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The common and specific components of inflation expectation across European countries *

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Abstract

Inflation expectation (IE) is often considered to be an important determinant of actual inflation in modern economic theory, we are interested in investigating the main risk factors that determine its dynamics. We first apply a joint arbitrage-free term structure model across different European countries to obtain estimate for country-specific IE. Then we use the two-component and three-component models to capture the main risk factors. We discover that the extracted common trend for IE is an important driver for each country of interest. Moreover a spatial-temporal copula model is fitted to account for the non-Gaussian dependency across countries. This paper aims to extract informative estimates for IE and provide good implications for monetary policies.

Keywords: inflation expectation; joint yield-curve modeling; factor model; common trend; spatial-temporal copulas

JEL classification: C02, C13, C38, E31, E43

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1 Introduction

In modern economic theory, inflation expectation (IE) is an important determinant of realized inflation and therefore is often considered to be an important indicator for policy makers and financial investors. For example, some economists favor a low and steady rate of inflation because it facilitates real wage adjustments in the presence of downward nominal wage rigidity. Hence, one of the major objectives of modern monetary policy is to bring IE under control, as this is considered to be the first step in controlling inflation/deflation. Meanwhile, hedging the risk around the inflation forecast becomes more attractive in financial markets, as many investors rely on the stability and predictability of future inflation levels. Moreover, price stability is of immense importance to sustain social welfare, job opportunities, and economic upturn. The objective of price stability refers to the general level of prices in the economy, which implies avoiding both prolonged inflation and deflation. IE that is involved in a contemporary macroeconomic framework anticipates future economic trend and it will further affect monetary decisions.

Since there is considerable demand for having reasonable IE estimates, a sizable literature has focused on analyzing the government's conventional and inflation-indexed bonds, which can implicitly provide a vast amount of information about the expectations of nominal and real interest rates obtained from the market. Such estimates are also known to be an important complement to the IE estimates provided by the survey data (see Nautz et al. (2017), Golosnoy and Roestel (2018)). Despite the fact that inflation-indexed bonds have been more frequently and widely issued in recent times, one would still have great difficulties in integrating market information from multiple countries. The major problems lie in the relatively short period of data availability and the existence of missing values. Therefore most existing literature mainly focuses on a single country such as United States and United Kingdom. In this paper, we provide a multi-country factor model for joint estimation of nominal and real yields. The proposed estimation framework allows us to conduct cross-country learning, and study the common and specific components across different countries. In particular, we investigate the main factors determining the IE dynamics using the two-component and three-component models.

The starting point of our research is to analyze the breakeven inflation rate (BEIR), which is known to be the difference between the yield on a nominal fixed-rate bond and the real yield on an inflation-linked bond of the same maturity and similar credit quality. The BEIR can generally indicate how IE is priced in the market. However, BEIR is not a perfect measure for IE, as it may encompass other risk premiums. For example, Joyce et al. (2010) develops an affine term structure model to decompose forward rates to obtain the risk premium. Notably, Christensen et al. (2010) uses an affine arbitrage-free (AF) model of term structure to decompose BEIR that captures the pricing of both nominal and inflation-indexed securities. To adopt a real-time approach to access the term structure of nominal and inflation-linked yields, we use the three-factor term structure model motivated by Nelson and Siegel (1987). The attractiveness of Nelson-Siegel (NS) factor models is due to their convenient affine function structure and good empirical performance. Diebold and Li (2006) extends the original NS model to a dynamic environment. Theoretically, the NS model does not ensure the absence of arbitrage opportunities, as shown by Bjork and Christensen (1999). Christensen et al. (2011) develops the NS model to an arbitrage-free Nelson-Siegel (AFNS) model by imposing the arbitrage free hypothesis, which reflects most of the real activities of financial markets. The standard approaches for pricing forwards or swaps are all derived from such arbitrage arguments for both complete and incomplete markets. This work is then extended to a four-factor joint AFNS model by Christensen et al. (2010), and is proved to be efficient for fitting and forecasting analysis.

However, previous research has paid relatively little attention to the story of multiple countries. Therefore we study the the.dynamics of IE in a multi-country setting, and examine the common and specific components of IE in five European countries. We aim to determine the main factors that driving the IE dynamics. Diebold et al. (2008) is the first to consider a global multiple-country model for nominal yield curves. There are a few European central bank reports focusing on household and expert IE and the anchoring of IE in the two currency areas before and during the 2008 crisis, for example Pflueger and Viceira (2011) and Jarocinski and Lenza (2015), the latter one considers a specification involving a factor model of economic activity, for the purpose of estimating the output gap. Moreover, the spatial-temporal model allows us to further understand the non-Gaussian dependency structure across countries. The use of spatial-temporal copula provides useful insights to the regional effects by including spatial information, empirical examples include Schenker and Straub (2011), Holly et al. (2010) and Bai et al. (2012). To sum up, this paper contributes new insights to the literature by providing a multi-country factor model framework for joint estimation of nominal and real yields. Based on the country-specific IE estimates, we identify the common and specific components that driving the IE dynamics, the empirical findings can be of interest to investors and policy makers. We further consider the regional effects by fitting a spatial-temporal copula model to account for the non-Gaussian dependency structure across different countries.

The rest of the paper is structured as follows. Section 2 outlines the methodology by first introducing the joint AFNS to obtain the IE estimates, and then presenting a multi-country factor model framework which makes it possible to further analyze the IE dynamics. In Section 3, we describe the data and report the country-specific IE estimates. Section 4 discusses the empirical results. Section 5 concludes. The ethical standards are stated in Section 6. Finally, the technical details are reported in Section 7.

2 Methodology

2.1 A four-factor AFNS model

Following the work of Nelson and Siegel (1987), Christensen et al. (2010) and Christensen et al. (2011), we fit the yield curves using a four-factor AFNS model together with the corresponding real-world dynamics (under P-measure), which are given by,

$$\begin{aligned}
\begin{pmatrix} y^N_{it,\tau} \\ y^R_{it,\tau} \end{pmatrix} &= \begin{pmatrix} 1 & \frac{1-e^{-\lambda_i\tau}}{\lambda_i\tau} & \frac{1-e^{-\lambda_i\tau}}{\lambda_i\tau} - e^{-\lambda_i\tau} & 0 \\ 0 & \alpha_i^S\frac{1-e^{-\lambda_i\tau}}{\lambda_i\tau} & \alpha_i^C(\frac{1-e^{-\lambda_i\tau}}{\lambda_i\tau} - e^{-\lambda_i\tau}) & 1 \end{pmatrix} X_{it,\tau} \\
&+ \begin{pmatrix} \varepsilon^N_{it,\tau} \\ \varepsilon^R_{it,\tau} \end{pmatrix} - \begin{pmatrix} \frac{A^N_{i,\tau}}{\tau} \\ \frac{A^R_{i,\tau}}{\tau} \end{pmatrix} \\
dX_{it,\tau} &= K_i(\theta_i - X_{it,\tau})dt + \Sigma_i dW_t \qquad (2.1)
\end{aligned}$$

where $X_{it,\tau}$ consists of four latent factors that dependent on maturity τ and evolve dynamically

$$X_{it,\tau}^{\top} = \left(L_{it,\tau}^{N}, S_{it,\tau}^{N}, C_{it,\tau}^{N}, L_{it,\tau}^{R}\right) \tag{2.2}$$

with $y_{it,\tau}^{N}$ and $y_{it,\tau}^{R}$ are the nominal and inflation-linked yields with maturity τ at time t for country i respectively and τ determines the decay speed of parameters. $\varepsilon_{it,\tau}^{N}, \varepsilon_{it,\tau}^{R}$ are the innovation terms. α_i^S, α_i^C are country-specific coefficients of the latent time-varying slope and curvature factors. $\frac{A_{i,\tau}}{\tau}$ is an unavoidable yield-adjustment term depending on maturity τ. K_i and θ_i correspond to drifts and dynamics terms, and are both allowed to vary freely. Σ_i is a diagonal volatility matrix.

For the NS-type models, there are three time-varying factors known as level L_t, slope S_t, and curvature C_t. Such a choice of the latent factors is motivated by principal component analysis, which gives us three principal components corresponding to the latent factors. For instance, the most variation of yields is accounted for by the first principal component, i.e. level factor L_t. With the NS-type models, we first have (L_t^N, S_t^N, C_t^N) for the nominal bond, and (L_t^R, S_t^R, C_t^R) for the inflation-linked bond. As it is shown that the correlation between the two slope/curvature factors are quite high, this motivates the following assumption,

$$S_{it}^{R} = \alpha_i^S S_{it}^{N}, \quad C_{it}^{R} = \alpha_i^C C_{it}^{N}$$

By imposing the above assumption, we then have the four-factor AFNS model in 2.1. This assumption simplifies model estimation by limiting the number of parameters, and improves model performance. Moreover, this assumption is also supported by our empirical results. In this paper, the Kalman filter and maximum likelihood estimation are implemented for estimating the state variables and unknown parameters. More details can be found in Appendix A.

In order to obtain the IE estimate, it is necessary to understand the components of the bond yields. A considerable amount of literature has adopted a parametric approach to estimate the IE and its corresponding risk premiums (RP) using nominal and indexed bond data, such as Adrian and Wu (2009), Campbell and Viceira (2009), and Pflueger and Viceira (2011). They decompose the yield of an inflation-linked bond into current expectation of a future real interest rate and a real interest rate premium. The yield on a nominal bond can be decomposed into parts of the yield on a real bond, expectations of future inflation, and RP. Therefore, the spread between both yields, the BEIR, reflects the level of IE and RP. Mathematically, it can be expressed as,

$$\widehat{BEIR}_{it,\tau} = \hat{y}_{it}^{N}(\tau) - \hat{y}_{it}^{R}(\tau) = \hat{\pi}_{it,\tau} + \hat{rp}_{it,\tau} \tag{2.3}$$

where $\hat{\pi}_{it,\tau}$ is the model-implied IE and $\hat{rp}_{it,\tau}$ is the corresponding RP. More details are available in Appendix B.

The above decomposition is also supported by our estimation results as shown in Figure 2.1. We plot the $\widehat{BEIR}_{it,3}$ for five European countries - United Kingdom (UK), Germany, France, Italy, and Sweden - with maturity of three years. A fall in the consumer prices appears since September 2009 due to a drop in energy costs, which exhibits some degree of co-movement. It is known that the euro-zone annual inflation rate was recorded at -0.2 percent in December of 2014 which matches, but is slightly higher than the overall BEIR. Moreover, we also observe a co-movement in the figure, this motivates us to extract a time-varying common component of IE estimated from country-specific BEIR in a multiple-country framework.

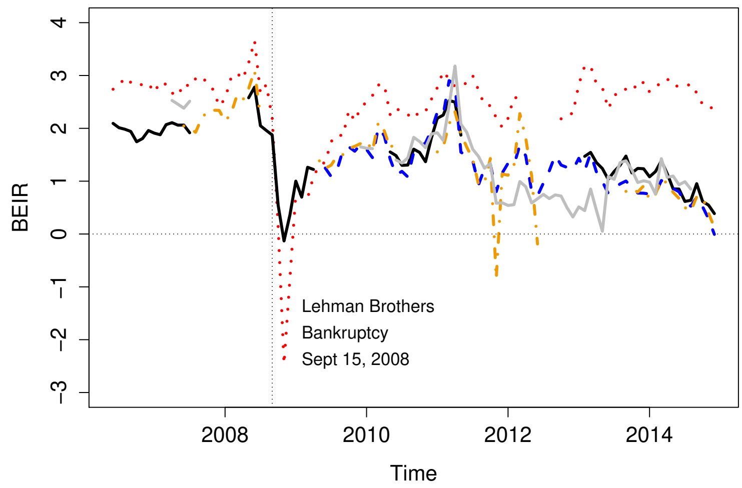

Figure 2.1: BEIR for five industrialized European countries—UK (red dotted line), Germany (blue dashed line), France (black line), Italy (orange dot-dashed line), and Sweden (grey line)—with maturity of three years.

2.2 A multi-country factor model for joint estimation of nominal and real yields

As far as we have obtained the country-specific IE estimates, we build a multi-country factor model in this subsection. In particular, we allow the country-specific IE to load on a common time-varying factor and country-specific factors.

2.2.1 Estimating a two-component model for country-specific IE

We construct the two-component model as follows,

$$\hat{\pi}_{it,\tau} = m_{i,\tau} + n_{i,\tau}\Pi_{t,\tau} + u_{it,\tau} \quad (2.4)$$

$$\Pi_{t,\tau} = p_\tau + q_\tau \Pi_{t-1,\tau} + \nu_{t,\tau} \quad (2.5)$$

where $\Pi_{t,\tau}$ is the time-varying dynamic factor that determines the IE dynamics across countries at maturity τ. For example when $\tau = 3$ years, $\Pi_{t,3}$ represents the common trend for 3-year IE estimates. $m_{i,\tau}$, $n_{i,\tau}$, p_τ, and q_τ are unknown parameters. The innovation $\nu_{t,\tau}$ is assumed to be independent and identically distributed (i.i.d.).

For the innovation term $u_{it,\tau}$ in (2.4), we further include the spatial information to account for the effects of different regions on the common IE trend. This non-Gaussian dependency structure is modelled by a spatial-temporal copula process, technical details will be introduced in subsection 2.2.3. To capture the regional effects, we therefore consider the following two settings:

1. $u_{it,\tau}$ is assumed to be i.i.d.
2. $u_{it,\tau}$ is assumed to follow a non-Gaussian spatial-temporal copula process

2.2.2 Estimating a three-component model for country-specific IE

Since there is a dynamic interaction between the macro-economy and the yield curve, as evidenced by the literature, for example Diebold et al. (2006). A straightforward extension of the two-component model in 2.4 is to add a proxy of the macroeconomic factor, the three-component model is then defined as,

$$\begin{aligned}\hat{\pi}_{it,\tau} &= m_{i,\tau} + n_{i,\tau}\Pi_{t,\tau} + l_{i,\tau}d_{it,\tau} + u_{it,\tau} \\ \Pi_{t,\tau} &= p_\tau + q_\tau\Pi_{t-1,\tau} + \nu_{t,\tau}\end{aligned} \tag{2.6}$$

where $\Pi_{t,\tau}$ has the same dynamic form as given in (2.5). $d_{it,\tau}$ is a macroeconomic factor varying over time, it can a proxy of macro-economy such as Consumer Price Index (CPI), default risk factor et al. $m_{i,\tau}$, $n_{i,\tau}$, p_τ, and q_τ are unknown coefficients. $\nu_{t,\tau}$ is i.i.d. innovations. For the $u_{it,\tau}$ in (2.6), we also consider the two settings stated above, i.e.,

1. $u_{it,\tau}$ is assumed to be i.i.d.

2. $u_{it,\tau}$ is assumed to follow a non-Gaussian spatial-temporal copula process

As shown in Figure 2.1, we observe a decrease of BEIR appears around 2012 due to the European sovereign debt crisis. The impact caused by this crisis is quite significant, therefore we use a default proxy as the macroeconomic factor in our paper. By this way, we could integrate macroeconomic information for analyzing the co-movement of the IE dynamics. Our study can also be applied to investigate the impacts caused by other macroeconomic factors, we leave this to future work.

2.2.3 Spatial-temporal copula

In this subsection, we explain the spatial-temporal copula model for estimating the non-Gaussian dependency across countries. For the error term $u_{it,\tau}$ in (2.4) and (2.6) with $t = 1, \ldots, T$, $i = 1, \ldots, N$ and maturity τ, we assume $u_{it,\tau}$ follows a spatial-temporal copula process given by,

$$u_{it,\tau} = \alpha_{it,\tau} + \xi_{it,\tau}, \tag{2.7}$$

where $\alpha_{it,\tau}$ captures the spatial temporal variation and $\xi_{it,\tau}$ is the Gaussian noise with mean 0 and variance σ_ξ (see Bai et al. (2014)).

It can then be formulated as,

$$F_{it}^{(\tau)}(\alpha) = \Phi_{NT,\tau}\{\Phi_{NT,\tau}^{-1}(F_{11}^{(\tau)}(\alpha_{11})), \cdots, F_{NT}^{(\tau)}(\alpha_{NT})|\Sigma^{(\tau)}\}, \tag{2.8}$$

where $\Phi_{NT}^{(\tau)}(\cdot)$ is the cumulative distribution function (c.d.f.) of a multivariate Gaussian distribution, its variance-covariance matrix $\Sigma^{(\tau)}$ captures the spatial-temporal dependence. Note that this framework can also be generalized to a t-distribution. According to the marginal closure property of Gaussian copula, we marginalize the high-dimensional c.d.f. in equation (2.8) into two-dimensional marginals. Thus, for any spatial-temporal coordinate t_1, n_1 and t_2, n_2, the dependency can be expressed by the following pair-wise copula model,

$$F_{t,n}^{(\tau)}(\alpha_{t_1,n_1}, \alpha_{t_2,n_2}) = \Phi_2(\Phi^{-1}(\alpha_{t_1,n_1}), \Phi^{-1}(\alpha_{t_2,n_2})|\Sigma_{t_1,n_1,t_2,n_2}^{(\tau)}), \tag{2.9}$$

where $\Sigma_{t_1,n_1,t_2,n_2}^{(\tau)}$ is a submatrix of $\Sigma^{(\tau)}$. In this paper, we parameterize Σ_{t_1,n_1,t_2,n_2} as follows,

$$\begin{aligned}\sigma(v,u) &= \sigma(n_2 - n_1, t_2 - t_1) \\ &\stackrel{\text{def}}{=} \begin{cases} \frac{2\sigma^2\beta}{(a^2u^2+1)^\eta(a^2u^2+\beta)\gamma(\eta)}\left(\frac{b}{2}\left(\frac{a^2u^2+1}{a^2u^2+\beta}\right)^{1/2}v\right)^\eta K_\eta\left(b\left(\frac{a^2u^2+1}{a^2u^2+\beta}\right)^{1/2}v\right) & \text{if } v > 0 \\ \frac{\sigma^2\beta}{(a^2u^2+1)^\eta(a^2u^2+\beta)\gamma(\eta)} & \text{if } v = 0 \end{cases}\end{aligned}$$

where a, b, β, η are parameters, $\gamma(\eta)$ is the gamma function, and $K_\eta(\cdot)$ is the Bessel function of the second kind. Denote $f_{t,n}(\alpha_{t_1,n_1}, \alpha_{t_2,n_2})$ as the joint density of two random variables, the estimator can be attained by maximizing the joint composite log likelihood (see Varin (2008)),

$$l(\theta, d_1, d_2) = \sum_{t_1,t_2,n_1,n_2:\|t_1-t_2\|\leq d_1,\|n_1-n_2\|\leq d_2} \log f_{\alpha_{t_1 n_1},\alpha_{t_2 n_2}}, \tag{2.10}$$

where $\theta \stackrel{\text{def}}{=} (a, b, \beta, \eta)^\top$, and d_1, d_2 are the cut-off points. In particular, the density is given as

$$f_{\alpha_{t_1,n_1},\alpha_{t_2,n_2}} \stackrel{\text{def}}{=} c_\Phi\{F(\alpha_{t_1,n_1}), F(\alpha_{t_2,n_2})\}f(\alpha_{t_1,n_1})f(\alpha_{t_2,n_2})$$

with

$$\begin{aligned} c_\Phi\{F(\alpha_{t_1,n_1}), F(\alpha_{t_2,n_2})\} &= |\Sigma_{t_1,n_1,t_2,n_2}|^{-1/2}\exp\{q^\top(I_2 - \Sigma_{t_1,n_1,t_2,n_2}^{-1})q\} \\ q &\stackrel{\text{def}}{=} (q_{t_1,n_1}, q_{t_2,n_2}) \\ q_{t_i,n_i} &= \Phi^{-1}\{\hat{F}(x_{t_i,n_i})\} \end{aligned}$$

3 Data and IE estimation

3.1 Data

We take monthly nominal and inflation-linked yield data of zero-coupon government bonds from both Bloomberg and Datastream. The research databases are supported by the Collaborative Research Center 649, Humboldt University of Berlin. In this paper, we consider five developed European countries - UK, France, Germany, Italy, and Sweden - all of which are member states of the European Union (EU) during the studied period. The motivation for selecting these countries is the availability of inflation-indexed bond data. It should be noted that two of the selected countries are outside the euro-zone (UK and Sweden) and, thus, with their own currencies, have independent central bank and monetary policy. While the Swedish krona is loosely tied to the euro, the exchange rate between the euro and the pound is more flexible.

The sample period we study is from 2006 to 2014, and covers the subprime crisis in 2008 and the European sovereign debt crisis in 2011. Due to the lack of the inflation-indexed bond data, the sample period for each country is sightly different. We collect treasury bond data for each country, that is, the gilt bonds for the UK, the OATs bonds for France, the bunds for Germany, the BTPs for Italy and the index-linked bonds for Sweden. Moreover, the UK real gilt bonds are linked to the UK Retail Prices Index (RPI); the real OATs bonds are linked to the France consumer price index, excluding tobacco (France CPI ex Tobacco); the German real bonds - Bundei and BT Pei in Italy - are both linked to the European Harmonised Index of Consumer Prices, excluding tobacco (EU HICP ex Tobacco); the real SGBi bonds in Sweden are linked to Sweden consumer price index (Sweden CPI). The lack of short-maturity inflation-linked bonds of the sample countries indicates that inflation-linked yield at short-maturity tends to be less reliable. We, therefore, select three maturities for each country to ensure that enough observations are available. The surfaces of the yield data are plotted in Figure 3.1, with the blank areas represent small proportion of missing values. We further report summary statistics in Table 3.1. It is obvious that the yields of nominal bonds is higher than that of the inflation-indexed bonds over all maturity levels. While Germany has the smallest standard deviation (s.d.), the s.d. of UK is the largest.

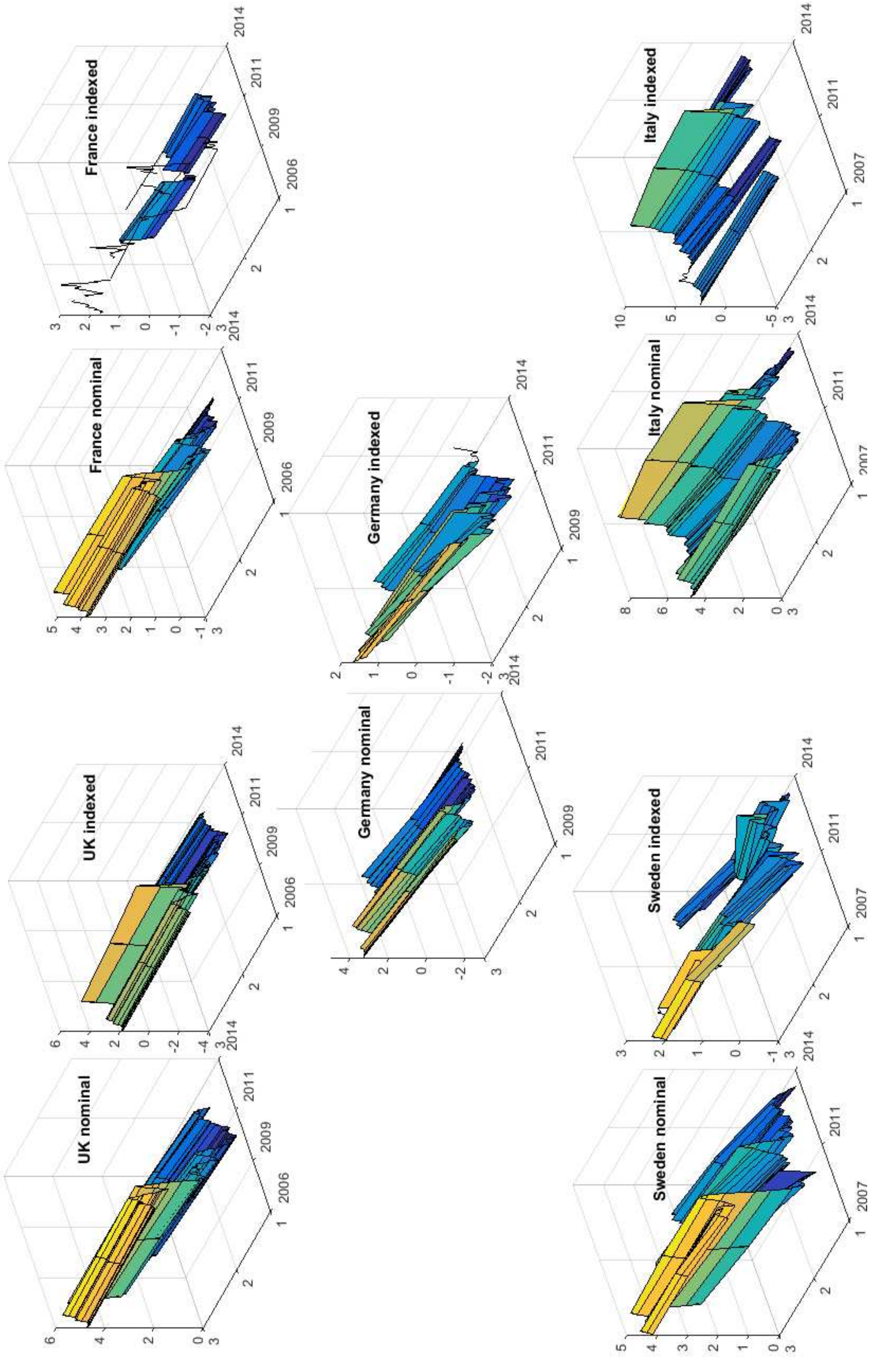

Figure 3.1: Term structures of nominal and inflation-linked bond yields for five European countries.

	period	maturity	type	min	mean	max	s.d.
UK	30.06.2006	3	nominal	0.16	2.20	5.69	1.73
			inflation-indexed	-2.87	-0.14	5.35	1.81
	—	4	nominal	0.35	2.44	5.62	1.59
			inflation-indexed	-2.62	-0.04	4.74	1.64
	31.12.2014	5	nominal	0.57	2.66	5.56	1.48
			inflation-indexed	-2.37	0.11	4.27	1.49
Sweden	30.04.2007	3	nominal	0.18	1.80	4.67	1.27
			inflation-indexed	-0.71	0.15	1.83	0.54
	—	5	nominal	0.58	2.33	4.71	1.11
			inflation-indexed	-0.84	0.51	2.33	0.79
	29.08.2014	10	nominal	0.88	2.59	4.61	1.03
			inflation-indexed	-0.30	0.98	2 29	0.64
France	30.06.2006	3	nominal	-0.02	1.86	4.74	1.46
			inflation-indexed	-1.19	0.43	2.75	1.29
	—	5	nominal	0.06	2.10	4.80	1.37
			inflation-indexed	-1.29	-0.40	1.06	0.60
	31.12.2014	10	nominal	0.18	2.34	4.80	1.28
			inflation-indexed	-1.09	1.03	2.66	1.03
Germany	30.06.2009	3	nominal	-0.07	0.90	2.38	0.77
			inflation-indexed	-1.39	-0.35	1.00	0.54
	—	5	nominal	0.05	1.37	2.85	0.86
			inflation-indexed	-1.16	0.05	1.36	0.65
	31.12.2014	10	nominal	0.39	1.94	3.29	0.84
			inflation-indexed	-0.53	0.33	1.67	0.67
Italy	30.06.2007	3	nominal	0.55	2.94	7.37	1.33
			inflation-indexed	-0.34	1.51	8.21	1.46
	—	5	nominal	0.95	3.53	7.54	1.20
			inflation-indexed	0.20	2.00	7.84	1.29
	31.12.2014	10	nominal	1.89	4.45	7.11	0.93
			inflation-indexed	1.02	2.77	6.72	1.06

Table 3.1: Descriptive statistics of the monthly yields data.

3.2 IE estimation

With the four-factor AFNS model, country-specific IE estimates can be easily obtained from decomposing the model-implied BEIR. In Figure 3.2, we compare the revolution of IE estimates $\hat{\pi}_{it,\tau}$ with maturity of $\tau = 3, 5$ years. In general, the IE estimates over different maturities follow a similar pattern for each country of interest. We can observe an obvious decrease of the expected inflation for the UK following the subprime crisis, which is also seemingly present in France and Sweden. Since the beginning of 2009, the IE levels have been on a declining trend for all sample countries. The long period of downward trend is associated with an increased probability of low inflation or even falling into deflation, in which prices and wages are declining on average.

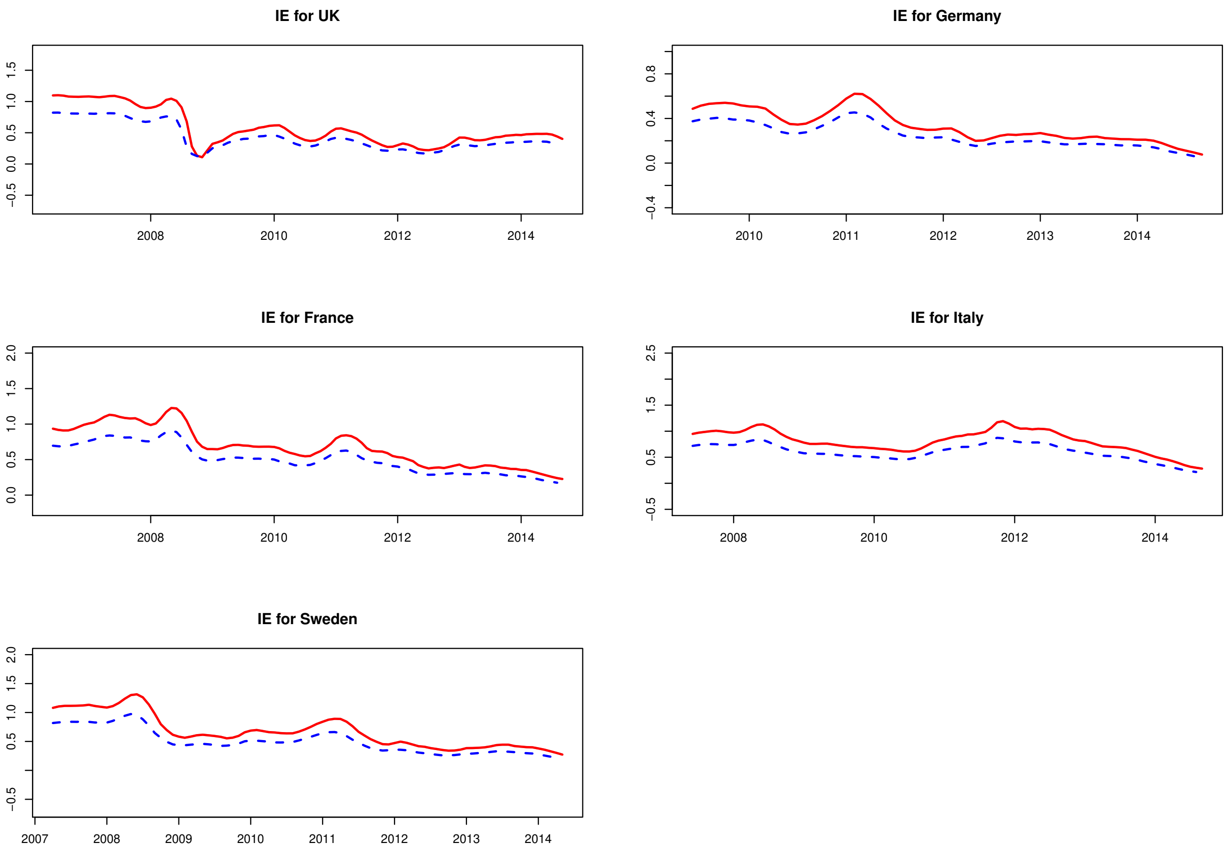

Figure 3.2: The IE estimates over two different maturities for each country. The three-year IE is the red line and the five-year IE is dashed blue.

It is worth mentioning that, even with limited data availability, model (2.1) still achieves

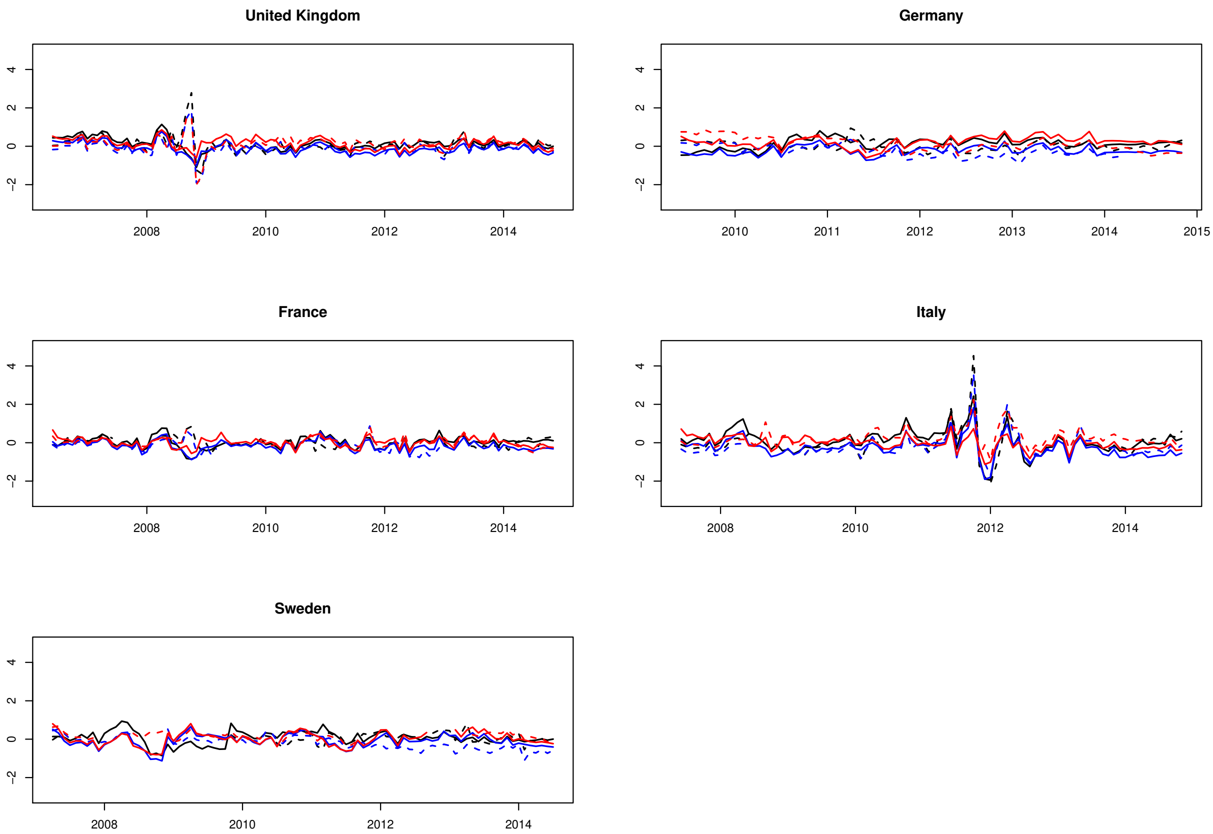

Figure 3.3: The model residuals for the four-factor AFNS model over three maturities τ_1 (red) $< \tau_2$ (blue) $< \tau_3$ (black), with nominal yield is solid and real yield in dashed line.

good performance. Figure 3.3 exhibits the residuals of the four-factor AFNS model over three maturities ($\tau_1 < \tau_2 < \tau_3$). The overall level of the residuals is small with average absolute value at around 0.09, and averaged root mean squared error (RMSE) around 0.1. However, we notice that the model residuals have some jumps. For example, the outliers observed for Italy happened to be during the sovereign default crisis in 2012. We can also observe larger residuals around September 2008 for the UK and Sweden due to the well-documented sub-prime crisis.

4 Empirical results

4.1 Common and specific components of IEs

With the multi-country factor model presented in section 2.2, we are able to conduct joint estimation of nominal and inflation-indexed yields and identify the common factors that determine the IE dynamics. In this section, we will compare and discuss the results of the two-component and three-component models without considering the regional effects across different countries. The effects of the spatial information will be discussed in the following section 4.2.

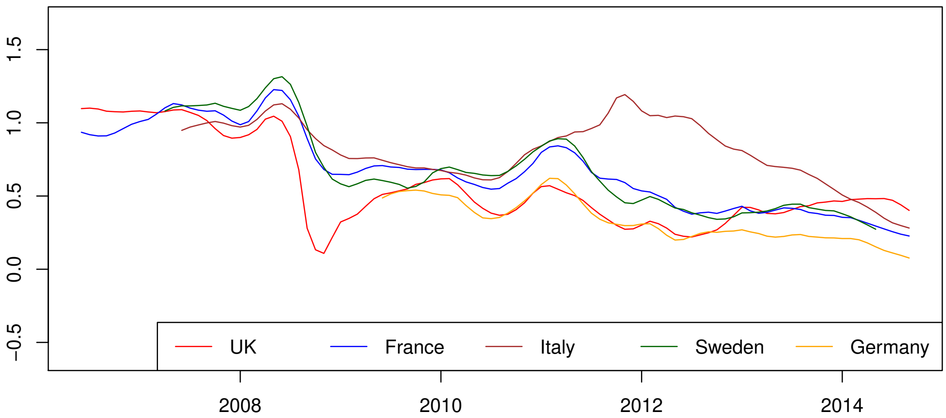

Figure 4.1: Country-specific IE estimates at $\tau = 3$ years

Two-component model To motivate the research, we first plot the three-year IE estimates in Figure 4.1. We find that the three-year IE estimates can track the realized inflation level closely. For instance, the realized inflation level of Sweden had two small peaks at around the third quarter of 2008 and 2011, which can also be observed in our results. Moreover, the estimated IE for Italy increased significantly since the beginning of 2012 and went back to almost zero towards the end of 2014. It is interesting to observe that the inflation rate is not lower in the countries most affected by the sovereign crisis like Italy, this is due to country-specific macroeconomic situation such as the increase of labour cost and the drop in productivity. As can be seen in Figure 3.2, for each sample country, the IE estimates over different maturities in general have similar patterns. We therefore choose three-year IE estimates as the input for empirical analysis in the following.

The estimation results for the two-component model (2.4) and (2.5) are reported in Table 4.1, with all the coefficients significantly different from zero, indicating that the common trend $\Pi_{t,\tau}$ is not negligible. The coefficients in the country-specific regressions, ranging from 0.379 of UK to 0.521 of Germany, and the intercept coefficients, on the other hand is the opposite, with the 0.307 of Germany is the smallest and 0.377 of France is the largest. The extracted common IE factor is presented in Figure 4.2, which experienced a drop since the beginning of 2009 and increased gradually to a small peak around 2012, then it went back to zero towards the end of 2014. Moreover, the model fit results are displayed in Figure 4.3 to show the goodness of fit of the two-component model.

Country-specific equation	$m_{i,\tau}$ (95% CI)	$n_{i,\tau}$ (95% CI)
UK	0.357 (0.198, 0.515)	0.379 (0.143, 0.616)
France	0.377 (0.337, 0.416)	0.518 (0.300, 0.736)
Italy	0.359 (0.214, 0.505)	0.421 (0.160, 0.683)
Sweden	0.329 (0.165, 0.492)	0.464 (0.170, 0.759)
Germany	0.307 (0.132, 0.482)	0.521 (0.182, 0.860)
Common effect equation	p_τ (95% CI)	q_τ (95% CI)
common factor $\Pi_{t,\tau}$	0.244 (0.041, 0.446)	0.406 (-0.178, 0.989)

Table 4.1: Estimated coefficients and 95% confidence intervals (CI) of the two-component model for the IE dynamics.

	UK	France	Italy	Sweden	Germany
Common effect	26.59	34.66	30.93	28.93	32.91
Country-specific effect	73.41	65.34	69.07	71.07	67.09

Table 4.2: Variance decomposition result of the two-component model (in percentage).

To measure how much variability in the outcome variable, the country-specific IE estimates, is explained by the common effect and the country-specific effect, we conduct

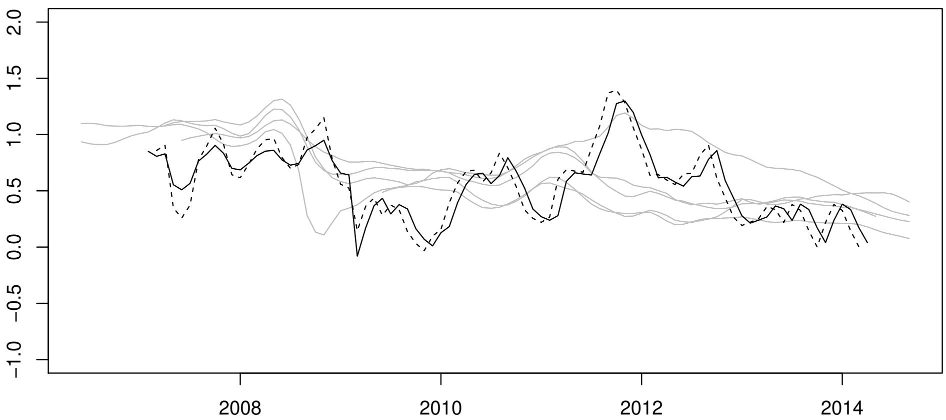

Figure 4.2: The evolution of the common factor using the two-component model, with the predicted value in black solid line and the filtered value in black dashed line.

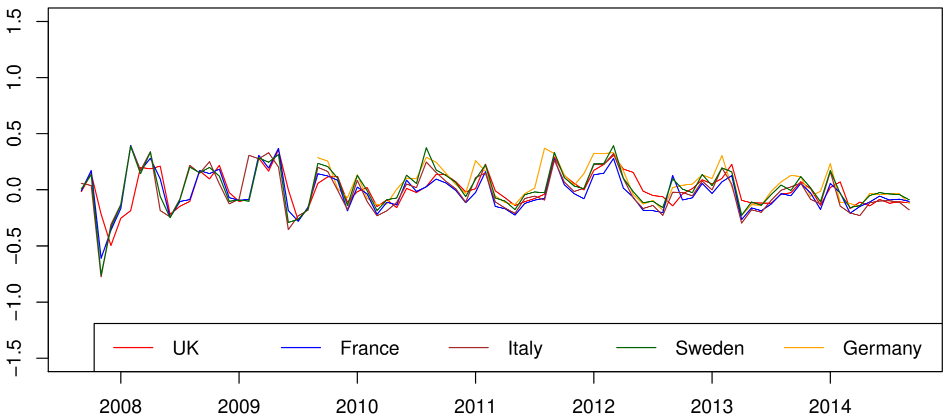

Figure 4.3: Model residual for the two-component modeling of IE dynamics over different countries.

variance decomposition and present the result in Table 4.2. In general, the common effect explains roughly 30% of the total variation of each country with Sweden and UK having relatively smaller share of explained variation. The reason may be these two countries are not members of the euro area, and therefore operate their own monetary policy. To support this hypothesis, we perform a robustness check to examine the role of the common inflation factor by comparing the estimates for different groupings of the countries, i.e.

euro countries and non-euro countries. The estimation results are provided in Table 4.3. The upper panel shows that the common effects across the euro countries - France, Italy, Germany - are quite high, ranging from 64% to 74%. This is due to all euro countries belong to the Economic and Monetary Union (EMU) with a single currency and EMU brings economic integration. In the lower panel of Table 4.3, two non-euro countries are considered. We find that the common effect accounts for 30% of the variation of UK IE estimates and 36% of Sweden IE estimates respectively, which are slightly higher than the joint estimation results (see Table 4.2) of both euro and non-euro countries. The above findings indicate that the common effect is an important factor that driving the IE dynamics, especially for the countries within the EMU. Therefore, policy makers also need to take into account the economic and fiscal policies operate in other countries. As a consequence, economic policy-making becomes a coordination process for all sample countries.

euro countries	$m_{i,\tau}$ (95% CI)	$n_{i,\tau}$ (95% CI)	p_τ (95% CI)	q_τ (95% CI)	common effect (in %)	country-specific effect (in %)
France	0.467	0.208			74.04	25.96
	(0.196, 0.738)	(0.088,0.329)				
Italy	0.457	0.203	-0.014	0.973	70.58	29.42
	(0.251, 0.763)	(0.084,0.322)	(-0.157,	(0.873,		
Germany	0.456	0.208	0.128)	1.074)	64.46	35.54
	(0.325, 0.775)	(0.088,0.329)				
non-euro countries	**$m_{i,\tau}$ (95% CI)**	**$n_{i,\tau}$ (95% CI)**	**p_τ (95% CI)**	**q_τ (95% CI)**	**common effect (in %)**	**country-specific effect (in %)**
UK	0.287	0.140	0.233	0.850	30.52	69.48
	(0.132, 0.442)	(-0.011, 0.291)	(0.032,	(0.598,		
Sweden	0.283	0.146	0.434)	1.102)	36.22	63.78
	(0.122, 0.444)	(-0.009,0.301)				

Table 4.3: Comparison of the estimated coefficients and 95% confidence intervals (CI) for different groups of countries using the two component model for the IE dynamics.

Three-component model With the two-component model, we find that the common IE factor accounts for a large share of the country-specific IE dynamics. Now we further include the macroeconomic factor to understand the interaction between macro-economy and inflation.

In the three-component model (2.6), we choose a default risk factor as $d_{it,\tau}$ because the sample contains two crises during our sample period. In particular, we use a default risk proxy by adopting the three-year credit default swaps (CDS) obtained from Bloomberg. The estimation result of the three-component model is reported in Table 4.4. The last column of Table 4.4 contains the coefficient estimates for the default risk factor, with the largest value of 0.466 for Italy and the smallest value of 0.195 for Germany. Figure 4.4 reveals the evolution of the common IE factor for all sample countries over our sample period and the model residuals are displayed in Figure 4.5. In general, this common trend moves closely and at a similar level of the country-specific IE estimates. It can be seen that the common IE factor decreases substantially during the period spanning the last quarter of 2008 and the first quarter of 2009 due to the subprime crisis initiated in the US, and remains low afterwards. The impact of these negative shocks on the common IE factor also appears to be persistent over time, and a persistent decline in the level of prices can have a negative impact on the economy. Therefore the estimated common IE factor can be used by policy makers and investors as an early signal for the long-run inflation trend .

The next step is to look at the variance decomposition (see Grömping (2007)) result presentd in Table 4.5. The common effect in the first row shows that it can explain roughly 30% of the variation for all sample countries except for Italy, this result in general is consistent with the findings of the two-component model. On the other hand, the variation explained by the chosen macroeconomic factor suggests that Italy is most affected by the crises. Moreover, this default risk effect explains the least variation in Germany during the sample period, it indicates that German economy proved to be resistant in a difficult economic environment.

Country-specific equation	$m_{i,\tau}$ (95% CI)	$n_{i,\tau}$ (95% CI)	$l_{i,\tau}$ (95% CI)
UK	0.198 (0.054, 0.342)	0.267 (0.127, 0.407)	0.283 (0.170, 0.549)
France	0.260 (0.069, 0.451)	0.266 (0.100, 0.433)	0.325 (0.104, 0.545)
Italy	0.280 (0.128, 0.431)	0.232 (0.161, 0.302)	0.466 (0.315, 0.617)
Sweden	0.277 (0.129, 0.424)	0.203 (0.069, 0.338)	0.395 (0.245, 0.542)
Germany	0.156 (0.074, 0.239)	0.262 (0.086, 0.439)	0.195 (0.113, 0.278)
Common effect equation	p_τ (95% CI)	q_τ (95% CI)	
common factor $\Pi_{t,\tau}$	0.322 (0.152, 0.493)	0.788 (0.587, 0.988)	

Table 4.4: Estimated coefficients and 95% confidence intervals (CI) of the three-component model for the IE dynamics.

Figure 4.4: The evolution of the common factor using the three-component model, with the predicted value in black solid line and the filtered value in black dashed line.

	UK	France	Italy	Sweden	Germany
Common effect	30.51	31.05	6.27	34.18	34.25
Country-specific effect	58.72	67.76	50.38	57.49	65.22
Default risk effect	10.77	1.19	43.35	8.33	0.54

Table 4.5: Variance decomposition result of the three-component model (in percentage)

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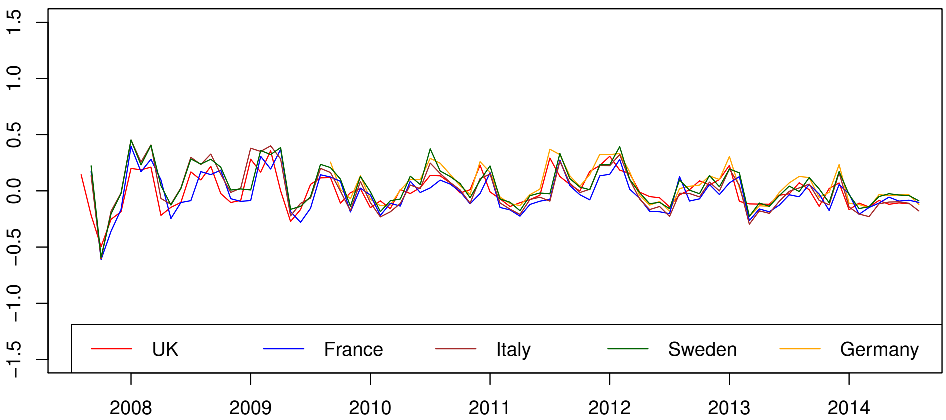

Figure 4.5: Model residual for the three-component modeling of IE dynamics over different countries.

4.2 Spatial-temporal copula for non-Gaussian structure

In this section, we further consider the regional effects across different countries with the spatial-temporal copula model introduced in section 2.2.3. Let the marginal distribution function be estimated non-parametrically as the following empirical distribution function,

$$\hat{F}_{\alpha_{i.}}(u) = (T+1)^{-1} \sum_{t}^{T} \mathbf{I}\{\alpha_{it} \leq u\},$$

and the marginal probability density function is estimated as kernel density estimator,

$$\hat{f}_{\alpha_{i.}}(u) = T^{-1} \sum_{t}^{T} K_h(\alpha_{it} - u),$$

where $K_h(\cdot)$ is the kernel density function. In the spatial-temporal covariance function, the parameters a, b, β and σ are unknown. The parameter v characterizes the behavior of the correlation function near the origin, a grid search method is conducted for a pre-estimation of the parameters. The optimal η is estimated as $\eta = 0.5$, indicating that the spatial correlation is an exponential function of u. Moreover, a is the scaling parameter of time, b is the spatial scaling parameter, and β corresponds to the spatial temporal interaction. We then have the following interpretation for the estimated parameters: Given $\hat{\beta} = 0.1958$, $\hat{a} = 0.02277$ means that the marginal temporal correlation decreases by around 2% with 1 month increase in time , and $\hat{b} = 0.00137$ indicates that

	France	Germany	Italy	Sweden	UK
longitude	2.2	13.3	12.3	18.0	0.09
latitude	48.5	52.3	41.5	59.1	51.3

Table 4.6: The longitude and latitude of the sample countries (in degree).

the marginal space correlation decays by around 1% with a 100-km increase in space. Moreover, we adopt a cross-validation method for selecting the cut-off points d_1 and d_2, it can be observed that the estimated cut-off points correspond to the points where the fitted variogram shows the correlation starts to vanish.

The longitude and latitude of the capital city for each country in Table 4.6 are used to measure the geographical distance. Define a variogram as $\Gamma(t_1 - t_2, n_1 - n_2) \stackrel{\text{def}}{=} \frac{1}{2}E(\alpha_{t_1n_1} - \alpha_{t_2n_2})^2$. It is worth noting that the concept is closely related to the covariance function. For a stationary process with σ^2 being the variance, we then have

$$\hat{\Gamma}(t_1 - t_2, n_1 - n_2) \stackrel{\text{def}}{=} \sigma^2 - Cov(\alpha_{t_1n_1}, \alpha_{t_2n_2})$$

Therefore the empirical variogram is defined as

$$\Gamma(d_1, d_2) \stackrel{\text{def}}{=} \frac{1}{N_{d_1,d_2}} \sum_{t_1,t_2,n_1,n_2:\|t_1-t_2\|\leq d_1,\|n_1-n_2\|\leq d_2} (\alpha_{t_1n_1} - \alpha_{t_2n_2})^2,$$

where N_{d_1,d_2} is the number of observations within the local region indexed by d_1 and d_2. The parametrically fitted variogram is defined as $\Gamma_{\hat{\theta}}(d_1, d_2)$, with the estimates of the parameters $\hat{\theta}$ plugged in,

$$\hat{\theta} \stackrel{\text{def}}{=} \arg\min_\theta(\hat{\Gamma}(t_1 - t_2, n_1 - n_2) - \hat{\Gamma}_\theta(t_1 - t_2, n_1 - n_2))^2.$$

The fitted empirical and parameterized variograms on the transformed data are shown in Figure 4.6 (the transformed data are $\Phi(\hat{F}^{-1}(u_{it,\tau}))$, with $\Phi(\cdot)$ as the c.d.f. of a standard normal distribution). One axis represents the time lags, while another represents the spatial lags. We observe that the vertical axis is increasing with both time lag and distance, which means a decrease in the variance-covariance matrix in time and in distance. The one-step ahead forecasts for the sample countries are plotted in Figure 4.7, where the blue solid line corresponds to the prediction of residuals coming from our copula model. The forecast is later integrated to forecast IE, i.e., we use $\tilde{u}_{it,\tau} = \Phi(\hat{F}^{-1}(u_{it,\tau}))$ to replace the $u_{it,\tau}$.

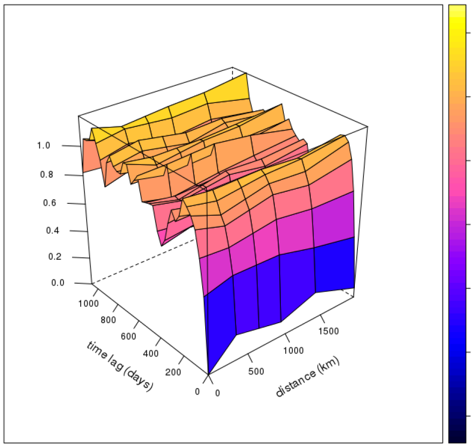

Figure 4.6: The empirical fitted variogram (left) and the parametrically fitted variogram (right).

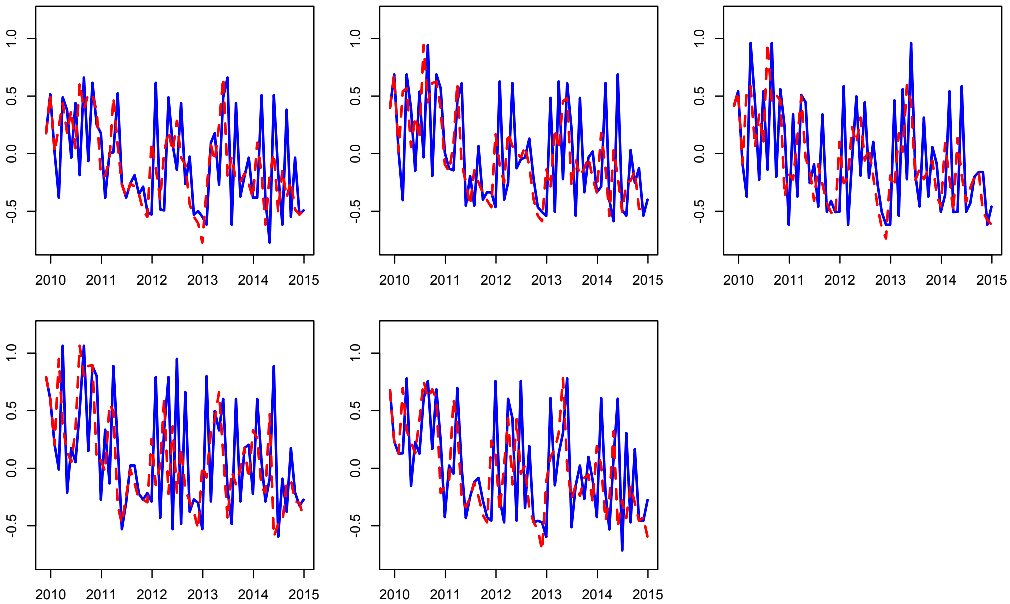

Figure 4.7: The one-month ahead forecast for the residuals without a spatial-temporal copula model (red dotted) and the residual with the spatial-temporal copula model (blue solid).

To examine the regional effects, we compare the out-of-sample forecasting performance for the three-component model with and without including the spatial information. The

	UK	France	Italy	Sweden	Germany
μ_u	0.009	0.005	0.011	0.007	0.009
$RMSE_u$	0.114	0.135	0.146	0.151	0.157
$\mu_{\tilde{u}}$	0.020	0.043	0.025	0.026	0.048
$RMSE_{\tilde{u}}$	0.115	0.140	0.147	0.152	0.167

Table 4.7: The mean μ and RMSE for the out-of-sample forecast errors with the three-component model for all sample countries. $\tilde{u}_{it,\tau}$ is the filtered model residuals using spatial-temporal copula.

rolling sample selected for calculating the forecast errors is from June 2009 to July 2013, and the test sample is from August 2013 to July 2014. The mean and root mean square error (RMSE) for the out-of-sample forecast errors are listed in Table 4.7. Clearly, we observe better performance of $u_{it,\tau}$ than $\tilde{u}_{it,\tau}$. The model with residuals following spatial-temporal copula process do not have better IE estimates, indicating that the geographical correlation is not an important factor accounting for country-specific IE.

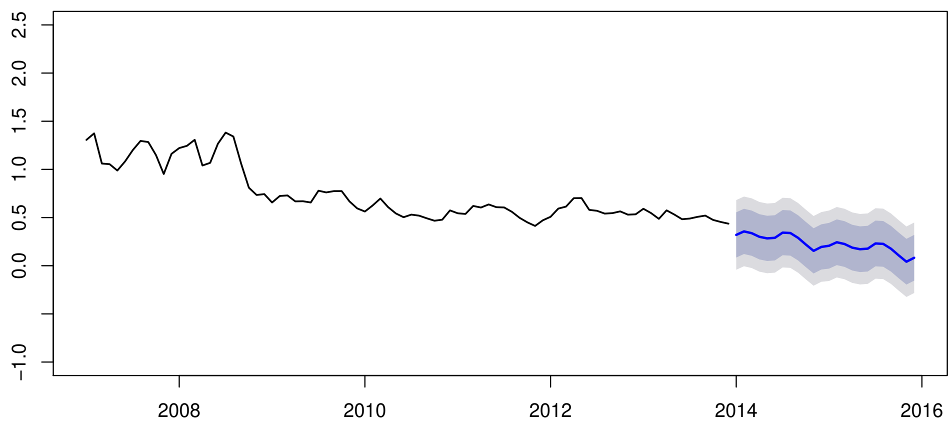

Figure 4.8: The 24-months ahead forecast using the three-component model, the 80% and 95% CI are marked in the shaded area.

Furthermore, in Figure 4.8, we plot the 24-months ahead forecast of the estimated com-

	HICP	SPF1	SPF2
Spearman rank correlation ρ	0.370	0.732	0.528
$p - value$	$1.12e - 2$	$1.44e - 5$	$4.62e - 3$

Table 4.8: The mean μ and RMSE for the out-of-sample forecast errors with the three-component model for all sample countries. $\tilde{u}_{it,\tau}$ is the filtered model residuals using spatial-temporal copula.

mon IE factor estimated from the three-component model, with the corresponding CI of 80% in dark grey and 95% in light grey. Our estimates show that the fall in inflation has started since the beginning of 2009 and the common IE factor is on downward trend afterwards. This suggests that, risks of long-run lowflation should be taken into account by policy makers and investors. To reduce the risk of long-run lowflation and sustain economic stability, the central banks may respond with rate cuts to give economy a boost, or quantitative easing (QE) by which the government buys both sovereign and private debt securities, or commitments to keep rates low and raise them only gradually over time.

In addition, we calculate the Spearman rank correlation coefficient to evaluate the degree of linear association or correlation between our common IE estimate and other inflation measures. This is a nonparametric technique so it is unaffected by the distribution of the population and there is no requirement that the data be collected over regularly spaced intervals. We collect the quarterly euro HICP Inflation forecasts (defined as the year on year percentage change), and the data of the one-year (SPF1) and two-year (SPF2) Survey Professional Forecast on euro. The result in Table 4.8 reports the Spearman rank correlation coefficients and the corresponding p-values. The $p-value$ of the correlation is very small in the last two columns. We therefore reject the null hypothesis of $H_0 : \rho = 0$, indicating that there is strong evidence of linear correlation between our common IE estimates and survey professional forecasts.

5 Conclusion

This paper provides a multi-country factor model framework for joint estimation of nominal and real yields. We first apply a joint arbitrage-free term structure model across different European countries to obtain estimates for country-specific IE. Then we investigate the main factors determining the IE dynamics using the two-component and three-component models.

We find that the estimated common trend is an important driver for each country of interest, especially for the countries within the EMU. The negative impacts caused by the crises have a persistent impact on this common IE factor, its downward trend is associated with an increased probability of low inflation or even falling into deflation. This can be viewed as an early signal for policy decision-making. To reduce the lowflation risk, the central banks may respond with monetary policies such as rate cuts, and QE. Moreover, policy makers also need to take into account the economic and fiscal policies operate in other countries due to the co-movement of country-specific IE dynamics. When there are negative shocks on the economy, Italy is most affected by the crises while Germany appears to be more resistant in a difficult economic environment.

We further consider the regional effects across different countries by applying the spatial-temporal copula model, this allows us to understand the non-Gaussian dependency structure across countries by including the spatial information. However we find that the geographical correlation is not a driving factor for country-specific IE. Based on our empirical results, the paper extracts informative estimates for IE, which may serve as good implications for monetary policies.

6 Compliance with Ethical Standards

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Conflict of Interest: No conflict exists: Shi Chen declares that she has no conflict of interest. Wolfgang K. Härdle declares that he has no conflict of interest. Weining Wang declares that she has no conflict of interest.

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

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7 Appendix

A Estimation of multiple-yield-curve Modelling

The analysis starts by introducing the yield-adjustment term proposed in the original AFNS model. Derived in an analytical form, the yield-adjustment term $\frac{A(\tau)}{\tau}$ with τ months to maturity can be written as

$$\begin{aligned}
\frac{A(\tau)}{\tau} &= \bar{A}\frac{\tau^2}{6} + \bar{B}\left\{\frac{1}{2\lambda^2} - \frac{1}{\lambda^3}\frac{1-\exp(-\lambda\tau)}{\tau} + \frac{1}{4\lambda^3}\frac{1-\exp(-2\lambda\tau)}{\tau}\right\} \\
&+ \bar{C}\left\{\frac{1}{2\lambda^2} + \frac{1}{\lambda^2}\exp(-\lambda\tau) - \frac{1}{4\lambda}\tau\exp(-2\lambda\tau) - \frac{3}{4\lambda^2}\exp(-2\lambda\tau)\right\} \\
&+ \bar{C}\left\{-\frac{2}{\lambda^3}\frac{1-\exp(-\lambda\tau)}{\tau} + \frac{5}{8\lambda^3}\frac{1-\exp(-2\lambda\tau)}{\tau}\right\} \\
&+ \bar{D}\left\{\frac{1}{2\lambda}\tau + \frac{1}{\lambda^2}\exp(-\lambda\tau) - \frac{1}{\lambda^3}\frac{1-\exp(-\lambda\tau)}{\tau}\right\} \\
&+ \bar{E}\left\{\frac{3}{\lambda^2}\exp(-\lambda\tau) + \frac{1}{2\lambda}\tau + \frac{1}{\lambda}\exp(-\lambda\tau) - \frac{3}{\lambda^3}\frac{1-\exp(-\lambda\tau)}{\tau}\right\} \\
&+ \bar{F}\left\{\frac{1}{\lambda^2} + \frac{1}{\lambda^2}\exp(-\lambda\tau) - \frac{1}{2\lambda^2}\exp(-2\lambda\tau) - \frac{3}{\lambda^3}\frac{1-\exp(-\lambda\tau)}{\tau} + \frac{3}{4\lambda^3}\frac{1-\exp(-2\lambda\tau)}{\tau}\right\}
\end{aligned} \tag{A.1}$$

where $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$, $\bar{E}$, and $\bar{F}$ can be identified by the volatility matrix Σ defined in the dynamics equation under the P-measure. The value of the adjustment term is constant in time t but depends on τ, the coefficient λ governs the mean reversion rate of slope and curvature factors and the volatility parameters $\bar{A}$, $\bar{D}$, and $\bar{F}$.

The four latent factors defined in the state variable $X_{it}^{\top} = \left(L_{it}^N, S_{it}^N, C_{it}^N, L_{it}^R\right)$ evolve dynamically according to,

$$\begin{pmatrix} dL_{it}^N \\ dS_{it}^N \\ dC_{it}^N \\ dL_{it}^R \end{pmatrix} = \begin{pmatrix} \kappa_{i,11} & \kappa_{i,12} & \kappa_{i,13} & \kappa_{i,14} \\ \kappa_{i,21} & \kappa_{i,22} & \kappa_{i,23} & \kappa_{i,24} \\ \kappa_{i,31} & \kappa_{i,32} & \kappa_{i,33} & \kappa_{i,34} \\ \kappa_{i,41} & \kappa_{i,42} & \kappa_{i,43} & \kappa_{i,44} \end{pmatrix} \begin{pmatrix} L_{it}^N \\ S_{it}^N \\ C_{it}^N \\ L_{it}^R \end{pmatrix} dt + \Sigma_i \begin{pmatrix} dW_t^{L^N} \\ dW_t^{S^N} \\ dW_t^{C^N} \\ dW_t^{L^R} \end{pmatrix} \tag{A.2}$$

where $W_t^{L^N}$, $W_t^{S^N}$, $W_t^{C^N}$, and $W_t^{L^R}$ are independent Brownian motions.

We estimate the parameters in (A.2) using the Kalman filter technique. The Kalman filter recursion is a set of equations that allow for an estimator to be updated once a new observation y_{it} becomes available. It first forms an optimal predictor of the unobserved state variable vector given its previously estimated value. This prediction is obtained using the distribution of unobserved state variables, conditional on the previous estimated values. These estimates for unobserved state variables are then updated using the information provided by the observed variables.

The transition equation derived from Christensen et al. (2011) takes the form,

$$X_{i,t} = \left[I - expm\left(-K_i \Delta t\right)\right]\theta_i + expm\left(-K_i \Delta t\right) X_{i,t-1} + \eta_{it} \tag{A.3}$$

where *expm* stands for matrix exponential. The measurement and transition equations are assumed to have the following error structure,

$$\begin{pmatrix} \eta_{it} \\ \varepsilon_{it} \end{pmatrix} = N\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Q_i & 0 \\ 0 & H_i \end{pmatrix} \right\}$$

where Q_i has a special structure as follows,

$$Q_i = \int_0^{\Delta t} e^{-K_i s} \Sigma \Sigma^\top e^{-(K_i)^\top s} ds$$

In particular, the transition and measurement errors are assumed to be orthogonal to the initial state. The initial value of the filter is given by the unconditional mean and variance of the state variable $X_{it}^\top$ under the P measure, that is,

$$\begin{aligned} X_{i,o} &= \theta_i \\ \varSigma_{i,o} &= \int_o^{\infty} e^{-K_i s} \varSigma \varSigma^\top e^{-(K_i)^\top s} ds \end{aligned}$$

which can be calculated using the analytical solution provided in Fisher and Gilles (1996).

B BEIR Decomposition

In the context of an AF model, it is assumed that investors have no opportunities to make risk-free profits. Thus, the bonds can be priced by basic pricing equations according to Cochrane (2009),

$$P_{it} = \mathsf{E}_t \left\{ M_{i,t+1} x_{i,t+1} \right\}, \tag{B.1}$$

To estimate the expected value of inflation using the country-specific stochastic discount factor (SDF) $M_{i,t}$, we first use the Taylor series to approximate the moments of the logarithm. Assuming that M_t is significant from zero, the yield for a nominal bond can be extended as follows,

$$\log\left(M_{i,t+1}^N M_{i,t+2}^N \cdots M_{i,t+\tau}^N\right) = \log\left\{\left(\mu_{i,M} + M_{i,t+1}^N M_{i,t+2}^N \cdots M_{i,t+\tau}^N - \mu_{i,M}\right)\right\} \tag{B.2}$$

where

$$\begin{aligned} \mu_{i,M} &\stackrel{\text{def}}{=} \mathsf{E}_t\left(M_{i,t+1}^N M_{i,t+2}^N \cdots M_{i,t+\tau}^N\right) \\ \mathbf{M}_{i,1:\tau}^N &\stackrel{\text{def}}{=} \left(M_{i,t+1}^N M_{i,t+2}^N \cdots M_{i,t+\tau}^N\right). \end{aligned}$$

Take a Taylor expansion of equation (B.2) and a conditional expectation E_t on both sides, we have

$$\mathsf{E}_t\left(\mathbf{M}_{i,1:\tau}^N\right) = \log \mu_{i,M} - (2\mu_{i,M}^2)^{-1}\mathsf{Var}_t\left(\mathbf{M}_{i,1:\tau}^N\right) + o_p(\mathsf{Var}_t\mathbf{M}_{i,1:\tau}^N), \tag{B.3}$$

and further as

$$\mathsf{Var}\log\mathbf{M}_{i,1:\tau}^N = (\mu_{i,M}^2)^{-1}\,\mathsf{Var}\,\mathbf{M}_{i,1:\tau}^N + o_p(\mathsf{Var}_t\mathbf{M}_{i,1:\tau}^N), \tag{B.4}$$

$$\mathsf{E}_t\left(\log\mathbf{M}_{i,1:\tau}^N\right) = \log\mu_{i,M} - 2^{-1}\mathsf{Var}_t\left(\log\mathbf{M}_{i,1:\tau}^N\right) + o_p(\mathsf{Var}_t\mathbf{M}_{i,1:\tau}^N). \tag{B.5}$$

Therefore,

$$\begin{aligned} y_{it}^N(\tau) &= -\frac{1}{\tau}\log\mathsf{E}_t\left(\mathbf{M}_{i,1:\tau}^N\right) \\ &= -\frac{1}{\tau}\mathsf{E}_t\left(\log\mathbf{M}_{i,1:\tau}^N\right) - \frac{1}{2\tau}\mathsf{Var}_t\left(\log\mathbf{M}_{i,1:\tau}^N\right) + o_p(\mathsf{Var}_t\mathbf{M}_{i,1:\tau}^N) \end{aligned} \tag{B.6}$$

Similar solutions could be obtained for the inflation-indexed bonds by the same logic, i.e., define

$$\mathbf{M}_{i,1:\tau}^N \stackrel{\text{def}}{=} \left(M_{i,t+1}^N M_{i,t+2}^N \cdots M_{i,t+\tau}^N\right) \tag{B.7}$$

$$\mathbf{M}_{,i1:\tau}^R \stackrel{\text{def}}{=} \left(M_{i,t+1}^R M_{i,t+2}^R \cdots M_{i,t+\tau}^R\right), \tag{B.8}$$

where the nominal and the real (for inflation-linked bond) SDF at time t for country i are denoted by M_{it}^N and M_{it}^R respectively. Then the prices of the zero-coupon bonds at time t, which pay one unit measured by the consumption basket at the time of maturity $t+\tau$, are formed as follows:

$$\begin{aligned} P_{it}^N(\tau) &= \mathsf{E}_t\left(\mathbf{M}_{i,1:\tau}^N\right) \\ P_{it}^R(\tau) &= \mathsf{E}_t\left(\mathbf{M}_{i,1:\tau}^R\right) \end{aligned} \tag{B.9}$$

where $P_{it}^N(\tau)$ and $P_{it}^R(\tau)$ represent the prices of nominal and real bonds respectively. The price of the consumption basket, which is known as the overall price level $Q_{i,t}$, has the following link with SDFs given the no-arbitrage assumption,

$$\frac{M_{it}^N}{M_{it}^R} = \frac{Q_{i,t-1}}{Q_{i,t}} \tag{B.10}$$

Converting the price into the yield by the equation of $y_{it}(\tau) = -\frac{1}{\tau}\log P_{it}(\tau)$,

$$\begin{aligned}
y_{it}^N(\tau) &= -\frac{1}{\tau}\log \mathsf{E}_t\left(\mathbf{M}_{i,1:\tau}^N\right) \\
&\approx -\frac{1}{\tau}\mathsf{E}_t\left(\log \mathbf{M}_{i,1:\tau}^N\right) - \frac{1}{2\tau}\mathsf{Var}_t\left(\log \mathbf{M}_{i,1:\tau}^N\right) \\
y_{it}^R(\tau) &= -\frac{1}{\tau}\log \mathsf{E}_t\left(\mathbf{M}_{i,1:\tau}^R\right) \\
&\approx -\frac{1}{\tau}\mathsf{E}_t\left(\mathbf{M}_{i,1:\tau}^R\right) - \frac{1}{2\tau}\mathsf{Var}_t\left(\log \mathbf{M}_{i,1:\tau}^R\right),
\end{aligned}$$

Therefore,

$$\begin{aligned}
y_{it}^N(\tau) - y_{it}^R(\tau) &\approx -\frac{1}{\tau}\mathsf{E}_t\left(\log\frac{\mathbf{M}_{i,1:\tau}^N}{\mathbf{M}_{i,1:\tau}^R}\right) + \frac{1}{2\tau}\mathsf{Var}_t\left(\log\frac{\mathbf{M}_{i,1:\tau}^N}{\mathbf{M}_{i,1:\tau}^R}\right) \\
&- \frac{1}{\tau}\mathsf{Cov}_t\left(\log\frac{\mathbf{M}_{i,1:\tau}^N}{\mathbf{M}_{i,1:\tau}^R}, \log \mathbf{M}_{i,1:\tau}^R\right).
\end{aligned}$$

Given the log inflation is $\pi_{i,t+1} = \log\frac{Q_{i,t+1}}{Q_{i,t}}$ and (B.10), the $BEIR$ can be decomposed as

$$\begin{aligned}
y_{it}^N(\tau) - y_{it}^R(\tau) &\approx \frac{1}{\tau}\mathsf{E}_t\left(\log \pi_{i,t+1}\pi_{i,t+2}\cdots\pi_{i,t+\tau}\right) - \frac{1}{2\tau}\mathsf{Var}_t\left(\log \pi_{i,t+1}\pi_{i,t+2}\cdots\pi_{i,t+\tau}\right) \\
&+ \frac{1}{\tau}\mathsf{Cov}_t\left(\log \pi_{i,t+1}\pi_{i,t+2}\cdots\pi_{i,t+\tau}, \log \mathbf{M}_{i,1:\tau}^R\right)
\end{aligned} \tag{B.11}$$

that is,

$$\widehat{BEIR}_{it,\tau} = y_{it}^N(\tau) - y_{it}^R(\tau) = \hat{\pi}_{it,\tau} + \hat{rp}_{it,\tau} \tag{B.12}$$

where $\hat{\pi}_{it,\tau}$ is the model-implied IE, $\hat{rp}_{it,\tau}$ is the RP. To link the $BEIR_{it}(\tau)$ with the estimated state variables in equation 2.1, we assume that the P-dynamics of the SDF are,

$$\begin{aligned}
\frac{dM_{it}^N}{M_{it}^N} &= -(r_i^N(t) - r_i^N(t-1))dt - (\Gamma_{it}^N - \Gamma_{i,t-1}^N)dW_t \\
\frac{dM_{it}^R}{M_{it}^R} &= -(r_i^R(t) - r_i^R(t-1))dt - (\Gamma_{it}^R - \Gamma_{i,t-1}^R)dW_t,
\end{aligned} \tag{B.13}$$

where $r_i^N(t) = L_{it}^N + S_{it}^N$, $r_i^R(t) = L_{it}^R + \alpha_i^S S_{it}^N$, and Γ_{it}^N and Γ_{it}^R represent the corresponding risk premium. Hence, the dynamics of the overall price level are given by

$$\begin{aligned}
d\log\left(\frac{Q_{i,t-1}}{Q_{i,t}}\right) &= -\{r_i^N(t) - r_i^R(t)\}dt + \{r_i^N(t-1) - r_i^R(t-1)\}dt \\
d\log(Q_{i,t}) &= \{r_i^N(t) - r_i^R(t)\}dt
\end{aligned} \tag{B.14}$$

Then the IE is,

$$\hat{\pi}_{it}(\tau) = -\frac{1}{\tau} \log \mathsf{E}_t^P \left[\exp\left\{ -\int_t^{t+\tau} r_i^N(s)ds + \int_t^{t+\tau} r_i^R(s)ds \right\} \right] \tag{B.15}$$

with $r_i^N(t), r_i^R(t)$ correspond to the instantaneous risk-free rates of the nominal and inflation-indexed bonds respectively. Furthermore, $r_i(t)$ and the risk premium Γ_{it} are given as,

$$r_i(t) = \rho_{i,0}(t) + \rho_{i,1}(t) X_{it} \tag{B.16}$$

$$\Gamma_{it} = \gamma_{i,0} + \gamma_{i,1} X_{it} \tag{B.17}$$

where $\rho_{i,0}(t)$, $\rho_{i,1}(t)$, $\gamma_{i,0}$, and $\gamma_{i,1}$ are bounded, continuous functions for each country i. X_{it} is the state variable and Y_{it} is the realized observations. The solutions can be solved by a system of ordinary differential equations using the fourth-order Runge-Kutta method.