in $\mathcal{O}_k^{(1)}$ rooted on the cut-vertex. This structural information will be useful in the later sections.

We also observe the following.

Lemma 9. *For every $k \geq 1$ every graph $\mathbf{G} \in \mathcal{O}_k^{(1)}$ has a biconnected underlying graph.*

Proof. Let $\mathbf{G} = (G, \langle v \rangle) \in \mathcal{O}_k^{(1)}$ and let $H = \mathbf{G} \oplus \mathbf{G}$. From Lemma 8, $H \in \mathcal{O}_k$. As v is a cut-vertex of H, from Lemma 7, G cannot contain any cut-vertex. □

5 Obstructions for ctvs at most 2

This section is devoted to the proof of Theorem 3 that characterizes the set $\mathcal{O}_2$. To state this characterization, a few more notations and definitions have to be introduced. First, Figure 7 depicts some rooted graphs that will be central to the description of $\mathcal{O}_2$. Among those, there are three 2-rooted graphs, namely $\mathbf{R}_{xy} = (R, \langle x, y \rangle)$, $\mathbf{R}_x^y = (R', \langle x, y \rangle)$ and $\mathbf{R}_y^x = (R', \langle y, x \rangle)$. Observe that $\mathbf{R}_x^y$ and $\mathbf{R}_y^x$ are not isomorphic as their root vertices x and y are switched. Among the 1-rooted graphs described in Figure 7, $\mathbf{Y}_x$ and $\mathbf{Y}_x^{(2)}$ are to be distinguished. Indeed, observe that, for any $k \geq 2$, we have $\mathbf{Y}_x^{(k)} = (k \times \mathbf{R}_x^y, \langle x \rangle)$.

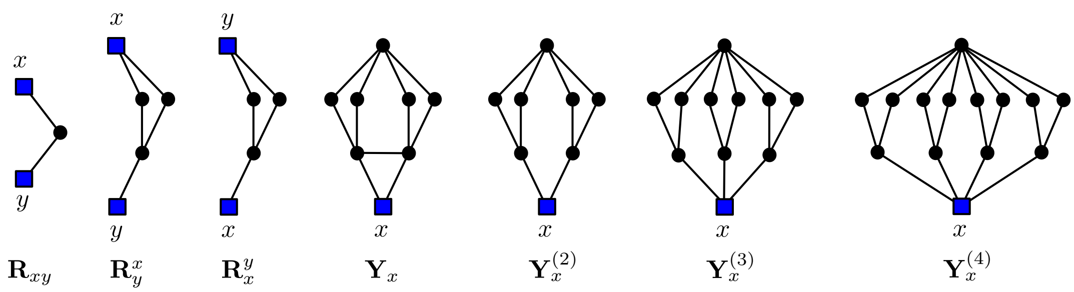

Figure 7: The rooted graphs $\mathbf{R}_{xy}$, $\mathbf{R}_x^y$, $\mathbf{R}_y^x$, $\mathbf{Y}_x$, $\mathbf{Y}_x^{(2)}$, $\mathbf{Y}_x^{(3)}$, and $\mathbf{Y}_x^{(4)}$.

Figure 8 describes three biconnected graphs. As we will see later, these three graphs are the only biconnected graphs belonging to $\mathcal{O}_2$. From the graphs of Figure 7 and Figure 8, we define the following two sets respectively containing 1-rooted graphs and graphs:

$$\mathcal{B}_2^{(1)} = \{\mathbf{Y}_x\} \cup \{\mathbf{Y}_x^{(k)} \mid k \geq 2]\} \quad \text{and} \quad \mathcal{B}_2 = \{K_4, W_1, W_2\} \cup \{\mathbf{A} \oplus \mathbf{B} \mid \mathbf{A}, \mathbf{B} \in \mathcal{B}_2^{(1)}\}$$

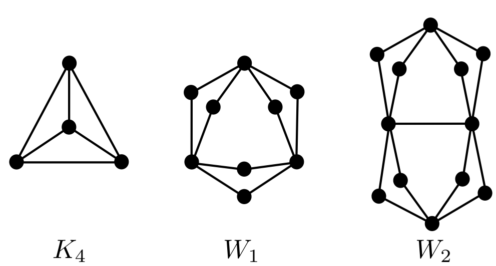

Figure 8: The graphs K_4, W_1, and W_2.

Let $\mathbf{G} = (G, \mathbb{R})$ be a rooted graph and let $S \subseteq V(G)$. We say that S is a 2-*twin family* of $\mathbf{G}$ if $S \cap V(\mathbb{R}) = \emptyset$, $|S| \geq 2$ and there are two distinct vertices $a, b \in V(G)$ such that $\forall s \in S, N_G(s) = \{a, b\}$. We call the vertices a, b the *bases* of the 2-twin family S. We say that a graph $\mathbf{G}' = (G', \mathbb{R}')$ is a 2-*twin expansion of* G if $\mathbb{R} = \mathbb{R}'$ and G' is obtained from G by adding vertices such that each additional vertex is made adjacent with the base vertices of some of the 2-twin families of $\mathbf{G}$. Given a class of rooted graphs $\mathcal{C}$ we define its 2-*twin expansion* $\mathsf{texp}(\mathcal{C})$ as the class of rooted graphs containing all 2-twin expansions of all the graphs in $\mathcal{G}$. We say that a rooted graph $\mathbf{G}$ is *simplified* if all its 2-twin families have size 2.

Given a rooted graph $\mathbf{G}$ we denote by $\tilde{\mathbf{G}}$ the unique simplified rooted graph such that $\mathbf{G} \in \mathsf{texp}(\{\tilde{\mathbf{G}}\})$. Given a set $\mathcal{C}$ of rooted graphs, we define $\tilde{\mathcal{C}} = \{\tilde{\mathbf{G}} \mid \mathbf{G} \in \mathcal{C}\}$. Observe that every graph of $\mathcal{B}_2$ is simplified.

All the above definitions apply to graphs as well as we treat graphs as a rooted graphs with empty set of roots.

We are now ready to state the main result of this section.

Theorem 3. $\mathcal{O}_2 = \mathsf{texp}(\mathcal{B}_2)$.

Outline of the proof of Theorem 3. As a first step we show in Subsection 5.1 that the sets $\mathcal{O}_2$ and $\mathcal{O}_2^{(1)}$ are closed under 2-twin expansion, thereby it is enough to prove that $\tilde{\mathcal{O}}_2 = \mathcal{B}_2$. For this we show that $\{K_4, W_1, W_2\}$ are the biconnected graphs in $\tilde{\mathcal{O}}_2$ while $\{\mathbf{A} \oplus \mathbf{B} \mid \mathbf{A}, \mathbf{B} \in \mathcal{B}_2^{(1)}\}$ are the non-biconnected graphs in $\tilde{\mathcal{O}}_2$. The conclusion of the first part appears in Subsection 5.6. Because of Lemma 8, the non-biconnected case boils down to the identification of $\tilde{\mathcal{O}}_2^{(1)}$. This is achieved by considering two cases. If the obstruction contains a separating edge, then we prove that it is the 1-rooted graph Y_r, depicted in Figure 7 (this is concluded in Subsection 5.4). Otherwise, we prove that the obstruction the graph $\mathbf{Y}_x^{(k)}$, for some $k \geq 2$, that is illustrated in Figure 7 (we prove that in Subsection 5.5 and in Subsection 5.7).

5.1 The set $\mathcal{O}_2$ is closed under 2-twin expansion.

Lemma 10. *Let G be a graph containing a 2-twin family S and such that $\mathsf{ctvs}(G) \leq 2$. Suppose G contains a path P between the bases a and b of S, such that P is disjoint from S. If G' is obtained by adding a vertex z adjacent to a and b, then $\mathsf{ctvs}(G') \leq 2$.*

Proof. Observe that $S' = S \cup \{z\}$ is a 2-twin family of G'. As $\mathsf{ctvs}(G) \leq 2$, there exists σ such that $\mathsf{tcost}(G, \sigma) \leqslant 2$. Suppose without loss of generality that $\sigma(a) < \sigma(b)$. Consider $\sigma' = \sigma \odot \langle z \rangle$. Clearly σ' is a connected layout of G'. Observe that, by construction, for every vertex $u \neq b$, $S_{\sigma'}^{(t)}(u) = S_{\sigma}^{(t)}(u)$ and that $|S_{\sigma'}^{(t)}(z)| = 2$. We also observe that, as z is the mid vertex of a length-two path between a and b, $S_{\sigma'}^{(t)}(\sigma'(b)) = S_{\sigma}^{(t)}(\sigma'(b)) \cup \{a\}$. However, we now argue that $a \in S_{\sigma}^{(t)}(\sigma(b))$, implying that $S_{\sigma'}^{(t)}(\sigma'(b)) = S_{\sigma}^{(t)}(\sigma(b))$ and thereby that $\mathsf{tcost}(G, \sigma') \leqslant 2$. To see this, consider x and y the first two vertices of S in σ (that is for every $v \in S \setminus \{x, y\}$, if any, $\max\{\sigma(x), \sigma(y)\} < \sigma(v)$). Suppose for a contradiction that $a \notin S_{\sigma}^{(t)}(b)$. Then $\sigma(x) < \sigma(b)$, $\sigma(y) < \sigma(b)$ and there exists a vertex $v \in P$ such that $\sigma(v) < \sigma(b)$. But this would imply that $|S_{\sigma}^{(t)}\sigma(b))| > 3$: a contradiction to the hypothesis that $\mathsf{tcost}(G, \sigma) \leqslant 2$. ☐

Lemma 11. *Let $G \in \mathcal{O}_2$ be a graph containing a 2-twin family S. If a and b are the bases of S, then G contains a path P from a to b disjoint from S.*

Proof. First observe that $V(G) \neq \{a, b\} \cup S$ and that G has a biconnected component C containing $\{a, b\} \cup S$. By Lemma 7, at least one of a and b, say a, is not a cut-vertex. Observe that $(G[S \cup$

$\{a,b\}], \langle b\rangle)$ does not belong to $\mathcal{O}_2^{(1)}$. It follows that $C \neq \{a,b\} \cup S$ and thereby G contains a path P from a to b that is disjoint from S. □

Lemma 12. *A graph G belongs to $\mathcal{O}_2$ iff $\tilde{G}$ belongs to $\mathcal{O}_2$.*

Proof. We may assume that G and $\tilde{G}$ are different graphs as otherwise it is trivial. This implies that G contains a 2-twin family of size at least 3 and that for every such family S, $|V(\tilde{G}) \cap S| = 2$. In other words, there is a subset $X \subseteq V(G)$ such that $\tilde{G}$ is the induced subgraph $G[X]$.

Let us first assume that $G \in \mathcal{O}_2$. Let S be a 2-twin family of size at least 3 of G and let a and b be the bases of S. By Lemma 11, G contains a path P from a to b disjoint from S. As G is a 2-twin expansion of $\tilde{G}$, by Lemma 10, $\mathsf{ctvs}(\tilde{G}) \leq 2$ would imply that $\mathsf{ctvs}(G) \leq 2$, contradicting $G \in \mathcal{O}_2$. To prove that $\tilde{G} \in \mathcal{O}_2$, it remains to show that if e is an edge of $\tilde{G}$, then $\mathsf{ctvs}(\tilde{G}/e) \leq 2$. As e is also an edge of G and as $G \in \mathcal{O}_2$, there exists $\sigma \in \mathcal{L}^c(G/e)$ such that $\mathsf{tcost}(G/e, \sigma) \leq 2$. The set X such that $\tilde{G} = G[X]$ can be obtained by removing from $V(G)$ every vertex z belonging to a 2-twin family T such that T contains two vertices x and y distinct from z with $\max\{\sigma(x), \sigma(y)\} < \sigma(z)$. Observe that e is an edge between to vertices u and v of X. Let w be the vertex resulting from the contraction of e and set $X/e = X \setminus \{u, v\} \cup \{w\}$. Then the sequence $\tilde{\sigma} = \sigma \cap X/e$ is a layout of $\tilde{G}/e$. As removing vertices cannot increase the cost of a layout, $\mathsf{tcost}(G/e, \sigma) \leq 2$ implies $\mathsf{tcost}(\tilde{G}/e, \tilde{\sigma}) \leq 2$. It remains to prove that the layout $\tilde{\sigma}$ of $\tilde{G}/e$ is connected. This follows from the constuction of X. Observe indeed that if, for some vertex $u \in X/e$, $S_\sigma^{(t)}(\sigma(u))$ contains a vertex $z \notin X$, then $S_\sigma^{(t)}(\sigma(u))$ contains a vertex $x \in X/e$ that belongs to the same 2-twin family as z. It follows that $S_{\sigma'}^{(t)}(\sigma'(u))$ also contains x and thereby $\tilde{\sigma}$ is a connected layout of $\tilde{G}$.

Let us now assume that $\tilde{G} \in \mathcal{O}_2$. As observed before $\tilde{G}$ contains a 2-twin family S. We prove the statement for $V(G) = V(\tilde{G}) \cup \{z\}$, that is there exist two vertices x and y such that $S = \{x, y, z\}$ is a 2-twin family of G. The fact that $G \in \mathcal{O}_2$ follows by applying inductively this argument to all the vertices in $|V(G) \setminus V(\tilde{G})|$. We first prove that $\mathsf{ctvs}(G) > 2$. For the sake of contradiction, suppose that there exists $\sigma \in \mathcal{L}^c(G)$ such that $\mathsf{tcost}(G, \sigma) \leq 2$. Observe that one can select such a layout σ such that z is the last vertex of S in σ. Let us consider $\sigma^- = \sigma_{<i} \odot \sigma_{>i}$, where $i = \sigma(z)$. Observe that $\mathsf{tcost}(G, \sigma) \leq 2$ implies $\mathsf{tcost}(\tilde{G}, \sigma^-) \leq 2$. Moreover $\sigma^- \in \mathcal{L}^c(\tilde{G})$ since if $z \in S_\sigma^{(\mathsf{t})}(\sigma(v))$ for some vertex v, then x and y, the twins of z, belong to $S_\sigma^{(\mathsf{t})}(\sigma(v))$ as well and the connectivity of σ^- is preserved. It follows that $\mathsf{ctvs}(\tilde{G}) \leqslant 2$: a contradiction to $\tilde{G} \in \mathcal{O}_2$.

It remains to show that for every edge $e \in E(G)$, $\mathsf{ctvs}(G/e) \leq 2$. We now examine two different cases depending on e. We let a and b denote the bases of S.

- Suppose first that e is incident to one of the vertices of S, that is in G/e vertices a and b are adjacent. Observe that G/e can be obtained from $H = \tilde{G}/\tilde{e}$, where $\tilde{e}$ is incident to x or y, by adding the degree-two vertex z adjacent to a and b. As H is a contraction of $\tilde{G}$, there exists $\sigma_H \in \mathcal{L}^c(H)$ such that $\mathsf{tcost}(H, \sigma_H) \leq 2$. Consider $\sigma_{/e} = \sigma_H \odot \langle z\rangle$. We claim that $\mathsf{tcost}(G/e, \sigma_{/e}) \leq 2$. Suppose without loss of generality that $\sigma_{/e}(a) < \sigma_{/e}(b)$. Observe that, by construction, for every vertex $u \neq b$, $S_{\sigma_H}^{(t)}(\sigma_H(u)) = S_{\sigma_{/e}}^{(t)}(\sigma_{/e}(u))$. As a and b are adjacent in H, we also have $a \in S_{\sigma_H}^{(t)}(\sigma_H(b))$, Thereby padding z at the end of σ_H does not augment $S_{\sigma_H}^{(t)}(\sigma_H(b))$ and $\mathsf{tcost}(G/e, \sigma_{/e}) = \mathsf{tcost}(H, \sigma_H) \leq 2$.

- Suppose e is not incident to a vertex of S but is not the edge ab. Thereby e is an edge common to $\tilde{G}$ and G and G/e is a twin-expansion of $H = \tilde{G}/e$. By Lemma 11, $\tilde{G}$ contains a path from

a to b disjoint from S. Observe that H contains such a path P. As G/e is a 2-twin expansion of H and $\mathsf{ctvs}(H) \leq 2$, by Lemma 10, $\mathsf{ctvs}(G/e) \leq 2$.

- Suppose $e = ab$ and let v_e be the result of the contraction of ab in $\tilde{G}$. Then $\mathsf{ctvs}(\tilde{G}/e) \leq 2$ and it is easy to see that G/e is the graph obtained from $\tilde{G}/e$ after adding in it the vertex z and the edge zv_e. Notice that if $\sigma \in \mathcal{L}^c(\tilde{G}/e)$ such that $\mathsf{tcost}(\tilde{G}/e, \sigma) \leq 2$, then $\sigma' = \sigma_H \odot \langle z \rangle \in \mathcal{L}^c(G/e)$ and $\mathsf{tcost}(G/e, \sigma') \leq 2$, as required. □

Using Lemma 8, we can directly extend Lemma 12 to 1-rooted obstructions:

Lemma 13. *Let $\mathbf{H} = (H, \langle v \rangle)$ be a 1-rooted graph. Then $\mathbf{H} \in \mathcal{O}_2^{(1)}$ if and only if $\tilde{\mathbf{H}} \in \mathcal{O}_2^{(1)}$.*

Proof. From Lemma 8, $\mathbf{H} \in \mathcal{O}_2^{(1)}$ iff $G = 2 \times \mathbf{H}$, where $\mathbf{H} \in \mathcal{O}_2$. From Lemma 12, $G \in \mathcal{O}_2 \Leftrightarrow \tilde{G} \in \tilde{\mathcal{O}}_2$ and the Lemma follows if we notice that $\tilde{G} = 2 \times \tilde{\mathbf{H}}$ and apply again Lemma 8. □

5.2 Subsets of (rooted) obstructions

Recall the definition of the sets $\mathcal{B}_2^{(1)}$ and $\mathcal{B}_2$ built from the graphs of Figure 7 and Figure 8. We first show that these graphs (and their 2-twin expansions) are 1-rooted obstructions (Lemma 15) and obstructions (Lemma 16) respectively. We will then study the set of 2-rooted obstructions.

Lemma 14. *Let $\mathbf{G}$, $\mathbf{H} \in \mathcal{O}_2$ (resp. $\mathbf{G}$, $\mathbf{H} \in \mathcal{O}_2^{(1)}$). If $\mathbf{G}$ is simplified and there exist some graph (resp. rooted) graph $\mathbf{F} \in \mathsf{texp}(\{\mathbf{H}\})$ such that $\mathbf{F}$ is a contraction of $\mathbf{G}$, then $\mathbf{G}$ is isomorphic to $\mathbf{H}$.*

Proof. We provide the proof for $\mathbf{G}, \mathbf{H} \in \mathcal{O}_2$. For the sake of contradiction, suppose that $\mathbf{F}$ is a proper contraction of $\mathbf{G}$. Then $\mathbf{F} \notin \mathcal{O}_2$ and thereby Lemma 12 implies that $\mathbf{H} \notin \mathcal{O}_2$: a contradiction. The same arguments apply to $\mathbf{G}$, $\mathbf{H} \in \mathcal{O}_2^{(1)}$ using Lemma 13. □

Lemma 15. *Let $\mathbf{G} = (G, \langle x \rangle)$ be a 1-rooted graph. If $\mathbf{G} \in \mathsf{texp}(\mathcal{B}_2^{(1)})$, then $\mathbf{G} \in \mathcal{O}_2^{(1)}$.*

Proof. From Lemma 13, $\mathcal{O}_2^{(1)} = \mathsf{texp}(\tilde{\mathcal{O}}_2^{(1)})$. Therefore, it is enough to prove that $\mathcal{B}_2^{(1)} \subseteq \tilde{\mathcal{O}}_2^{(1)}$. Let $\mathbf{G} = (G, \langle x \rangle)$ be $\mathbf{Y}_x$ or $\mathbf{Y}_x^{(\ell)}$ for some $\ell \geq 2$. Let y be the unique vertex of G at maximum distance from x.

We first prove that $\mathsf{ctvs}(\mathbf{G}) > 2$. Observe that for every layout $\sigma \in \mathcal{L}^c(\mathbf{G})$, there exist adjacent vertices a and b such that a is adjacent to x, b is adjacent to y (that is, $\{x, a, b, y\}$ induces a path of $\mathbf{G}$) and $\sigma(x) < \sigma(a) < \sigma(b) < \sigma(y)$. Indeed, this is a direct consequence of the fact that the layout σ is connected and the distance in G between x and y is 3. Observe that $\mathbf{G}$ contains three disjoint paths P_b from b to y, P_a from a to y and P_x from x to y. Notice then that $S_\sigma^{(t)}(y)$ contains one vertex from each of P_a, P_b and P_x, implying that $\mathsf{tcost}(\mathbf{G}, \sigma) \geq 3$ and hence $\mathsf{ctvs}(\mathbf{G}) > 2$.

Let now consider $\mathbf{G}' = (G', \langle x \rangle)$ where G' is the result of the contraction of an edge e in G. We consider two different cases.

Case 1. Suppose the contraction of e creates a cut-vertex z, that is $\mathbf{G} = \mathbf{Y}_x$, and e is the edge between the two neighbours of x. Then observe that the connected layout $\sigma = \langle x, z, a, y, b, c, d \rangle$, with a, b, c and d being the degree-two vertices, satisfies $\mathsf{tcost}(\mathbf{G}', \sigma) = 2$.

Case 2. Suppose now that the contraction of e creates a path P of length two between x and y. Let u be the mid vertex of P. Observe that $\{x, y, u\}$ is a separator of $\mathbf{G}'$. Let $\{H_0, H_1, \ldots, H_q\}$ be the connected components of $G - \{x, y, u\}$. Observe that exactly one of these components, say H_0, contains a single vertex z of degree two in $\mathbf{G}'$ and, for every $1 \leq i \leq q$, $(G'[H_i \cup \{x, y\}], \langle x, y\rangle)$ is isomorphic to $\mathbf{R}_x^y$. As $\mathsf{ctvs}(\mathbf{R}_x^y) \leq 2$, for every $i \in [q]$, there is a layout $\sigma^{(i)} \in \mathcal{L}^c(\mathbf{H}_i)$ such that $\mathsf{tcost}(\mathbf{H}_i, \sigma^{(i)}) \leq 2$. We set $\sigma = \langle x, u, y, z\rangle \odot \sigma^{(1)}_{>2} \odot \cdots \odot \sigma^{(q)}_{>2}$. As $\{x, u, y, z\}$ induces a 4-cycle, we have $|S^{(t)}_\sigma(y)| \leq 2$ and $|S^{(t)}_\sigma(z)| \leq 2$. It follows that $\mathsf{tcost}(\mathbf{G}', \sigma) \leq 2$ and hence $\mathsf{ctvs}(\mathbf{G}') \leq 2$. □

Lemma 16. *If a graph G belongs to* $\mathsf{texp}(\mathcal{B}_2)$*, then G belongs to $\mathcal{O}_2$.*

Proof. From Lemma 13, $\mathcal{O}_2 = \mathsf{texp}(\mathcal{O}_2)$. Therefore, it is enough to prove that $\mathcal{B}_2 \subseteq \mathcal{O}_2$. For this, one can verify that $K_4, W_1, W_2 \in \mathcal{O}_2$ by exhaustive check. The fact that $\{\mathbf{A} \oplus \mathbf{B} \mid \mathbf{A}, \mathbf{B} \in \mathcal{B}_2^{(1)}\} \subseteq \mathcal{O}_2$ follows from Lemma 15 and Lemma 8. □

Let us now study 2-rooted obstructions. We define the set $\mathcal{M} = \{\mathbf{R}^{xy}, \mathbf{R}^{xy+}, \mathbf{K}_4^{xy-}, \mathbf{K}_4^{xy}\}$ of 2-rooted graphs depicted in Figure 9. We say that a biconnected 2-rooted graph $\mathbf{H} = (H, \langle x, y\rangle)$ is *elementary* if it is $\mathsf{texp}(\mathcal{M})$-free.

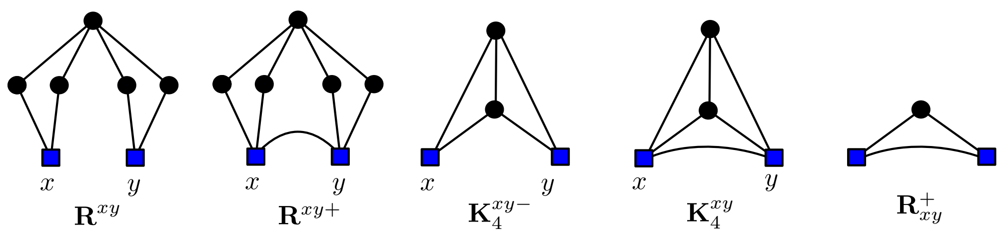

Figure 9: The 2-rooted graphs $\mathbf{R}^{xy}, \mathbf{R}^{xy+}, \mathbf{K}_4^{xy-}, \mathbf{K}_4^{xy}$ and $\mathbf{R}_{xy}^+$.

Lemma 17. *Let (G, x, y) be an s-triple of a graph G, and let $\mathbf{H} = (H, \langle x, y\rangle)$ be a 2-component of (G, x, y). If G is K_4-free, then either $(H, \langle x, y\rangle)$ is isomorphic to $\mathbf{R}_{xy}$ or $\mathbf{R}_{xy}^+$, or G contains a vertex $z \in C(H) \setminus \{x, y\}$ and, for each such z, at least one of (G, z, x) and (G, y, z) is an s-triple.*

Proof. We set $H^- = (V(H), E(H) \setminus \{xy\})$ and $H^+ = (V(H), E(H) \cup \{xy\})$. We may assume that H^- contains at least two vertices different than x and y. Indeed if this is not the case, we are done as then $(H, \langle x, y\rangle)$ is isomorphic either to $\mathbf{R}_{xy}$ (if $xy \notin E(G)$) or to $\mathbf{R}_{xy}^+$ (if $xy \in E(G)$).

We next prove that H^- contains a cut-vertex z. Indeed if this is not the case then H^- contains two internally vertex disjoint paths P_1 and P_2 from x to y. As $(H, \langle x, y\rangle)$ is a 2-component of (G, x, y), it follows that $H \setminus \{x, y\}$ and, therefore also $H^- \setminus \{x, y\}$, is connected. This, in turn, implies that there is a path in H^- joining two internal vertices of P_1 and P_2, avoiding x and y and containing no other internal vertices of P_1 and P_2. But then, H^+ (and thus G as well) can be contracted to K_4, a contradiction.

Notice now that, as $H^- \setminus \{x, y\}$ contains at least two vertices different than x and y, then some of (G, z, x) and (G, y, z) is an s-triple. □

Lemma 18. *If $\mathbf{G}$ is an elementary 2-rooted graph, then* $\mathsf{ctvs}(\mathbf{G}) \leq 2$.

Proof. Let $\mathbf{G} = (G, \langle x, y\rangle)$. Observe first that if $xy \in E(G)$, then $\mathbf{G}^- = ((V(G), E(G) \setminus \{xy\}), \langle x, y\rangle)$, is an elementary 2-rooted graph. Moreover, if $\mathsf{ctvs}(\mathbf{G}^-) \leq 2$, then $\mathsf{ctvs}(\mathbf{G}) \leq 2$. These observations permit us to additionally assume that $xy \notin E(G)$.

Notice that every biconnected 2-rooted graph (with at least 3 vertices) contains $\mathbf{G} = \mathbf{R}_{xy}$ or $\mathbf{G} = \mathbf{R}^+_{xy}$ as a contraction. Also if $\mathbf{G} = \mathbf{R}_{xy}$ or $\mathbf{G} = \mathbf{R}^+_{xy}$ the lemma holds trivially. Let $\mathbf{G}$ be a 2-rooted graph that is contraction minimal counterexample, that is:

(i) $\mathbf{G}$ is elementary (biconnected and $\mathcal{M}$-free),

(ii) $\mathsf{ctvs}(\mathbf{G}) > 2$,

(iii) if $\mathbf{G}'$ is a connected proper contraction of $\mathbf{G}$, then $\mathsf{ctvs}(\mathbf{G}) \leq 2$.

Observe that (i)–(iii) hold for $\mathbf{G}^+ = (G^+, \langle x, y\rangle)$ as well (recall that $\mathbf{G}^+$ if obtained from $\mathbf{G}$ after making its roots adjacent, if they are not already so). We also set $Z = \mathbf{R}_{xy} \oplus \mathbf{G}$. As $\mathbf{G}$ is $\mathcal{M}$-free, Z is K_4-free. Let $C = C(G) \cup \{x, y\}$ and let P be a path in G between x and y. As $\mathbf{G}$ is biconnected (from (i)), all vertices of C are vertices of P. Given a pair $\langle a, b\rangle \in C \times C$, we first observe that (Z, a, b) is an s-triple and we denote by $\mathbf{G}_{ab} = (G_{ab}, a, b)$ the union of all $V(Z) \setminus V(G)$-avoiding 2-components of (Z, a, b). Notice that each $\mathbf{G}_{ab}$ is biconnected and is a contraction of $\mathbf{G}$ or $\mathbf{G}^+$ (depending on whether $ab \in E(G)$ or not). As both $\mathbf{G}$ and $\mathbf{G}^+$ are $\mathcal{M}$-free, then each $\mathbf{G}_{a,b}$ is also $\mathcal{M}$-free.

Claim 1. $|\mathcal{F}(G, \{x, y\})| = 1$.

Proof of Claim 1: Suppose that $\mathcal{F}(\mathbf{G}, \{x, y\}) = \{F_1, \ldots, F_q\}$ with $q \geq 2$. For every $i \in [q]$, we define the 2-rooted graph $\mathbf{F}_i = (F_i, \langle x, y\rangle)$. Observe that for every $i \in [q]$, $\mathbf{F}_i$ is a proper contraction of $\mathbf{G}^+$. As property (iii) holds for $\mathbf{G}^+$ as well, it holds that, for each $i \in [q]$, there is a $\sigma^{(i)} \in \mathcal{L}^c(\mathbf{F}_i)$ such that $\mathsf{tcost}(\mathbf{F}_i, \sigma^{(i)}) \leq 2$. We set $\sigma = \sigma^{(1)} \odot \sigma^{(2)}_{>2} \odot \cdots \odot \sigma^{(q)}_{>2}$. Observe that $\sigma \in \mathcal{L}^c(\mathbf{G})$ and $\mathsf{tcost}(\mathbf{G}, \sigma) \leq 2$, a contradiction to (ii). ◇

Claim 2. If $\langle a, b\rangle \in C \times C$, then one of the following holds:

(a) $|\mathcal{F}(G_{ab}, \{a, b\})| \geq 2$;

(b) $ab \in E(G)$;

(c) there is a $z \in C \setminus \{a, b\}$ such that z belongs to the subpath of P between a and b.

Proof of Claim 2: We assume that $|\mathcal{F}(G_{ab}, \{a, b\})| = 1$ and $ab \notin E(G)$. This implies that $\mathbf{G}_{ab}$ is the unique $V(Z) \setminus V(G)$-avoiding 2-component of (Z, a, b). As $ab \notin E(G)$, $\mathbf{R}^+_{ab}$ cannot be isomorphic to $\mathbf{G}_{ab}$. If $\mathbf{R}_{xy}$ is isomorphic to $\mathbf{G}$, then the claim follows trivially. We now apply Lemma 17 on Z and $\mathbf{G}_{ab}$ and obtain that there is some $z \in C(G_{ab}) \setminus \{a, b\}$ for which (c) holds. ◇

From Claim 1 and Claim 2 (applied for $a = x$ and $b = y$) we obtain that $|C| \geq 3$. Let a_x and a_y be two vertices of $C \setminus \{x, y\}$ with the constraint that a_y (resp. a_x) is the one that is closest to the vertex y (resp. x) in P. Notice that a_x and a_y may be the same vertex.

Claim 3. $xa_x, a_y y \notin E(G)$.

Proof of Claim 3: Suppose that xa_x is an edge of G. The proof for $a_y y$ is symmetric. We distinguish two cases (see Figure 10):

Case 1. xa_x is a separating edge of G^+.
Let $\mathbf{G}_x = (G_x, \langle x, a_x\rangle)$ be union of the y-avoiding 2-components of (G^+, x, a_x). Notice that $\mathbf{G}_x$ is a proper contraction of $\mathbf{G}^+$. Therefore, from (iii), there is a $\sigma \in \mathcal{L}^c(\mathbf{G}_x)$ such that $\mathsf{tcost}(\mathbf{G}_x, \sigma) \leq 2$.

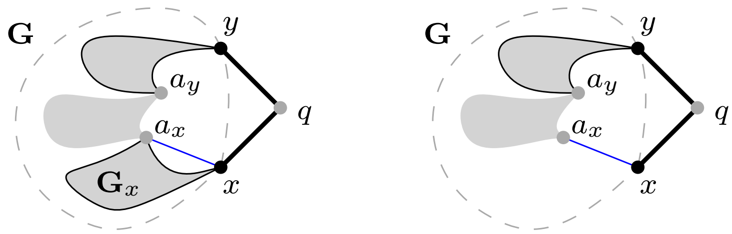

Figure 10: The two cases of the proof of Claim 3.

Let $\mathbf{G}'_x$ be the rooted graph obtained after removing from G all vertices in $V(G_x) \setminus \{x, a_x\}$. Notice that $\mathbf{G}'_x = (G'_x, \langle x, y\rangle)$ is also a proper contraction of $\mathbf{G}$. Therefore, again from (iii), there is a $\sigma' \in \mathcal{L}^c(\mathbf{G}'_x)$ such that $\mathsf{tcost}(\mathbf{G}'_x, \sigma') \leq 2$. We observe that $\sigma'' = \sigma' \odot \sigma'_{>2} \in \mathcal{L}^c$ and $\mathsf{tcost}(\mathbf{G}, \sigma'') \leq 2$. Therefore $\mathsf{ctvs}(\mathbf{G}) \leq 2$, a contradiction to (ii).

Case 2. xa_x is not a separating edge of G^+.
Let us consider the graphs $G_y = G \setminus x$ and $\tilde{G} = G/xa_x$. We let v_{xa_x} denote the vertex of $\tilde{G}$ resulting from the contraction of the edge xa_x. As a_x is a cut-vertex and xa_x is not a separating edge, there exists an isomorphism from G_y to $\tilde{G}$ that maps $a_x \in V(G_y)$ to $v_{xa_x} \in V(\tilde{G})$ and $y \in V(G_y)$ to $y \in V(\tilde{G})$. It follows that $\tilde{\mathbf{G}} = (\tilde{G}, \langle v_{xa_x}, y\rangle)$ and $\mathbf{G}_y = (G_y, \langle a_x, y\rangle)$ are isomorphic and thereby $\mathsf{ctvs}(\tilde{\mathbf{G}}) = \mathsf{ctvs}(\mathbf{G}_y)$. As $\tilde{\mathbf{G}}$ is a proper contraction of $\mathbf{G}$, because of (iii), we have that there exists $\sigma \in \mathcal{L}^c(G_y)$, such that $\mathsf{tcost}(G_y, \sigma) \leq 2$. We now set $\sigma' = \langle x, y, a_x\rangle \odot \sigma_{>2}$. Observe that $\sigma' \in \mathcal{L}^c(\mathbf{G})$ and that $\mathsf{tcost}(\mathbf{G}, \sigma') \leq 2$. Therefore $\mathsf{ctvs}(\mathbf{G}) \leq 2$, a contradiction to (ii). ◇

Applying Claim 2 with $\langle x, a_x\rangle$, we obtain that $|\mathcal{F}(\mathbf{G}_{xa_x}, \{x, a_x\})| \geq 2$ (notice that (b) is excluded by Claim 3 and (c) is excluded by the choice of a_x). Similarly, applying Claim 2 with $\langle y, a_y\rangle$, we obtain that $|\mathcal{F}(\mathbf{G}_{ya_y}, \{y, a_y\})| \geq 2$. It remains to observe that $|\mathcal{F}(\mathbf{G}_{xa_x}, \{x, a_x\})| \geq 2$ and $|\mathcal{F}(\mathbf{G}_{ya_y}, \{y, a_y\})| \geq 2$ along with the fact that $xa_x, a_y y \notin E(G)$ (from Claim 3), imply that $\mathbf{G}$ can be contracted to some graph in $\mathsf{texp}(\{\mathbf{R}^{xy}, \mathbf{R}^{xy+}\})$, a contradiction to (i). □

The following corollary is an immediate consequence of Lemma 18. It constitutes a first step in the characterization of the set $\mathcal{O}_2^{(2)}$ of 2-rooted obstructions.

Corollary 1. *The set of biconnected graphs in* $\mathcal{O}_2^{(2)}$ *is* $\mathsf{texp}(\mathcal{M})$.

Proof. It is easy to verify that for each of the 2-rooted graphs in $\mathsf{texp}(\mathcal{M})$ the value of ctvs is 3, while this value becomes 2 for every proper contraction of a graph of this set. This proves that $\mathsf{texp}(\mathcal{M}) \subseteq \mathcal{O}_2^{(2)}$. Suppose now that there is an elementary 2-rooted graph $\mathbf{G} = (G, \langle x, y\rangle) \in \mathcal{O}_2^{(2)} \setminus \mathsf{texp}(\mathcal{M})$. As $\mathsf{ctvs}(\mathbf{G}) > 3$ and $\mathbf{G}$ is elementary, we have a contradiction to Lemma 18. □

5.3 Structure of 2-rooted obstructions

Lemma 19. *Let* $\mathbf{H} = (H, \langle x, y\rangle)$ *be a 2-rooted graph where* x *is not a cut-vertex and where* H *contains a path from* x *to* y. *Let also* $G = \mathbf{H} \oplus (2 \times \mathbf{R}_{xy})$ *and* $G' = \mathbf{H} \oplus \mathbf{R}_x^y$. *Then* $\mathsf{ctvs}(G) \leq 2 \Rightarrow \mathsf{ctvs}(G') \leq 2$

Proof. The graphs G and G' are depicted in Figure 11. We let denote x, y, a and a' the vertices of $2 \times \mathbf{R}_{xy}$, and x, y, z, a, and a' the vertices of $\mathbf{R}_x^y$ (see Figure 11). Observe that G is obtained by the contraction the edge xz of G' into the vertex x of G. Let I be the set of vertices distinct from x that belong to the connected component of $H - y$ containing x. We also define the set of vertices J as the union of the x-avoiding 1-components of (H, y) minus the vertex y. Observe that $\{I, J, \{y, a, a'\}\}$ is a partition of $V(G)$.

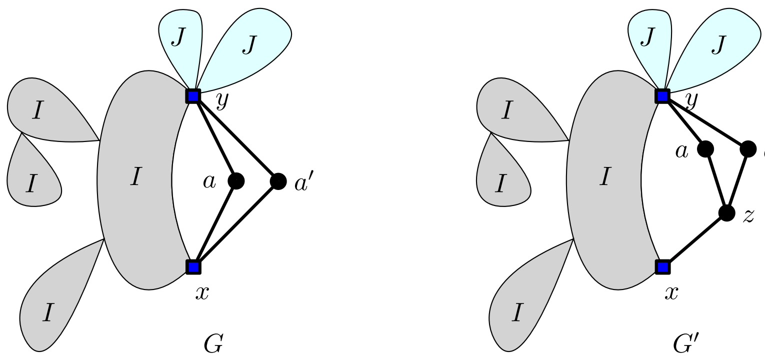

Figure 11: The graphs G and G'.

Let $\sigma \in \mathcal{L}^c(G)$ such that $\mathsf{tcost}(G, \sigma) \leq 2$. W.l.o.g,. we assume that $\sigma(a) < \sigma(a')$. We set $l = \min\{\sigma(x), \sigma(y)\}$ and $m = \max\{\sigma(x), \sigma(y)\}$. Notice that $\sigma(a') > m$ as otherwise $S^{(t)}_\sigma(m)$ contains a, a' and a vertex of $V(H)$ from a path between x and y. We distinguish four cases. In each of these cases, we build from σ a connected layout $\sigma' \in \mathcal{L}^{(c)}(G')$ such that $\mathsf{tcost}(G', \sigma') \leq 2$.

Case 1. $y <_\sigma a <_\sigma x <_\sigma a'$. As $N(a') = \{x, y\}$, we can assume that $\sigma(a') = \sigma(x) + 1 = m + 1$. Notice that $I \subseteq \sigma_{>\sigma(a')} = \sigma_{>m+1}$, i.e,. all the vertices in I appear after a' in σ. Indeed, if this is not the case, then the tree-supporting set $S^{(t)}_\sigma(m)$ would contain vertices a, y and a vertex of I, a contradiction. Notice also that the vertices of J do not appear in the tree-supporting set of a vertex in $I \cup \{x, a, a'\}$. Therefore, we may assume that the vertices y, a, x, a' appear consecutively in σ. See Figure 12. We then observe that $\sigma' = \sigma_{<l} \odot \langle y, a, z, x, a' \rangle \odot \sigma_{\geq l+4}$ is a connected layout of G' and that $\mathsf{tcost}(G', \sigma') = \mathsf{tcost}(G, \sigma) \leq 2$.

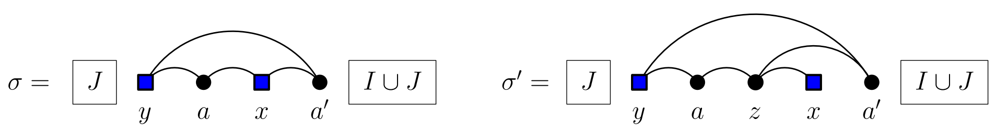

Figure 12: *Case 1.* $y <_\sigma a <_\sigma x <_\sigma a'$.

Case 2. $a <_\sigma y <_\sigma x <_\sigma a'$. Again we can assume that $\sigma(a') = m + 1$ and, by the same argument as in the previous case, we deduce that $I \subseteq \sigma_{>m+1}$. As σ is connected, and $N(a) = \{x, y\}$, we have that $\sigma(a) = 1$. As σ is a connected layout and as y separates J from the rest of the graph, $a <_\sigma y$ implies that y is visited before all the vertices of J. We can thereby assume that J appears after a' in σ. So we have $\sigma_{>4} = I \cup J$. See Figure 13. We then observe that $\sigma' = \langle a, y, z, x, a' \rangle \odot \sigma_{>4}$ is a connected layout of G' such that $\mathsf{tcost}(G', \sigma') = \mathsf{tcost}(G, \sigma) \leq 2$.

Case 3. $x <_\sigma a <_\sigma y <_\sigma a'$ or $a <_\sigma x <_\sigma y <_\sigma a'$. Again, we can assume that $\sigma(a') = m + 1$. We reduce this case to the previous one. To that aim we prove that $I \cup J = \sigma_{>m+1}$. This implies

Figure 13: *Case 2.* $a <_\sigma y <_\sigma x <_\sigma a'$.

that the layout $\tau = \langle a, y, x, a' \rangle \odot \sigma_{>4}$ belongs to $\mathcal{L}^{(c)}(G)$ and that $\mathsf{tcost}(G, \tau) \leq 2$. Observe that τ satisfies the condition of case 2.

We now prove that $I \cup J = \sigma_{>m+1}$. First observe that as $x <_\sigma y$ and y separate the vertices of J from those of I, the connectivity of σ implies that the vertices of J appear after y in σ. So we can assume that $J \subseteq \sigma_{>m+1}$. Suppose now that there exists a vertex $v \in I$ such that $v <_\sigma y$. As x is not a cut-vertex, there exists a path P from v to y avoiding x. It then follows that the tree-supporting set $S^{(t)}_\sigma(m)$ contains a, x and a vertex of the path P, contradicting $\mathsf{tcost}(G, \sigma) \leq 2$. Therefore we have $\sigma_{>4} = \sigma_{>m+1} = I \cup J$.

Case 4. $x <_\sigma y <_\sigma a <_\sigma a'$ or $y <_\sigma x <_\sigma a <_\sigma a'$. We can assume that $\sigma(a') = \sigma(a) + 1 = m + 2$. See Figure 14. In both cases we observe that $\sigma' = \sigma_{\leq m} \odot \langle z \rangle \odot \sigma_{>m}$ is a connected layout of G' such that $\mathsf{tcost}(G', \sigma') = \mathsf{tcost}(G, \sigma) \leq 2$.

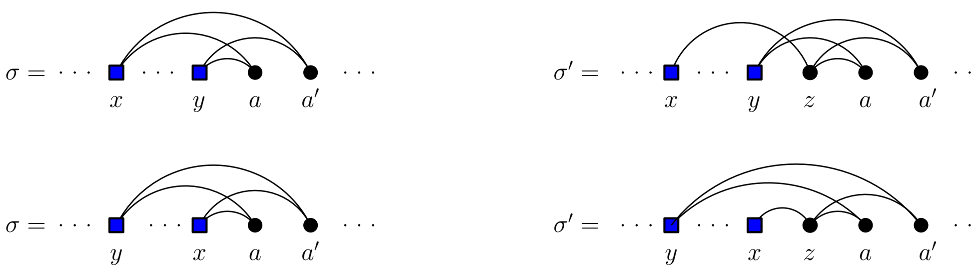

Figure 14: *Case 4.* $x <_\sigma y <_\sigma a <_\sigma a'$ at the top line and $y <_\sigma x <_\sigma a <_\sigma a'$ at the bottom line.

□

We observe that in the previous lemma, the assumption that x is not a cut-vertex is only used in the third case of the proof.

Lemma 20. *Let (G, x, y) be an s-triple of a graph $G \in \mathcal{O}_2$. If $(H, \langle x, y \rangle)$ is a 2-component of (G, x, y) that is isomorphic to $\mathbf{R}^y_x$, then x is a cut-vertex of G.*

Proof. Let $\mathbf{W} = (W, \langle x, y \rangle)$ be the 2-rooted graph such that $G = \mathbf{W} \oplus (H, \langle x, y \rangle)$. Notice that $G' = \mathbf{W} \oplus (2 \times \mathbf{R}_{xy})$ is a proper contraction of G, therefore $\mathsf{ctvs}(G') \leq 2$. Suppose that x is not a cut-vertex of G. Then it is not a cut-vertex in G' neither. It follows that Lemma 19 applies, implying that $\mathsf{ctvs}(G) \leq 2$, a contradiction. □

5.4 Components with separating edges

Lemma 21. *Let G be a K_4-free graph and let $\mathbf{H}$ be a 2-component of the s-triple (G, x, y). If $\mathbf{H}$ is not elementary, then $\mathbf{H}$ contains some graph in $\mathsf{texp}(\{\mathbf{R}^{xy}, \mathbf{R}^{xy+}\})$ as a contraction.*

Proof. As $\{x, y\}$ is a minimal separator, G contains a paths from x to y avoiding $\mathbf{H}$. As G is K_4-free but $\mathbf{H}$ is not $\mathsf{texp}(\mathcal{M})$-free, $\mathbf{H}$ contains some graph in $\mathsf{texp}(\{\mathbf{R}^{xy}, \mathbf{R}^{xy+}\})$ as a contraction. □

Lemma 22. *Let G be a graph in $\tilde{\mathcal{O}}_2$. If G contains a separating edge xy, then either G is isomorphic to W_2 or G contains a cut-vertex r and the 1-component of the s-pair (G, r) containing xy is isomorphic to $\mathbf{Y}_r$.*

Proof. As the lemma holds trivially when $G \in \mathcal{B}_2$, we can assume that G is $\mathcal{B}_2$-free which implies that G is K_4-free and W_2-free. We first prove the following:

Claim 1: If $\mathbf{U} = (U, \langle x, y\rangle)$ is a biconnected 2-component of the s-triple (G, x, y), then $\mathbf{U}$ is not elementary.

Proof of Claim 1: Suppose $\mathbf{U}$ is elementary. Then, by Lemma 18, we have $\mathsf{ctvs}(\mathbf{U}) \leq 2$. Observe that as xy is an edge, the induced subgraph $G^- = G \setminus (V(U) \setminus \{x, y\})$ is a proper contraction of G. It follows that $\mathsf{ctvs}(G^-) \leq 2$. Now consider $\sigma \in \mathcal{L}^c(\mathbf{U})$ such that $\mathsf{tcost}(\mathbf{U}, \sigma) \leq 2$ and $\sigma^- \in \mathcal{L}^c(G^-)$ where $\mathsf{tcost}(G^-, \sigma^-) \leq 2$. As xy is an edge of G, observe that $\sigma' = \sigma^- \odot \sigma_{>2}$ belongs to $\mathcal{L}^c(G)$ and that $\mathsf{tcost}(G, \sigma') \leq 2$. Therefore $\mathsf{ctvs}(G) \leq 2$, a contradiction. ◇

Claim 2: At most one 2-component of (G, x, y) is biconnected. *Proof of Claim 2:* Suppose (G, x, y) contains at least two 2-components. By Claim 1, they are both not elementary. As G is K_4-free, from Lemma 21 and the fact that $xy \in E(G)$, they both contain some 2-rooted graph in $\mathsf{texp}(\{\mathbf{R}^{xy+}\})$ as a contraction, therefore G can be contracted to some 2-rooted graph in $\mathsf{texp}(\{W_2\}) \in \mathcal{O}_2$. Because of Lemma 14, as G is simplified and $W_2, G \in \mathcal{O}_2$, we have that G is isomorphic to W_2, a contradiction. ◇

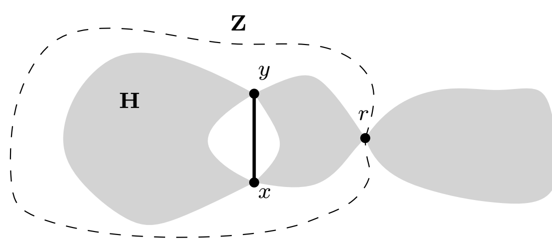

Figure 15: The graph G after Claim 2.

A direct consequence of Claim 2 is that every 2-component of (G, x, y) , but at most one, contains a cut-vertex. By Lemma 7 G contains a unique cut-vertex. It follows that (G, x, y) contains two 2-components, one of which contains a cut-vertex r. Observe that the uniqueness of r implies that neither x nor y is a cut-vertex. Recall that by Lemma 7, the pair s-pair (G, r) contains exactly two 1-components and, among them, let $\mathbf{Z} = (Z, \langle r\rangle)$ be the one containing the edge xy (see Figure 15).

Let $\mathbf{H} = (H, \langle x, y\rangle)$ be the biconnected 2-component of the s-triple (G, x, y). $\mathbf{H}$ is not elementary, because of Claim 1. Recall that G is K_4-free. From Lemma 21 and the fact that $xy \in E(G)$, $\mathbf{H}$ contains as a contraction some 2-rooted graph in $\mathsf{texp}(\{\mathbf{R}^{xy+}\})$. As $\mathbf{Z}$ is biconnected, there is a path between x and y that contains r as an internal vertex and with no internal vertex in H. This means that $\mathbf{Z}$ contains some graph $\mathbf{Y}$ in $\mathsf{texp}(\{\mathbf{Y}_r\})$ as a contraction. As $\mathbf{Z}$ is simplified, $\mathbf{Z} \in \mathcal{O}_2^{(1)}$ and $\mathbf{Y} \in \mathcal{O}_2^{(1)}$ (because of Lemma 15), it follows, from Lemma 14, that $\mathbf{Z}$ is isomorphic to $\mathbf{Y}_r$, as required. □

5.5 Components without separating edge

We say that a path in a graph G is *long* if all its internal vertices have degree 2 and if its length is at least 3.

Lemma 23. *Let G be a graph containing a long path P. If x and y are two adjacent internal vertices of P, then* $\mathsf{ctvs}(G) \leq \mathsf{ctvs}(G/xy)$.

Proof. We let v_{xy} denote the vertex of $G' = G/xy$ resulting from the contraction of the edge xy in G. Let $\sigma' \in \mathcal{L}^c(G')$ such that $\mathsf{tcost}(G', \sigma')$=$\mathsf{ctvs}(G')$. As σ' is connected, v_{xy} has a neighbor u such that $\sigma'(u) < \sigma'(v_{xy})$. Suppose without loss of generality that u is adjacent to x in G. We define $\sigma = \sigma'_{<i} \odot \langle x, y\rangle \odot \sigma'_{>i}$ where $i = \sigma'(v_{xy})$.

Observe that, by construction, σ is a connected layout of G such that $\mathsf{tcost}(G, \sigma) \leq \mathsf{tcost}(G', \sigma')$. Therefore $\mathsf{ctvs}(G) \leq \mathsf{ctvs}(G')$, as required. □

Lemma 24. *Let G be a graph that is biconnected, K_4-free and has no separating edges. If (G, x, y) is an s-triple and there exists a 2-component of (G, x, y) denoted by $\mathbf{H} = (H, \langle x, y\rangle)$ that is not elementary, and the 2-rooted graph $\mathbf{Z} = (G \setminus (V(H) \setminus \{x, y\}), \langle x, y\rangle)$ can be contracted to $\ell \times \mathbf{R}_{xy}$ for some $\ell \geq 2$, then G can be contracted to some graph in* $\mathsf{texp}(\{W_1\})$.

Proof. Let (G, x, y) be a s-triple where G is K_4-free graph and does not contain any separating edge. For the sake of contradiction, suppose there exists a 2-component $\mathbf{H} = (H, \langle x, y\rangle)$ of (G, x, y) such that H is not elementary, $\mathbf{Z} = (G \setminus (V(H) \setminus \{x, y\}), \langle x, y\rangle)$ can be contracted to $\ell \times \mathbf{R}_{xy}$ for some $\ell \geq 2$, and $G = \mathbf{H} \oplus \mathbf{Z}$ cannot be contracted to some graph in $\mathsf{texp}(\{W_1\})$.

Observe that as G is K_4-free, the property of not being elementary implies that $\mathbf{R}^{xy+} \leq (H, \langle x, y\rangle)$ or $\mathbf{R}^{xy} \leq (H, \langle x, y\rangle)$. If $\mathbf{R}^{xy} \leq (H, \langle x, y\rangle)$, then $\mathbf{H} \oplus \mathbf{Z}$ contains some graph in $\mathsf{texp}(\{W_1\})$ as a contraction and we are done. Therefore we have that:

(1) $\mathbf{R}^{xy} \not\leq (H, \langle x, y\rangle)$,

(2) $\mathbf{R}^{xy+} \leq (H, \langle x, y\rangle)$, and

(3) $\mathbf{Z}$ can be contracted to $\ell \times \mathbf{R}_{xy}$, for some $\ell \geq 2$.

Suppose the s-triple (G, x, y) is chosen *extremal* with respect to $\mathbf{H}$ satisfying properties (1)–(3), that is for every s-triple (G, x', y'), containing a 2-component $\mathbf{H'} = (H', \langle x', y'\rangle)$ and such that $\mathbf{H'}$ as a proper contraction of $\mathbf{H}$, the 2-rooted graphs $(H', \langle x', y'\rangle)$ and $\mathbf{Z'} = (G \setminus (V(H') \setminus \{x', y'\}), \langle x', y'\rangle)$ do not satisfy properties (1)–(3).

As, by (2), $\mathbf{H}$ can be contracted to $\mathbf{R}^{xy+}$, it cannot be isomorphic to $\mathbf{R}_{xy}$, neither to $\mathbf{R}^+_{xy}$. So from Lemma 17, there is vertex $z \in C(H) \setminus \{x, y\}$ such that at least one of (G, x, z) and (G, y, z) is an s-triple. Moreover, as G has no separating edges, $xz \notin E(G)$ and $zy \notin E(G)$. As z is a cut-vertex of $\mathbf{H}$ and $\mathbf{R}^{xy+}$ is biconnected, one of the 2-components of of the triples in $\{(G, x, z), (G, z, y)\}$ that is an s-triple contains $\mathbf{R}^{xz+}$ as a contraction. W.l.o.g. assume that this 2-component is a 2-component of the s-triple (G, x, z) and we denote it by $\mathbf{H'} = (H', x, z)$ (otherwise, we work with a 2-component $\mathbf{H'} = (H', z, y)$ of the s-triple (G, z, y)). Observe now that, because $xz \notin E(G)$, the 2-rooted graph $(G \setminus (V(H') \setminus \{x, z\}), \langle x, z\rangle)$ can be contracted to $\ell \times \mathbf{R}_{xz}$, for some $\ell \geq 2$. By the extremal choice of (G, x, y), one of the three Conditions (1), (2), and (3) does not hold for (G, z, x) and $\mathbf{H'}$. As we just verified that (2) and (3) hold, we obtain that $\mathbf{R}^{xz} \leq (H', x, z)$. This, in turn, implies that G contains some graph in $\mathsf{texp}(\{W_1\})$ as a contraction, as yielding a contradiction. □

Lemma 25. *Let $G \in \tilde{\mathcal{O}}_2$ and let (G,x,y) be an s-triple, where $x,y \notin C(G)$. If $\mathbf{H} = (H, \langle x,y \rangle)$ is an elementary $C(G)$-avoiding 2-component of (G,x,y), then $\mathbf{H}$ is isomorphic to $\mathbf{R}_{xy}$.*

Proof. We first examine the case where $xy \in E(G)$. Then, from Lemma 22, either G is biconnected and is isomorphic to W_2 or G has a cut-vertex r and xy belongs to some 1-component of the s-pair $(G, \langle r \rangle)$ that is isomorphic to $\mathbf{Y}_r$. In both cases, by exhaustive checking, one may verify that the lemma is correct in this case.

We now assume that $xy \notin E(G)$. Suppose to the contrary that there is an s-triple (G,x,y) and an elementary $C(G)$-avoiding 2-component $\mathbf{H} = (H, \langle x,y \rangle)$ of (G,x,y) such that $\mathbf{H}$ is not isomorphic to $\mathbf{R}_{xy}$. We choose the s-triple (G,x,y) to be *extremal* in the sense that there is no other minimal separator $\{x', y'\}$ of G and an elementary $C(G)$-avoiding s-component $(H', \langle x', y' \rangle)$ of the s-triple (G, x', y') such that $(H', \langle x', y' \rangle)$ is not isomorphic to $\mathbf{R}_{xy}$ and H' is proper contraction of H.

Recall that $\mathbf{H}$ is not isomorphic to $\mathbf{R}_{xy}$ and that, from Lemma 5, $\mathbf{H}$ is not isomorphic to $\mathbf{R}^+_{xy}$. Therefore, from Lemma 17, G contains a vertex $z \in C(H) \setminus \{x,y\}$ such that one of (G,z,x) and (G,y,z) is an s-triple.

By the extremal choice of (G,x,y) and $\mathbf{H}$, if (G,x,z) is an s-triple then each $C(G)$-avoiding 2-component of (G,x,z) is isomorphic to $\mathbf{R}_{xz}$. By the same argument, if (G,z,y) is an s-triple then all $C(G)$-avoiding 2-components of (G,z,y) are isomorphic to $\mathbf{R}_{zy}$. We distinguish the following cases:

- (G,y,z) is an s-triple but not (G,x,z). We observe that as (G,x,z) is not a s-triple and as z is a cut-vertex of H, xz is an edge. So the existence of a unique $C(G)$-avoiding 2-component of (G,y,z), isomorphic to $\mathbf{R}_{yz}$, would imply the existence of a long path from x to y, a contradiction to Lemma 23. If there are more than one $C(G)$-avoiding 2-components of (G,y,z), then, as G is simplified, the union of those 2-components is isomorphic to $2 \times \mathbf{R}_{yz}$. This implies that $\mathbf{H}$ is isomorphic to $\mathbf{R}^y_x$. As by assumption $\{x,y\} \cap C(G) = \emptyset$, we have a contradiction to Lemma 20.

- (G,x,y) is an s-triple but not (G,z,y). This case is symmetric to the previous one.

- Both (G,z,x) and (G,y,z) are s-triples. Suppose that there exists a unique $C(G)$-avoiding 2-component $\mathbf{H}' = (H', \langle x,z \rangle)$ of (G,x,z). Let a be the vertex of $V(H') \setminus \{x,z\}$. Then observe that (G,a,y) is a s-triple and that $(H \setminus x, \langle a,z \rangle)$ is an elementary $C(G)$-avoiding component, contradicting the extremal choice of (G,x,y). It follows that there are at least two $C(G)$-avoiding 2-components of (G,x,z). Moreover, as G is simplified, there are exactly two such components. By the same argument, there are exactly two $C(G)$-avoiding 2-components of (G,y,z). As these 2-components are respectively isomorphic to $\mathbf{R}_{xy}$ and $\mathbf{R}_{zy}$,we deduce that $\mathbf{H} = \mathbf{R}^{xy}$, a contradiction to the fact that $\mathbf{H}$ is elementary. □

Lemma 26. *Let $\mathbf{G} = (G, \langle x \rangle) \in \tilde{\mathcal{O}}^{(1)}_2$. There is no s-triple (G,x,y) such that one of its 2-components is isomorphic to $\mathbf{R}_{xy}$.*

Proof. Let $J = 2 \times \mathbf{G}$. From Lemma 8, $J \in \tilde{\mathcal{O}}_2$. As the lemma holds trivially when $\mathbf{G} \in \mathcal{B}^{(1)}_2$, we assume that $\mathbf{G}$ is $\mathcal{B}^{(1)}_2$-free. As the underlying graphs of the 1-rooted graphs in $\mathcal{B}^{(1)}_2$ are $\{K_4, W_1, W_2\}$-free, Lemma 8 implies that J is $\mathcal{B}_2$-free. From Lemma 9, G is biconnected. Moreover G does not contains separating edges. To see this observe first that by construction J is not isomorphic to W_2. Then by Lemma 22 $\mathbf{G}$ is isomorphic to $\mathbf{Y}_x$, contradicting the fact that $\mathbf{G}$ is $\mathcal{B}^{(1)}_2$-free.

Let (G, x, y) be a s-triple and let $\mathcal{H} = \{\mathbf{H}_0, \mathbf{H}_1, \ldots, \mathbf{H}_q\}$ be its 2-components. Suppose without loss of generality that $\mathbf{H}_0 = (H, \langle x, y\rangle)$ is isomorphic to $\mathbf{R}_{xy}$. We denote by a the vertex of H that is not x or y.

Claim 1 : There exists $i \in [q]$ such that $\mathbf{H}_i$ is not elementary

Proof of Claim 1: Suppose to the contrary that for every $i \in [q]$ $\mathbf{H}_i$ is elementary. From Lemma 18, $\mathsf{ctvs}(\mathbf{H}_i) \leq 2$ and therefore there is a $\sigma^{(i)} \in \mathcal{L}^c(\mathbf{H}_i)$ such that $\mathsf{tcost}(\mathbf{H}_i, \sigma^{(i)}) \leq 2$. We now set $\sigma = \langle x, a, y\rangle \odot \sigma^{(1)}_{>2} \odot \cdots \odot \sigma^{(q)}_{>2}$. Observe that $\sigma \in \mathcal{L}^c(\mathbf{G})$ and $\mathsf{tcost}(\mathbf{G}, \sigma) \leq 2$. This implies that $\mathsf{ctvs}(\mathbf{G}) \leq 2$, a contradiction, as $\mathbf{G} \in \mathcal{O}_2^{(1)}$. $\diamond$

Suppose that $\mathbf{H}_1 = (H_1, \langle x, y\rangle)$ is not elementary.

Claim 2: $\mathcal{H}$ contains a 2-component distinct from $\mathbf{H}_0$ and $\mathbf{H}_1$, that is $q \geq 2$.

Proof of Claim 2: Suppose to the contrary that $\mathcal{H} = \{\mathbf{H}_0, \mathbf{H}_1\}$. Observe that $\mathbf{H}_1$ is not isomorphic to $\mathbf{R}_{xy}$ nor $\mathbf{R}^+_{xy}$. Thereby by Lemma 17 applied on G and $\mathbf{H}_1$, there exists a vertex $z \in C(H_1)\backslash\{x, y\}$ such that one of (G, z, x) and (G, y, z) is an s-triple.

- Suppose that (G, y, z) is a not a s-triple. As z is a cut-vertex in H_1, yz is an edge. It follows that $\{z, y, a, x\}$ induces a long path in J, a contradiction to Lemma 23.
- Suppose that (G, y, z) is a s-triple and all its x-avoiding 2-components are elementary. Then, from Lemma 25, and the fact that x is distinct from y and z every x-avoiding 2-component of (G, y, z) is isomorphic to $\mathbf{R}_{zy}$. As J is simplified, there cannot be more that two such 2-components. If there is only one, then again J contains a long path, a contradiction to Lemma 23. If they are two, then the s-triple (G, z, a) contains only one x-avoiding 2-component that is isomorphic to $\mathbf{R}^z_a$. As a is not a cut-vertex of J, we have a contradiction to Lemma 20.
- (G, y, z) is a s-triple and at least one, say $\mathbf{A}$, of its x-avoiding 2-components is not elementary. As J is K_4-free and $\mathbf{A}$ is also a non-elementary 2-component of the s-triple (J, y, z), Lemma 21, implies that $\mathbf{A}$ can be contracted to some $\mathbf{Y}$ in $\mathsf{texp}(\{\mathbf{Y}^{(2)}_x\})$ (if $\mathbf{R}^{xy} \leq \mathbf{A}$) or in $\mathsf{texp}(\{\mathbf{Y}_x\}$ (if $\mathbf{R}^{xy+} \leq \mathbf{A}$). This, in turn, implies that $\mathbf{G}$ could be contracted to some graph $\mathbf{Y}$ in $\mathsf{texp}(\{\mathbf{Y}^{(2)}_x, \mathbf{Y}_x\}$. As $\mathbf{G}$ is simplified, $\mathbf{G} \in \mathcal{O}_2^{(1)}$ and $\mathbf{Y} \in \mathcal{O}_2^{(1)}$ (because of Lemma 15), it follows, from Lemma 14, that $\mathbf{G}$ is isomorphic to $\mathbf{Y}^{(2)}_r$, a contradiction. $\diamond$

From Claim 1 and Claim 2, the union of all the 2-components of the s-triple (G, x, y), except from $\mathbf{H}_1$, can be contracted to $\ell \times \mathbf{R}_{xy}$ for some $\ell \geq 2$. Recall that G has no separating edges. Moreover, G is $\mathcal{B}_2$-free and therefore K_4-free. As $\mathbf{H}_1$ is not elementary we can apply Lemma 24 and deduce that G can be contracted to some graph in $\mathsf{texp}(\{W_1\})$. This means that J contains some graph in $\mathsf{texp}(\{W_1\}) \in \mathcal{O}_2$ as a proper contraction, a contradiction as $J \in \tilde{\mathcal{O}}_2 \subseteq \mathcal{O}_2$. $\square$

Lemma 27. *Let* $\mathbf{G} = (G, \langle x\rangle) \in \tilde{\mathcal{O}}_2^{(1)}$. *If* $\mathbf{H} = (H, \langle x, y\rangle)$ *is an elementary 2-component of an s-triple* (G, x, y), *then* $\mathbf{H}$ *is isomorphic to* $\mathbf{R}^y_x$.

Proof. Let $J = 2 \times \mathbf{G}$. Copying the arguments in the beginning of the proof of Lemma 26, J is $\mathcal{B}_2$-free, $\mathbf{G}$ is $\mathcal{B}_2^{(1)}$-free, and G is biconnected without separating edges.

Suppose in contrary that there is an s-triple $(G, \langle x, y\rangle)$ and an elementary 2-component $\mathbf{H} = (H, \langle x, y\rangle)$ of $(G, \langle x, y\rangle)$ such that $\mathbf{H}$ is not isomorphic to $\mathbf{R}^y_x$. We choose $(G, \langle x, y\rangle)$ and $\mathbf{H}$ to be *extremal* in the sense that there is no other minimal separator $\{x', y'\}$ of G and an elementary biconnected 2-component $(H', \langle x', y'\rangle)$ of the s-triple (G, x', y') such that $(H', \langle x', y'\rangle)$ is not isomorphic to $\mathbf{R}_x^y$ and H' is proper subgraph of H.

As $K_4 \in \mathcal{O}_2$, G cannot be contracted to K_4. From Lemma 5, $\mathbf{H}$ is not isomorphic to $\mathbf{R}_{xy}^+$. Also, from Lemma 26, $\mathbf{H}$ is neither isomorphic to $\mathbf{R}_{xy}$. Therefore, we can apply Lemma 17, and deduce that G contains a vertex $z \in C(H) \setminus \{x, y\}$ such that one of (G, x, z) and (G, y, z) is an s-triple of G.

Claim 1: If (G, y, z) is a s-triple, then the union $\mathbf{U}_x$ of all the x-avoiding 2-components of (G, y, z) is isomorphic to $2 \times \mathbf{R}_{yz}$.

Proof of Claim 1: Let $\mathbf{A} = (A, \langle y, z\rangle)$ be a x-avoiding 2-component of (G, y, z). For the sake of contradiction suppose that $\mathbf{A}$ is not elementary. Observe that $\mathbf{A}$ is also a 2-component of the s-triple (J, y, z). As J is K_4-free, Lemma 21 implies that $\mathbf{A}$ can be contracted to some 2-rooted graph in $\mathsf{texp}(\{\mathbf{R}^{yx}, \mathbf{R}^{xy+}\})$. This, in turn, implies that $\mathbf{G}$ could be contracted to some graph $\mathbf{Y}$ in $\mathsf{texp}(\{\mathbf{Y}_x^{(2)}\})$ (if $\mathbf{R}^{xy} \leq \mathbf{A}$) or in $\mathsf{texp}(\{\mathbf{Y}_x\}$ (if $\mathbf{R}^{xy+} \leq \mathbf{A}$). As $\mathbf{G}$ is simplified, $\mathbf{G} \in \mathcal{O}_2^{(1)}$ and $\mathbf{Y} \in \mathcal{O}_2^{(1)}$ (because of Lemma 15), it follows, from Lemma 14, that $\mathbf{G}$ is isomorphic to $\mathbf{Y}_r^{(2)}$ or $\mathbf{Y}_r$, a contradiction as $\mathbf{G}$ is $\mathcal{B}_2^{(1)}$-free and $\mathbf{Y}_r, \mathbf{Y}_r^{(2)} \in \mathcal{B}_2^{(1)}$.

So every x-avoiding 2-component of (G, y, z) is elementary. As none of z and y are cut-vertices of J, by Lemma 25, each of these 2-components is isomorphic to $\mathbf{R}_{zy}$. Observe that if $\mathbf{A}$ is the unique x-avoiding component, then this would violate the extremal choice of $(G, \langle x, y\rangle)$. Indeed, observe that if v is the unique vertex of $V(A) \setminus \{y, z\}$, then $(G, \langle x, v\rangle)$ and $\mathbf{H}$ certifies that $(G, \langle x, y\rangle)$ and $\mathbf{H}$ are not extremal. Finally, as G is simplified, (G, y, z) has at most two x-avoiding 2-components. We conclude that if (G, y, z) is a s-triple, then $\mathbf{U}_x$ is isomorphic to $2 \times \mathbf{R}_{yz}$. ◇

Claim 2: If (G, z, x) is a s-triple, then the union $\mathbf{U}_y$ of all the y-avoiding 2-components of (G, z, x) is isomorphic to $\ell \times \mathbf{R}_x^z$ for some $\ell \geq 1$.

Proof of Claim 2: By the extremal choice of $(G, \langle x, y\rangle)$ and $\mathbf{H}$, every y-avoiding 2-component of the s-triple (G, z, x) is isomorphic to $\mathbf{R}_x^z$. This implies that $\mathbf{U}_y$ is isomorphic to $\ell \times \mathbf{R}_x^z$ for some $\ell \geq 1$.◇

We arrive to a contradiction by distinguishing the following cases.

- Both (G, y, z) and (G, x, z) are s-triples. By Claim 1, $\mathbf{U}_x$ is isomorphic to $2 \times \mathbf{R}_{yz}$. So if we contract the z-avoiding 2-components of (G, x, y) to a single edge $e = xy$, then the union of e and $\mathbf{U}_x$ is a 2-rooted graph $\mathbf{B}$ isomorphic to $\mathbf{R}_x^z$. According to Claim 2, $\mathbf{U}_y$ is isomorphic to $\ell \times \mathbf{R}_x^y$ for some $\ell \geq 1$. It follows that $\mathbf{Y}_x^{\ell+1} \in \mathcal{B}_2^{(1)}$, obtained as the union of $\mathbf{U}_y$ and $\mathbf{B}$, is a contraction of $\mathbf{G}$, a contradiction.

- (G, y, z) is an s-triple but (G, x, z) is not. Recall that, by Claim 1, $\mathbf{U}_x$ is isomorphic to $2 \times \mathbf{R}_{yz}$. As G does not have separating edges, xy is not an edge of G. Moreover as $z \in C(H)$ and (G, x, z) is not an s-triple, xz is an edge. It follows that $\mathbf{H}$ is isomorphic to $\mathbf{R}_x^y$, a contradiction to the choice of $(G, \langle x, y\rangle)$ and $\mathbf{H}$.

- (G, y, z) is not an s-triple but (G, x, z) is. From Claim 2, the union $\mathbf{U}_y = (U_y, \langle x, z\rangle)$ of the y-avoiding 2-components of the s-triple (G, x, z) is isomorphic to $\ell \times \mathbf{R}_x^z$ for some $\ell \geq 1$. We prove that $\ell \geq 2$. For the sake of contradiction, assume that $\mathbf{U}_y$ is isomorphic to $\mathbf{R}_x^z$. Let z' be the unique neighbour of x in $\mathbf{U}_y$. Observe that (G, y, z') forms an s-triple of G. As $z \in C(H)$ and (G, y, z) is not an s-triple, yz is an edge. It follows that the unique 2-component of (G, y, z') containing z is isomorphic to $\mathbf{R}_y^{z'}$. As y is not a cut-vertex of J, this contradicts Lemma 20. It follows that $\ell \geq 2$ and that the vertex set of graph $\mathbf{H}$ is $\{y\} \cup V(\mathbf{U}_y)$ and its edge set is $\{yz\} \cup E(\mathbf{U}_y)$ (see Figure 16, where H is depicted outside the shadow area of the left handside graph). We let $A = \{a_1, \ldots, a_q\}$ denote the neighbours of x in $\mathbf{U}_y$ and let $B = \{b_1, b'_1, \ldots, b_q, b'_q\}$ denote the vertices of $\mathbf{U}_y$ such that for each $i \in [q]$, b_i, b'_i are the common neighbors of a_i and z.

Claim 3: There exists $\sigma \in \mathcal{L}^c(\mathbf{U}_y)$ such that $\mathsf{tcost}(\mathbf{U}_y, \sigma) \leq 2$.

Proof of Claim 3: Observe that $\sigma = \langle x, z, a_1, b_1, b'_1, \ldots, a_q, b_q, b'_q \rangle$ is the claimed layout. $\diamond$

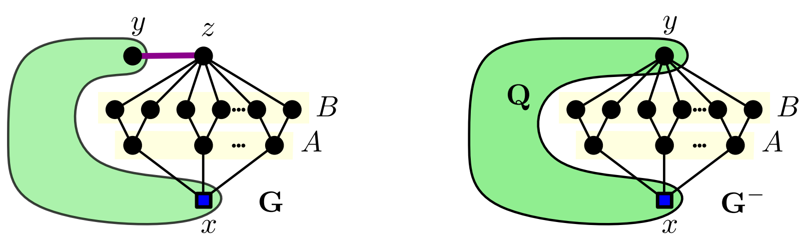

Figure 16: The 1-rooted graphs $\mathbf{G}$ and $\mathbf{G}^-$.

Let $\mathbf{G}^- = (G^-, \langle x \rangle)$ be the 1-rooted graph where G^- is obtained from G by contracting the edge zy to the vertex y. We denote by $\mathbf{Q} = (Q, \langle x, y \rangle)$ the 2-component of (G^-, x, y) that avoids the vertices in $A \cup B$ (in the rightmost graph of Figure 16, $\mathbf{Q}$ is represented by the shadowed part). As $\mathbf{G}^-$ is a proper contraction of $\mathbf{G}$, $\mathsf{ctvs}(\mathbf{G}^-) \leq 2$. Let $\sigma' \in \mathcal{L}^c(\mathbf{G}^-)$ where $\mathsf{tcost}(\mathbf{G}^-, \sigma') \leq 2$. Recall that σ' starts with x.

Claim 4: Every vertex of $A \cup B$ appears after y in σ'.

Proof of Claim 4: Suppose that some vertex of $A \cup B$ appears in σ' before y and we denote by a the first such vertex that appears in σ'. Notice that $a \in A$ as the vertices of A are the only vertices of U_y that are adjacent to x in U_y (recall that σ' is connected).

Let X be a set containing at most one of the vertices of B that appear before y in σ'. Notice that X is empty or a singleton, depending whether there are vertices of B appearing before y in σ.

Notice that there is a path P in G^- from x to y and a path P' in G^- from a to y such that $V(P) \cap V(P') = \{y\}$ and such that none of the internal vertices of P and P' belongs to $V(Q)$. Notice that P and P' can be chosen so that they avoid all vertices (if any) in X.

Let also $I = N_Q(y)$ and $i = \sigma'(y)$.

Case 1: $|B| = 1$. Then $S^{(\mathsf{t})}_{\sigma'}(i)$ contains one vertex in $V(P) \setminus \{y\}$, one vertex in $V(P') \setminus \{y\}$, and the unique vertex of X. This means that $|S^{(\mathsf{t})}_{\sigma'}(i)| > 2$, therefore $\mathsf{tcost}(\mathbf{G}^-, \sigma') > 2$, a contradiction.
Case 2: $|B| = 0$. As σ' is connected there exists some vertex $m \in Q \setminus \{x, y\}$ that appears before y in σ'. Then let P'' be a path in $\mathbf{G}^-$ from m to y whose internal vertices belong to $Q \setminus \{x, y\}$. As $V(P) \cap V(P') \cap V(P'') = \{y\}$ we have that $S^{(\mathsf{t})}_{\sigma'}(i)$ contains one vertex in $V(P) \setminus \{y\}$, one vertex in $V(P') \setminus \{y\}$, and one vertex in $V(P'') \setminus \{y\}$. This means that $|S^{(\mathsf{t})}_{\sigma'}(i)| > 2$, therefore $\mathsf{tcost}(\mathbf{G}^-, \sigma') > 2$, a contradiction. $\diamond$

Consider the layout σ^* obtained by removing the vertices of $A \cup B$ from σ', that is $\sigma^* = \sigma' \setminus (A \cup B)$. Observe that by Claim 4, $\sigma^* \in \mathcal{L}^c(Q, \langle x \rangle)$. We set $\sigma'' = \sigma^* \odot \sigma_{>2}$. Observe that $\sigma'' \in \mathcal{L}^c(\mathbf{G})$ and that, by Claim 3, $\mathsf{tcost}(\mathbf{G}, \sigma'') \leq 2$. It follows that $\mathsf{ctvs}(\mathbf{G}) \leq 2$, a contradiction as $\mathbf{G} \in \mathcal{O}_2^{(1)}$. $\square$

5.6 Biconnected graphs in $\mathcal{O}_2$.

Let G be a graph and let $e = xy$ be an edge of G. We say that e is a *marginal* edge of G if it is not a separating edge and there is a vertex z such that both (G, x, z) and (G, y, z) are s-triples. We call z the *base* of the marginal edge xy.

Lemma 28. *If G is a biconnected graph in $\tilde{\mathcal{O}}_2$, then G does not contain marginal edges.*

Proof. From Lemma 22, if G is biconnected and has separating edges then it is isomorphic to W_2 and, as W_2 has no marginal edges, we are done. We can now assume that G has no separating edges. As the lemma holds for all biconnected graphs in $\mathcal{B}_2$, we can assume that G excludes all of them as a contraction.

Suppose to the contrary that xy is a marginal edge of G. We denote by $\mathbf{U}_x$ the union of the y-avoiding 2-components of (G, x, z) and by $\mathbf{U}_y$ the union of the x-avoiding 2-components of (G, y, z).

Notice that it is not possible that both $\mathbf{U}_x$ and $\mathbf{U}_y$ are non-elementary. Indeed, if this is the case, then $\mathbf{U}_x$ can be contracted to $\mathbf{R}^{xz}$ or $\mathbf{R}^{xz+}$ or $\mathbf{K}_4^{xz-}$ or $\mathbf{K}_4^{xz}$ and $\mathbf{U}_y$ can be contracted to $\mathbf{R}^{yz}$ or $\mathbf{R}^{yz+}$ or $\mathbf{K}_4^{yz-}$ or $\mathbf{K}_4^{yz}$. This means that G contains one of K_4, W_1, W_2 as a contraction, a contradiction.

W.l.o.g,. we assume that $\mathbf{U}_x$ is elementary. This means that all 2-components of (G, x, z) are elementary. From Lemma 25, all of them are isomorphic to $\mathbf{R}_{xz}$. If they are only one, then G contains a long path, a contradiction to Lemma 23. If they are more than one, then they are two as G is simplified. But then the 2-component of (G, y, z) that contains x is isomorphic to $\mathbf{R}_y^z$ while y is not a cut-vertex, a contradiction to Lemma 20. □

Lemma 29. *The biconnected graphs in $\tilde{\mathcal{O}}_2$ are the graphs K_4, W_1, and W_2.*

Proof. Suppose to the contrary that there is a biconnected graph $G \in \tilde{\mathcal{O}}_2 \setminus \{K_4, W_1, W_2\}$. Notice that G has no separating edges, because, otherwise, from Lemma 22 it is isomorphic to W_2, a contradiction.

As G excludes K_4 as a contraction, it contains some vertex a of degree 2. Let x and y be the neighbors of a. Observe that (G, x, y) is an s-triple. We let $\mathcal{H} = \{\mathbf{H}_0, \ldots, \mathbf{H}_q\}$ be the 2-components of (G, x, y) with $\mathbf{H}_0 = (H_0, \langle x, y\rangle)$ and $V(H_0) = \{x, a, y\}$. As G is biconnected, every 2-rooted graph in $\mathcal{H}$ is biconnected.

Claim. Exactly one of the 2-rooted graphs $\{\mathbf{H}_1, \ldots, \mathbf{H}_q\}$ is not elementary.

Proof of claim. Suppose that for every $i \in [q]$, $\mathbf{H}_i$ is elementary. From Lemma 18, $\mathsf{ctvs}(\mathbf{H}_i) \leq 2$, for $i \in [2, q]$ therefore, there is a $\sigma^{(i)} \in \mathcal{L}^c(\mathbf{H}_i)$ where $\mathsf{tcost}(\mathbf{H}_i, \sigma^{(i)}) \leq 2$. Let $\sigma = \langle x, a, y\rangle \odot \sigma^{(2)}_{>2} \odot \cdots \odot \sigma^{(q)}_{>2}$. Observe that $\mathsf{tcost}(G, \sigma) \leq 2$, therefore $\mathsf{ctvs}(G) \leq 2$, a contradiction. So there exists $i \in [q]$ such that $\mathbf{H}_i$ is elementary.

Suppose there exist distinct $i \in [q]$ and $j \in [q]$ such that $\mathbf{H}_i$ and $\mathbf{H}_j$ are not elementary. As G is K_4-free, Lemma 21 implies that both $\mathbf{H}_i$ and $\mathbf{H}_j$ can be contracted to some 2-rooted graph in $\mathsf{texp}(\{\mathbf{R}^{yx}, \mathbf{R}^{xy+}\})$. But then G can be contracted to some graph in $\mathsf{texp}(\{W_1, W_2\})$. As $G \in \tilde{\mathcal{O}}_2$ and $\mathsf{texp}(\{W_1, W_2\}) \subseteq \mathcal{O}_2$ (because of Lemma 16), it follows, from Lemma 14, that $\mathbf{G}$ is isomorphic to a graph in $\{W_1, W_2\}$, a contradiction. ◇

Assume without loss of generality that $\mathbf{H}_1$ is not elementary. From Lemma 25, every $\mathbf{H}_j \in \mathcal{H}$, distinct from $\mathbf{H}_1$, is isomorphic to $\mathbf{R}_{xy}$. As G is simplified, we have $q \leq 2$.

- Suppose that $q = 2$. Then the union of $\mathbf{H}_0$ and $\mathbf{H}_2$ is isomorphic to $2 \times \mathbf{R}_{xy}$. As G is K_4-free, $\mathbf{H}_1$ is not elementary and G does not contain a separating edge, Lemma 24 implies that G is either W_1 or it can be properly contracted to some graph in $\mathsf{texp}(\{W_1\}) \in \mathcal{O}_2$, a contradiction as $G \in \tilde{\mathcal{O}}_2 \subseteq \mathcal{O}_2$.
- Suppose that $q = 1$. From Lemma 17, H_1 contains a cut-vertex z such that one of (G, z, x) and (G, y, z) is an s-triple. W.l.o.g,. we assume that (G, z, x) is an s-triple. But then the edge xa would be a marginal edge (with base z), a contradiction to Lemma 28. □

5.7 Non-biconnected graphs in $\tilde{\mathcal{O}}_2$.

The class of all *2-trees* is recursively defined as follows. K_3 is a 2-tree and a graph H with more than 3 vertices is a 2-tree if it contains some vertex v of degree 2 such that its two neighbours are adjacent and $H \setminus v$ is a 2-tree. Given a 2-tree G and an edge $e \in E(G)$, we say that e is a *simplicial* edge of G if it is incident to a simplicial vertex.

Lemma 30. *Let G be a 2-tree. Then sets of marginal, simplicial, and separating edges of G form a partition of $E(G)$.*

Proof. Let G be a counterexample with a minimum number of vertices. Clearly G cannot be isomorphic to K_3 as every edge of K_3 is simplicial. A 2-tree that is not isomorphic to K_3 contains a vertex v such that $G^- = G \setminus v$ is a 2-tree. By the minimality of G, the edge set of G^- can be partitioned to the sets of the marginal, simplicial, and separating edges. Let x, y be the neighbors of v and let $e \in E(G)$. If $e = vx$ or $e = vy$, then e is a simplicial edge of G. If $e = xy$, then it is a separating edge of G. So suppose that $e \notin \{xy, vx, vy\}$. In that case, observe that e has the same type in G as in G^-. It follows that the marginal, simplicial, and separating edges of G form a partition of $E(G)$. □

Lemma 31. *Every biconnected K_4-free graph G is the spanning subgraph of a 2-tree T with the following properties:*

(D1) *If an edge is marginal in T then it is also marginal in G.*

(D2) *If an edge is simplicial in T then one of its endpoints has degree 2 in G.*

(D3) *If an edge is a separating edge of G, then it is also a separating edge in T.*

Proof. It is known that each K_4-free graph is the spanning subgraph of a 2-tree (this fact can be easily derived by Dirac's theorem [21], asserting that every K_4-free graph contains some vertex of degree ≤ 2). Let T be such a 2-tree. All properties (D1)-(D3) follow directly by the fact that G is biconnected. □

Lemma 32. *The non-biconnected graphs in $\tilde{\mathcal{O}}_2$ are the graphs in $\{\mathbf{A} \oplus \mathbf{B} \mid \mathbf{A}, \mathbf{B} \in \mathcal{B}_2^{(1)}\}$.*

Proof. Let $J = 2 \times \mathbf{G}$. Copying the arguments in the beginning of the proof of Lemma 26, J is $\mathcal{B}_2$-free, $\mathcal{B}_2^{(1)}$-free, and G is biconnected without separating edges.

From Lemma 8 and Lemma 15, it is enough to prove that $\mathcal{O}_2^{(1)} \subseteq \mathcal{B}_2^{(1)}$. We assume, towards a contradiction, that there is some 1-rooted graph $\mathbf{G} = (G, \langle r \rangle) \in \mathcal{O}_2^{(1)} \setminus \mathcal{B}_2^{(1)}$. Clearly, $\mathbf{G}$ is $\mathcal{B}_2^{(1)}$-free and, from Lemma 9, G is biconnected. From Lemma 22, we can assume that G does not have separating edges. As the underlying graphs of the 2-rooted graphs in $\mathcal{B}_2^{(1)}$ are $\{K_4, W_1, W_2\}$-free, Lemma 8 implies that J is $\mathcal{B}_2$-free. We will make use the fact that J is K_4-free.

Claim 1: r has more than two neighbors.

Proof of Claim 1: Suppose to the contrary that r has only two neighbors, say x and y in G. Let $H = G \setminus \{r\}$ and $\mathbf{H} = (H, \langle x, y\rangle)$ and let $\mathcal{A}$ be the set of all r-avoiding 2-component of the s-triple (G, x, y).

We first claim that every 2-rooted graph in $\mathcal{A}$ is elementary. For the sake of contradiction, suppose some $\mathbf{A} \in \mathcal{A}$ is not elementary. As J is K_4-free and $\mathbf{A}$ is an r-avoiding 2-component of the s-triple (J, x, y), Lemma 21 implies that $\mathbf{A}$ can be contracted to some 2-rooted graph in $\mathsf{texp}(\{\mathbf{R}^{yx}, \mathbf{R}^{xy+}\})$. We now observe that applying the same contractions in $\mathbf{G}$ and contracting every 2-rooted graphs in $\mathcal{A} \setminus \{\mathbf{A}\}$ to a single edge yields some graph $\mathbf{Y}$ in $\mathsf{texp}(\{\mathbf{Y}_r\})$ as a contraction of $\mathbf{G}$. As $\mathbf{G}$ is simplified, by Lemma 15, $\mathbf{G} \in \mathcal{O}_2^{(1)}$ and $\mathbf{Y} \in \mathcal{O}_2^{(1)}$. It follows, from Lemma 14, that $\mathbf{G}$ is isomorphic to $\mathbf{Y}_r^{(2)}$, a contradiction as $\mathbf{G}$ is $\mathcal{B}_2^{(1)}$-free and $\mathbf{Y}_r^{(2)} \in \mathcal{B}_2^{(1)}$.

Let $\mathbf{H}_1, \ldots, \mathbf{H}_q$ be the r-avoiding 2-components of the s-triple (G, x, y). As G is biconnected, each $\mathbf{H}_i$ is biconnected. We can now apply Lemma 18 and deduce that $\mathsf{ctvs}(\mathbf{H}_i) \leq 2$, for every $i \in [q]$, therefore there is some $\sigma^{(i)} \in \mathcal{L}^c(\mathbf{H_i})$ where $\mathsf{tcost}(\mathbf{H_i}, \sigma) \leq 2$. We set $\sigma' = \langle r, x, y\rangle \odot \sigma_{>2}^{(1)} \odot \cdots \odot \sigma_{>2}^{(q)}$ and observe that $\sigma' \in \mathcal{L}^c(G)$ and $\mathsf{tcost}(\mathbf{G}, \sigma') \leq 2$, a contradiction. ⋄

Because of Claim 1, r has at least 3 neighbors in G. Let T be a 2-tree that contains G as a spanning subgraph and satisfies properties (D1)–(D3) of Lemma 31. Let z be a neighbor of r. Because of (D3), the edge $e = rz$ is either a marginal or a simplicial edge of T. We claim that e is marginal. Indeed, if e is simplicial, then from (D2) z has degree 2. Let w be the other neighbor of z. Notice that one of the 2-components of the s-triple (G, r, w) is isomorphic to $\mathbf{R}_{rw}$, a contradition to Lemma 26.

We now know that $e = rz$ is a marginal edge. Let t be the base of e. Clearly (G, r, t) is an s-triple and $tr \notin E(G)$ as G does not have separating edges. We denote by $\mathcal{U} = \mathbf{U}_1, \ldots, \mathbf{U}_q$ the 2-components of (G, r, t).

Claim 2: All 2-rooted graphs in $\mathcal{U}$ are simple.

Proof of Claim 2: Suppose to the contrary that one, say $\mathbf{U}_1$, of the 2-rooted graphs in $\mathcal{U}$ is non-simple and let $\mathbf{U}'$ be the union of all the rest. Notice that as $q > 2$, $\mathbf{U}'$ can be contracted to $\ell \times \mathbf{R}_{r,t}$ for some $\ell \geq 2$. Notice that (J, r, t) is an s-triple such that one of its 2-components is $\mathbf{U}_1$ (that is non-simple) and the union of the rest is $\mathbf{U}'$ which can be contracted to $\ell \times \mathbf{R}_{xy}$ for some $\ell \geq 2$. Therefore, Lemma 24 applies on G and yields that G can be contracted to some graph in $\mathsf{texp}(\{W_1\})$. This means that J contains some graph in $\mathsf{texp}(\{W_1\}) \in \mathcal{O}_2$ as a proper contraction, a contradiction as $J \in \tilde{\mathcal{O}}_2 \subseteq \mathcal{O}_2$. ⋄

From Lemma 27 and Claim 2, all graphs in $\mathcal{U}$ are isomorphic to $\mathbf{R}_r^t$. This implies that $\mathbf{G}$ contains as a contraction some $\mathbf{Y}_t^{(\ell)}$ for some $\ell \geq 3$. As each such $\mathbf{Y}_t^{(\ell)}$ belongs to $\mathcal{B}_2^{(1)}$ we have a contradiction. □

Proof of Theorem 3. From Lemma 29, and Lemma 32, we obtain that $\tilde{\mathcal{O}}_2 = \tilde{B}_2$. The result follows from Lemma 13. □

6 Members of $\mathcal{O}_k$ for every k

In this section we make some observations on the general structure of the graphs in $\mathcal{O}_k$ and $\mathcal{O}_k^{(1)}$ for higher values of k.

Let $\mathcal{Y}_k, k \geq 2$, be a set containing every 1-rooted graph that can be constructed as follows: take a tree, rooted at r, where the distance of all its leaves from the root is k, and where each non-leaf vertex has at least two children. Then make all its leaves adjacent with a new vertex z (we call z *top* vertex) and we root the resulting graph to r. The graph with the least possible number of vertices in $\mathcal{Y}_4$ is depicted in Figure 17.

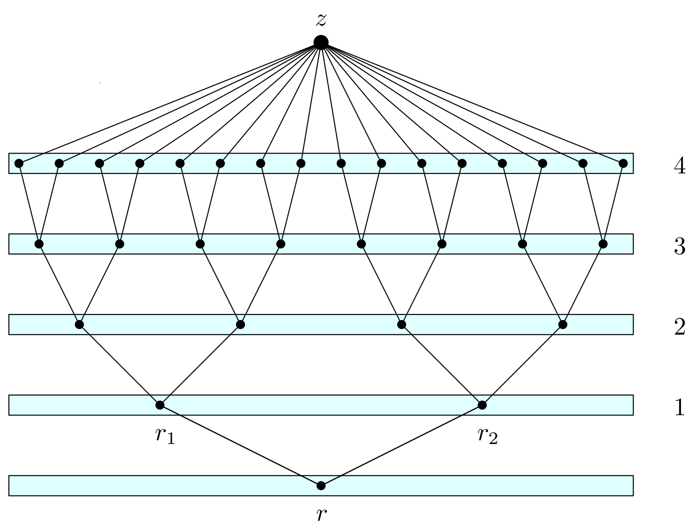

Figure 17: A 1-rooted graph in $\mathcal{Y}_4$.

Lemma 33. *For every $k \geq 2$, $\mathcal{Y}_k \subseteq \mathcal{O}_k^{(1)}$.*

Proof. Let $\mathbf{Y} = (Y, \langle r \rangle)$ be a 1-rooted graph in $\mathcal{Y}_k$ for some $k \geq 2$. For each such $\mathbf{Y}$, we define $\bar{\mathbf{Y}} = (\bar{Y}, \langle r, z \rangle)$ as the 2-rooted graph obtained if in $\mathbf{Y}$ we also root the top vertex z. We also denote by d_r the degree of r. We can easily verify that, for $k \geq 3$, $\mathbf{Y}$ can be constructed if we take d_r 1-rooted graphs $\mathbf{Y}_1, \ldots, \mathbf{Y}_{d_r}$ (for some $d_r \geq 2$) in $\mathcal{Y}_{k-1}$, rooted to $r_1, \ldots, r_{d_r}$ respectively, then identify their top vertices to a single vertex z, add a new root r, and make it adjacent to $r_i, i \in [d_r]$.

Claim 1: $\mathsf{ctvs}(\bar{\mathbf{Y}}) \leq 2$, for every $k \geq 2$.

Proof of Claim 1: Suppose to the contrary that this is not the case for some $k \geq 2$ and let k be the minimum such integer. It is easy to verify that if $\mathbf{Y} \in \mathcal{Y}_2$, then $\mathsf{ctvs}(\bar{\mathbf{Y}}) \leq 2$, therefore, $k \geq 3$. Moreover, by the minimality of k, $\mathsf{ctvs}(\bar{\mathbf{Y}}_i) \leq 2$, for every $i \in [d_r]$, therefore there is a $\sigma_i \in \mathcal{L}^c(\bar{\mathbf{Y}}_i)$, where $\mathsf{tcost}(\bar{\mathbf{Y}}_i, \sigma_i) \leq 2$, $i \in [2]$. We now consider the layout $\sigma = \langle r, z \rangle \odot (\sigma_1 \setminus \{z\}) \odot \cdots, \odot (\sigma_{d_r} \setminus \{z\})$ and observe that $\sigma \in \mathcal{L}^c(\bar{\mathbf{Y}})$ and that $\mathsf{tcost}(\bar{\mathbf{Y}}, \sigma) \leq 2$, a contradiction.

Claim 2: $\mathsf{ctvs}(\mathbf{Y}) > k$, for every $k \geq 0$.

Proof of Claim 2: Let $\sigma \in \mathcal{L}^c(\mathbf{Y})$ and assume that $z = \sigma_i$. As $Y[\sigma_{\leq i}]$ is connected, it contains a path, say P, from r to z. Clearly P has $k+2$ vertices (including its endpoints). We call the first

$k+1$ vertices $r = v_1, \ldots, v_{k+1}$. Notice that there are $k+1$ internally vertex disjoint paths from z to these vertices. We call these paths $P_1, \ldots, P_{k+1}$. For each $j \in [k+1]$, let x_j be the last vertex of P_j, that appears in $\sigma_{<i}$. Notice that $\{x_1, \ldots, x_{k+1}\} \in S_\sigma^{(\mathsf{t})}(i)$, therefore $\mathsf{tcost}(\mathbf{Y}, \sigma) > k$. As σ has been chosen arbitrarily, we conclude that $\mathsf{ctvs}(\mathbf{Y}) > k$.

Claim 3: For every $e \in E(Y)$, $\mathsf{ctvs}(Y/e, \langle r \rangle) \leq k$, for every $k \geq 1$.

Proof of Claim 3: Let $Y' = Y/e$ be the result of the contraction of an edge e in Y and let $\mathbf{Y}' = (Y', \langle r \rangle)$. Notice that $\bar{\mathbf{Y}}'$ is a contraction of $\bar{\mathbf{Y}}$. From Lemma 1 and Claim 1, it follows that $\mathsf{ctvs}(\bar{\mathbf{Y}}') \leq 2$. Let $\sigma' \in \mathcal{L}^c(\bar{\mathbf{Y}}')$, where $\mathsf{tcost}(\bar{\mathbf{Y}}', \sigma') \leq 2$. We also agree that if e has r or z as an endpoint, then the result of its contraction is also r or z, respectively. Notice that because of the contraction of e, the distance in the resulting Y' between z and r is k. Therefore Y' contains a path P on k edges from z to r (clearly this path passes from the vertex that is created because of the contraction of e). Let $I = \{v_0, v_1, \ldots, v_{k-1}, v_k\}$ be the vertices of this path, agreeing that $v_0 = r$ and $v_k = z$. We now define $\sigma = \langle v_0, \ldots, v_k \rangle \odot (\sigma' \setminus I)$ and observe that $\sigma \in \mathcal{L}^c(\mathbf{Y}')$, where $\mathsf{tcost}(\mathbf{Y}', \sigma) \leq k$. Therefore, $\mathsf{ctvs}(\mathbf{Y}') \leq k$, as required.

Using now Claims 2 and 3, we have that $\mathbf{Y} \in \mathcal{O}_k^{(1)}$ and we are done. □

Consider now the graph class $\mathcal{B}_k = \{2 \times \mathbf{Y} \mid \mathbf{Y} \in \mathcal{Y}_k\}$ (see Figure 18 for an example of a graph in $\mathcal{Q}_k$). Because of Lemma 8, we conclude to the following.

Lemma 34. *For every $k \geq 2$, $\mathcal{B}_k \subseteq \mathcal{O}_k$.*

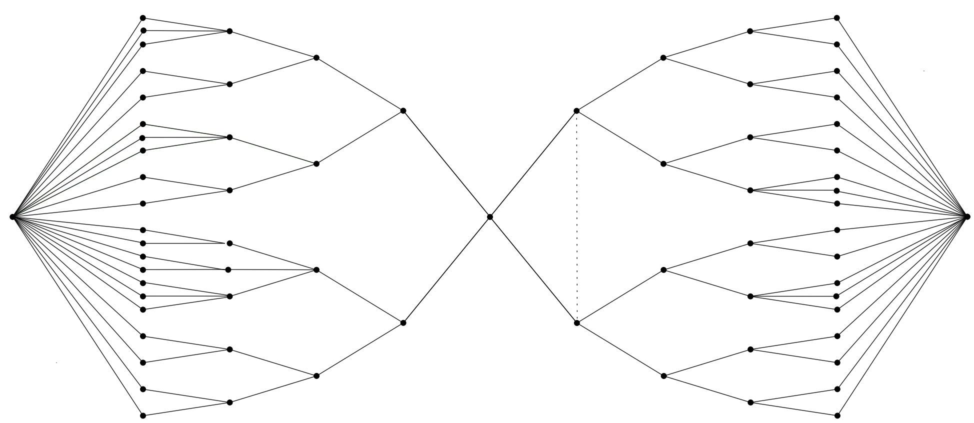

Figure 18: A member of $\mathcal{B}_4$.

As all graphs in $\mathcal{B}_k$ have treewidth 2, the above lemma has the following consequence.

Corollary 2. *For every $k \geq 2$, $\mathcal{O}_k$ contains infinitely many graphs of treewidth 2.*

7 Open problems

The main result of this paper was the identification of $\mathbf{obs}_{\preceq}(\mathcal{T}_2)$. As this set is infinite, we provided a "regular description" of this set where parts of the identified graphs can be "multiplied" arbitrarily. It is interesting to investigate whether similar behavior is met for other contraction obstruction sets of contraction-closed graph classes, for instance, those where the bag-connected treewidth by [20] and [33] is bounded. Is there a way to systematize this notion of "regular description"? Are there natural graph parameters that are contraction-closed and whose contraction obstruction have no such description?

In [31], Fraigniaud and Nisse studied the connected variant of the node search game where the robber is agile and visible. In this variant the search strategy is not an ad-hoc sequence of the cop's positions. Instead it is a function that, given the current position of the cops and the robber, returns the next move of the cops. As proved in [16], the search game against an agile and visible robber is equivalent to the one against an inert and invisible robber that is considered in the present paper. Also, because of the monotonicity results in [51], the graph searching parameters corresponding to both these two versions of graph searching are also equal to treewidth plus one. It is easy to prove that this equivalence, under the monotonicity assumption, transfers also to the connected versions of both agile & visible and lazy & invisible variants that are both equivalent to connected treewidth plus one. This means that all of our results can be transferred to the agile and visible setting, under the monotonicity assumption. Using this, the discrepancy of treewidth and connected treewidth can also be certified by a construction given in [31]. On the other hand, it is interesting to examine whether the equivalence of the two variants (agile/visible and inert/invisible) transfers also in the non-monotone setting. The results of [31] could be a good starting point in this direction.

Another interesting question is the algorithmic complexity of connected treewidth. Treewidth can be straightforwardly reduced to connected treewidth: given a graph G we denote by G^* the graph obtained from G after adding a new vertex and making it adjacent to all the vertices of G. It is easy to see then $\mathsf{tw}(G) = \mathsf{ctw}(G^*)$. Therefore, the problem of deciding, whether the connected treewidth of a graph G is at most k is NP-complete. An interesting question is whether this problem, when parameterized by k, is fixed parameter tractable, i.e,. it can be answered in $f(k) \cdot n^{O(1)}$ time for some function f. Notice that the non-connected counterpart of this problem is fixed parameter tractable. Actually, for both treewidth and pathwidth we immediately know that there are $f(k) \cdot n$ step algorithms, because of the finiteness of $\mathbf{obs}_{\leq}(\mathcal{T}_k)$ and $\mathbf{obs}_{\leq}(\mathcal{P}_k)$, for every k [11]. However, we cannot proceed like this for the connected treewidth or the connected pathwidth, as $\mathbf{obs}_{\preceq}(\mathcal{T}_k^c)$ and $\mathbf{obs}_{\preceq}(\mathcal{P}_k^c)$ are not finite in general (here $\mathcal{P}_k^c$ is the class of graphs with connected pathwidth at most k). Very recently, an $O(f(k) \cdot n)$ time algorithm has been devised in [35] for connected pathwidth (see also [19] for earlier results).

Lastly, it is an interesting question whether the ideas of this paper can lead to the identification, *for every* k, of the set of all obstructions that have treewidth at most 2, i.e., the set $\mathcal{D}_k = \mathbf{obs}_{\preceq}(\mathcal{T}_k^c) \cap \mathcal{T}_2$. From the constructions behind Corollary 2 we already know an infinite subset of this set. We believe that $\mathcal{B}_k$ is not far from $\mathcal{D}_k$. Based on the proof ideas of this paper, we conjecture that the whole $\tilde{\mathcal{D}}_k$ is generated if we enhance the construction of $\mathcal{Y}_k$ so that, in case $d_r = 2$, an edge may be added between the two neighbors of r (visualized by the dotted edge in Figure 18). From the algorithmic point of view, it was recently proved that connected treewidth can be computed in $O(n^2 \log n)$ time on graphs of treewidth at most 2 [43].

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