Two classes of β-perfect graphs that do not necessarily have simplicial extremes

Jake Horsfield* Kristina Vušković†

February 11, 2021

Abstract

For a graph G, $\beta(G) = \max\{\delta(G')+1 \mid G'$ is an induced subgraph of $G\}$. This parameter is an upper bound on the chromatic number of a graph. A graph G is β-perfect if $\chi(G') = \beta(G')$ for all induced subgraphs G' of G. A number of classes have been shown in literature to be β-perfect, but for the class of all β-perfect graphs, the complexity of their recognition and characterization in terms of forbidden induced subgraphs remain open.

It is known that minimally β-imperfect graphs cannot have a simplicial extreme, i.e. a vertex whose neighborhood is a clique or of size 2. β-perfection of all the known β-perfect classes of graphs was shown through the existence of simplicial extremes. In this paper we study two classes of β-perfect graphs that do not necessarily have this property.

A hole is a chordless cycle of length at least 4, and it is even or odd depending on the parity of its length. β-perfect graphs cannot contain even holes as induced subgraphs. We prove that graphs that do not contain an even hole, a twin wheel nor a cap as an induced subgraph are β-perfect (where a twin wheel is a graph that consists of a hole and a vertex that has exactly 3 neighbors on the hole, that are furthermore consecutive on the hole; and a cap is a graph that consist of a hole and a vertex that has exactly 2 neighbors on the hole, that are furthermore consecutive on the hole). This class properly contains chordal graphs, and is the only known generalization of chordal graphs that is shown to be β-perfect.

A hyperhole is a graph that is obtained from a hole by a sequence of clique substitutions. We give a complete structural characterization of β-perfect hyperholes, which we then use to give a linear-time algorithm to recognize whether a hyperhole is β-perfect. We also obtain a complete list of minimally β-imperfect hyperholes.

*School of Computing, University of Leeds, Leeds, UK. E-mail: sc15jh@leeds.ac.uk

†School of Computing, University of Leeds, Leeds, UK. Partially supported by EPSRC grant EP/N019660/1. E-mail: k.vuskovic@leeds.ac.uk

1 Introduction

Throughout this paper, all graphs are finite and simple. We say that a graph G *contains* a graph F if some induced subgraph of G is isomorphic to F, and it is F*-free* if it does not contain F. A graph G is $\mathcal{F}$*-free*, for a family of graphs $\mathcal{F}$, if G is F-free for every $F \in \mathcal{F}$. A *hole* in a graph is a chordless cycle of length at least 4. A hole is *even* (resp. *odd*) if it has an even (resp. odd) number of vertices.

For a graph G, let

$$\beta(G) = \max\{\delta(G') + 1 \mid G' \text{ is an induced subgraph of } G\},$$

where $\delta(G)$ is the minimum degree of a vertex in G. A graph G is β*-perfect* if $\chi(G') = \beta(G')$ for all induced subgraphs G' of G. A k*-coloring* of a graph G is a function $c\colon V(G) \to \{1, \dots, k\}$ such that for adjacent vertices $u, v \in V(G)$, $c(u) \neq c(v)$. The chromatic number of a graph G, denoted by $\chi(G)$, is the smallest integer k for which there exists a k-coloring of G.

For a graph G, consider the ordering $v_1, \dots, v_n$ of $V(G)$ such that for $i \in \{1, \dots, n\}$, the vertex v_i is of minimum degree in $G[\{v_1, \dots, v_i\}]$. Coloring greedily with respect to this ordering gives the bound $\chi(G) \leq \beta(G)$. It follows that, given such an ordering, the greedy coloring algorithm produces an optimal coloring for β-perfect graphs. Since this ordering can be produced in polynomial time, β-perfect graphs can be colored in polynomial time.

Observe that since any even hole H has $\chi(H) = 2$ and $\beta(H) = 3$, any β-perfect graph must be even-hole-free. Since there are even-hole-free graphs that are not β-perfect (see Figure 1), the class of all β-perfect graphs forms a proper subclass of even-hole-free graphs. β-perfect graphs were introduced by Markossian, Gasparian and Reed in [7], where the following class of graphs was shown to be β-perfect. A *cap* is a hole H together with an additional vertex whose neighbourhood in H induces an edge. A *diamond* is a K_4 with one edge removed.

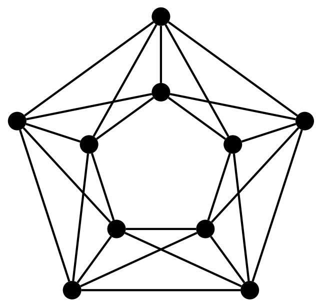

Figure 1: An even-hole-free graph that is not β-perfect.

Theorem 1.1 (Markossian, Gasparian and Reed [7]). *If G is an (even hole, diamond, cap)-free graph, then G is β-perfect.*

This result was then extended to the following.

Theorem 1.2 (de Figueiredo and Vušković [4]). *If G is an (even hole, diamond, cap-on-6-vertices)-free graph, then G is β-perfect.*

In the paper of de Figueiredo and Vušković, they conjectured that excluding only even holes and diamonds is sufficient for β-perfection. This conjecture was answered positively in [6].

Theorem 1.3 (Kloks, Müller and Vušković [6]). *If G is an (even hole, diamond)-free graph, then G is β-perfect.*

A graph is *minimally β-imperfect* if it is not β-perfect but all of its proper induced subgraphs are β-perfect. A vertex is *simplicial* if its neighborhood is a clique, and is a *simplicial extreme* if either it is simplicial or it has degree 2. The proofs of Theorems 1.1, 1.2 and 1.3 used the following result on minimally β-imperfect graphs.

Lemma 1.4 (Markossian, Gasparian and Reed [7]). *If G is a minimally β-imperfect graph that is not an even hole, then G does not have a simplicial extreme.*

In view of Lemma 1.4, in order to show that every graph in a class $\mathcal{C}$ of even-hole-free graphs is β-perfect, it suffices to show that every graph in $\mathcal{C}$ contains a simplicial extreme.

Lemma 1.5. *If G is a minimally β-imperfect graph, then $\beta(G) = \delta(G) + 1$.*

PROOF — Let H be an induced subgraph of G such that $\beta(G) = \delta(H) + 1$. Suppose that H is a proper induced subgraph of G. By our choice of H we have that $\beta(H) = \beta(G)$, so $\chi(H) = \beta(H) = \beta(G)$. But then $\chi(G) \geq \beta(G)$, contradicting our assumption that G is not β-perfect. So $H = G$ and $\beta(G) = \delta(G) + 1$. □

A graph is *chordal* if it is hole-free. It is well known that every chordal graph has a simplicial vertex, and hence it follows directly from Lemma 1.4 that chordal graphs are β-perfect.

Theorem 1.6. *If G is a chordal graph, then G is β-perfect.*

It is not known whether β-perfect graphs are recognizable in polynomial time. It also remains open to characterise the class of β-perfect graphs in terms of forbidden induced subgraphs.

We observe that for all of the above mentioned classes of graphs, their β-perfection was shown by proving that every graph in the class must have a simplicial extreme. In this paper we study two classes of β-perfect graphs that do not necessarily have simplicial extremes.

A *twin wheel* is a graph comprised of a hole and a vertex that has exactly three neighbors on this hole, that are furthermore consecutive vertices of the hole. In Section 2 we prove that (even hole, twin wheel, cap)-free graphs are β-perfect, hence generalizing both Theorems 1.1 and 1.6.

In Section 3 we study hyperholes, i.e. graphs that can be obtained from holes by clique substitutions (they are defined more formally later). So, for example, a twin wheel is also a hyperhole. We give a complete structural characterization of β-perfect hyperholes, which we then use to give a linear-time algorithm to recognize whether a hyperhole is β-perfect. We also obtain a complete list of minimally β-imperfect hyperholes.

1.1 Terminology and notation

Let G be a graph. The vertex set of G is denoted by $V(G)$. For a vertex $x \in V(G)$, $N_G(x)$ (or simply $N(x)$ when clear from context) denotes the set of all neighbors of x in G, and $N_G[x]$ (or simply $N[x]$ when clear from context) denotes the closed neighborhood of x, i.e. the set $N_G(x) \cup \{x\}$.

Let A and B be disjoint subsets of $V(G)$. We say that A is *complete* (resp. *anticomplete*) to B if every vertex of A is adjacent (resp. nonadjacent) to every vertex of B.

A *clique* (resp. *stable set*) is a (possibly empty) set of pairwise adjacent (resp. nonadjacent) vertices of G. The size of a maximum clique in G is denoted by $\omega(G)$, and the size of a maximum stable set in G is denoted by $\alpha(G)$. A *complete graph* is a graph whose vertex set is a clique. We refer to a complete graph on 3 vertices also as a *triangle*.

Let S be a subset of $V(G)$. $G[S]$ denotes the subgraph of G induced by S, and $G \setminus S = G[V(G) \setminus S]$. We say that S is a *cutset* if $G \setminus S$ is disconnected. A cutset S is a *clique cutset* if S is a clique.

A *cycle* is a graph C with vertex set $V(C) = \{x_1, \dots, x_k\}$ (where $k \geq 3$) and edge set $E(C) = \{x_1x_2, x_2x_3, \dots, x_{k-1}x_k, x_kx_1\}$. Under these circumstances, we say that the *length* of C is k. A cycle of length k is denoted by C_k. A *chord* of a cycle C in graph G is an edge of G that is not an edge of C but both of whose ends belong to C, and C is *chordless* if no edge of G is a chord of C.

2 β-perfection of (even hole, twin wheel, cap)-free graphs

In this section we prove that (even hole, twin wheel, cap)-free graphs are β-perfect (Theorem 2.7). Note that chordal graphs are properly contained in the class of (even hole, twin wheel, cap)-free graphs. Furthermore, since diamond-free graphs do not contain twin wheels, this result generalises Theorem 1.1.

The following well-known characterisation of chordal graphs, and the construction of (cap, 4-hole)-free graphs with a hole and no clique cutset given by Theorem 2.2 are used to prove Lemma 2.3, a decomposition theorem for (even hole, twin wheel, cap)-free graphs.

Theorem 2.1 (Dirac [5]). *If G is a chordal graph, then either G is a complete graph or it has a clique cutset.*

Given graphs G and F, we say that G is *obtained by blowing up vertices of F into cliques* provided that there exists a partition $\{X_v\}_{v\in V(F)}$ of $V(G)$ into nonempty cliques such that for all distinct $u, v \in V(F)$, if $uv \in E(F)$ then X_u is complete to X_v in G, and if $uv \notin E(F)$ then X_u is anticomplete to X_v in G. A *universal clique* is a (possibly empty) clique K in a graph G such that K is complete to $V(G) \setminus K$.

Theorem 2.2 (Cameron, da Silva, Huang and Vušković [2]). *Let G be a (cap, 4-hole)-free graph that contains a hole and has no clique cutset. Let F be any maximal induced subgraph of G with at least 3 vertices that is triangle-free and has no clique cutset. Then G is obtained from F by first blowing up vertices of F into cliques, and then adding a universal clique. Furthermore, any graph obtained by this sequence of operations starting from a (triangle, 4-hole)-free graph with at least 3 vertices and no clique cutset is (cap, 4-hole)-free and has no clique cutset.*

Lemma 2.3. *If G is an (even hole, twin wheel, cap)-free graph, then one of the following holds:*

(i) G is a complete graph;
(ii) G consists of a triangle-free graph on at least 3 vertices that has no clique cutset together with a (possibly empty) universal clique; or
(iii) G has a clique cutset.

PROOF — Let G be an (even hole, twin wheel, cap)-free graph and assume that (i) and (iii) do not hold. By Theorem 2.1, G contains a hole. Let F be any maximal induced subgraph of G with at least 3 vertices that is triangle-free and has no clique cutset. By Menger's theorem, every vertex of F is contained in a hole. Since G does not contain a twin wheel, Theorem 2.2 implies that (ii) holds, i.e. $V(G) \setminus V(F)$ is a clique that is complete to $V(F)$. □

Lemma 2.4. *If G is a graph whose vertex set can be partitioned into (possibly empty) sets A and B so that:*

- *A is a clique of G,*
- *if $B \neq \varnothing$ then $G[B]$ is a (triangle, even hole)-free graph, and*

- *A is complete to B,*

then G is β-perfect.

PROOF — It suffices to show that $\chi(G) = \beta(G)$. Clearly we may assume that $B \neq \varnothing$. By Theorem 1.1, $\chi(G[B]) = \beta(G[B])$. But then $\beta(G) = \beta(G[B]) + |A| = \chi(G[B]) + |A| = \chi(G)$. $\square$

Let $S \subseteq V(G)$ be a clique cutset of G and let $C_1, \dots, C_k$ be the connected components of $G \setminus S$. The *blocks of decomposition* of G with respect to the clique cutset S are graphs $G_i = G[V(C_i) \cup S]$, for $i = 1, \dots, k$. If, for some i, G_i has no clique cutset then G_i is an *extreme block* and S is an *extreme clique cutset*. To complete our proof we will use the following well-known property of clique cutsets.

Lemma 2.5. *If a graph G has a clique cutset, then it has an extreme clique cutset.*

Lemma 2.6 (Markossian, Gasparian and Reed [7])**.** *Let G be a (triangle, even hole)-free graph. Let x be a vertex of G. Then either $\{x\}$ is complete to $V(G) \setminus \{x\}$ or there is some vertex y in $G \setminus N[x]$ such that y has degree at most 2 in G.*

A graph is *k-degenerate* if every one of its subgraphs has a vertex of degree at most k. It is well known that the chromatic number of a k-degenerate graph is at most $k + 1$.

Theorem 2.7. *If G is an (even hole, twin wheel, cap)-free graph, then G is β-perfect.*

PROOF — Suppose not and let G be a minimally β-imperfect (even hole, twin wheel, cap)-free graph. By Lemma 1.5, $\beta(G) = \delta(G) + 1$.

Lemmas 2.3 and 2.4 together imply that G has a clique cutset. Let K be an extreme clique cutset of G (it exists by Lemma 2.5). Let $C_1, \dots, C_k$ be the connected components of $G \setminus K$, and $G_1, \dots, G_k$ their respective blocks of decomposition. Without loss of generality, let $G_1 = G[V(C_1) \cup K]$ be an extreme block. Since G_1 has no clique cutset, by Lemma 2.3 G_1 is either a complete graph or a 2-connected triangle-free graph together with a universal clique.

If G_1 is a clique, then every vertex of C_1 is a simplicial extreme in G_1 and hence in G, contradicting Lemma 1.4. So $V(G_1)$ may be partitioned into sets A_1 and B_1 such that A_1 is a clique, $G[B_1]$ is 2-connected triangle-free, and A_1 is complete to B_1. Since $G[B_1]$ is 2-connected triangle-free, by Lemma 2.6 B_1 contains 2 nonadjacent distinct vertices y_1 and y_2 that are both of degree 2 in $G[B_1]$. It follows that y_1 and y_2 are both of degree $2 + |A_1|$ in G_1. Without loss of generality, assume that $y_1 \in V(C_1)$. It follows that $d_G(y_1) = 2 + |A_1|$. Since $G[B_1]$ contains an odd hole and is 2-degenerate, $\chi(G[B_1]) = 3$ and so $\chi(G_1) = 3 + |A_1|$. But then

$$\beta(G) = \delta(G) + 1 \leq d_G(y_1) + 1 = 3 + |A_1| = \chi(G_1) \leq \chi(G),$$

and hence $\beta(G) = \chi(G)$, contradicting our assumption that G is minimally β-imperfect. $\square$

We observe that the class of (even hole, twin wheel, cap)-free graphs can be recognized in polynomial time. In [3] it was shown that (even hole, cap)-free graphs can be recognized in polynomial time. To recognize whether a graph contains a twin wheel it suffices to test for every $\{u, v, x, y\} \subseteq V(G)$ that induces a diamond, with say $uv \notin E(G)$, whether there is a path from u to v in $G \setminus ((N[x] \cup N[y]) \setminus \{u, v\})$.

3 Hyperholes

A *hyperhole* is a graph H whose vertex set can be partitioned into $k \geq 4$ nonempty sets $X_1, \ldots, X_k$ such that for all $i \in \{1, \ldots, k\}$, X_i is a clique that is complete to $X_{i-1} \cup X_{i+1}$ and anticomplete to $V(H) \setminus (X_{i-1} \cup X_i \cup X_{i+1})$. Under these circumstances, we say that H is a *k-hyperhole*, and write $H = (X_1, \ldots, X_k)$. We refer to $(X_1, \ldots, X_k)$ as the *hyperhole partition* of H, and to sets X_i as the *bags* of H. Throughout, indices of bags are assumed to be taken modulo k (e.g. $X_{k+1} = X_1$). A k-hyperhole is *odd* if k is odd, and it is *even* otherwise. It is easy to see that for a hyperhole $H = (X_1, \ldots, X_k)$, $\omega(H) = \max_{1 \leq i \leq k} |X_i \cup X_{i+1}|$. Furthermore, observe that every induced subgraph of a k-hyperhole is either a chordal graph or a k-hyperhole.

The following result on the chromatic number of a hyperhole is used throughout this section.

Theorem 3.1 (Narayanan and Shende [8]). *If H is a hyperhole, then*

$$\chi(H) = \max \left\{ \omega(H), \left\lceil \frac{|V(H)|}{\alpha(H)} \right\rceil \right\}.$$

Corollary 3.2. *If H is a minimally β-imperfect k-hyperhole with k odd, then*

$$|V(H)| \leq \frac{(\beta(H) - 1)(k - 1)}{2}.$$

PROOF — Since k is odd, $\alpha(H) = \frac{k-1}{2}$, and hence by Theorem 3.1, $\chi(H) \geq \frac{2|V(H)|}{k-1}$. Since H is minimally β-imperfect, $\beta(H) > \chi(H)$. It follows that $\beta(H) - 1 \geq \frac{2|V(H)|}{k-1}$. $\square$

The following lemma will be used repeatedly throughout this paper.

Lemma 3.3. *Let $H = (X_1, \ldots, X_k)$ be a minimally β-imperfect k-hyperhole. Then for all $i \in \{1, \ldots, k\}$, the following hold:*

(i) if $|X_{i-1}| = |X_{i+1}| = 1$, then $|X_i| \geq \beta(H) - 2$;

(ii) if $|X_i| = 1$, then $|X_{i-2} \cup X_{i-1} \cup X_i| \geq \beta(H)$ and $|X_i \cup X_{i+1} \cup X_{i+2}| \geq \beta(H)$;

(iii) if $|X_i| = |X_{i+1}| = 1$, then $|X_{i+2}| \geq \beta(H) - 2$ and $|X_{i-1}| \geq \beta(H) - 2$.

PROOF — Suppose that for some $i \in \{1, \ldots, k\}$, $|X_{i-1}| = |X_{i+1}| = 1$ and $|X_i| \leq \beta(H) - 3$. Fix a vertex $x \in X_i$. Then $d(x) \leq \beta(H) - 2$ and so $\beta(H) \geq d(x) + 2 \geq \delta(H) + 2$, contradicting Lemma 1.5. So (i) holds.

To prove (ii), by symmetry it suffices to prove that if $|X_1| = 1$, then $|X_1 \cup X_2 \cup X_3| \geq \beta(H)$. Suppose that $|X_1| = 1$ but $|X_1 \cup X_2 \cup X_3| \leq \beta(H) - 1$. Fix a vertex $x \in X_2$. Then $d(x) \leq \beta(H) - 2$ and so $\beta(H) \geq d(x) + 2 \geq \delta(H) + 2$, contradicting Lemma 1.5. So (ii) holds.

It follows directly from (ii) that (iii) holds. □

Lemma 3.4. *Let $H = (X_1, \ldots, X_k)$ be a k-hyperhole such that $|X_i| \geq 2$ for all $i \in \{1, \ldots, k\}$. Then H is not β-perfect.*

PROOF — If k is even then clearly H is not β-perfect, so we may assume that k is odd, and hence $k \geq 5$. Consider a k-hyperhole $H' = (X'_1, \ldots, X'_k)$ such that for all $i \in \{1, \ldots, k\}$, $|X'_i| = 2$. Clearly $\delta(H') = 5$, and so $\beta(H') \geq 6$. Using Theorem 3.1 we obtain $\chi(H') = \max\{4, \lceil \frac{4k}{k-1} \rceil\}$. Since $\lceil \frac{4k}{k-1} \rceil = 5$ for all $k \geq 5$, $\chi(H') = 5 < \beta(H')$. Therefore H' is not β-perfect. Since we may find an induced subgraph of H that is isomorphic to H', it follows that H is not β-perfect. □

3.1 The 5-hyperholes and 7-hyperholes

We begin by characterising β-perfect 5-hyperholes and 7-hyperholes.

Theorem 3.5. *Let $H = (X_1, \ldots, X_5)$ be a 5-hyperhole. Then H is β-perfect if and only if some bag of H is of size 1.*

PROOF — If H is β-perfect, then some bag of H is of size 1, for otherwise Lemma 3.4 is contradicted.

Now suppose, without loss of generality, that $|X_1| = 1$ but that H is not β-perfect. Since every induced subgraph of H is either chordal or is a 5-hyperhole, by Theorem 1.6 we may assume that H is minimally β-imperfect. By Lemma 3.3, $|X_1 \cup X_2 \cup X_3| \geq \beta(H)$ and $|X_4 \cup X_5 \cup X_1| \geq \beta(H)$. So $|V(H)| \geq 2\beta(H) - 1 = \frac{4(\beta(H)-1)}{2} + 1$, contradicting Corollary 3.2. □

So the graph in Figure 1 is the only minimally β-imperfect 5-hyperhole.

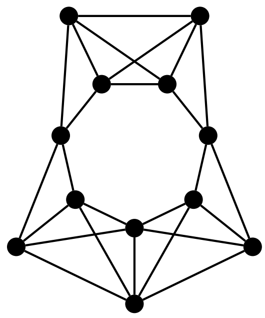

Figure 2: A minimally β-imperfect 7-hyperhole.

Theorem 3.6. *Let $H = (X_1, \ldots, X_7)$ be a 7-hyperhole. Then H is β-perfect if and only if for some $i \in \{1, \ldots, 7\}$, either $|X_i| = |X_{i+1}| = 1$ or $|X_i| = |X_{i+2}| = 1$.*

PROOF — Suppose that H is β-perfect but $(|X_i|, |X_{i+1}|) \neq (1, 1)$ and $(|X_i|, |X_{i+2}|) \neq (1, 1)$ for all $i \in \{1, \ldots, 7\}$. By Lemma 3.4, we may assume without loss of generality that $|X_1| = 1$. We begin by claiming that we may assume that for all $i \in \{1, \ldots, 7\} \backslash \{1, 4\}$, $|X_i| \geq 2$. Suppose that $|X_j| = 1$ for some $j \in \{1, \ldots, 7\} \backslash \{1, 4\}$. From our assumption that $(|X_i|, |X_{i+1}|) \neq (1, 1)$ and $(|X_i|, |X_{i+2}|) \neq (1, 1)$ for all $i \in \{1, \ldots, 7\}$, it follows that $j \in \{4, 5\}$. So, without loss of generality, we may assume that $|X_4| = 1$. It follows from the same assumption that all remaining bags are of size at least 2.

Therefore H contains a hyperhole $H' = (X'_1, \ldots, X'_7)$ such that $|X'_1| = |X'_4| = 1$ and with all remaining bags being of size 2. By Theorem 3.1, $\chi(H') = \max\{\omega(H'), \left\lceil \frac{|V(H')|}{3} \right\rceil\} = \max\{4, \lceil \frac{12}{3} \rceil\} = 4$. But $\beta(H') \geq \delta(H') + 1 = 5 > \chi(H')$, and hence H' is not β-perfect, contradicting our assumption that H is β-perfect.

Suppose now that $|X_1| = |X_2| = 1$ but that H is not β-perfect. Since every induced subgraph of H is chordal or a 7-hyperhole with two consecutive bags of size 1, by Theorem 1.6 we may assume that H is minimally β-imperfect. By Lemma 3.3, $|X_3| \geq \beta(H) - 2$ and $|X_7| \geq \beta(H) - 2$. If $|X_4| \geq 2$, then $\chi(H) \geq \omega(H) \geq |X_3 \cup X_4| \geq \beta(H)$ and so $\chi(H) = \beta(H)$, contradicting our assumption that H is minimally β-imperfect. So $|X_4| = 1$, and by symmetry $|X_6| = 1$. It then follows from Lemma 3.3 that $|X_5| \geq \beta(H) - 2$. But now $|V(H)| \geq 3(\beta(H) - 2) + 4 = \frac{6(\beta(H)-1)}{2} + 1$, contradicting Corollary 3.2.

Finally, suppose that $|X_1| = |X_3| = 1$ but that H is not β-perfect. As before, by Theorem 1.6 we may assume that H is minimally β-imperfect. It follows from Lemma 3.3 that $|X_2| \geq \beta(H) - 2$, $|X_3 \cup X_4 \cup X_5| \geq \beta(H)$, and $|X_6 \cup X_7 \cup X_1| \geq \beta(H)$. Therefore $|V(H)| \geq 3\beta(H) - 2 = \frac{6(\beta(H)-1)}{2} + 1$, contradicting Corollary 3.2. This completes the proof. $\square$

It follows that the graph in Figure 2 is the only minimally β-imperfect

7-hyperhole.

3.2 Odd hyperholes of length at least 9

Let $H = (X_1, \ldots, X_k)$ be an odd hyperhole. For $i, j, m \in \{1, \ldots, k\}$, $(X_i, \ldots, X_j)$ is a *sequence of m bags of H* if for $l \in \{1, \ldots, m\}$, the l-th element of the sequence is X_{i+l-1} (and in particular the m-th element of the sequence is the bag X_j). A *sector of H* is a sequence of at least 2 bags $(X_i, \ldots, X_j)$ such that $|X_i| = |X_j| = 1$ and all the other bags in the sequence have size at least 2. We say that X_i and X_j are the *end bags* of the sector, and all the other bags are called the *interior bags* of the sector. The *length* of a sector is the number of its interior bags. A sector is an *n-sector*, for an integer $n \geq 0$, if it is of length n. A sector is *safe* if it has length 1 or length at least 3. A *super-sector of H* is a sequence of at least 5 bags $(X_i, \ldots, X_j)$ such that $|X_i| = |X_{i+1}| = |X_{j-1}| = |X_j| = 1$ and for $h \in \{i+1, \ldots, j-2\}$, $(|X_h|, |X_{h+1}|) \neq (1, 1)$. If $(X_i, \ldots, X_j)$ is a super-sector of H then we say that H *contains* a super-sector. We say that a super-sector $(X_i, \ldots, X_j)$ *contains an n-sector* if some subsequence of $(X_{i+1}, \ldots, X_{j-1})$ is an n-sector of H.

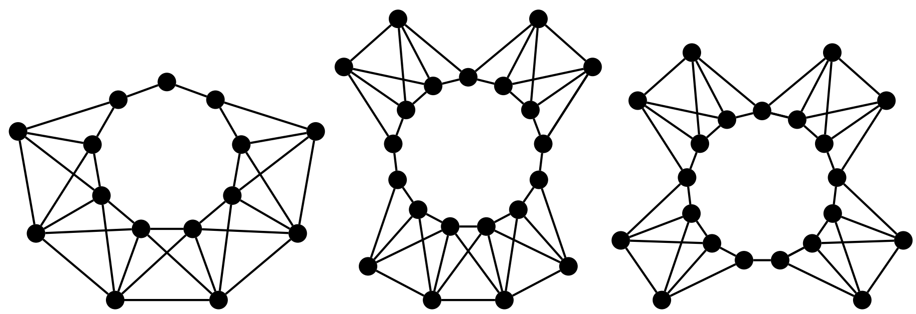

Figure 3: From left to right: hyperholes satisfying parts (i), (ii), and (iii) of the definition of a trivial hyperhole.

A k-hyperhole $H = (X_1, \ldots, X_k)$ is *trivial* if at least one of the following holds:

(i) for some $i \in \{1, \ldots, k\}$, $|X_i| = |X_{i+1}| = |X_{i+2}| = 1$;

(ii) H contains a super-sector that contains only 2-sectors;

(iii) H contains exactly one 0-sector, and all its other sectors are of length 2.

See Figure 3 for examples of trivial hyperholes. A *nontrivial* hyperhole is a hyperhole that is not trivial.

Lemma 3.7. *Let $H = (X_1, \ldots, X_k)$ be an odd k-hyperhole. If $|X_i| = |X_{i+1}| = 1$ and $|X_{i+2}|, |X_{i+3}| \geq 2$ for some $i \in \{1, \ldots, k\}$, then H is not minimally β-imperfect.*

PROOF — We may assume, by symmetry, that $|X_1| = |X_2| = 1$ and $|X_3|, |X_4| \geq 2$. Let x be the vertex in X_2, and suppose that H is minimally β-imperfect. Then, by Lemma 1.5, $\beta(H) = \delta(H) + 1$. Therefore $\beta(H) \leq d(x) + 1 = |X_3| + 2 \leq |X_3| + |X_4| \leq \omega(H) \leq \chi(H)$. But then $\chi(H) = \beta(H)$, a contradiction. □

Lemma 3.8. *If H is a trivial odd hyperhole, then H is β-perfect.*

PROOF — Let $H = (X_1, \ldots, X_k)$ be a trivial odd hyperhole, and assume that H is not β-perfect. Since every induced subgraph of H is either chordal or a trivial hyperhole, by Theorem 1.6 we may assume that H is minimally β-imperfect. By Lemma 3.7, H cannot satisfy (ii) or (iii) of the definition of a trivial hyperhole, and hence for some $i \in \{1, \ldots, k\}$, $|X_i| = |X_{i+1}| = |X_{i+2}| = 1$. Let x be the vertex of X_{i+1}. Then $d(x) = 2$, and hence x is a simplicial extreme of H, contradicting Lemma 1.4. □

A *base hyperhole* is any odd hyperhole $H = (X_1, \ldots, X_k)$ such that for all $i \in \{1, \ldots, k\}$: $|X_i| \leq 2$, $(|X_i|, |X_{i+1}|, |X_{i+2}|) \neq (1, 1, 1)$, and $(|X_i|, |X_{i+1}|) \neq (2, 2)$. It follows that every sector of H is of length 0 or 1, and hence every proper induced subgraph of a base hyperhole is either chordal or a trivial hyperhole. Note that if H is a base hyperhole, then $\omega(H) = 3$ and $\beta(H) = 4$. We say that a base hyperhole H is *good* if it has exactly one sector of length 0, and *bad* otherwise. Note that, up to isomorphism, there is only one good base hyperhole of length k. Also, observe that since k is odd, every base hyperhole must have a sector of length 0, and hence bad base hyperholes have at least two sectors of length 0. See Figure 4 for examples of base hyperholes.

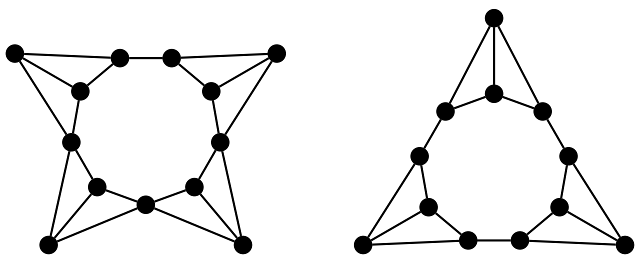

Figure 4: The unique (up to isomorphism) good base hyperhole of length 9 (left) and a bad base hyperhole that has three sectors of length 0 (right).

We now characterise β-perfect base hyperholes. First, we prove the following useful lemma on the number of vertices in a base hyperhole.

Lemma 3.9. *Let H be a base hyperhole of length k. The following hold.*

(i) If H is good, then $|V(H)| = \frac{3(k-1)}{2} + 1$.

(ii) If H is bad, then $|V(H)| \leq \frac{3(k-1)}{2}$.

PROOF — Suppose that H is good. Then H contains exactly one 0-sector, and all other sectors are of length 1. It follows that H has $\frac{k-1}{2}$ bags of size 2 and $\frac{k-1}{2}+1$ bags of size 1. Therefore $|V(H)| = \frac{3(k-1)}{2} + 1$, and (i) holds.

Now suppose that H is bad. Let (X_i, X_{i+1}) and (X_j, X_{j+1}) be distinct 0-sectors of H (their existence follows from the definition of a bad base hyperhole). Let m denote the number of bags in the sequence $(X_{i+2}, \dots, X_{j-1})$, and let m' denote the number of bags in the sequence $(X_{j+2}, \dots, X_{i-1})$. Since $k = m + m' + 4$, we may assume without loss of generality that m is even and m' is odd. It follows from H being a base hyperhole and from $|X_{i+1}| = |X_j| = 1$ that $|X_{i+2} \cup \cdots \cup X_{j-1}| \leq 2\,(m/2) + m/2 = 3m/2$. Similarly, we can obtain the bound $|X_{j+2} \cup \cdots \cup X_{i-1}| \leq 2\lceil m'/2 \rceil + \lfloor m'/2 \rfloor = m' + 1 + (m'-1)/2$. Now, using these bounds together with the fact that $m + m' = k - 4$ and $|X_i \cup X_{i+1} \cup X_j \cup X_{j+1}| = 4$, we obtain

$$\begin{aligned} |V(H)| &\leq \frac{3m}{2} + m' + \frac{m'-1}{2} + 5 \\ &= \frac{3m}{2} + \frac{3m'}{2} + \frac{9}{2} \\ &= \frac{3(k-4)}{2} + \frac{9}{2} \\ &= \frac{3(k-1)}{2}. \end{aligned}$$

Therefore (ii) holds. □

Lemma 3.10. *Let $H = (X_1, \dots, X_k)$ be a base hyperhole. Then H is β-perfect if and only if H is good. Furthermore, if H is bad, then H is minimally β-imperfect.*

PROOF — Since H is a base hyperhole, $\omega(H) = 3$ and $\beta(H) = 4$. Suppose that H is bad. Substituting the upper bound on $|V(H)|$ given by Lemma 3.9 (ii) into the equation in Theorem 3.1 (observing that since k is odd, $\alpha(H) = \frac{k-1}{2}$) gives

$$\chi(H) \leq \max\left\{\omega(H), \left\lceil \frac{2\left(\frac{3(k-1)}{2}\right)}{k-1} \right\rceil\right\} = \max\left\{3, \left\lceil \frac{3(k-1)}{k-1} \right\rceil\right\} = 3.$$

Therefore $\chi(H) < \beta(H)$, and hence H is not β-perfect. Since any proper induced subgraph of H is either a chordal graph or a trivial hyperhole, by Theorem 1.6 and Lemma 3.8, every proper induced subgraph of H is β-perfect. Therefore H is minimally β-imperfect.

Now suppose that H is good but not β-perfect. Since every proper induced subgraph of H is chordal or a trivial hyperhole, by Theorem 1.6 and Lemma 3.8, H is minimally β-imperfect. By Corollary 3.2,

$$\frac{2|V(H)|}{k-1} \leq \beta(H) - 1 = 3,$$

which implies that $|V(H)| \leq \frac{3(k-1)}{2}$. But this contradicts Lemma 3.9 (i). □

Lemma 3.11. *Every nontrivial odd hyperhole contains a base hyperhole.*

PROOF — Let $H = (X_1, \ldots, X_k)$ be a nontrivial odd hyperhole. Suppose that at most one bag of H is of size 1. Without loss of generality, we may assume that for $i \in \{2, \ldots, k\}$, $|X_i| \geq 2$. For $i \in \{1, \ldots, k\}$, if i is odd then let X'_i be any one-element subset of X_i, and otherwise let X'_i be any two-element subset of X_i. Then clearly $H' = (X'_1, \ldots, X'_k)$ is a base hyperhole that is contained in H.

So we may assume that there are at least two distinct bags of H that are of size 1, and hence H contains at least 2 sectors. Let $j_1, \ldots, j_t$ be the indices of the bags of H that are of size 1, ordered such that $j_1 < \cdots < j_t$, and let $S_1, \ldots, S_t$ be the sectors of H such that for $i = 1, \ldots, t-1$, $S_i = (X_{j_i}, \ldots, X_{j_{i+1}})$, and $S_t = (X_{j_t}, \ldots, X_{j_1})$.

We construct a hyperhole $H' = (X'_1, \ldots, X'_k)$, where for $i = 1, \ldots, k$, $X'_i \subseteq X_i$, as follows. For $i = 1, \ldots, t$, $X'_{j_i} = X_{j_i}$. For every safe sector S_i, we reduce the interior bags of S_i according to the following rules:

- If the length of S_i is odd, then for $h \in \{j_i + 1, \ldots, j_{i+1} - 1\}$,

$$|X'_h| = \begin{cases} 1 & \text{if } h - j_i \text{ is even,} \\ 2 & \text{otherwise.} \end{cases}$$

- If the length of S_i is even, then $|X'_{j_{i+1}-1}| = 2$, $|X'_{j_{i+1}-2}| = 1$, and for $h \in \{j_i + 1, \ldots, j_{i+1} - 3\}$,

$$|X'_h| = \begin{cases} 1 & \text{if } h - j_i \text{ is even,} \\ 2 & \text{otherwise.} \end{cases}$$

To finish off the construction of H' we now need to reduce all 2-sectors. To do so, we consider the following cases.

Case 1: *H has no 0-sectors.*

For every 2-sector S_i, we ensure that $|X'_{j_i+1}| = 2$ and $|X'_{j_i+2}| = 1$. The resultant hyperhole H' is clearly a base hyperhole.

Case 2: *H has exactly one 0-sector.*

Since H is nontrivial, it must contain a safe sector. Without loss of generality, let us say that S_1 is the 0-sector and S_s is a safe sector. For every 2-sector S_i we reduce its interior bags according to the following rule:

- If $1 < i < s$, then $|X'_{j_i+1}| = 2$ and $|X'_{j_i+2}| = 1$, and otherwise $|X'_{j_i+1}| = 1$ and $|X'_{j_i+2}| = 2$.

The resultant hyperhole H' is clearly a base hyperhole.

Case 3: *H has at least two 0-sectors.*

Since H is nontrivial, any two consecutive 0-sectors together with intermediate bags forms a super-sector. We consider each super-sector $S = (X_l, \ldots, X_r)$ separately. Since H is nontrivial, S contains a safe sector S_s. Let S_i be a 2-sector contained in S. If S_i is contained in the subsequence $(X_l, \ldots, X_{j_s})$ of S, then $|X'_{j_i+1}| = 2$ and $|X'_{j_i+2}| = 1$, and otherwise $|X'_{j_i+1}| = 1$ and $|X'_{j_i+2}| = 2$. The resultant hyperhole H' is clearly a base hyperhole. □

Lemma 3.12. *Let $H = (X_1, \ldots, X_k)$ be a minimally β-imperfect k-hyperhole. For integers $i, j, m \in \{1, \ldots, k\}$ and with $m \geq 3$ and m odd, let $(X_i, \ldots, X_j)$ be a sequence of m bags of H such that for all $h \in \{i, \ldots, j\}$, $|X_h| = 1$ if $h-i$ is even and $|X_h| \geq 2$ otherwise. Then $|X_i \cup \cdots \cup X_j| \geq \frac{(\beta(H)-1)(m-1)}{2} + 1$.*

PROOF — Observe that in the sequence $(X_i, \ldots, X_j)$ there are $\frac{m-1}{2}$ bags of size at least 2 and $\frac{m+1}{2}$ bags of size 1. It now follows from Lemma 3.3 that

$$\begin{aligned} |X_i \cup \cdots X_j| &\geq (\beta(H) - 2)\left(\frac{m-1}{2}\right) + \frac{m+1}{2} \\ &= \frac{(\beta(H) - 1)\,(m-1)}{2} - \frac{m-1}{2} + \frac{m+1}{2} \\ &= \frac{(\beta(H)-1)(m-1)}{2} + 1, \end{aligned}$$

as required. □

Let $H' = (X'_1, \ldots, X'_k)$ be a good base hyperhole with k odd and $k \geq 9$. Without loss of generality, let us assume that $|X'_3| = |X'_4| = 1$. Let A be one of the sets $\{1, 3\}$, $\{1, 6\}$, $\{3, 4\}$, $\{4, 6\}$. We call A a *free set* of H'. A hyperhole $H = (X_1, \ldots, X_k)$ is an *extension* of H' if for all $i \in \{1, \ldots, k\}$, $X'_i \subseteq X_i$ and for some free set A of H', for all $i \in \{1, \ldots, k\} \setminus A$, $|X'_i| = 1$ if and only if $|X_i| = 1$. (Such a free set A of H' is called a *free set of H*.) We now give a structural characterisation of β-perfect k-hyperholes with k odd and $k \geq 9$.

Lemma 3.13. *Let H be a k-hyperhole with k odd and $k \geq 9$. If H is nontrivial and is not an extension of a good base hyperhole, then H contains a bad base hyperhole.*

PROOF — Suppose that H is neither trivial nor an extension of a good base hyperhole. Since H is nontrivial, by Lemma 3.11 it contains a base hyperhole $H' = (X'_1, \dots, X'_k)$ (with $X'_i \subseteq X_i$ for all $i \in \{1, \dots, k\}$). We may assume that H' is good, for otherwise we are done. Without loss of generality we assume that $|X'_3| = |X'_4| = 1$. Since H is not an extension of H', one of the following holds:

- for some even $i \in \{8, \dots, k\}$, $|X_i| \geq 2$;
- $|X_1| \geq 2$ and $|X_4| \geq 2$;
- $|X_3| \geq 2$ and $|X_6| \geq 2$.

First, let us suppose that for some even $i \in \{8, \dots, k\}$, $|X_i| \geq 2$. Consider a hyperhole $H'' = (X''_1, \dots, X''_k)$ such that for all $h \in \{1, \dots, k\}$:

$$|X''_h| = \begin{cases} 1 & \text{if } h \in \{1, 3, i-1, i+1\}, \\ 1 & \text{if } h \in \{4, \dots, i-2\} \cup \{i+2, \dots, k\} \text{ and } h \text{ is even}, \\ 2 & \text{otherwise.} \end{cases}$$

Clearly H'' is a base hyperhole that is contained in H. Furthermore, (X''_{i-2}, X''_{i-1}) and (X''_{i+1}, X''_{i+2}) are two distinct 0-sectors of H'', and hence H'' is bad.

Suppose that $|X_1| \geq 2$ and $|X_4| \geq 2$. Consider a hyperhole $H'' = (X''_1, \dots, X''_k)$ such that for all $h \in \{1, \dots, k\}$:

$$|X''_h| = \begin{cases} 1 & \text{if } h \in \{2, 3, 5, k\}, \\ 1 & \text{if } h \in \{6, \dots, k-1\} \text{ and } h \text{ is even}, \\ 2 & \text{otherwise.} \end{cases}$$

Clearly H'' is a base hyperhole that is contained in H. Furthermore, (X''_2, X''_3) and (X''_5, X''_6) are two distinct 0-sectors of H'', and hence H'' is bad. A symmetric argument may be used for the case where $|X_3| \geq 2$ and $|X_6| \geq 2$. $\square$

Theorem 3.14. *A k-hyperhole H with k odd and $k \geq 9$ is β-perfect if and only if it is trivial or it is an extension of a good base hyperhole.*

PROOF — Let $H = (X_1, \dots, X_k)$ be a k-hyperhole with k odd and $k \geq 9$. By Lemma 3.10 and Lemma 3.13 it follows that if H is β-perfect then it is trivial or an extension of a good base hyperhole.

Now suppose that H is either trivial or is an extension of a good base hyperhole, but H is not β-perfect. By Lemma 3.8, H is an extension of a good base hyperhole $H' = (X'_1, \dots, X'_k)$. Furthermore, since every induced subgraph of H is either chordal, a trivial hyperhole, or an extension of a good base hyperhole, we may assume that H is minimally β-imperfect. Let A be a free set of H. By symmetry, it suffices to consider the following three cases. In each case, we obtain a lower bound on $|V(H)|$ which contradicts the bound given in Corollary 3.2.

Case 1: $A = \{1, 3\}$.

Applying Lemma 3.12 to the sequence $(X_4, \dots, X_{k-1})$ gives the lower bound

$$|X_4 \cup \dots \cup X_{k-1}| \geq (\beta(H) - 1)\left(\frac{k-5}{2}\right) + 1.$$

It follows from Lemma 3.3 applied to X_{k-1}, X_k, X_1 and to X_2, X_3, X_4 that $|X_{k-1} \cup X_k \cup X_1| \geq \beta(H)$ and $|X_2 \cup X_3 \cup X_4| \geq \beta(H)$. Adding together these bounds, and subtracting 2 to account for double counting, we obtain

$$\begin{aligned}
|V(H)| &\geq (\beta(H) - 1)\left(\frac{k-5}{2}\right) + 2\beta(H) - 1 \\
&= (\beta(H) - 1)\left(\frac{k-1}{2}\right) - 2\left(\beta(H) - 1\right) + 2\beta(H) - 1 \\
&= (\beta(H) - 1)\left(\frac{k-1}{2}\right) + 1,
\end{aligned}$$

contradicting Corollary 3.2. This completes Case 1.

Case 2: $A = \{1, 6\}$.

Using Lemma 3.3, we easily obtain the following bounds: $|X_{k-1} \cup X_k \cup X_1| \geq \beta(H)$, $|X_2 \cup X_3 \cup X_4| \geq \beta(H)$, $|X_3 \cup X_4 \cup X_5| \geq \beta(H)$, and $|X_6 \cup X_7 \cup X_8| \geq \beta(H)$. If $k = 9$, then adding together these bounds and subtracting 3 to account for double counting gives the bound

$$|V(H)| \geq 4\beta(H) - 3 = (\beta(H) - 1)\left(\frac{k-1}{2}\right) + 1,$$

contradicting Corollary 3.2. So $k > 9$. By applying Lemma 3.12 to the sequence $(X_8, \dots, X_{k-1})$, we obtain the additional bound

$$|X_8 \cup \dots \cup X_{k-1}| \geq (\beta(H) - 1)\left(\frac{k-9}{2}\right) + 1.$$

Adding this to the above bounds, and subtracting 4 to account for double counting, we obtain

$$
\begin{aligned}
|V(H)| &\geq (\beta(H)-1)\left(\frac{k-9}{2}\right) + 4\beta(H) - 3 \\
&= (\beta(H)-1)\left(\frac{k-1}{2}\right) - 4\left(\beta(H)-1\right) + 4\beta(H) - 3 \\
&= (\beta(H)-1)\left(\frac{k-1}{2}\right) + 1,
\end{aligned}
$$

contradicting Corollary 3.2. This completes Case 2.

Case 3: $A = \{3, 4\}$.

Applying Lemma 3.12 to the sequence $(X_6, \ldots, X_k, X_1)$ gives the lower bound

$$|X_6 \cup \cdots \cup X_k \cup X_1| \geq (\beta(H)-1)\left(\frac{k-5}{2}\right) + 1.$$

It follows from Lemma 3.3 applied to the sets X_1, X_2, X_3 and to X_4, X_5, X_6 that $|X_1 \cup X_2 \cup X_3| \geq \beta(H)$ and $|X_4 \cup X_5 \cup X_6| \geq \beta(H)$. Adding together these bounds, and subtracting 2 to account for double counting, we obtain

$$
\begin{aligned}
|V(H)| &\geq (\beta(H)-1)\left(\frac{k-5}{2}\right) + 2\beta(H) - 1 \\
&= (\beta(H)-1)\left(\frac{k-1}{2}\right) - 2\left(\beta(H)-1\right) + 2\beta(H) - 1 \\
&= (\beta(H)-1)\left(\frac{k-1}{2}\right) + 1,
\end{aligned}
$$

contradicting Corollary 3.2. This completes Case 3. □

3.3 Forbidden induced subgraph characterisation

By putting together previously obtained results we obtain the following forbidden induced subgraph characterisation of β-perfect hyperholes (in which all the excluded induced subgraphs are minimally β-imperfect). Let H_1 be the graph in Figure 1, and let H_2 be the graph in Figure 2.

Theorem 3.15. *A hyperhole is β-perfect if and only if it is (even hole, bad base hyperhole, H_1, H_2)-free*

PROOF — Suppose that H is a β-perfect hyperhole. Clearly H must be even-hole-free. It follows from Lemma 3.4 that H is H_1-free, from Theorem 3.6 that H is H_2-free, and from Theorem 3.10 that H does not contain a bad base hyperhole.

Now suppose that $H = (X_1, \ldots, X_k)$ is a k-hyperhole that does not contain an even hole, H_1, H_2, or a bad base hyperhole. In particular, k is odd. If $k = 5$, then it follows from H being H_1-free that some bag of H has size 1. Therefore H is β-perfect by Theorem 3.5. If $k = 7$, then it follows from H being H_2-free that for some $i \in \{1, \dots, k\}$, either $|X_i| = |X_{i+1}| = 1$, or $|X_i| = |X_{i+2}| = 1$. Therefore, by Theorem 3.6, H is β-perfect. Now suppose that $k \geq 9$. Since H contains no bad base hyperhole, by Lemma 3.13 it is either trivial or an extension of a good base hyperhole, and hence it is β-perfect by Theorem 3.14. $\square$

3.4 A recognition algorithm for β-perfect hyperholes

In this section, we give a linear-time algorithm that decides whether an input hyperhole is β-perfect. We observe that in [1] it is shown that hyperholes can be recognized in linear-time, and that a hyperhole partition can be found also in linear-time.

Theorem 3.16. *There is an algorithm with the following specifications:*

> **Input:** A k-hyperhole $H = (X_1, \dots, X_k)$.
> **Output:** YES if H is β-perfect, and NO otherwise.
> **Running time:** $\mathcal{O}(k)$.

PROOF — Consider the following algorithm:

1. If k is even, then return NO.
2. If $k = 5$, then check whether some bag of H is of size 1. If so, then return YES. Otherwise, return NO.
3. If $k = 7$, then check whether $|X_i| = |X_{i+1}| = 1$ or $|X_i| = |X_{i+2}| = 1$ for some $i \in \{1, \dots, k\}$. If so, then return YES. Otherwise, return NO.
4. From now on, we may assume that k is odd and $k \geq 9$. Check whether H is trivial. If so, then return YES.
5. We may now assume that H is nontrivial. Using the reduction rules given in Lemma 3.11, construct a base hyperhole $H' = (X'_1, \dots, X'_k)$ such that $V(H') \subseteq V(H)$.
6. Check whether H' is bad. If so, then return NO.
7. Check whether H is an extension of H'. If so, then return YES. Otherwise, return NO.

We now prove that the algorithm is correct. Suppose that the algorithm returns YES when given a k-hyperhole $H = (X_1, \dots, X_k)$ as input. As a result of Step 1, we may assume that k is odd. If $k = 5$, then by Step 2, $|X_i| = 1$ for some $i \in \{1, \dots, k\}$. It follows from Theorem 3.5 that

H is β-perfect. If $k = 7$, then by Step 3, either $|X_i| = |X_{i+1}| = 1$ or $|X_i| = |X_{i+2}| = 1$ for some $i \in \{1, \ldots, k\}$. It follows from Theorem 3.6 that H is β-perfect. If $k \geq 9$, then by Steps 4 and 7, either H is trivial or H is an extension of a good base hyperhole. It follows from Theorem 3.14 that H is β-perfect.

Suppose now that the algorithm returns NO, but that H is β-perfect. Since even holes are not β-perfect, we may assume that k is odd. If $k = 5$, then by Theorem 3.5, $|X_i| = 1$ for some $i \in \{1, \ldots, k\}$. But then the algorithm returns YES in Step 2. If $k = 7$, then by Theorem 3.6, $|X_i| = |X_{i+1}| = 1$ or $|X_i| = |X_{i+2}| = 1$ for some $i \in \{1, \ldots, k\}$. But then the algorithm returns YES in Step 3. So we may assume that $k \geq 9$. By Theorem 3.14, H is either trivial or an extension of a good base hyperhole. If H is trivial, then the algorithm returns YES in Step 4. If H is an extension of a good base hyperhole, then H is an extension of the hyperhole H' constructed in Step 5 of the algorithm. But then the algorithm returns YES in Step 7. This completes the proof that the algorithm is correct.

Each step of the algorithm can clearly be performed in $\mathcal{O}(k)$ time. □

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