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Measuring welfare, inequality and poverty with ordinal variables:

the univariate case.

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Abstract

The key challenge in making distributional comparisons with ordinal data is the lack of commensurability of the distances between the ordered categories. This chapter provides a critical review of the most recent theoretical developments addressing this challenge and providing methods for ethical poverty, welfare, and inequality comparisons with univariate ordered multinomial distributions.

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1. Introduction

Ordinal data fits somewhere between nominal and quantitative data. Both nominal and ordinal data come in the form of multinomial distributions; but only in the latter the categories are ordered. That is, in some meaningful sense one can talk about better and worse categories. However, unlike continuous data (discrete or continuous variables), the distances between the ordered categories are not inherently commensurable. That is, if there are self-reported health with three categories: "bad", medium", and "good"; without further information it is impossible to tell whether the difference between "medium" and "bad" is greater, smaller or equal to the difference between "good" and "medium".

Yet despite these measurement limitations, ordinal data has become more prominent in the social sciences during the last few decades, partly owing to the popularity of self-reported data in the health, subjective-wellbeing and life satisfaction literature (including job satisfaction), which adds to a long list of much older attempts to describe human phenomena with ordered categories (e.g. Cantril ladders; social stratification models, sanitation ladders, etc.).

Notwithstanding the appeal, distributional assessments of poverty, welfare and inequality with ordered categorical variables (henceforth "ordinal variables") face a fundamental challenge due to the aforementioned lack of commensurability in the distances between categories. This leaves mainly two ways of handling ordinal data: (1) assigning scales or numerical value judgments to the categories, respecting and reflecting their order, then applying statistical tools for quantitative data; or (2) just using the (absolute or relative) frequencies of the ordered multinomial distribution.

The first approach is controversial because it amounts to rendering the distances between categories commensurable with scales whose choice is ultimately *arbitrary*. Therefore, a minimum consistency property is that distributional comparisons with ordinal data should be robust to any admissible choice of scales; where admissibility in this context means that the scales respect the order of the categories. This is quite a demanding requirement since the set of admissible scales can be quite large. Indeed, as shown by Stevens (1946) and Mendelson (1987), it encompasses any subset of strictly ordered numbers as well as all its possible increasing monotonic functions bound to preserve the categories' ranks. For instance, even basic mean comparisons fail this test (Allison and Foster, 2004). Consider, for example, three ordered categories and the following two relative frequency distributions: $A = (0.4,0.2,0.4)$ and $B = (0.2,0.6,0.2)$. With linear scaling $L = (1,2,3)$ one gets $\mu^A = 2 = \mu^B$, where μ^X is the mean. However, if the scaling $L = (1,4,9)$ is used, the result is that $\mu^A = 4.8 > 4.4 = \mu^B$. Similar problems arise with any summary statistic or index originally devised for quantitative data.

In fact, all the "permissible statistics (invariantive [sic])" for ordinal data identified by Stevens (1946, p. 678) can be computed without resorting scales, e.g. percentiles and quantiles like the median, frequencies, the mode, and some correlation measures based on contingency tables. Yet this chapter will cover some recent proposals for more complex ethical comparisons of distributions relying on accepting a limited degree of commensurability.

In its applications to welfare, poverty and inequality comparisons, the second measurement approach, based on dismissing scales altogether, is a natural response to the arbitrariness/commensurability problem. Its key ingredient is almost invariably the cumulative distribution function (henceforth CDF). Due attention will be paid to these contributions as well.

More specifically, this chapter introduces the reader to the most recent proposals for performing ethically meaningful comparisons of welfare, poverty and inequality with ordinal data. The surge in novel contributions over the last couple of decades warrants focusing on one single variable.

However, there is a growing literature on distributional comparisons with multiple ordinal and binary variables, including the popular multidimensional poverty assessments. These would warrant at least another chapter and there already exist excellent review chapters on these methods (see, the surveys of the literature in Kakwani and Silber (2008), Chakravarty (2009, 2015), Atkinson and Bourguignon, (2014), Alkire, et al. (2015), D'Ambrosio (2018), among others). Likewise, this chapter does not dwell on techniques designed for combinations of quantitative and non-qualitative data (For a recent contribution see Bosmans et al. (2017).

Neither does it engage with polarization analysis based on exogenous groups (i.e. groups not defined by the ordinal variable itself). The topic is fascinating, but the axioms involved are far more complex than those considered in this paper, as they involve judgments on group identification, cohesive versus diffuse opposition between groups and so forth. The interested reader is advised to read Permanyer and D'Ambrosio (2015), the key work on this strand of the literature, as well as a recent elegant contribution by Mussini (2018). Finally, this chapter will not discuss the related literature on distributional assessments with nominal data, the so-called diversity measurement. The statistical tools for nominal data are also suitable for ordinal data because no scales are ever involved. On the other hand, they do not exploit all the embedded information in ordinal variables since, by necessity and construction, they neglect the order of the categories. The reader is invited to read Patil and Taillie (1982), the seminal contribution in the literature on distributional analysis with nominal data.

Though interesting and obviously relevant topics, neither does the present study discuss statistical inference (including hypothesis testing and cardinalisation of ordinal variables in parametric models) nor dwell significantly on empirical applications of the methods discussed in this chapter. The content involved in the most recent theoretical contributions is rich enough to warrant its own review chapter.

The chapter's notation will be introduced as and when it is needed. The rest of the chapter proceeds as follows. Section 2 discusses the measurement of poverty with ordinal variables. The chapter draws significantly on the comprehensive framework proposed by Seth and Yalonetzky (2020a). A subsection provides detail of the poverty partial orderings derived by Seth and Yalonetzky (2020a). Then section 3 presents and compares the proposals of Apouey et al. (2019), Gravel et al. (2020) and Seth and Yalonetzky (2020b) for measuring dispersion-sensitive welfare with ordinal variables. A subsection provides some key partial orderings for welfare measures, including those of Mendelson (1987), Gravel et al. (2020), and Seth and Yalonetzky (2020b). Section 4 discusses old and recent contributions to the measurement of inequality and bipolarisation with ordinal variables. After discussing the situations of equality and maximum inequality in most of the literature as well as some key inequality-altering transformations, the section presents inequality measures based on the attribution of natural numbers to the categories' ranks (mainly the work of Lv et al., 2015). Then the prominent measurement approach based on quantile-preserving spreads, chiefly median-preserving spreads, is described in detail, crucially distinguishing between explicit and implicit quantiles, as well as between inequality and bipolarisation concepts (works reviewed include Mendelson, 1987; Apouey, 2007; Abul Naga and Yalcin, 2008; Lazar and Silber, 2013; Kobus, 2015; Chakravarty and Maharaj, 2015). Drawing from Sarkar and Santra (2020), a special subsection explains proposals to measure inequality and bipolarisation when distributions do not share quantiles of interest (e.g. the median). This is followed by a subsection on a recent alternative approach based on the ordinal notion of status (Cowell and Flachaire, 2017; Jenkins, 2020). The section continues with a subsection on partial orderings reflecting the different measurement paradigms discussed in the preceding subsections. The chapter ends on section 5 with some concluding remarks.

2. Poverty measurement with ordinal variables

Most poverty assessments draw on continuous data (e.g. monetary indicators) or, more recently, multiple binary indicators in the burgeoning literature on multidimensional poverty measurement (e.g. see Alkire, et al. 2015 for an introduction). However, some data relevant to poverty assessments, like access to services and quality of dwelling materials, is ordinal. For instance, Seth and Yalonetzky (2020a) offer the example of the service ladder approach whereby access to sanitation is classified into five ordered categories: "open defecation", "unimproved", "limited", "basic unsafe", "improved". As long as one can make a case that some, but not all, of these categories represent different degrees of deprivation, then a poverty assessment (e.g. for tracking progress in access to safe sanitation) is more appropriate than a welfare assessment, precisely because only the former establishes a sharp conceptual distinction between deprivation and non-deprivation categories.

Bennett and Hatzimasoura (2011) were perhaps the first to propose a class of poverty measures for ordinal variables, together with some partial orderings. However, Yalonetzky (2012) noted that their proposal was unduly restrictive and generalised it. Yet neither of these contributions offered measurement tools that could explicitly and usefully prioritise the needs of the poorest among the poor, answering the call not to leave anybody behind (UN, 2018). Of course, this faced the challenge of how to measure the depth of deprivation, and prioritise those in the lowest categories, in the absence of categorical commensurability.

Seth and Yalonetzky (2020a) tackled this problem by first considering the poverty headcount ratio, which in the ordinal-variable context is simply the proportion of people in any of the deprivation categories (e.g. the proportion of people who do not have "improved" sanitation facilities). This measure provides potentially perverse incentives because if a policy-maker is to be judged by reductions in the headcount ratio, then marginal improvements benefitting the poorest among the poor may not be detected by the ratio, whereas efforts may be focused on the least poor (especially if the cost of poverty-alleviation programs does not decrease with the depth of deprivation). Alternatively, one could monitor headcount ratios, for each possible poverty line (e.g. the four deprivation categories in the sanitation example), or relative frequencies for each deprivation category. But this may only work efficiently if the combined number of categories and distributional comparisons is small. Or, if one would like to focus on the poorest among the poor, setting the poverty line at the lowest category would do the trick. But then one would lose information on changes among people in the other deprivation categories. Therefore, a single measure that summarises societal poverty by providing differential evaluations depending on the categories may be warranted.

The only compromise one will need to make is accepting ethical evaluations of the categories in order to arrive at a synthetic measure of poverty effectively involves taking up implicit scaling. On the plus side, Seth and Yalonetzky (2020a) proceeded with an axiomatic characterisation, so at least the classes of admissible scales (reflecting ethical judgments) are supposed to respect a set of desirable properties.

In order to proceed further, there is a need to introduce some notation. Let $C>1$ be a natural number representing the number of categories in an ordered multinomial distribution. Each category is denoted by a natural number i such that $i \in \mathcal{C}_C$ where $\mathcal{C}_C = \{1, \dots, C\}$ is the set of all categories and higher numbers sensibly reflect higher desirability among them. The population frequency of i is $p_i \geq 0$; therefore $\sum_{i=1}^{C} p_i = 1$. $\boldsymbol{p} = (p_1, \dots, p_C)$ is the discrete probability distribution in the population and $\boldsymbol{p} \in \mathbb{P}_C$, where $\mathbb{P}_C$ is the set of all possible discrete probability distributions of size C.

Now, for poverty measures a deprivation category, k, has to be chosen that will serve as the poverty line: all categories below (or at) the poverty line represent deprivation, whereas all categories above do not reflect deprivation. It will be assumed that at least one category is free from deprivation, therefore $k = 1, \dots, C-1$. Then a poverty measure $P(\boldsymbol{p}, k)$ is defined as a mapping from $\mathbb{P} \times \mathcal{C}_{C-1}$ onto the non-negative real line. Seth and Yalonetzky (2020) then propose four desirable properties for P which are jointly necessary and sufficient to characterise a class of poverty measures taking the form of weighted averages of the elements in $\boldsymbol{p}$.

The first desirable property is ordinal monotonicity (OM) stating that if a proportion of poor people is moved to a better category then poverty should decrease:

Ordinal Monotonicity (OM): For any $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{P}_C$ and for any $k \in \mathcal{C}_{C-1}$, if $q_i < p_i$ for some $i \leq k$ and $i < j$ but $p_s = q_s$ for all $s \neq \{i, j\}$, then $P(\boldsymbol{q}, k) < P(\boldsymbol{p}, k)$.

The second desirable property, single-category deprivation (SCD), reasonably requires that the poverty measure should boil down to the poverty headcount when the poverty line is set at the most deprived category:

Single-category deprivation (SCD): For any $\boldsymbol{p} \in \mathbb{P}_C$ and $k = 1$, $P(\boldsymbol{p}, 1) = p_1$.

The third desirable property, focus (FOC), is quite standard in the poverty measurement literature and demands that changes in the non-poor's situation should not alter societal poverty, as long as the non-poor remain in that status:

Focus (FOC): For any $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{P}_C$ and for any $k \in \mathcal{C}_{C-1}$, if $p_s = q_s$ for all $s \leq k$, then $P(\boldsymbol{q}, k) = P(\boldsymbol{p}, k)$.

Finally, the fourth desirable property, subgroup decomposability (SUD), states that the poverty measure should be expressible as the population-weighted sum of poverty measures in non-overlapping and fully exhaustive subgroups of the population. Let $\pi_m \geq 0$ for all $m = 1,2, \dots, M$ be the proportion of the population in group (e.g. region) m, such that $\sum_{m=1}^{M} \pi_m = 1$. Also let $\boldsymbol{p^m} = (p_1^m, p_2^m, \dots, p_C^m)$ be the probability distribution in m, so that $\sum_{m=1}^{M} \pi_m \boldsymbol{p^m} = \boldsymbol{p}$. Then the axiom can be stated the following way:

Subgroup decomposability (SUD): For any natural number $M > 1$ and $k \in \mathcal{C}_{C-1}$, for any $\boldsymbol{p} \in \mathbb{P}$ such that $\sum_{m=1}^{M} \pi_m \boldsymbol{p^m} = \boldsymbol{p}$, where (i) $\boldsymbol{p^m} \in \mathbb{P}_C$ for all $m = 1,2, \dots, M$, (ii) $\pi_m \geq 0$ for all $m = 1,2, \dots, M$, and (iii) $\sum_{m=1}^{M} \pi_m = 1$: $P(\boldsymbol{p}, k) = \sum_{m=1}^{M} \pi_m P(\boldsymbol{p^m}, k)$.

With these four axioms, Seth and Yalonetzky (2020; theorem 1) axiomatically characterise the following broad class of poverty measures for ordinal variables, denoted by $\mathcal{P}$:

$$P(\boldsymbol{p}, k) = \sum_{i=1}^{C} p_i \omega_i, \quad (1)$$

where $\omega_1 = 1$, $\omega_{i-1} > \omega_i > 0$ for all $2 \leq i \leq k$ when $k \geq 2$ and $\omega_i = 0$ for all $k < i \leq C$. Thus, class $\mathcal{P}$ is sensitive to the depth of deprivations (as required by ordinal monotonicity) through a weighting schedule, $\boldsymbol{\omega} = (\omega_1, \omega_2, \dots, \omega_C)$, which is strictly decreasing in the domain of deprivation categories. In a sense the weighting schedule acts like an implicit scale. But at least only a set of ordering weights is being admitted by the axioms imposed. Additionally, as a consequence of the four desirable properties, $P(\boldsymbol{p}, k) = 0$ if and only if nobody is poor and $P(\boldsymbol{p}, k) = 1$ if and only if everybody is in the worst deprivation category. Hence, the indices in $\mathcal{P}$ feature convenient normalisation properties in terms of comparability and capturing meaningful limiting situations.

Some interesting examples of measures in $\mathcal{P}$, include the class based on ordering weights representing relative deprivation ranks (Bennett and Hatzimasoura, 2011), namely using the function:

$$\omega_i^{BH} = \left[\frac{k-i+1}{k}\right]^\gamma, \gamma > 0 \qquad \text{for } i \leq k \tag{2}$$

While every measure in $\mathcal{P}$ is sensitive to the depth of deprivation, not all of them prioritise improvements among the poorest. In order to do so, Seth and Yalonetzky (2020a) adopt the ethical notion of prioritarianism (Parfitt, 1997), which commands granting higher importance to the wellbeing improvements of the poorest, and operationalise different degrees of it in the context of poverty measures.

On one extreme, a minimum degree of precedence to poorer people would demand that a person moving from the worst category to the second-worst should elicit a greater reduction in societal poverty than a single-category improvement benefitting someone starting on the second-worst category (which, in turn, should elicit even further reduction than single-category improvements enjoyed by people departing from better categories). Seth and Yalonetzky (2020a) call it *minimum degree of precedence* (PRE-M). On the other extreme, the *greatest degree of precedence* (PRE-G) stipulates that a person moving from the worst category to the second-worst should elicit a greater reduction in societal poverty than the maximum possible improvement benefitting someone starting on the second-worst category.

More generally, Seth and Yalonetzky (2020a) consider a whole range of degrees of precedence to the poorest among the poor, with PRE-M and PRE-G as the two limiting cases and potentially several intermediate cases in between, depending on the number of deprivation categories and the location of the poverty line. Formally (Seth and Yalonetzky, 2020a, pp. 10-11):

Precedence of poorer people of order α (PRE-α): for any $\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{q}' \in \mathbb{P}_C$, any $k = 2,3, \dots, C-1$ and some $\alpha \in \mathbb{N}$ such that $1 \leq \alpha \leq k-1$, some $s < t \leq k < C$, and some $\epsilon \in (0,1)$; if (1) $\boldsymbol{p}'$ is obtained from $\boldsymbol{p}$ such that $p'_s = p_s - \epsilon$, $p'_{s+1} = p_{s+1} + \epsilon$, while $p'_u = p_u$ for all $u \neq \{s, s+1\}$ and (ii) $\boldsymbol{q}'$ is obtained from $\boldsymbol{p}$ such that $q'_t = p_t - \epsilon$, $q'_{min\{(t+\alpha),C\}} = p'_{min\{(t+\alpha),C\}} + \epsilon$, while $q'_u = p_u$ for all $u \neq \{t, \min\{t+\alpha, C\}\}$, then $P(\boldsymbol{p}'; k) < P(\boldsymbol{q}'; k)$.

Note that α measures the social planner's degree of poverty aversion, which rises from a minimum of PRE-1, which is essentially PRE-M, to a maximum of PRE-$(k-1)$ which is identical to PRE-

G. Here one should highlight that PRE-G bears resemblance to the Hammond transfer introduced by Gravel et al. (2020). In both cases the minimum improvement of a poorer person is granted more importance than any sort of improvement experienced by a better-off person. However, there are some conceptual differences worth highlighting. PRE-G is the operationalisation in the context of ordinal variables of the maximalist version of the prioritarian principle put forward by Parfit (1997). By contrast, the Hammond transfer is an inequality principle stating that any rank-preserving increase in the categorical proximity between two people (starting in different categories) ought to reduce inequality. By definition, the Hammond transfer implies that the poorer person becomes less poor while a less poor person becomes worse-off. Thus an inequality index decreasing in the event of a Hammond transfer is implicitly placing more weight on the minimal improvement of a poor person than any wellbeing deterioration of a less poor person, as long as their ranks are preserved.

Back to PRE-α. Different classes of indices within $\mathcal{P}$ satisfy different degrees of precedence, i.e. they fulfil PRE-α for different values of α. Seth and Yalonetzky (2020a, theorem 2) identify all such classes. Essentially, the members of class $\mathcal{P}_\alpha \subset \mathcal{P}$ satisfying a specific PRE-α are characterised by combinations of the following two additional restrictions on the weighting vectors: (i) $\omega_{s-1} - \omega_s > \omega_s - \omega_{s+\alpha}$ for all $s = 2, \dots k - \alpha$ and (ii) $\omega_{s-1} > 2\omega_s$ for all $s = k - \alpha + 1, \dots, k$ when $\alpha \leq k - 2$. When $\alpha = k - 1$, namely the PRE-G case, the weighting vector takes on just one additional restriction: $\omega_{s-1} > 2\omega_s$ for all $s = 2, \dots, k$. By way of corollary, Seth and Yalonetzky (2020a) show that the class satisfying PRE-M is characterised by the following restriction: (i) $\omega_{s-1} - \omega_s > \omega_s - \omega_{s+1}$ for all $s = 2, \dots k - 1$ and (ii) $\omega_{k-1} > 2\omega_k$.

An interesting feature of these subclasses of indices granting different degrees of precedence to the experience of the poorest, is that they are nested. To be more precise: $\mathcal{P}_{k-1} \subset \mathcal{P}_{k-2} \subset \cdots \subset \mathcal{P}_1 \subset \mathcal{P}$. That is, as expected, all indices satisfying PRE-G are also sensitive to lesser degrees of precedence, but the reverse is not true. Combining examples from Seth and Yalonetzky (2020a) consider this point with vectors $\boldsymbol{\omega}' = (1,0.8,0.5,0.4,0)$, $\boldsymbol{\omega}'' = (1,0.6,0.3,0.1,0)$, $\boldsymbol{\omega}''' = (1,0.48,0.23,0.1,0)$ and let $k = 4$. Clearly, all three weights are members of $\mathcal{P}$. However, $\boldsymbol{\omega}'$ does not really prioritise improvements among the poorest. In fact, the largest improvement occurs when leaving poverty from the least deprived category. Meanwhile both $\boldsymbol{\omega}''$ and $\boldsymbol{\omega}'''$ do attach greater improvements to single-category movements if they start at worst deprivation categories. Hence both weighting vectors are members of $\mathcal{P}_1$. However, $\boldsymbol{\omega}''$ does not provide the greatest precedence possible, whereas $\boldsymbol{\omega}'''$ does, since: 1>2*0.48>4*0.23>8*0.1>0. Therefore, of the latter two weighting vectors only $\boldsymbol{\omega}'''$ belongs in $\mathcal{P}_{k-1}$. If the weights are plotted on the vertical axis against the categories on the horizontal axis, one can clearly note how greater precedence to the poorest among the poor entails a more "convex" curvature of the weighting schedule.

Readers are invited to assess how different choices of the γ parameter affect the membership of rank-based weights such as ω_i^{BH} (equation 2) in each of the $\mathcal{P}_\alpha$ subclasses.

Partial orderings

Clearly, each subclass of poverty measures mentioned above contains many equally valid alternative weighting vectors. Hence the understandable interest in identifying comparisons which are robust to alternative poverty measures within a specific class. Seth and Yalonetzky (2020a) developed, for each class, stochastic dominance conditions whose fulfilment guarantees the robust poverty comparisons. As is well known, a crucial advantage of resorting to stochastic dominance conditions is that they involve testing over a finite domain, as opposed to probing the infinite possibilities within each class of poverty measures.

For the most general class $\mathcal{P}$, Seth and Yalonetzky (2020a, theorem 3) find the ordinal-variable equivalent of first-order stochastic dominance. For a given poverty line the condition reads as follows:

First-order stochastic dominance with fixed poverty line (Seth and Yalonetzky, 2020a; theorem 3): For any $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{P}_C$ and a *given* $k \in \{1,2,\dots,,C-1\}$, $P(\boldsymbol{p},k) < P(\boldsymbol{q},k)$ for all $P \in \mathcal{P}$ if and only if $\sum_{l=1}^{s}[p_l - q_l] \leq 0$ for all $s \leq k$ with at least one strict inequality.

Note how the first part of the theorem states the robustness condition, namely unambiguously less poverty in $\boldsymbol{p}$ than in $\boldsymbol{q}$ considering all poverty measures satisfying the four basic axioms characterising class $\mathcal{P}$ (namely OM, FOC, SUD and SCD); while the second part presents the testable condition based on cumulative frequencies. In fact, the testable condition can be restated in terms of poverty headcounts. That is, poverty in $\boldsymbol{p}$ is robustly lower than in $\boldsymbol{q}$ if and only if the headcounts in the former population are never higher (and at least once lower) than those in the latter for all possible poverty lines up to k. Thus, the test requires performing k comparisons.

For any conceivable poverty line, Seth and Yalonetzky, 2020a; corollary 2) provide the more stringent condition whereby $P(\boldsymbol{p},k) < P(\boldsymbol{q},k)$ for all $P \in \mathcal{P}$ and *any* $k \in \{1,2,\ldots,,C-1\}$, if and only if $\sum_{l=1}^{s}[p_l - q_l] \leq 0$ for all $s \leq C-1$ and $p_1 < q_1$. Hence the condition requires comparing every possible poverty headcount, i.e. $C-1$ comparisons, and verifying a strict inequality in the headcount with the lowest possible poverty line ($k=1$). Seth and Yalonetzky (2020a) note that both theorem 3 and its corollary (2) are the ordinal-variable versions of the headcount-ratio orderings for continuous variables derived by Foster and Shorrocks (1988)

For the classes $\mathcal{P}_\alpha$, Seth and Yalonetzky (2020a, theorem 4) provide the general condition, following a novel proof strategy for stochastic dominance conditions based on convex hulls.

Seth and Yalonetzky (2020a) present the dominance results for classes satisfying PRE-M and PRE-G respectively and for a fixed poverty line:

PRE-M stochastic dominance with fixed poverty line (Seth and Yalonetzky, 2020a; corollary 3): For any $\boldsymbol{p},\boldsymbol{q} \in \mathbb{P}_C$ and a *given* $k \in \{2,\ldots,C-1\}$, $P(\boldsymbol{p},k) < P(\boldsymbol{q},k)$ for all $P \in \mathcal{P}_1$ if and only if $\sum_{l=1}^{s}\sum_{i=1}^{l}[p_i - q_i] \leq 0$ for all $s \leq k$, with at least one strict inequality.

PRE-G stochastic dominance with fixed poverty line (Seth and Yalonetzky, 2020a; corollary 4): For any $\boldsymbol{p},\boldsymbol{q} \in \mathbb{P}_C$ and a *given* $k \in \{2,\ldots,C-1\}$, $P(\boldsymbol{p},k) < P(\boldsymbol{q},k)$ for all $P \in \mathcal{P}_{k-1}$ if and only if $\sum_{l=1}^{s} 2^{1-l}[p_l - q_l] \leq 0$ for all $s \leq k$, with at least one strict inequality.

Seth and Yalonetzky (2020a) note that their corollary 3 is the ordinal-variable versions of the "P_2" poverty orderings for continuous variables derived by Foster and Shorrocks (1988). Meanwhile the distributional condition of their corollary 4 is remarkably similar to the stochastic dominance condition of Gravel et al. (2020) for welfare functions.

Finally, Seth and Yalonetzky (2020a) generalise their partial ordering for any class $\mathcal{P}_\alpha$ and *any feasible poverty line*:

PRE-α stochastic dominance with any poverty line (Seth and Yalonetzky, 2020a; corollary 5): For any $\boldsymbol{p},\boldsymbol{q} \in \mathbb{P}_C$, some $\alpha \in \{1,\ldots,k-1\}$ and *all* $k \in \{2,\ldots,C-1\}$, $P(\boldsymbol{p},k) < P(\boldsymbol{q},k)$ for all $P \in \mathcal{P}_\alpha$ if and only if (a) $p_1 \leq q_1$ and $2p_1 + p_2 \leq 2q_1 + q_2$ with at least one strict inequality, and (b) $\sum_{s=1}^{C}\omega_s^r[p_s - q_s] \leq 0$ for all $r = 3,\ldots,C-1$ where $\boldsymbol{\omega^r}$ is defined in Seth and Yalonetzky (2020a).

3. Welfare measurement

Welfare measurement with ordinal data has gained interest alongside the increased popularity of subjective and self-reported wellbeing indicators (including life satisfaction and happiness whether as global life assessment or focused on areas of life such as occupation, relationships, health, etc.). The measurement proposals bear important similarities with the toolkit put forward for poverty assessments in the sense that both seek to gauge individual improvements, i.e. people moving to better categories, while passing ethical judgments on the dispersion in the multinomial distributions. However, welfare assessments typically set themselves apart for the difficulty (or irrelevance) in setting a poverty line separating deprivation from non-deprivation categories. Hence the focus axioms is no longer needed in welfare assessments.

Gravel et al. (2011) axiomatically characterised mean attainments as social welfare functions for distributional comparisons. The literature on welfare measurement for ordinal variables has followed this route. Specifically, both Apouey et al. (2019) and Gravel et al. (2020) independently characterised social welfare indices which are weighted averages of the relative frequencies in each category and where the weights essentially acts as implicit scales attributed to each category based on a set of desirable axioms. This is basically the same strategy followed by Seth and Yalonetzky (2020a), but without poverty lines and the concomitant focus axiom.

All the proposals in the literature either define from the outset or end up with a welfare measure $W(\boldsymbol{p})$ as a mapping from $\mathbb{P}_C$ onto the real line. Hence, they satisfy traditional properties of anonymity and the population principle (whereby cloning each person by the same natural number should render the welfare index unaltered). In addition, Apouey et al. (2019) proposed an independence axiom which in our notation reads as follows:

Independence (Apouey et al., 2019): For some $i \in \mathcal{C}_C$, let $\boldsymbol{p^{-i}}$ denote the vector of relative frequencies excluding category i. Then independence means: $W(p_i; \boldsymbol{p^{-i}}) - W(q_i; \boldsymbol{p^{-i}}) = W(p_i; \boldsymbol{q^{-i}}) - W(q_i; \boldsymbol{q^{-i}})$.

Now consider ordinal monotonicity (OM) for welfare indices: For any $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{P}$, if $q_i < p_i$ and $i < j$, but $p_s = q_s$ for all $s \neq \{i, j\}$, then $W(\boldsymbol{q}) > W(\boldsymbol{p})$. Satisfaction of ordinal monotonoicity is crucial if one wants W to increase in the aftermath of a so-called increment (Gravel et al. 2020), whereby a person moves to a better category.

Then independence coupled with ordinal monotonicity imply a class of linear welfare indices:

$$W(\boldsymbol{p}) = \sum_{i=1}^{C} p_i w_i \,, \tag{3}$$

where $w_i < w_j$ for any $1 \leq i < j \leq C$ and $\mathcal{W}$ denotes the corresponding class of welfare measures. Note that independence plays a similar role to subgroup decomposability in poverty measurement. The former property is more elegant (as it implies subgroup decomposability), though the latter may be more intuitive.

Now, in the case of poverty measurement, the properties of single-category deprivation (whereby the poverty measure should equate the headcount ratio whenever the poverty line is set at the lowest possible level) and focus naturally impose a normalisation on $P(\boldsymbol{p}; k)$ such that $P(p_1 = 1; k) = 1$ for any $k \leq C - 1$ and $P(p_i = 0; k) = 0$ for all $i \leq k$ and $k \leq C - 1$. This implied normalisation is not a logical consequence of the axioms used to characterise $W(\boldsymbol{p})$ in (8).

Therefore, if one wants to increase comparability with welfare indices applied to ordinal variables one needs to impose a normalisation axiom. This is precisely what Apouey et al. (2019) do:

Normalisation (Apouey et al. 2019): (1) $W(\boldsymbol{p}) = 0$ if and only if $p_1 = 1$; and (2) $W(\boldsymbol{p}) = 1$ if and only if $p_C = 1$.

That is, the welfare index achieves its maximum normalised value of 1 if and only if everyone in society enjoys the highest category. On the other extreme, the index achieves its lowest normalised value of 0 if and only if everyone in societies endures the worst category. Naturally, satisfaction of the normalisation axiom restricts the set of admissible value weights $\boldsymbol{w} = (w_1, \dots, w_C)$, such that: $0 = w_1 < \cdots < w_C = 1$.

However, just as in the case of poverty measurement, the aforementioned restriction on $\boldsymbol{w}$ is agnostic to any concerns over the dispersion of the categorical attainments. Incorporating value judgments favouring a more egalitarian distribution into $W(\boldsymbol{p})$, which already satisfies ordinal monotonicity (i.e. being sensitive to movements from worse to better categories), effectively means adopting the ordinal-variable equivalent of welfare functions consistent with the generalised Lorenz ordering (Shorrocks, 1983). Three main proposals exist in the literature.

Firstly, Seth and Yalonetzky (2020b) suggested applying their prioritarian approach for poverty measures (Seth and Yalonetzky, 2020a) in the realm of welfare measures. The idea behind it is, again, to render the welfare index more sensitive to a single-category improvement by a worse-off person than to an α-category improvement by a better-off person, where $\alpha = 1, \dots, C-2$ is a natural number, with higher values of α signalling higher degrees of prioritisation. The formal definition is:

Priority of order α to the worse-off (PRI-α) (Seth and Yalonetzky, 2020b): For any $\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{q}' \in \mathbb{P}$, for some $\alpha \in \mathbb{N}$ such that $1 \leq \alpha \leq C-2$, for some $s < t < C$, if (i) $p_s' = p_s - \epsilon, p_{s+1}' = p_{s+1} + \epsilon$ while $p_u' = p_u \; \forall u \neq \{s, s+1\}$, and (ii) $q_t' = p_t - \epsilon$, $q_{min\{(t+\alpha),C\}}' = p_{min\{(t+\alpha),C\}}' + \epsilon$ while $q_u' = p_u \forall u \neq \{t, \min\{t+\alpha, C\}\}$, then $W(\boldsymbol{p}') > W(\boldsymbol{q}')$ for all $W \in \mathcal{W}$.

Likewise, for any admissible value of α it is possible to identify subclasses of welfare measures satisfying a specific priority, i.e. PRI-α. In fact, Seth and Yalonetzky (2020b; theorem 2.1) stipulates that for any $C \geq 3$ and for some $\alpha \in \mathbb{N}$ such that $1 \leq \alpha \leq C-2$, $W \in \mathcal{W}$ satisfies PRI-α if and only if $2w_s > w_{s-1} + w_{s+\alpha} \; \forall s = 2, \dots, C-\alpha$ and $2w_s > w_{s-1} + w_C \; \forall s = C-\alpha+1, \dots, C-1$ whenever $\alpha \leq C-3$; and $2w_s > w_{s-1} + w_C \; \forall s = 2, \dots, C-1$ whenever $\alpha = C-2$. All welfare measures satisfying PRI-α are said to belong in subclass $\mathcal{W}_\alpha$.

By analogy to the poverty case, PRI-1, also known as PRI-MIN, provides the minimum degree of priority to the worse-off and members of $\mathcal{W}_1$ are characterised by $2w_s > w_{s-1} + w_{s+1} \; \forall s = 2, \dots, C-1$. On the other extreme, PRI-C-2, also known as PRI-MAX, provides the maximum degree of priority to the worse-off and members of $\mathcal{W}_{C-2}$ are characterised by $2w_s > w_{s-1} +$

w_C $\forall s = 2, \dots, C-1$. The different subclasses of indices are clearly nested, since those satisfying PRI-MAX also satisfy PRI-MIN, but the reverse is not true. Hence, generally: $\mathcal{W}_{C-2} \subset \mathcal{W}_{C-1} \subset \dots \subset \mathcal{W}_1 \subset \mathcal{W}$.

Secondly, Gravel et al (2020) proposed an alternative way of incorporating inequality concerns into welfare assessments with ordinal variables, pioneering the so-called Hammond transfer. Operationalising the ideas of Hammond (1976), Gravel et al. (2020) posit that given the problem of incommensurability of distances between categories, the only uncontroversial inequality-decreasing transformation involves two people who move to nearer categories without switching ranks. Such a transformation, named a Hammond transfer by them, ought to decrease inequality, irrespective of the difference in the number of categories transited by each person in the pair, given that those distances are inherently arbitrary. Only the increased proximity among the pair should matter. Formally, it can be stated that $\boldsymbol{q}$ is obtained from $\boldsymbol{p}$ through a Hammond transfer if there exist natural numbers i, j, k, l such that $1 \leq i < j \leq k < l \leq C$ and $\delta \in (0,1)$ such that $q_i = p_i - \delta$; $q_j = p_j + \delta, q_k = p_k + \delta, q_l = p_l - \delta$, and $q_u = p_u$ $\forall u \neq \{i, j, k, l\}$. That is, at least one pair of people are closer to each other in $\boldsymbol{q}$ vis-à-vis $\boldsymbol{p}$.

Then Gravel et al (2020) identify the class of linear welfare measures of the form (7) satisfying both ordinal monotonicity and sensitivity to Hammond transfers (whereby welfare measures increase in the event of such inequality-reducing transfers). Combining their theorem 3 with proposition 6 and using our notation, the first two parts of a three-statement equivalence result is the following:

Gravel et al (2020) (first two parts of Theorem 3 and proposition 6): For any $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{P}_C$, $W(\boldsymbol{p}) > W(\boldsymbol{q})$ for all $W \in \mathcal{W}_{C-2}$ if and only if $\boldsymbol{p}$ can be obtained from $\boldsymbol{q}$ through a finite sequence of Hammond transfers and/or increments.

Note the above statement "for all $W \in \mathcal{W}_{C-2}$". Essentially, the class of linear welfare functions satisfying ordinal monotonicity and PRI-MAX (Seth and Yalonetzky 2020b) and the class which is sensitive to Hammond transfers and increments are one and the same. The maximum priority that can be given to the worst-off is sensitivity to a Hammond transfer. The third part/statement is a dominance condition that is discussed in the next subsection on partial orderings.

Thirdly, Apouey et al. (2019) introduce the property of proportional equality, with which one can characterise a spectrum of classes of welfare measures varying in their implicit degree of prioritisation of the worst-off. In other words, it can be shown that their measures belong into different $\mathcal{W}_\alpha$ classes with $\alpha = 1, \dots, C-2$. Using the notation of the present chapter, Apouey et al. (2019) define proportional equality as a constant ratio of decrease in the marginal welfare improvement due to moving from one category to the adjacent next:

$$0 < \beta \equiv \frac{w_{s+1} - w_s}{w_s - w_{s-1}} < 1 \text{ for all } s = 2, \dots, C-1. \quad (4)$$

Then, Apouey et al (2019, proposition 3) show that, for any particular β the following welfare measure based on the frequency distribution and satisfying ordinal monotonicity, normalisation, independence and proportional equality, ensues, using our notation:

$$W_\beta(\boldsymbol{p}) = \sum_{s=1}^{C} p_s \frac{1-\beta^{s-1}}{1-\beta^{C-1}} \quad (5)$$

As stated by Apouey et al. (2019) lower values of β reflect increased inequality aversion, which means that, for a given C, the lower the β the higher the degree of priority to the less advantaged. That is, welfare measures with lower values of β belong to subclasses $\mathcal{W}_\alpha$ indexed by higher values of α.

3.1.Partial orderings

Just as in the case of poverty measurement, each subclass of welfare measures mentioned above contains many equally valid alternative weighting vectors ($\boldsymbol{w}$). Therefore, it is also possible to search for welfare comparisons which are robust to alternative welfare measures within a specific class.

The first-order dominance result for ordinal variables is quite well known (Gravel et al., 2020, provide the key references). Using the previous notations and considering strict inequalities in the welfare comparison (i.e. $W(\boldsymbol{p}) > W(\boldsymbol{q})$ instead of $W(\boldsymbol{p}) \geq W(\boldsymbol{q})$), the useful three-statement version proposed by Gravel et al. (2020) is now stated:

First-order stochastic dominance (Gravel et al. 2020; theorem 1): The following statements are equivalent: (1) $\boldsymbol{p}$ is obtained from $\boldsymbol{q}$ through a finite sequence of increments; (2) $W(\boldsymbol{p}) > W(\boldsymbol{q})$ for all $W \in \mathcal{W}$; and (3) $\sum_{l=1}^{s}[p_l - q_l] \leq 0$ for all $s = 1, \dots, C$ with at least one strict inequality.

The above result is quite demanding as it expects robustness across all $W \in \mathcal{W}$, whether they are sensitive to different inequality-decreasing transformations or not at all. Following Seth and Yalonetzky (2020a), Seth and Yalonetzky (2020b) proposed higher-order dominance conditions for different degrees of prioritisation of the worst-off; specifically, for each subclass $\mathcal{W}_\alpha$:

PRI-α stochastic dominance (Seth and Yalonetzky, 2020b; theorem 3.1): For any $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{P}_C$, some $C > 2$ and some $\alpha \in \{1, \dots, C-2\}$, $W(\boldsymbol{p}) > W(\boldsymbol{q})$ for all $W \in \mathcal{W}_\alpha$ if and only if, with at least one strict inequality, $\sum_{s=1}^{C} \omega_s^r [p_s - q_s] \leq 0$ for all $r = 1,2, \dots, C-1$, where ω_s^r is defined in Seth and Yalonetzky (2020b).

Note the resemblance between the PRI-α dominance conditions for welfare measures and the PRE-α dominance conditions for poverty measures when $k = C - 1$ (i.e. when the poverty line is set at its highest possible categorical value). As Seth and Yalonetzky (2020b) explain in the proof of their theorem 3.1, the two sets of dominance conditions match because $P(\boldsymbol{p}, C-1) = (w_C -$

$W(\boldsymbol{p}))/(w_C - w_1)$. Therefore $W(\boldsymbol{p}) - W(\boldsymbol{q}) = -(w_C - w_1)[P(\boldsymbol{p}, C-1) - P(\boldsymbol{q}, C-1)]$. So the conditions applying to the poverty comparison $(P(\boldsymbol{p}, C-1) - P(\boldsymbol{q}, C-1))$ must be the same as those relevant for the welfare comparison $(W(\boldsymbol{p}) - W(\boldsymbol{q}))$.

Note also that the dominance condition for PRI-MAX (i.e. for $\alpha = C-2$) coincides with the third part of theorem 3 in Gravel et al (2020); that is, their dominance condition for welfare measures sensitive to both increments and Hammond transfers. Indeed, Gravel et al (2020) (last two parts of Theorem 3 and proposition 6) state that, for any $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{P}$, $W(\boldsymbol{p}) > W(\boldsymbol{q})$ for all $W \in \mathcal{W}_{C-2}$ if and only if $\sum_{s=1}^{r} 2^{r-s}[p_s - q_s] \leq 0$ for all $r = 1, \dots, C-1$ (Gravel et al. (2020) always consider weak inequalities). Now let $C = 4$. Then this condition involves the comparisons

$p_1 - q_1 \leq 0$, $\sum_{s=1}^{2} 2^{2-s}[p_s - q_s] = 2[p_1 - q_1] + [p_2 - q_2] \leq 0$, and $\sum_{s=1}^{3} 2^{3-s}[p_s - q_s] = 4[p_1 - q_1] + 2[p_2 - q_2] + [p_3 - q_3] \leq 0$. Meanwhile, for $C = 4$ and $\alpha = C - 1 = 3$, one gets from (11) : $\sum_{s=1}^{C} \omega_s^1 [p_s - q_s] = [p_1 - q_1] \leq 0$; $\sum_{s=1}^{C} \omega_s^2 [p_s - q_s] = [p_1 - q_1] + 2^{-1}[p_2 - q_2] \leq 0$, and $\sum_{s=1}^{C} \omega_s^3 [p_s - q_s] = [p_1 - q_1] + 2^{-1}[p_2 - q_2] + 2^{-2}[p_3 - q_3] \leq 0$. Essentially the same conditions.

Mendelson (1987) took a different approach to welfare partial orderings with ordinal variables. Instead of looking into classes of implicit scales satisfying certain desirable properties and into conditions guaranteeing robust welfare comparisons across any choices within each such subclasses (as the aforementioned authors in this section did), he proposed that any welfare comparison (whether based on indices or partial orderings) should be consistent to any alternative choice of scales respecting the order of the categories. Now, first-order stochastic dominance for quantitative data is known to imply second-order dominance. Hence it stands to reason that if a first-order dominance ordering based on attributing scales to ordered categories is robust to any choice of order-respecting scales, then the pair of distributions will also be ordered according to the second-order dominance criterion for all admissible scales.

But Mendelson (1987, theorem 2.2) found that the reverse is also true. This is quite remarkable because with quantitative data, where there is no issue of choosing arbitrary scales, first-order dominance implies second-order dominance, but the reverse is not true. The gist of the proof is as follows. For any given scale, second-order dominance of distribution A over B implies that the mean of A is not lower than the mean of B (one is switching to weak inequalities as Mendelson, 1987, did). This is a standard result in literature. However, if second-order dominance of A over B held for every possible scale respecting the ordering of the categories then the mean of A should not be lower than the mean of B for every possible scale respecting the ordering of the categories. But the latter is simply stating that $W(\boldsymbol{p^A}) \geq W(\boldsymbol{p^B})$ for all $W \in \mathcal{W}$. And it is already known that the latter is true if and only if $\sum_{l=1}^{s}[p_l - q_l] \leq 0$ for all $s = 1, \dots, C$; that is, if there is first-order stochastic dominance (in its weak-inequality form).

4. Inequality measurement

4.1. Equality and maximum inequality in ordinal data.

As demonstrated by Mendelson (1987) and Allison and Foster (2004), inequality rankings generated by measures originally tailored for quantitative data (e.g. the coefficient of variation, the Gini index, etc.) are not robust to alternative choices of scales for the categories. On top of that, as mentioned in the cases of poverty and welfare measurement, such choices are essentially arbitrary. Hence the challenge of finding suitable measures for inequality measurement with ordinal variables.

This section discusses several groups of measurement proposals. First, it reviews those that use natural cardinalisations of categorical ranks. Then it discusses proposals for comparisons of distributions sharing a particular quantile (usually the median), but not requiring any information above and beyond the elements of the ordered multinomial distribution. Among these, a distinction is made between inequality and bipolarisation concepts; as well as between explicit and implicit common quantiles in the formulas. Since these proposals are not immediately suitable for comparisons involving distributions not sharing a quantile of interest (e.g. the median), this section first reviews a recent refinement which expands the reach of admissible comparisons based on ingenuous quantile-homogenising transformation. Then it discusses an alternative approach based on the notion of inequality of status (measured by cumulative frequencies). A similar order is followed when discussing proposals for partial orderings and dominance conditions in a separate subsection.

But first, it is worth discussing suitable benchmarks of equality and maximum inequality for ordinal variables, and the intuition behind them. Equality carries broad consensus in the literature and occurs whenever the whole population reports the same category (whichever that might be). Formally:

Equality: $\boldsymbol{p} \in \mathbb{P}_C$ is an egalitarian distribution if and only if there is a category $i \in \{1, \ldots, C\}$ such that $p_i = 1$.

Let the set of the only C possible egalitarian distributions be denoted as $\mathbb{E}_C \subset \mathbb{P}_C$. On the other extreme, identifying maximum inequality requires additional reasoning. The literature measuring inequality for nominal data identifies maximum so-called diversity whenever $p_i = \frac{1}{C}$ for all $i = 1, \ldots, C$; this is reasonable given that the categories are unordered. However, whenever the categories are ordered, maximum diversity does not concur with the intuition regarding how far inequality could go. For instance, let $C = 4$. Intuitively, starting from a situation of maximum diversity, one could take all the population in the second category and move it to the first category. Then take the population in the third category and move it to the fourth category. Most people would reckon an increase in inequality has occurred since now more members of the population are further apart than before. Indeed, inequality with ordinal variables seems to carry a meaning of bipolarization. As more people get concentrated in the two extreme categories, inequality ought to increase, according to this view. Moreover, if a comparison is made between distributions where only the extreme categories are populated, the intuition tells one that maximum inequality is found in those where exactly half the population is in each extreme. That is, maximum inequality coincides with maximum diversity among the two extreme distributions when the latter contain the whole population between themselves. This is by and large the consensus in the literature. Formally:

Maximum inequality: $\boldsymbol{p} \in \mathbb{P}_C$ is distribution representing maximum inequality if and only if $p_1 = p_C = 0.5$.

Clearly, for each C there is only one maximum unequal distribution which is denoted as $\boldsymbol{i_C} \in \mathbb{P}_C$.

4.2.Inequality-altering transformations

One of the reasons why inequality indices for quantitative data provide inconsistent rankings across alternative scales when applied to ordinal data is that many inequality indices compare distances from a measure of central tendency, usually the mean, which is also sensitive to alternative scales. This led Mendelson (1987) and others thereafter to consider alternative reference points. Mendelson (1987; theorem 2.3) showed that the only location parameters invariant to alternative scales were quantiles. Hence if two distributions shared a quantile, e.g. the median, with a particular scale; then they would always share the same quantile with any alternative scale. That is, the value of the quantile itself could change, but it would be identical across both distributions. By contrast, two distributions may have the same mean with one particular scale, but then have different means as soon as the scale is suitably altered.

Next, Mendelson (1987) adopted the measure of risk as dispersion around the mean pioneered by Rotschild and Stiglitz (1970), the famous mean-preserving spread, and proposed instead a quantile-preserving spread. That is, he proposed comparing distributions with the same summary measure of central tendency in terms of dispersion about it. The key ingredient is choosing a measure which is invariant to alternative scales, hence the recourse to quantiles. Then lower clustering around the quantile should signal higher inequality.

Before continuing one needs to add notation and more precise definitions. Though one could define quantiles in terms of a particular scale or just mention the corresponding percentile, the literature has reasonably preferred to express the quantile in the equally invariant form of the respective category. Let $P_i \equiv \sum_{s=1}^{i} p_i$ stand for the cumulative frequency and $\boldsymbol{P}$ be the C-sized cumulative distribution function (CDF). The quantile category m_α corresponding to percentile α is then defined as follows:

α-quantile: For any $\alpha \in [0,1]$, the α-quantile, $m_\alpha \in \{1, \dots, C\}$ is a category such that: $P_{m_\alpha - 1} < \alpha$ and $P_{m_\alpha} \geq \alpha$.

Since most of the literature has focused on the median, i.e. $\alpha = 0.5$, the shortcut notation $m = m_{0.5}$ will be used to refer to it. Now, the quantile preserving spread itself is a displacement of probability mass away from the common quantile, thereby generating higher spread around it. Formally, the quantile-preserving spread transformation is defined as follows:

α-quantile-preserving spread: For any $\alpha \in [0,1]$, an α-preserving spread is a transfer of population involving categories i and j such that: (i) either i is the recipient category and j is the donor category if $1 \leq i < j \leq m_\alpha$ or i is the donor category and j is the recipient category if $m_\alpha \leq i < j \leq 1$; and (ii) the value of m_α is preserved after the transfer (i.e. the α quantile remains in the same category).

If the median is considered, then reference will be made to median-preserving spreads. Now denote the set of ordered multinomial distributions of size C sharing a common quantile m_α by $\mathbb{P}_{C,\alpha} \subset \mathbb{P}_C$. Also define an inequality index as a non-negative real-valued function of the ordered multinomial distribution $I: [0,1]^C \rightarrow \mathbb{R}_+$. Then, with this definition the measure already satisfies traditional properties of anonymity (whereby category swaps between population pairs should not alter the value of the index) and the population principle (whereby equal replications of each person in the population should not alter the index' value, thereby enabling comparisons of distribution with different population sizes). Additionally, and crucially, one wants I not to decrease in the event of quantile-preserving spreads. Formally:

Aversion to quantile-preserving spreads: for every $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{P}_{C,\alpha}$, $I(\boldsymbol{p}) \geq I(\boldsymbol{q})$ if $\boldsymbol{p}$ is obtained from $\boldsymbol{q}$ through a sequence of α-quantile-preserving spreads.

Another very popular property in the literature (not only that based on quantile-preserving spreads) is normalisation, requiring the inequality index to attain a normalised minimum value of 0 only when the distribution is egalitarian and a normalised maximum of 1 only when the distribution features maximum inequality. Formally:

Normalisation: $I(\boldsymbol{p}) = 0$ if and only if $\boldsymbol{p} \in \mathbb{E}_C$ and: $I(\boldsymbol{p}) = 1$ if and only if $\boldsymbol{p} = \boldsymbol{i_C}$.

Mendelson (1987) proposed the first class of inequality indices for comparison of distributions within the same set $\mathbb{P}_\alpha$. Using the notation of the present chapter, the Mendelson class is the following:

$$W_{m_\alpha}(\boldsymbol{p}) = \sum_{i=1}^{C} p_i \, w_i, \qquad (6)$$

where $w_1 \geq \cdots \geq w_{m_{\alpha-1}}$ and $w_{m_\alpha} \leq \cdots \leq w_C$. When $w_i = -i$ for all $i < m_\alpha$, $w_i = i$ for all $i \geq m_\alpha$ and $\alpha = 0.5$, it is easy to show that one obtains the s index proposed by Allison and Foster (2004), which measures the difference between the average number of categories that the upper half is above the median and the average number of categories that the lower half of the distribution falls below the median.

Clearly, inequality rankings based on these indices are sensitive to changes in the chosen scale. Hence, they have mainly been considered to study partial orderings, i.e. situations where such measures robustly rank distributions regardless of the scale chosen. They will be revisited in the subsection on partial orderings.

4.3. Categorical rank-based measures

Start with categorical rank-based measures. The shared trait of these indices is their foundation on ranks allocated to the categories (hence "categorical rank-based"). In a sense, this is a departure from the principle of ditching scales when measuring inequality with ordinal variables. But at least one can make the case that it is harder to justify any ranking system different from that based on a set of equally distanced natural numbers. Then, categorical rank-based measures become variations on the theme of measuring inequality as some aggregation of rank distances. Very often, this aggregation bears resemblance to one of the formulas of the Gini coefficient for quantitative variables (the one in which a mean-normalised average of the distances between all "income" values is taken). Therefore, if most people are in fewer categories and closer by, inequality should decrease. In fact, in the limit, an egalitarian distribution is characterised by no rank distance whatsoever. On the other extreme, rank distances are maximised as the two halves of the population move further apart. In the limit, the aforementioned situation of maximum inequality ought to maximise any aggregation of categorical rank distances.

The pioneering contribution in this type of indices comes from Berry and Mielke (1992) who define an Index of Ordinal Variation (IOV) as:

$$I^{BM}(\boldsymbol{p}) = \frac{T}{T_{Max}} \qquad (7)$$

where $T = \sum_{i=1}^{C} \sum_{j \neq i}^{C} p_i p_j (j - i)$ and $T_{Max} = \frac{(C-1)}{4}$ if the population size is an even number. Let n be the population size and $n_i = np_i$ be the absolute frequency of people in category i. Then, if n is odd one gets $T_{Max} = \frac{(n^2-1)(C-1)}{4n^2}$. Interestingly, Berry and Mielke (1992) demonstrate that maximum inequality can be attained in at least two situations when n is odd. In terms of relative frequencies: first, when one extreme category has a proportion $(n+1)/2n$ and the other has $(n-1)/2n$; second, when $p_1 = p_C = (n-1)/2n$ and $p_i = 1/n$ for any $i \in \{2, \dots, C-1\}$.

While intuitive, a problem with contributions like I^{BM} is that the respective authors do not discuss the indices' sensitivity to any specific inequality-altering transformation (e.g. a quantile-preserving spread). However, it should not be difficult to show that I^{BM} is averse to median-preserving spreads when comparing distributions with a common quantile of interest (e.g. the median).

Kalmijn and Arends (2010) follow a similar path, making a distinction between the case where the various possible ratings of ordered responses are assumed to be equidistant, and that where such an assumption is not made. In the case of equidistance between ranks, they end up proposing a measure which aggregates rank distances (not unlike Berry an Mielke, 1992). In particular they consider (with the previous notations and a generalisation):

$$I^{KA}(\boldsymbol{p}, \theta) = \frac{S(\theta)}{S^{Max}(\theta)}, \qquad (8)$$

where $S(\theta) = \sum_{i=1}^{C} \sum_{j=1}^{C} p_i p_j |i - j|^{\theta}$ and $S^{Max}(\theta) = \sup S(\theta)$. Kalmijn and Arends (2010) consider the cases $\theta = 1,2$, but one can readily verify that, generally, $S^{Max}(\theta) = \frac{(C-1)^{\theta}}{2}$ when the population is of even size. The case of a population of odd size is left for the reader to work out. Besides the usual discussion of situations of equality and maximum inequality, the authors (like many others) do not discuss their indices' sensitivity to any specific inequality-altering transformation. However, again, it should not be difficult to show that I^{KA} is also averse to median-preserving spreads when comparing distributions with a common quantile of interest (e.g. the median).

Finally, Kalmijn and Arends (2010) extend their analysis to the case where equidistance is not assumed, borrowing ideas from Veenhoven (2009). However, once equidistance is dropped, one is back in the world of all possible scales respecting the order of the categories, hence concerns for robustness to alternative arbitrary scales re-emerge, but without any attempt at attenuating them through restrictions on the admissible set of scales, as done in the poverty and welfare analysis (presented in the previous sections).

Giudici and Raffinetti (2011) suggested drawing a rank-related Lorenz curve where the coordinates of the horizontal axis are given by the ordered elements of $\boldsymbol{P}$. Meanwhile for the vertical axis they propose a rank r_i for each category i as follows: $r_1 = 1, r_i = (r_{i-1} + n_{i-1})\ \forall i = 2, \ldots, C$. Then they propose computing the share of the bottom k categories in the total "rank sum", that is: $s_k = \frac{\sum_{i=1}^{k} r_i n_i}{\sum_{i=1}^{C} r_i n_i}$. Therefore, the coordinates of the rank-based Lorenz curve are (P_i, s_i) for all $i = 1, \ldots, C$. Finally, they propose computing a Gini-style index equal to twice the area between the rank-based Lorenz curve and the 45-degree diagonal line (with coordinates (P_i, P_i)). No major discussion of properties satisfied by this Gini index is provided, though the Gini is expected to be 0 when the distribution is egalitarian.

Lv et al. (2015) provide the first attempt at connecting a measure of inequality based on an aggregation of rank distances with some property reflecting sensitivity to a particular inequality-altering transformation. Moreover, they accomplished an axiomatic characterisation, perhaps the only one for inequality measures based on rank distances. Their class of categorical rank-based indices encompasses those proposed by Berry and Mielke (1992) as well as Kalmjin and Arends (2010) as specific cases.

Lv et al. (2015) consider, firstly, a weak normalisation axiom whereby $I(\boldsymbol{p} \in \mathbb{E}_C) = 0$. Secondly, they also take into account the aversion to median-preserving spreads. Thirdly, they consider a technical axiom of additivity that renders the index a sum of all possible individual pairwise rank inequalities (e.g. one person within category "1" against another one in category "3", etc.). Fourthly, they add independence, which is another technical axiom aimed at rendering the inequality index a linear function of the relative frequencies. Specifically, independence states that a change in a pairwise rank inequality element due to a change in the relative frequency of one of the two categories involved in the comparison should be independent from the level of that frequency. Finally, they use "invariance to parallel shifts" which works a bit like the translation invariance axiom in inequality measurement for quantitative data and states that adding constants to the ranks should not alter the inequality assessment. This latter axiom is interesting in the sense that it identifies a set of permissible ranks. Any choice of ranking system among them does not alter the inequality assessment.

Armed with the aforementioned axioms, Lv et al. (2015) characterise the following class:

$$I^{LWX}(\boldsymbol{p}) = \sum_{i=1}^{C} \sum_{j \neq i}^{C} g(|i-j|) p_i p_j, \qquad (9)$$

where $g(.)$ is an increasing function. Note that these indices can be applied to distributions without common quantiles. Though as in previous contributions, it is not clear which inequality-altering transformations affect the index when distributions do not share quantiles. Otherwise, Lv et al. (2015) do show, as part of their axiomatic characterisation, that I^{LWX} satisfies aversion to median-preserving spreads.

Two particular examples of I^{LWX} proposed by the authors include $I^{KA}(.,1)$, namely the Gini-style aggregator of absolute rank distances; and one based on the exponential function $g(|i-j|) = 2\psi^{C-1-|i-j|}$, where $0 < \psi < 1$.

4.4. Quantile and median-preserving spreads: explicit quantiles

Abul Naga and Yalcin (2008) were the first to propose alternative indices of inequality for ordinal variables which, in addition to their aversion to quantile-preserving spreads, were invariant to alternative scales. The key, they found, was to make the index dependent on the elements of the ordered multinomial distribution. In other words, scales need to be ditched altogether. In our notation and generalised presentation for any α quantile, the class axiomatically characterised by Abul Naga and Yalcin (2008) looks like this:

$$I^{AY}(\boldsymbol{P}) = \frac{\phi(\boldsymbol{P}) - \phi(\boldsymbol{P}|\boldsymbol{p} \in \mathbb{E}_C)}{\phi(\boldsymbol{P}|\boldsymbol{p} = \boldsymbol{i}_C) - \phi(\boldsymbol{P}|\boldsymbol{p} \in \mathbb{E}_C)} , \qquad (10)$$

where ϕ is increasing in all P_i such that $i < m_\alpha$ and decreasing in all P_i such that $i \geq m_\alpha$. Such curvature guarantees the satisfaction of the aversion/transfers axiom. Meanwhile the subtracted element in the numerator alongside the denominator ensure fulfilment of the normalisation axiom. The class I^{AY} also satisfies the standard property of continuity and, chiefly, is completely independent from any scale. Abul Naga and Yalcin (2008) provided a subclass example of (10) which proved quite popular in the literature based on two parameters, which is presented here in a generalised form for all possible α quantiles:

$$I^{AY}(\boldsymbol{P};\alpha,\beta,\gamma) = \frac{\sum_{i<m_\alpha}(P_i)^\gamma - \sum_{i\geq m_\alpha}(P_i)^\beta + (C+1-m_\alpha)}{k_{\gamma,\beta} + (C+1-m_\alpha)} \quad , \tag{11}$$

where:

$$k_{\gamma,\beta} = (m_\alpha - 1)\left(\frac{1}{2}\right)^\gamma - \left[1 + (C - m_\alpha)\left(\frac{1}{2}\right)^\beta\right] \tag{12}$$

Notably, when $\alpha = 0.5$ and $\gamma = \beta = 1$ in (14) and (15) one obtains a symmetric index, meaning that equal deviations from 0.5 below and above the median render the value of the inequality index unaltered:

$$I^{AY}(\boldsymbol{P}; 0.5,1,1) = 1 - \frac{1}{C-1}\left[2\sum_{i=1}^{C}|P_i - 0.5| - 1\right] \tag{13}$$

Otherwise, when $\alpha = 0.5$ but $\gamma \neq \beta$ the subclass in (15) is convenient to introduce asymmetry in order to get different results when deviations from 0.5 take place below or above the median. For example, assume that $\gamma \geq 1$ and $\beta \geq 1$. Then for a given value of β, the index $I^{AY}(.; 0.5, \beta, \gamma)$ becomes more sensitive to the cumulative probability mass at the bottom of the distribution as $\gamma \rightarrow 1$. On the contrary, as $\gamma \rightarrow \infty$ the index will ignore the dispersion below the median, Similar considerations hold evidently when varying β for a given value of γ.

Kobus and Milos (2012) contributed an interesting refinement by examining the class of inequality indices that would ensue if, on top of scale independence, normalisation, and aversion to quantile-preserving spreads, they also satisfied a property of subgroup decomposability. Fulfilment of this property means that the value of the inequality index in the whole population can be expressed as a function of inequality values in respective non-overlapping exhaustive group partitions. This is particularly useful when one considers the specific case of the linear decomposition rendering the index a population-weighted average of the index values in group partitions. In fact, the functional form eventually found by Kobus and Milos (2012) in their axiomatic characterisation is any monotonic transformation of such linear decomposition (which obviously includes the linear decomposition itself as a subclass).

Now, in its most stringent form, i.e. being independent of scales, the decomposition axiom reads as follows using our notation:

Subgroup decomposability (Kobus and Milos, 2012): $I(t\boldsymbol{p}+(\mathbf{1}-\boldsymbol{t})\boldsymbol{q})=f(I(\boldsymbol{p}),I(\boldsymbol{q}),t)$, for any $t\in(0,1)$, any $\boldsymbol{p},\boldsymbol{q}\in\mathbb{P}_C$ and any function $f:\mathbb{R}_+^2\times(0,1)\rightarrow\mathbb{R}_+$ continuous and strictly increasing on the first two arguments.

Then Kobus and Milos (2012, theorem 3 which is here generalised for all quantiles) prove that an inequality index for ordinal variables fulfils the axioms of continuity, normalisation, scale independence, aversion to quantile-preserving spreads and subgroup decomposability if and only if it is of the form:

$$I^{KM}=G\left(\textstyle\sum_{i=1}^{C}w_i p_i\right), \qquad (14)$$

where G is a strictly increasing function and $w_i\geq w_{i+1}$ when $i<m_\alpha$ and $w_i\leq w_{i+1}$ when $i\geq m_\alpha$. That is, in concordance with the principle of aversion to quantile-preserving spreads, the inequality indices in (17) give higher weight to a category, the further away it is from the median.

By way of providing useful examples in the form of linear decompositions, Kobus and Milos (2012) propose the following generalization of $I^{AY}(.;\alpha,1,1)$ (see (10), but here it is generalised for all quantiles):

$$I^{KM}(\boldsymbol{P};\alpha,a,b)=\frac{a\sum_{i<m_\alpha}P_i-b\sum_{i\geq m_\alpha}P_i+b(C+1-m_\alpha)}{(a(m_\alpha-1)+b(C-m_\alpha))/2} \text{ with } a\geq 0; b\geq 0. \qquad (15)$$

Clearly $I^{KM}(\boldsymbol{P};\alpha,1,1)=I^{AY}(\boldsymbol{P};\alpha,1,1)$. Finally, Kobus and Milos (2012, section 4) provide a useful identification of linearly decomposable indices in the literature.

Finally, Chakravarty and Marahaj (2015) show that one can also construct inequality indices for ordinal variables satisfying the median-preserving spreads axioms (when $\alpha = 0.5$) with a general formula like the following (which is generalised for any quantile α):

$$I^{CM}(\boldsymbol{P}; \alpha) = \sum_{i=1}^{m_\alpha - 1} w_i F_i + \sum_{i=m_\alpha}^{C-1} w_i (1 - F_i), \qquad (16)$$

where, crucially, $w_i > 0$ for all $i = 1, \dots, C - 1$.

There are many other inequality indices suitable for comparisons of distributions with common quantiles, i.e. satisfying the aversion-to-spreads axiom, but one will discuss them in the next subsections because, conveniently, they do not require computing the quantile of interest (e.g. the median), therefore their formulas tend to be simpler.

4.5. Quantile and median-preserving spreads: implicit quantiles

The family of inequality indices discussed in this subsection can be best described by the commonality of what they are not. They do not rely on ranks, whether categorical or personal (see subsection below). They do not depend on scales. And they do not include a quantile in their formulas. When applied to distributions sharing quantiles these measures also satisfy the axiom of aversion to quantile-preserving spreads.

Almost every single proposal in the literature of inequality indices for ordinal variables satisfying aversion to quantile-preserving spreads without the quantile of interest in their formulas can be deemed a member of just two classes. Note that one is not considering here the status inequality indices proposed by Cowell and Flachaire (2017) in this typology, because they map from a different domain.

One class was proposed by Lazar and Silber (2013), who noted that functional forms proposed by Reardon (2009) for the measurement of socioeconomic segregation were suitable for inequality measurement for ordinal variables respecting the median-preserving spreads partial ordering. The general functional form is the following:

$$I^{LS}(\boldsymbol{p}) = \left(\frac{1}{C-1}\right) \sum_{i=1}^{C-1} f(P_i) \qquad (17)$$

where $f: [0,1] \rightarrow [0,1]$ is a continuous function, such that $f(0) = f(1) = 0$, $f(P_i)$ is increasing for $P_i \in [0,0.5]$ and then decreasing for $P_i \in [0.5,1]$. Hence $f(0.5) = 1$ is the maximum value attainable. Then clearly $0 = I^{LS}(\boldsymbol{p} \in \mathbb{E}_C) \leq I^{LS}(\boldsymbol{p}) \leq I^{LS}(\boldsymbol{p} = \boldsymbol{i_C}) = 1$. That is, normalisation is satisfied. Moreover, the class also satisfies aversion to median-preserving spreads. Some useful functional examples provided by Lazar and Silber (2013) include: $f_2(P_i) = 4P_i(1 - P_i)$, $f_3(P_i) =$

$2\sqrt{P_i(1-P_i)}$ and $f_4(P_i) = 1 - |2P_i - 1|$. Lazar and Silber (2013) also note that f_4 in equation (20) yields an index mentioned by Apouey (2007) and Abul Naga and Yalcin (2008).

The other general class was proposed by Apouey (2007) and its general functional form can be represented as follows:

$$I^{AP}(\boldsymbol{p}, \alpha) = 1 - \frac{2^\alpha}{C-1} \sum_{i=1}^{C-1} |P_i - 0.5|^\alpha, \alpha > 0, \tag{18}$$

One can easily confirm that $0 = I^{AP}(\boldsymbol{p} \in \mathbb{E}_C) \leq I^{AP}(\boldsymbol{p}) \leq I^{AP}(\boldsymbol{p} = \boldsymbol{i_C}) = 1$; thereby satisfying normalisation. Aversion to median-preserving spreads is also satisfied. Moreover, pioneeringly, Apouey (2007) notes that the proposal class satisfies an axiom of increased bipolarity. In order to understand this property, there is a need to briefly mention the key distinction between inequality and bipolarisation with continuous data (Wolfson, 1994; Foster and Wolfson 2010; Wang and Tsui, 2002; Bossert and Schworm, 2008). In the tradition of inequality measurement based on majorization relationships, any rank-preserving progressive transfer (henceforth Pigou-Dalton transfer) should decrease inequality. Therefore, equality occurs when everyone in the population enjoys the same income. On the other extreme, maximum inequality requires only one person enjoying positive income (though the actual maximum will vary between relative, absolute and intermediate approaches to inequality; see Kolm, 1976).

By contrast, the bipolarisation paradigm is interested in departures from equality that lead toward bimodal clustering at the expense of the "middle class". Therefore bipolarisation approaches distinguish between Pigou-Dalton transfers across the median (i.e. involving a richer person above the median transferring income to a poorer person below the median) and Pigou-Dalton transfers on one side of the median (i.e. either involving two people below the median or two people above the median). The former transfers are deemed to decrease bipolarisation (since mean attainments of both halves get closer to each other) while also decreasing inequality; whereas the latter are deemed to increase bipolarisation since they produce clustering within each half, even though they also decrease inequality. Hence even though equality reasonably coincides with the complete absence of bipolarisation, maximum bipolarisation with quantitative data must differ from maximum inequality. In fact, maximum bipolarisation requires that the bottom half has zero income while the latter half has the same positive income (then the actual maximum will also vary between relative, absolute and intermediate approaches to bipolarisation).

Apouey (2007) was the first to implement these concepts of bipolarisation in the context of ordinal data. For the transformations affecting the spread between the two halves of the distributions she reasonably chose the median-preserving spread promoted by Allison and Foster (2004) and incorporated into scale-free inequality measures by Abul Naga and Yalcin (2008). Hence bipolarisation measures for ordinal data ought to fulfil the axiom of aversion to median-preserving spreads. In order to account for increased clustering within each half she adapted the version of the respective axiom from Wang and Tsui (2002). Effectively, it is possible to restate this axiom in the form of sensitivity to rank-preserving transfers on either side of the median that resemble Pigou-Dalton transfers and are defined the following way:

Simple progressive transfer on either side of the median (Apouey, 2007; Chakravarty and Maharaj, 2015): $\boldsymbol{p}$ is obtained from $\boldsymbol{q}$ through a sequence of simple progressive transfers on either side of the median if there is a pair $\{i,j\}$ and probability mass δ such that $i+1 \leq j-1 < m$ or $m < i+1 \leq j-1$ and $q_{i+1} = p_{i+1} + \delta$, $q_i = p_i - \delta$, $q_{j-1} = p_{j-1} + \delta$, $q_j = p_j - \delta$.

Then the following axiom can be stated:

Increased clustering (or increased bipolarity, e.g. Apouey, 2007; Chakravarty and Maharaj, 2015; Sarkar and Santra, 2020): for every $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{P}_{C,\alpha}$, $I(\boldsymbol{p}) \geq I(\boldsymbol{q})$ if $\boldsymbol{p}$ is obtained from $\boldsymbol{q}$ through a sequence of simple progressive transfers on either side of the median (but not across).

Apouey (2007) then showed that her proposal class (21) satisfies both transfers axioms (aversion to median-preserving spreads and increased clustering) and other standard desirable properties. She also offered useful advice for the choice of α. Specifically, she suggested that if one wanted to locate the uniform distribution in the middle of the complete ordering, then one should solve the following equation for α^* (when $C > 2$):

$$I^{AP}(\boldsymbol{u_C}, \alpha^*) = 1 - \frac{2^{\alpha^*}}{C-1} \sum_{i=1}^{C-1} \left| \frac{1}{C} - 0.5 \right|^{\alpha^*} = 0.5 \rightarrow \alpha^*(C) = \frac{\ln 0.5}{\ln(1-\frac{2}{C})}, \qquad (19)$$

where $\boldsymbol{u_C}$ is the uniform ordered multinomial distribution, whereby: $p_1 = \cdots = p_C = \frac{1}{C}$. Other authors have proposed similar indices. For instance, Blair and Lacy (1996, 2000) implicitly overturned the normalisation axiom (as they were implicitly equating less inequality with high concentration around one single category) and suggested both $1 - I^{AP}(.,2)$ and its square root.

Note that, despite satisfying bipolarisation properties, I^{AP} satisfies the same normalisation axiom as purported inequality indices, which suggests that the distinction between inequality and bipolarisation is not that crucial in the world of ordinal data. Indeed, in addition to the identity between equality and complete absence of bipolarisation, the situations of maximum inequality and maximum bipolarisation are also bound to coincide with ordinal variables, since $\boldsymbol{i_C}$ also represents a situation of complete clustering compounded by the farthest spread possible between the two halves.

This subsection concludes by noting that a subclass from the class $I^{CM}(.;0.5)$ of indices (equation 16) proposed by Chakravarty and Maharaj (2015) fulfils the median-preserving spreads and increased-clustering axioms if and only if $w_{i+1} > w_i > 0$ for all $i \leq m-2$ and $0 < w_{i+1} < w_i$ for all $m \leq i \leq C-1$ (Chakravarty and Maharaj, 2015; theorem 3).

4.6.Distributional comparisons without common quantiles of interest

Though a major breakthrough, the aforementioned inequality indices based on the concept of quantile-preserving spreads (subsections 4.4 and 4.5) have a potential problem: it is now known a priori whether they are suitable for comparisons involving distributions without common quantiles. With quantitative data, it is always possible to implement transformations, such as rescaling or translations, to equalise the means of two distributions. Since certain sets of inequality indices and partial-ordering curves (e.g. the Lorenz curve) are insensitive to such transformations, one can then use them to perform inequality comparisons. Unfortunately, those transformations will not be able to homogenise quantiles between two distributions, because they are rank-dependent and the ranks are not altered by alternative scales (as shown by Mendelson, 1987).

Thankfully, Sarkar and Santra (2020) made another breakthrough in the form of proposing categorical transformations which homogenise the quantile of interest across distributions. Then it is possible to resort to the aversion to quantile-preserving spreads axiom. That is, these transformations play the same role as rescaling, translations, and the like in the world of quantitative data, and without the need to rely on arbitrary scales for categories, which is most remarkable. However, not all inequality measures satisfying aversion to quantile-preserving spreads are insensitive to the transformations proposed by Sarkar and Santra (2020) to homogenise the quantile of interest. Only those inequality indices which are (1) averse to quantile-preserving spreads and (2) insensitive to quantile-homogenising transformations are suitable for inequality comparisons with distributions without common quantiles based on the notion of quantile-preserving spreads.

This subsection first explains the two quantile-homogenising transformations proposed by Sarkar and Santra (2020) followed by a typology of indices in terms of their suitability for comparisons without common quantiles based on Sarkar and Santra (2020, Table 1).

The first quantile-homogenising transformation is called a slide and it is used to displace a subset of adjacent frequencies toward an extreme if and when there are empty categories "available". For instance, consider $\boldsymbol{p_1} = (0.1,0.3,0.5,0.1,0,0)$. One could obtain $\boldsymbol{p_2} = (0,0.1,0.3,0.5,0.1,0)$ from $\boldsymbol{p_1}$through one "slide to the right", but also one could obtain $\boldsymbol{p_3} = (0,0,0.1,0.3,0.5,0.1)$ from $\boldsymbol{p_1}$through two "slides to the right". Formally:

Slides (Sarkar and Santra, 2020; definition 7): Let $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{P}_C$. Then $\boldsymbol{q}$ is obtained from $\boldsymbol{p}$ through $k < C$ slides to the right (or alternatively $\boldsymbol{p}$ is obtained from $\boldsymbol{q}$ through $k < C$ slides to the left) if and only if $p_i = 0$ for all $= C - k + 1, \dots, C$, $q_i = 0$ for all $i = 1, \dots, k$ and $q_i = p_{i-k}$ for all $i = k + 1, \dots, C$.

The second quantile-homogenising transformation is called "addition of unpopulated categories at extremities" (henceforth "additions") and, as the name says, simply involves obtaining one distribution from another one by adding any number of empty categories at either extreme of the categorical ordering. Formally:

Additions (Sarkar and Santra, 2020; definition 7): One says that $\boldsymbol{q} \in \mathbb{P}_{C+k}$ is obtained from $\boldsymbol{p} \in \mathbb{P}_C$ by adding k empty categories to the right (left) if and only if $\boldsymbol{q} = (\boldsymbol{p}, 0_k)$ $((0_k, \boldsymbol{p}))$.

Then, Sarkar and Santra (2020) show that different combinations of slides and additions can homogenise medians (and potentially other quantiles). In some cases, only slides may be needed, in others only additions, and in a third kind a specific combination thereof will be required. For example (Sarkar and Santra, 2020, p. 10), consider $\boldsymbol{p} = (0,0.1,0.2,0.1,0.2,0.2,0.2)$ with $m^p = 5$ and $\boldsymbol{q} = (0.2,0.4,0.3,0.1,0,0,0)$ with $m^q = 2$. If we slide $\boldsymbol{p}$ once toward the left and **q** twice toward the right then we end up with $\boldsymbol{p}' = (0.1,0.2,0.1,0.2,0.2,0.2,0)$ and $\boldsymbol{q}' = (0,0,0.2,0.4,0.3,0.1,0)$, both with $m' = 4$.

Often there will not be empty categories to start with. In those cases one needs to perform additions before resorting to slides. For instance, consider $\boldsymbol{p} = (0.4,0.05,0.25,0.2,0.1)$ with $m^p = 3$ and $\boldsymbol{q} = (0.1,0.2,0.15,0.25,0.3)$ with $m^q = 4$ (p. 11). Then if one adds to both distributions one empty category to the left and then slides just $\boldsymbol{q}$ one category to the right, one will end up with: $\boldsymbol{p}' = (0,0.4,0.05,0.25,0.2,0.1)$ and $\boldsymbol{q}' = (0.1,0.2,0.15,0.25,0.3,0)$, both with $m^p = 4$.

Hence, once one obtained the median-homogenised distributions, one should be able to make comparisons with any indices which, in addition to fulfilling the spread axioms (and any other bipolarisation-related axiom if making bipolarisation comparisons), are insensitive to both slides and additions. Sarkar and Santra (2020, table 1) show that $I^{AP}(., \alpha)$ and $I^{CM}(.; \alpha)$ for all α, as well as $I^{LS}(.)$ for all the functional forms proposed by Reardon (2009), and an extension by Lazar and Silber (2013) of a class of indices proposed by Abul Naga and Yalcin (2008), all satisfy the required insensitivities and are, therefore, suitable for comparisons of distributions without common medians, after the appropriate slide and/or addition transformations. Meanwhile, $I^{AY}(.; \alpha, \beta = \gamma)$ and $I^{KM}(.; \alpha, a = b)$ are also suitable (note the parametric restrictions involved).

Sarkar and Santra (2020, table 1) also provided a helpful taxonomy differentiating between pure inequality indices (i.e. those satisfying only the spreads axiom) and bipolarisation axioms (i.e. those satisfying both the aversion to spreads and increased-clustering axioms). For instance, among the indices reviewed above, only $I^{AP}(., 0.5)$ and $I^{AY}(.; 0.5, \gamma > 1, \beta > 1)$, and $I^{CM}(.; 0.5)$ when $w_{i+1} > w_i > 0$ for all $i \leq m-2$ and $0 < w_{i+1} < w_i$ for all $m \leq i \leq C-1$, are proper bipolarisation indices. The extensions (not reviewed here) by Lazar and Silber (2013) of a class of indices proposed by Abul Naga and Yalcin (2008) also need a parametric restriction in order to operate as a bipolarisation index.

4.7. Individual rank-based measures.

Cowell and Flachaire (2017) provide an ingenious alternative solution to the problem of performing inequality comparisons without common quantiles, which also serves as an appealing alternative to the controversial cardinalisation involved in assigning natural numbers to the categories' ranks. Their key ingredient is a notion of personal status which they operationalise, after a careful discussion, in four possible ways: (1) peer-inclusive downward-looking status which takes the form of the proportion of people in the same category or worse, i.e. associated to the cumulative relative frequency of the person's category ; (2) peer-exclusive downward-looking status which is the proportion of people in any worse category; (3) peer-inclusive upward-looking status which is measured by the proportion of people in the same category or better, i.e. associated to the survival frequency of the person's category; (4) peer-exclusive upward-looking status which is the proportion of people in any better category.

Then they propose measuring inequality in ordinal data as the level of dispersion in personal status. This means that their inequality indices map from a set of dimension n; i.e. $I: [0,1]^n \rightarrow \mathbb{R}$. For that purpose, the second key ingredient is the reference point determining the equality situation and around which dispersion manifests. For peer-inclusive statuses (whether upward-looking or downward-looking) Cowell and Flachaire (2017) convincingly propose using the value 1 as the reference point, since only when everyone is in the same category, personal status is equal to 1; that is, everyone is in the same category or worse (in the case of downward-looking status), or everyone is in the same category of better (in the case of upward-looking status), from everyone's "point of view" at the same time. Likewise, for peer-exclusive status, the authors sensibly propose using 0 as the reference point; since only when everyone is in the same category it is the case that nobody is in a worse category (in the case of downward-looking status) or in a better category (in the case of upward-looking status). Hence everyone's peer-exclusive status must be 0 in that situation.

The class of status-inequality measures proposed by Cowell and Flachaire (2017) is axiomatically characterised using five axioms: (1) a traditional continuity axiom (so that small changes in personal status do not generate major changes in the inequality measure); (2) a traditional anonymity axiom (whereby only the status information matters, so the inequality assessment is not altered if two individuals swap categories); (3) a typical technical property of independence (to render the inequality index a function of a sum of functions of status); (4) scale invariance (which is meant to render inequality comparisons unaltered when all statuses are rescaled by the same scalar); and (5) monotonicity in distance (whereby changes in status that increase the distance between the status and the reference point should increase inequality).

Note that the inequality-increasing transformation is provided by the latter axiom of monotonicity in distance. This is one of the few alternatives to quantile-preserving spreads in the literature, as far as inequality-altering properties go. Among its merits, it can be meaningfully applied to distributional comparisons without common quantiles. Also, interestingly, the five axioms imply weak normalisation, meaning that the indices in the class are bound to be equal to 0 if and only if all statuses are equal to the reference point. The characterised class bears striking similarities to the generalised-entropy class of inequality measures for quantitative data, which in turn can be represented as Atkinson inequality indices. For the case of downward-looking peer-inclusive status the axiomatically characterised class is the following:

$$I^{CF}(\boldsymbol{p};\alpha) = \frac{1}{\alpha(1-\alpha)}\left[\sum_{i=1}^{C} p_i (P_i)^{\alpha} - 1\right], \text{ for } \alpha \in (-\infty, 1)/\{0\}, \quad (20)$$

$$I^{CF}(\boldsymbol{p};0) = -\sum_{i=1}^{C} p_i \log P_i\,, \quad (21)$$

where one should note that for every category i there are bound to be p_i people all sharing the same status P_i (hence expressions like $\sum_{i=1}^{C} p_i (P_i)^{\alpha}$ or $\sum_{i=1}^{C} p_i \log P_i$ in the formulas). The parameter α is reasonably restricted to be strictly lower than 1 in order to rule out negative values for I^{CF}. For upward-looking peer-inclusive status, P_i needs to be replaced by $S_i \equiv \sum_{j=i}^{C} p_j$ in equations (20) and (21). For downward-looking peer-exclusive status P_i needs to be replaced by $P_i - p_i$ in equations (20) and (21) and "-1" needs to be deleted in equation (20). Finally, for upward-looking peer-exclusive status P_i needs to be replaced by $S_i - p_i$ in equations (20) and (21) and, again, "-1" needs to be deleted in equation (20).

On the significantly plus side, inequality measurement for ordinal variables based on personal status is intuitively appealing and enables comparisons of distributions without common quantiles.

On the other hand, the rankings generated by status-inequality measures may not be robust to alternative choices among the four possible ways of representing status with cumulative frequencies.

4.8. Partial orderings

As in the cases of poverty and welfare comparisons, one is entitled to ask whether certain inequality comparisons are robust to alternative choices of indices among those capturing the same notion of inequality and satisfying the same set of desirable properties. Our review of the main inequality partial orderings for ordinal variables in this subsection roughly follows the typology of inequality indices discussed in the previous subsections.

4.8.1. Lorenz curves in the face of arbitrary scales

As mentioned previously, Mendelson (1987) showed that, if one is going to compare ordered multinomial distributions using scales respecting the order of the categories, then first-order stochastic dominance of $\boldsymbol{p}$ over $\boldsymbol{q}$ for all admissible scales holds if and only if second-order dominance (i.e. generalised-Lorenz dominance) holds for all possible scales. Zheng (2008) extended this analysis to the assessment of Lorenz and absolute-Lorenz dominance, and concluded that neither ever holds across all admissible scales. The message is stark but intuitive: when one considers inequality partial orderings which ought to be robust to all admissible alternative scales, Lorenz and absolute Lorenz (and surely also intermediate Lorenz) curves have "no applicability in ranking inequality" (Zheng, 2008, p. 177).

4.8.2. Categorical rank-based measures

The vast majority of categorical rank-based measures are some specific case or subclass of the class $I^{LWX}(\boldsymbol{p}) = \sum_{i=1}^{C}\sum_{j\neq i}^{C} g(|i-j|)p_i p_j$ in (21) axiomatically characterised by Lv et al. (2015). Since many functional forms for $g(.)$ satisfying desirable properties are possible, Yalonetzky (2016) set out to derive a stochastic-dominance condition whose fulfilment would guarantee the robustness of an inequality comparison to all those possible choices pertaining to $g(.)$. Yalonetzky (2016), from the intuitive notion of the probability of finding two people in the population exhibiting an absolute gap in the ranks of their categories of exactly $\delta = |i-j|$, where $\delta \in \{0,\ldots,C-1\}$, showed that this probability is given by:

$$\pi(\boldsymbol{p},\delta) = \sum_{i=1}^{C-1} p(i)p(i+\delta),\ \ \delta = 0,1,\ldots,C-1 \quad (22)$$

Now consider the following accumulation of π: $\Pi(\boldsymbol{p},\delta)=\pi(0)+2\sum_{i=1}^{\delta}\pi(i)$, $\delta=0,\dots,C-1$. Then the dominance condition reads as follows:

First-order stochastic dominance for measures of categorical-rank inequality (Yalonetzky, 2016, proposition 1): For any $\boldsymbol{p},\boldsymbol{q}\in\mathbb{P}_C$, $I(\boldsymbol{p})>I(\boldsymbol{q})$ for all $I\in I^{LWX}$ if and only if $\Pi(\boldsymbol{p},\delta)\leq\Pi(\boldsymbol{q},\delta)$ for all $\delta=0,1,\dots,C-1$, with at least one strict inequality.

Intuitively, the population with a lower proportion of pairs of people with more similar categories will be robustly ranked more unequal than a population with a higher proportion of pairs of people with less dissimilar categories.

4.8.3. Individual rank-based measures.

Jenkins (2020) derived a partial ordering which is respected by all measures of inequality based on peer-inclusive downward-looking status proposed by Cowell and Flachaire (2017). The partial-ordering is based on generalised Lorenz curves (Shorrocks, 1983) mapping percentiles of the ordered multinomial distribution onto cumulative sums of individual statuses measured by peer-inclusive cumulative frequencies and ordered from lowest to highest. Formally and in our notation:

$$GL(\boldsymbol{p};k)=\sum_{i=1}^{c-1}p_iF_i+(k-F_{c-1})F_c,\ k\in[0,1],\ c=\arg\max F_{c-1}\ s.t.\ F_{c-1}<k\leq F_c \quad (23)$$

Clearly, $GL(\boldsymbol{p};k)\leq GL(\boldsymbol{p}\in\mathbb{E}_C;k)=k$ for all $k\in[0,1]$. That is, the generalised Lorenz curve maximises its height for any point in its domain only when evaluated at any equality distribution. Then Jenkins (2020) crucially proves (in our notation):

Stochastic dominance for status inequality comparisons (Jenkins 2020, Result 2): $I(\boldsymbol{p})\leq I(\boldsymbol{q})$ for any inequality index I that is a decreasing convex function of individual status (as measured by cumulative relative frequency of the individual's category) if and only if $GL(\boldsymbol{p};k)\geq GL(\boldsymbol{q};k)$ for all $k\in[0,1]$.

That is, if the generalised Lorenz curve of $\boldsymbol{p}$ is never below that of $\boldsymbol{q}$**,** then $\boldsymbol{p}$ is unambiguously less unequal than $\boldsymbol{q}$ for any measure of inequality in peer-inclusive downward-looking status operationalised by cumulative relative frequencies (including, of course, the class axiomatically characterised by Cowell and Flachaire 2017).

As Jenkins (2020) points out, readers should be forgiven for being surprised at seeing the generalised-Lorenz curve involved in robust inequality comparisons, since these curves are more appropriate for dispersion-sensitive welfare assessments when dealing with quantitative data (as opposed to pure inequality assessments when the means differ). One explanation (slightly different from the author's) for this seemingly odd result is that with status variables one does not really have a mean that may or may not vary across compared distributions. Instead one has a constant reference point across distributions (either 1 or 0). This is akin to comparing distributions with the same mean, in which case the generalised-Lorenz curve suffices.

Jenkins (2020) also attains two interesting corollaries. First, he shows that $GL(\boldsymbol{u_c}; k) \leq GL(\boldsymbol{i}; k)$ for all $k \in [0,1]$. That is, in status inequality comparisons, the maximum-bipolarisation distribution (where half of the population is in the bottom category and the other half is in the top) does not represent the situation of maximum inequality, since the uniform distribution attains an even lower generalised Lorenz curve. Secondly, he shows that the uniform distribution does not represent maximum inequality either, because any distribution obtained from the uniform distribution with a single transfer of people between two categories will have a generalised Lorenz curve which crosses that of the uniform distribution.

3.8.4 Quantile-preserving spreads and bipolarisation with common quantiles.

The first partial ordering for comparisons sensitive to quantile-preserving spreads was proposed by Mendelson (1987), who, as was previously mentioned, was interested in the robustness of partial orderings to alternative scales respecting the order of the categories. In our notation and interpretation, Mendelson's partial ordering works as follows:

Quantile-preserving spreads partial ordering (Mendelson, 1987, theorem 3.1): Let S_C be any collection of C numbers such that $s_1 < s_2 < \cdots < s_C$, and define the expected value $\mu(\boldsymbol{p}, g(S_c)) = \sum_{i=1}^{C} p_i g(s_i)$. Let $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{P}_{C,m_\alpha}$, for some $\alpha \in (0,1)$. Then the following statements are equivalent:

(1) $P_i \leq Q_i$ for all $i < m_\alpha$ and $P_i \geq Q_i$ for all $i \geq m_\alpha$.
(2) $\mu(\boldsymbol{p}, g(S_c)) \leq \mu(\boldsymbol{q}, g(S_c))$ for all $g(.)$ non-increasing in $s_i \leq s_{m_\alpha}$ and non-decreasing in $s_i \geq s_{m_\alpha}$.

There is a third equivalent statement related to a data-generating-process argument, but it is omitted here as it does not add much to the discussion.

Clearly, $\mu(., g(.))$ acts as an inequality index when $g(.)$ is asked to behave as in the second statement, placing more weight on the values away from the common quantile. The first statement is the classic distributional condition for quantile-preserving spreads. A more useful restatement of the quantile-preserving spread partial ordering was provide by Kobus (2015) who, focusing on median-preserving spreads partial orderings, concocted a three-statement theorem linking the above statement (1) from Mendelson's theorem 3.1 with median-preserving spreads and with inequality indices evaluated at the elements of the ordered multinomial distribution without need to rely on arbitrary scales. Here a generalised version of Kobus' theorem 1 for α-quantiles is presented:

Quantile-preserving spreads partial ordering (Kobus, 2015, definitions 1,2,3 and theorem 1, generalised): Let $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{P}_{C,m_\alpha}$, for some $\alpha \in (0,1)$. Then the following statements are equivalent:

(1) $P_i \leq Q_i$ for all $i < m_\alpha$ and $P_i \geq Q_i$ for all $i \geq m_\alpha$.
(2) $I(\boldsymbol{p}) \leq I(\boldsymbol{q})$ for any inequality index I satisfying aversion to α-quantile-preserving spreads.
(3) $\boldsymbol{q}$ can be obtained from $\boldsymbol{p}$ through a sequence of α-quantile-preserving spreads (see definition above).

This is a very useful results because statement (1) is a relatively easy testable condition and its implications are fleshed out in statement (2): fulfilment of condition (1) is necessary and sufficient to guarantee the robustness of an inequality comparisons between any two distributions sharing a particular α-quantile to all possible inequality indices satisfying the key property of aversion to α-quantile-preserving spreads.

When the median is chosen, i.e. $m = m_{0.5}$, one can also identify the bipolarisation partial ordering for distributions with common median:

Bipolarisation partial ordering (Sarkar and Santra 2020, theorem 1): Let $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{P}_{C,m}$. Then the following statements are equivalent:

(1) $\sum_{i=1}^{k} P_{m-i} \leq \sum_{i=1}^{k} Q_{m-i}$ for every $k = 1, \dots, m-1$ and $\sum_{i=m}^{k} P_i \geq \sum_{i=m}^{k} Q_i$ for every $k = m, \dots, C-1$.
(2) $I(\boldsymbol{p}) \leq I(\boldsymbol{q})$ for any inequality index I satisfying aversion to median-preserving spreads and increased clustering.
(3) $\boldsymbol{q}$ can be obtained from $\boldsymbol{p}$ through a sequence of median-preserving spreads and/or simple progressive transfers on either side of the median (see definitions above).

3.8.5 Bipolarisation and median-preserving spreads without common quantiles

In previous discussion it was shown how Sarkar and Santra (2020) proved that inequality comparisons without common quantiles are still possible if the distribution's quantile of interest is homogenised using combinations of slides and "additions", followed by the application of inequality indices which are insensitive to those homogenising transformations. Likewise, Sarkar and Santra (2020) showed that comparisons are possible with partial orderings applying specific combinations of these homogenising transformations. More specifically, the authors attain two "possibility" results and one "impossibility" result. One presents them starting with the good news:

Bipolarisation partial ordering with exclusive resort to slides (Sarkar and Santra 2020, theorem 3): Let $\boldsymbol{p} \in \mathbb{P}_{C,m+k}, \boldsymbol{q} \in \mathbb{P}_{C,m}$ and $k \in \{1, \dots, C-m\}$. If $\boldsymbol{p}', \boldsymbol{q}' \in \mathbb{P}_{C,m'}$ are obtained from $\boldsymbol{p}$ and $\boldsymbol{q}$ respectively, in the first case by k_1 slides to the left and in the second case by k_2 slides to the right such that $k_1 + k_2 = k$. Then the following conditions hold:

(1) $P_i' \leq Q_i'$ for all $i < m'$ and $P_i' \geq Q_i'$ for all $i \geq m'$ if and only if $I(\boldsymbol{p}') \leq I(\boldsymbol{q}')$ for any inequality index I satisfying aversion to median-preserving spreads and insensitivity to slides.

(2) $\sum_{i=1}^{t} P_{m-i} \leq \sum_{i=1}^{t} Q_{m-i}$ for every $t = 1, \dots, m-1$ and $\sum_{i=m}^{t} P_i \geq \sum_{i=m}^{t} Q_i$ for every $t = m, \dots, C-1$ if and only if $I(\boldsymbol{p}) \leq I(\boldsymbol{q})$ for any inequality index I satisfying aversion to median-preserving spreads, increased clustering and insensitivity to slides.

Theorem 3 essentially states that both inequality and bipolarisation partial orderings are feasible after median homogenisations when the latter are based solely on slides. Naturally, these partial orderings ensure robustness only among classes of inequality or bipolarisation indices which are insensitive to slides.

Bipolarisation partial ordering with additions to both distributions and slide to one distribution (Sarkar and Santra 2020, theorem 4): Let $\boldsymbol{p} \in \mathbb{P}_{C,m+k}, \boldsymbol{q} \in \mathbb{P}_{C,m}$ and $k \in \{1, \dots, C-m\}$ such that slides without additions cannot homogenise the median. If $\boldsymbol{p}', \boldsymbol{q}' \in \mathbb{P}_{C+k,m'}$ are obtained from $\boldsymbol{p}$ and $\boldsymbol{q}$ respectively, in the first case by adding k empty categories to the left followed by k slides to the left and in the second case by only adding k empty categories to the left. Then the following conditions hold:

(1) $\sum_{i=1}^{t} P_{m-i} \leq \sum_{i=1}^{t} Q_{m-i}$ for every $t = 1, \dots, m-1$ and $\sum_{i=m}^{t} P_i \geq \sum_{i=m}^{t} Q_i$ for every $t = m, \dots, C-1$ if and only if $I(\boldsymbol{p}) \leq I(\boldsymbol{q})$ for any inequality index I satisfying aversion to median-preserving spreads, increased clustering, insensitivity to slides and insensitivity to additions of empty categories.

Theorem 4 essentially states that only bipolarisation partial orderings are feasible after median homogenisations when the latter are based solely on additions in the case of one distribution and a combination of slides and equal additions in the other distribution. Naturally, these partial orderings ensure robustness only among classes of bipolarisation indices which are insensitive to both slides and additions.

Finally, Sarkar and Santra (2020) also identify an "impossibility result":

Impossibility of median-preserving-spread partial ordering with additions to both distributions and slide to one distribution (Sarkar and Santra 2020, theorem 2): Let $\boldsymbol{p} \in \mathbb{P}_{C,m+k}, \boldsymbol{q} \in \mathbb{P}_{C,m}$ and $k \in \{1, \dots, C-m\}$ such that slides without additions cannot homogenise the median. If $\boldsymbol{p}', \boldsymbol{q}' \in \mathbb{P}_{C+k,m'}$ are obtained from $\boldsymbol{p}$ and $\boldsymbol{q}$ respectively, in the first case by adding k empty categories to the left followed by k slides to the left and in the second case by only adding k empty categories to the left. Then $\boldsymbol{p}', \boldsymbol{q}' \in \mathbb{P}_{C+k,m'}$ cannot be ordered by the median-preserving spread partial ordering.

In a nutshell, combinations of additions and slides may homogenise the median and enable bipolarisation partial orderings, but they will never enable median-preserving spread partial orderings. As explained by Sarkar and Santra (2020, figure 1), this happens because one can obtain $\boldsymbol{q}'$ from $\boldsymbol{p}'$ (in theorem 2) through a combination of median-preserving spreads and median-preserving clusterings around the median (i.e. the opposite of spreads).

3.8.6 Hammond inequality partial orderings

This review ends with a brief discussion of the approach proposed by Gravel et al. (2020) for robust inequality comparisons with ordinal data based on their concept of Hammond transfers. As mentioned previously, Hammond transfers in the context of ordered categorical variables are rank-preserving movements of pairs of people that render them closer to each other. Crucially, the two people involved do not need to transit the same number of categories. The only thing that matters is that, besides being closer to each other after the movements, they do not swap ranks.

Gravel et al. (2020) consider the class of weighted sum of relative frequencies discussed in equation (3) and repurpose it for inequality measurement based on sensitivity to Hammond transfers. Let $I^{GMM}(\boldsymbol{p}) = \sum_{i=1}^{C} w_i p_i$. Then Gravel et al. (2020, proposition 5) show that I^{GMM} is an equality index increasing in the event of Hammond transfers if and only if there exists $t \in \{1, \dots, C\}$ such that $w_{i+1} - w_i \geq w_t - w_{i+1}$ for all $i \in \{1, \dots, t-1\}$ (if any) and $w_{j+1} - w_j \leq w_j - w_t$ for all $j \in \{t, \dots, C-1\}$ (if any). Note the concavity involved.

It is actually possible to transform I^{GMM} into a proper inequality index decreasing in the event of the Hammond transfers if one states that there exists $t \in \{1, \dots, C\}$ such that $w_i - w_{i+1} \geq w_{i+1} - w_t$ for all $i \in \{1, \dots, t-1\}$ (if any) and $w_j - w_t \leq w_{j+1} - w_j$ for all $j \in \{t, \dots, C-1\}$ (if any). Note the convexity involved and that there is no need to have common quantiles in an inequality comparison using this criterion.

Though Grave et al (2020) do not discuss specific choices for I^{GMM}, they do provide a useful three-part theorem stating what one could call the Hammond inequality partial ordering (which one has tweaked in order to work with proper inequality indices):

Hammond inequality partial ordering (Gravel et al. 2020, theorem 5, tweaked): Let $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{P}_C$ and $I^{GMM}(\boldsymbol{p}) = \sum_{i=1}^{C} w_i p_i$. Then the following statements are equivalent:

(1) $\boldsymbol{p}$ is obtained from $\boldsymbol{q}$ through a sequence of Hammond transfers.

(2) $I^{GMM}(\boldsymbol{p}) \leq I^{GMM}(\boldsymbol{p})$ for all I^{GMM} characterised by the existence of some $t \in \{1, \ldots, C\}$ such that $w_i - w_{i+1} \geq w_{i+1} - w_t$ for all $i \in \{1, \ldots, t-1\}$ (if any) and $w_j - w_t \leq w_{j+1} - w_j$ for all $j \in \{t, \ldots, C-1\}$ (if any).

(3) $\sum_{i=1}^{k} 2^{k-i}[p_i - q_i] \leq 0$ and $\sum_{i=k+1}^{C} 2^{i-k-1}[p_i - q_i] \leq 0$ for all $k = 1, \ldots, C-1$.

The third statement provides the stochastic-dominance condition that can be tested statistically (also called implementation criteria by Gravel et al., 2020). Its fulfilment guarantees the robustness of the inequality comparison to any alternative choice of Hammond-transfer-sensitive inequality index satisfying the parametric restrictions laid out in the second statement.

Summary

This chapter provided a brief review of the key recent contributions to the measurement of poverty, welfare, and inequality with ordinal variables. A common theme in all these proposals is the way they seek to answer the challenge posed by the lack of commensurability in the distances between the ordered categories.

It was shown that, when measuring poverty (except when merely using headcounts) and welfare one ends up using scales despite the inherent arbitrariness. The same happens with inequality measurement based on categorical ranks. At least one tried to restrict and inform the choice of scales through the imposition of desirable properties. Also, in the case of inequality measurement with categorical ranks, only the most intuitive ranks were admitted (e.g. equally spaced natural numbers from one until the total number of categories). While a difficult compromise, this problem is less serious in partial orderings where it is expected that a pairwise comparison is robust to all possible admissible scales within a class delimited by the aforementioned axioms. However, as is well known, by definition partial orderings are limited in their ability to compare all possible pairs of distributions. Moreover, while they will tell us whether some distribution is robustly better than another one (in terms of lower poverty, higher welfare or less inequality); partial orderings will not provide any measure of the gap (in poverty, welfare, or inequality) since the latter requires a cardinal measure.

Unlike poverty and welfare measurement, the literature does offer proposals for inequality measurement with ordinal variables sparing the need to resort to arbitrary scaling. In those cases, the recommendation is not to work with scales if they can be dismissed. Generally, if scales are unavoidable (e.g. in poverty and welfare assessments), it was strongly recommended to use, by way of second-best, either subsets of scales restricted by axiom fulfilment, or natural/intuitive ranks, or focusing on partial orderings.

Specifically concerning inequality measurement, it was found that a large part of the literature consists of authors proposing classes of indices that are either generalisations or subclasses of other contributions. There seems to be plenty of independent simultaneous discovery and "reinvention of the wheel". However, many proposals do not go beyond considering the benchmarks of equality and maximum inequality. Implicitly these proposals deem inequality as a distance away from reference points of equality (whichever way it is construed) and/or toward the proposed notion of maximum inequality.

By contrast, more elaborate proposals devote attention to the underlying inequality-altering transformations. By far the most popular transformation is the quantile-preserving spread; particularly the median-preserving spread. Second in popularity is, perhaps, the clustering transformation which, together with quantile-preserving spreads characterises the popular notion of bipolarisation with ordinal variables. The key difference between inequality and bipolarisation in this specific literature is that the latter concept explicitly incorporates the two transformations and expects its respective indices to be sensitive to both, whereas the inequality proposals only focus on the spreads property. Remarkably both notions, inequality and bipolarisation, are indistinguishable regarding their situations of equality and maximum inequality. This stands in sharp contrast with the quantitative-data literature, where the two notions of dispersion also differ in their maxima.

Unfortunately, the paradigm of median-preserving spreads is limited by the difficulties inherent in comparing distributions with different medians. Without any data transformations, it is impossible to make meaningful inequality comparisons with indices complying with an aversion to the median-preserving spreads (unless they implicitly satisfy an alternative inequality-altering axiom which does not require equal quantiles of interest). Additionally, the medians cannot be homogenised with scaling transformations as it is done with the means in the quantitative-data literature. Yet the literature has attempted to circumvent this problem.

Crucially, Sarkar and Santra (2020) showed that median homogenisation is feasible in nearly every pairwise comparison using combinations of additions of empty categories on the extremes with so-called slides of the entire vector of relative frequencies in the direction of the empty categories. Then they identified inequality and bipolarisation indices from the literature which are insensitive to these median-homogenisation transformations. These are precisely the indices that should be used for comparisons with different medians. They should be applied once the distributions' medians are homogenised. Interestingly, the authors showed that partial orderings can also be undertaken following these procedures, but some combinations of additions and slides may foreclose median-preserving partial orderings.

Cowell and Flachaire (2017) as well as Jenkins (2020) took an entirely different route which also enables inequality comparisons without common medians (or any different quantile of interest), but it entails a completely different, yet useful and interesting, notion of inequality with ordinal variables. Essentially, they propose measuring inequality in a conception of status which relates to the proportion of people below or above one's category, either in a so-called peer-inclusive or peer-exclusive way.

As a final reflection, it should be stressed that there is at least one remaining gap in the literature worth exploring pertains to distributional comparisons involving distributions with different number of categories. One could not find work addressing this issue, and one suspects that the status-inequality approach may be the most amenable to these comparisons. Even the transformations proposed by Sarkar and Santra (2020) render the ensuing distributions with the same number of categories, which is thoroughly sensible. But the question remains whether it is possible to make meaningful poverty, welfare, and inequality comparisons of distributions with different numbers of categories when the indices and partial orderings map from a set whose dimension is given by the number of categories itself (as opposed to status inequality measures and curves which map from a set whose dimension is given by the number of people).

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