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Lifschitz Realizability as a Topological Construction

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Abstract

We develop a number of variants of Lifschitz realizability for **CZF** by building topological models internally in certain realizability models. We use this to show some interesting metamathematical results about constructive set theory with variants of the lesser limited principle of omniscience including consistency with unique Church's thesis, consistency with some Brouwerian principles and variants of the numerical existence property.

1 Introduction

In [23] and [24], Van Oosten shows how the Lifschitz realizability topos can be viewed as a category of sheaves over a particular Lawvere-Tierney topology constructed in the effective topos. Although a remarkable result, it has some shortcomings:

1. The construction refers explicitly to computable functions and Lifschitz's encoding of finite sets. This makes it appear that the construction is unique to the effective topos and cannot be carried out in other toposes.

2. The construction relies on many technical definitions and techniques from topos theory.

3. The construction is not guaranteed to work predicatively.

In this paper we will give a new presentation of this result. Instead of topos theory we work in the set theory **CZF**, which is regarded as a predicative theory for mathematics. Instead of Lawvere-Tierney topologies, we will use formal topologies and a predicative notion of topological model due to Gambino.

Aside from this difference in presentation, our results are more general than Van Oosten's in two ways (although the first of these does relate to some more recent results by Lee and Van Oosten in [11]).

Firstly, instead of considering just one formal topology, we will consider an infinite family of formal topologies $\mathcal{L}_n$ for each natural number $n \geq 2$, with the original Lifschitz realizability model just corresponding to the formal topology $\mathcal{L}_2$. The topologies $\mathcal{L}_n$ correspond to certain variants of the lesser limited principle of omniscience, **LLPO**, which were first studied by Richman in [20], and are denoted $\mathbf{LLPO}_n$. We will use these models to give a new proof of a theorem due to Hendtlass and Lubarsky in [9]: $\mathbf{LLPO}_{n+1}$ is strictly weaker than $\mathbf{LLPO}_n$. This answers positively a question raised by Hendtlass: is there a variant of Lifschitz realizability that separates $\mathbf{LLPO}_n$ from $\mathbf{LLPO}_{n+1}$?

Secondly, we identify axioms, $\mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^{\mathbb{N}}}$ that hold in the McCarty realizability model $V(\mathcal{K}_1)$ that suffice to carry out internally the construction of the formal topologies $\mathcal{L}_n$ we will use in the models. This can be done entirely in $\mathbf{CZF} + \mathbf{MP} + \mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^{\mathbb{N}}}$, without any explicit reference to computable functions. This enables us to easily generate variants of Lifschitz realizability by simply checking that the same axioms $\mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^{\mathbb{N}}}$ hold in other realizability models. By using realizability with truth in this way we will show that the theories $\mathbf{CZF}+\mathbf{MP}+\mathbf{LLPO}_n$ have certain variants of the numerical existence property. By using realizability over $\mathcal{K}_2$ in this way we will show that $\mathbf{CZF}+\mathbf{LLPO}$ is consistent with certain (but not all) Brouwerian continuity principles.

A more traditional version of Lifschitz realizability for $\mathbf{CZF}+\mathbf{LLPO}+\mathbf{CT}_!$ similar to that in [5] can be recovered by a two step process of interpreting the topological model $V^{(\mathcal{L}_2)}$ in the McCarty realizability model $V(\mathcal{K}_1)$, itself constructed in $\mathbf{CZF}+\mathbf{MP}$ as illustrated below.

$$\begin{array}{r|ccccc}
\text{Theory} & \begin{array}{c}\mathbf{CZF}+\mathbf{MP}\\ +\mathbf{LLPO}+\mathbf{CT}_!\end{array} & & \begin{array}{c}\mathbf{CZF}+\mathbf{MP}\\ +\mathbf{IP}_{\mathcal{F}_2,\mathbb{N}^{\mathbb{N}}}+\mathbf{CT}_0\end{array} & & \mathbf{CZF}+\mathbf{MP} \\
 & & \hookrightarrow & & \hookrightarrow & \\
\text{Model} & V^{(\mathcal{L}_2)} & & V(\mathcal{K}_1) & & V
\end{array}$$

2 Constructive Set Theory

We will consider the intuitionistic set theories **CZF** and **IZF**, as described for instance in [1] or [2].

We will use the following set theoretic formulations of Markov's principle and Church's thesis.

Definition 2.1. *Markov's principle*, **MP**, is the following axiom. Let $\alpha : \mathbb{N} \to 2$ be a function. Then,

$$\neg\neg\,(\exists n \in \mathbb{N})\,\alpha(n) = 1 \quad \to \quad (\exists n \in \mathbb{N})\,\alpha(n) = 1$$

Definition 2.2. *Church's thesis*, $\mathbf{CT}_0$ is the following axiom. Let $\phi(x, y)$ be any formula. Then, writing $\{e\}(n)$ to mean the result of running the eth Turing machine with input n,

$$(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})\,\phi(n, m) \quad \to \quad (\exists e \in \mathbb{N})(\forall n \in \mathbb{N})\,\phi(n, \{e\}(n))$$

Church's thesis for functions, $\mathbf{CT}_!$ is the axiom that every function from $\mathbb{N}$ to $\mathbb{N}$ is computable.

We recall the following definitions and theorems on finite sets, as appear in [2, Chapters 6 and 8]. The theorems will often be used implicitly while working with finitely enumerable sets.

Definition 2.3. A set X is *finite* if for some $n \in \mathbb{N}$ there exists a bijection from n to X.

A set X is *finitely enumerable* if for some $n \in \mathbb{N}$ there exists a surjection from n to X.

Theorem 2.4 (**CZF**). *Suppose that $\phi(x_1, \ldots, x_n)$ is a formula of arithmetic, where all quantifiers are bounded, and the only free variables are amongst $x_1, \ldots, x_n$. Then we can prove the following instance of excluded middle.*

$$(\forall x_1, \ldots, x_n \in \mathbb{N})\ \phi(x_1, \ldots, x_n) \vee \neg\phi(x_1, \ldots, x_n)$$

Proof. See [2, Theorem 6.6.2]. □

Theorem 2.5 (**CZF**). *"The Pigeonhole Principle for Finitely Enumerable Sets." Let A be a finitely enumerable set. Every injective function $f : A \rightarrowtail A$ is also a surjection.*

Proof. See [2, Theorem 8.2.10]. □

Theorem 2.6 (**CZF**). *"The Finite Axiom of Choice." Suppose A is a finite set, B is any set, and $R \subseteq A \times B$ is a relation such that $(\forall a \in A)(\exists b \in B)\, \langle a, b\rangle \in R$.*

Then there is a function $f : A \to B$ such that for all $a \in A$, $\langle a, f(a)\rangle \in R$.

Proof. See [2, Theorem 8.2.8]. □

We can also prove a finite version of **LPO**:

Theorem 2.7 (**CZF**). *For every finitely enumerable set X and every $f : X \to 2$, either there exists some $x \in X$ such that $f(x) = 1$ or for all $x \in X$, $f(x) = 0$.*

Proof. Show by induction on n that if there is a surjection $n \twoheadrightarrow X$ then the result holds for X. □

3 Formal Topologies and Heyting Valued Models of CZF

3.1 Basic Definitions

We recall the basic definitions of formal topology and Gambino's Heyting valued interpretation of **CZF**. For details see [8]. The basic idea here is that to each formula in set theory, we assign an open set, which we think of as the "truth value" of the formula. We use Gambino's presentation of topological models since it can be formalised in, and provides models for **CZF**.

Definition 3.1. If $\langle S, \leq\rangle$ is a poset, and p is a subset of S, we write $p\downarrow$ for the downwards closure of p. That is,

$$p\downarrow \quad := \quad \{x \in S \mid (\exists y \in p)\, x \leq y\}$$

Definition 3.2. A *formal topology* is $\langle S, \leq, \lhd\rangle$ such that $\langle S, \leq\rangle$ is a poset, and $\lhd$ is a (class) relation between elements and subsets of S, such that

1. if $a \in p$, then $a \lhd p$
2. if $a \leq b$ and $b \lhd p$, then $a \lhd p$
3. if $a \lhd p$ and $(\forall x \in p)(x \lhd q)$, then $a \lhd q$
4. if $a \lhd p$ and $a \lhd q$, then $a \lhd \downarrow p \cap \downarrow q$

Definition 3.3. Let $\mathcal{S} := \langle S, \leq, \lhd \rangle$ be a formal topology. A *set-presentation* for $\mathcal{S}$ is a (set) function $R : S \to \mathcal{P}(\mathcal{P}S)$ such that

$$a \lhd p \leftrightarrow (\exists u \in R(a)) u \subseteq p$$

If $(S, \leq, \lhd)$ has a set-presentation, we say it is *set-presentable*.

Definition 3.4. Let $\mathcal{S} := \langle S, \leq, \lhd \rangle$ be a set presentable formal topology. We define the *nucleus* of $\mathcal{S}$ to be the following class function $j : \mathcal{P}(S) \to \mathcal{P}(S)$. For $p \subseteq S$,

$$j(p) := \{a \in S \mid a \lhd p\}$$

We extend j to an operation, J, on subclasses of S by

$$J(P) := \bigcup \{j(v) \mid v \subseteq P\}$$

Definition 3.5. We say a formal topology $\langle S, \leq, \lhd \rangle$ is *proper* if for all $a \in S$, $\neg a \lhd \emptyset$. (Or equivalently if $j(\emptyset) = \emptyset$.)

Definition 3.6. Let $\mathcal{S} = \langle S, \leq, \lhd \rangle$ be a set presentable formal topology. The class $V^{(\mathcal{S})}$ is defined inductively as the smallest class such that $f \in V^{(\mathcal{S})}$ whenever f is a function with $\mathrm{dom}(f) \subseteq V^{(\mathcal{S})}$ and for all $x \in \mathrm{dom}(f)$, $f(x)$ is a $\lhd$-closed subset of S.

For each sentence ϕ in the language of set theory with parameters from $V^{(\mathcal{S})}$, we assign a $\lhd$-closed class denoted $[\![\phi]\!]$, which we define by induction on formulas as follows. For bounded ϕ, $[\![\phi]\!]$ will be a set.

We first define a complete Heyting algebra structure on the class of $\lhd$-closed classes as follows. For P and Q $\lhd$-closed classes,

$$\begin{aligned}
\top &:= S \\
\bot &:= J(\emptyset) \\
P \wedge Q &:= P \cap Q \\
P \vee Q &:= J(P \cup Q) \\
P \to Q &:= \{a \in S \mid a \in P \to a \in Q\} \\
\bigvee_{x \in U} P_x &:= J\left(\bigcup_{x \in U} P_x\right) \\
\bigwedge_{x \in U} P_x &:= \bigcap_{x \in U} P_x
\end{aligned}$$

We define the interpretation of atomic sentences $a \in b$ and $a = b$ by simultaneous induction on a and b:

$$\begin{aligned}
a \in b &:= \bigvee_{c \in \mathrm{dom}(b)} b(c) \wedge [\![a = c]\!] \\
a = b &:= \bigwedge_{c \in \mathrm{dom}(a)} a(c) \to [\![c \in b]\!] \quad \wedge \quad \bigwedge_{c \in \mathrm{dom}(b)} b(c) \to [\![c \in b]\!]
\end{aligned}$$

We then extend this to all formulas as below.

$$
\begin{aligned}
[\![\perp]\!] &:= \perp \\
[\![\phi \wedge \psi]\!] &:= [\![\phi]\!] \wedge [\![\psi]\!] \\
[\![\phi \vee \psi]\!] &:= [\![\phi]\!] \vee [\![\psi]\!] \\
[\![\phi \to \psi]\!] &:= [\![\phi]\!] \to [\![\psi]\!] \\
[\![(\exists x \in a)\, \phi]\!] &:= \bigvee_{x \in \mathrm{dom}(a)} [\![\phi]\!] \\
[\![(\forall x \in a)\, \phi]\!] &:= \bigwedge_{x \in \mathrm{dom}(a)} [\![\phi]\!] \\
[\![(\exists x)\, \phi]\!] &:= \bigvee_{x \in V^{(\mathcal{S})}} [\![\phi]\!] \\
[\![(\forall x)\, \phi]\!] &:= \bigwedge_{x \in V^{(\mathcal{S})}} [\![\phi]\!]
\end{aligned}
$$

We write $V^{(\mathcal{S})} \models \phi$ to mean $[\![\phi]\!] = \top$. For a collection of formulas, Φ, we write $V^{(\mathcal{S})} \models \Phi$ to mean $V^{(\mathcal{S})} \models \phi$ for all $\phi \in \Phi$.

Theorem 3.7 (Gambino)**.** *Let $\mathcal{S}$ be a set presentable formal topology. Then*

$$V^{(\mathcal{S})} \models \mathbf{CZF}$$

Proof. See [8]. □

3.2 Some Absoluteness Lemmas

For some of the results later, it will be important that under certain conditions statements that hold in the background universe also hold internally in the topological model and vice versa. To this end, we prove a series of absoluteness lemmas below.

First note that any set x can be viewed as an element of $V^{(\mathcal{S})}$, $\hat{x}$ as follows.

$$
\begin{aligned}
\mathrm{dom}(\hat{x}) &:= x \\
\hat{x}(y) &:= \top \qquad\qquad \text{for all } y \in x
\end{aligned}
$$

Lemma 3.8. *In the below, let ϕ and ψ be any formulas, possibly with parameters from $V^{(\mathcal{S})}$.*

1. *We can prove in* **CZF** *that for any set x, $[\![\phi(\hat{y})]\!] = \top$ holds for all y in x if and only if $[\![(\forall y \in \hat{x})\, \phi(y)]\!] = \top$ holds.*
2. *$[\![\phi]\!] \subseteq [\![\psi]\!]$ if and only if $[\![\phi \to \psi]\!] = \top$.*
3. *$[\![\phi]\!] = \top$ and $[\![\psi]\!] = \top$ if and only if $[\![\phi \wedge \psi]\!] = \top$.*
4. *For proper formal topologies, $[\![\perp]\!] = \emptyset$.*
5. *If $(\exists y \in x)\, [\![\phi(\hat{y})]\!] = \top$ then $[\![(\exists y \in \hat{x})\, \phi(y)]\!] = \top$.*
6. *If $[\![\phi]\!] = \top$ or $[\![\psi]\!] = \top$ then $[\![\phi \vee \psi]\!] = \top$.*

Proof. For 1, 2 and 3 note that joins and implications in the Heyting algebra on $\lhd$-closed classes are exactly the usual joins and implications for the Heyting algebra of subsets of a set. 1, 2 and 3 follow by the basic properties of Heyting algebras.

4 is just by unfolding definitions.

For 5, note that we have

$$[\![(\exists y \in \hat{x})\, \phi(y)]\!] = J\left(\bigcup_{y \in x} [\![\phi(\hat{y})]\!]\right)$$

However, we also have

$$\bigcup_{y \in x} [\![\phi(\hat{y})]\!] \subseteq J\left(\bigcup_{y \in x} [\![\phi(\hat{y})]\!]\right)$$

Then 5 easily follows.

One can then prove 6 by a similar argument. $\square$

Lemma 3.9. *Suppose that* $(\bigcup_x [\![\phi(\hat{x})]\!]) \subseteq [\![\psi]\!]$*. Then* $[\![((\exists x)\, \phi(x)) \rightarrow \psi]\!] = \top$*. Suppose that* $[\![\phi]\!] \vee [\![\psi]\!] \subseteq [\![\chi]\!]$*. Then* $[\![\phi \vee \psi \rightarrow \chi]\!] = \top$*.*

Proof. Suppose that $(\bigcup_x [\![\phi(\hat{x})]\!]) \subseteq [\![\psi]\!]$. Then we have

$$J\left(\bigcup_{x} [\![\phi(\hat{x})]\!]\right) \subseteq J\left([\![\psi]\!]\right)$$

However, $[\![\psi]\!]$ is already $\lhd$-closed, so $J([\![\psi]\!]) = [\![\psi]\!]$. But then it easily follows that $[\![(\exists x)\, \phi(x)]\!] \subseteq [\![\psi]\!]$ and so $[\![((\exists x)\, \phi(x)) \rightarrow \psi]\!] = \top$.

The other part can be proved by a similar argument. $\square$

Lemma 3.10. *Let x and y be sets and let $z \in V^{(\mathcal{S})}$. Then,*

$$[\![z \in \widehat{\{x, y\}} \leftrightarrow z = \hat{x} \vee z = \hat{y}[\![= \top \tag{1}$$

$$[\![z \in \widehat{\bigcup x} \leftrightarrow (\exists w \in \hat{x})\, z \in w]\!] = \top \tag{2}$$

Proof. We first check (1). Unfolding definitions we have that both $[\![z \in \widehat{\{x, y\}}]\!]$ and $[\![z = \hat{x} \vee z = \hat{y}]\!]$ are equal to $j([\![z = \hat{x}]\!] \cup [\![z = \hat{y}]\!])$. It easily follows that (1) holds.

We now check (2). Unfolding definitions we have the following.

$$[\![z \in \widehat{\bigcup x}]\!] = j(\bigcup_{v \in x} \bigcup_{w \in v} [\![z = \hat{w}]\!])$$

$$[\![(\exists w \in \hat{x})\, z \in w]\!] = j(\bigcup_{v \in x} j(\bigcup_{w \in v} [\![z = \hat{w}]\!]))$$

By monotonicity of j and union we have $[\![z \in \widehat{\bigcup x}]\!] \subseteq [\![(\exists w \in \hat{x})\, z \in w]\!]$. We now check $[\![z \in \widehat{\bigcup x}]\!] \supseteq [\![(\exists w \in \hat{x})\, z \in w]\!]$. By axiom 3 of the definition of formal topology, it suffices to check that $\bigcup_{v \in x} j(\bigcup_{w \in v} [\![z = \hat{w}]\!]) \subseteq j(\bigcup_{v \in x} \bigcup_{w \in v} [\![z = \hat{w}]\!])$. Let $a \in \bigcup_{v \in x} j(\bigcup_{w \in v} [\![z = \hat{w}]\!])$. Then for some $v \in x$, we have $a \in j(\bigcup_{w \in v} [\![z = \hat{w}]\!])$. But now $a \in j(\bigcup_{v' \in x} \bigcup_{w \in v'} [\![z = \hat{w}]\!]$ by monotonicity of j, as required. $\square$

Lemma 3.11. *The natural numbers are absolute, in the following sense.*

$$[\![(\forall u)\ [u \in \hat{\mathbb{N}} \leftrightarrow (\emptyset = u \vee (\exists v \in \hat{\mathbb{N}})\, u = v \cup \{v\})]]\!] = \top$$

Proof. First note that $[\![(\forall u \in \hat{\mathbb{N}})\, u \cap \{u\} \in \hat{\mathbb{N}}]\!] = \bigcap_{u \in \mathbb{N}} [\![\hat{u} \cup \{\hat{u}\} \in \hat{\mathbb{N}}]\!]$ but this is equal to $\top$ by lemma 3.10 and the fact that $\{u\} \cup u \in \mathbb{N}$ for every $u \in \mathbb{N}$. We also easily have $[\![\emptyset \in \hat{\mathbb{N}}]\!]$. But we have now shown one half of the bi-implication:

$$[\![(\forall u)\ [u \in \hat{\mathbb{N}} \rightarrow (\emptyset = u \vee (\exists v \in \hat{\mathbb{N}})\, u = v \cup \{v\})]]\!] = \top$$

Now assume that for some $v \in \mathbb{N}$, $a \in [\![u = \hat{v} \cup \{\hat{v}\}]\!]$. Then using the soundness of the laws of equality, we have $[\![u = \hat{v} \cup \{\hat{v}\}]\!] \cap [\![\hat{v} \cup \{\hat{v}\} \in \hat{\mathbb{N}}]\!] \subseteq [\![u \in \hat{\mathbb{N}}]\!]$. Hence $a \in [\![u \in \hat{\mathbb{N}}]\!]$. But we now apply both parts of lemma 3.9 to deduce

$$[\![(\forall u)\ [u \in \hat{\mathbb{N}} \leftarrow (\emptyset = u \vee (\exists v \in \hat{\mathbb{N}})\, u = v \cup \{v\})]]\!] = \top$$

which is the other half of the bi-implication we require. □

Lemma 3.12. *Suppose that $\langle S, \leq, \lhd \rangle$ is a proper formal topology. Then equality and membership are absolute for the natural numbers in the following sense. For every $m, n \in \mathbb{N}$, we have that either $[\![\hat{m} = \hat{n}]\!] = \top$ or $[\![\hat{m} = \hat{n}]\!] = \emptyset$, $m = n$ if and only if $[\![\hat{m} = \hat{n}]\!] = \top$, either $[\![\hat{m} \in \hat{n}]\!] = \top$ or $[\![\hat{m} \in \hat{n}]\!] = \emptyset$ and $m \in n$ if and only if $[\![\hat{m} \in \hat{n}]\!] = \top$.*

Proof. These are proved simultaneously by induction on n and m. □

Lemma 3.13. *Finite tuples are absolute, in the following sense. We can show in* **CZF** *that for every set x and every $n \in \mathbb{N}$ and every set z,*

$$[\![z \in \widehat{x^n} \leftrightarrow z \in \hat{x}^{\hat{n}}]\!] = \top$$

Proof. This can be proved by induction on n. □

Lemma 3.14. *Let x be a set. Then function application for $\mathbb{N}^x$ is absolute, in the sense that for $f \in \mathbb{N}^x$, $z \in x$ and $n \in \mathbb{N}$, $f(z) = n$ if and only if $[\![\hat{f}(\hat{z}) = \hat{n}]\!] = \top$.*

Proof. Note that the formula $\hat{f}(\hat{z}) = \hat{n}$ is equivalent to the following

$$(\forall w \in \hat{f})(\forall v \in \hat{x})(\forall u \in \hat{\mathbb{N}})\, w = \langle v, u \rangle \rightarrow u = \hat{n}$$

This is clearly absolute by the previous lemmas. □

Remark 3.15. *In [8] it is stated that all restricted formulas are absolute. This is not provable in* **IZF** *or* **CZF**, *since the converses to parts 5 and 6 of lemma 3.8 do not hold in general and atomic formulas are not in general absolute. The double negation formal topology provides a counterexample, as do the formal topologies $\mathcal{L}_n$ considered in this paper. Also note that properness is necessary to show that $\perp$ is absolute.*

4 LLPO and LLPO$_n$

4.1 An Alternative Formulation of LLPO

We will first show how **LLPO** can be formulated in terms of the poset $\mathbb{N}_\infty$ defined below. This formulation will motivate the definition of the formal topology as the simplest one making **LLPO** true in the topological model (based on an observation of Van Oosten in [23]).

Definition 4.1. Let $\mathbb{N}_\infty$ be the set of decreasing binary sequences, i.e.

$$\mathbb{N}_\infty := \{\alpha : \mathbb{N} \to 2 \mid (\forall i \leq j)\, \alpha(j) \leq \alpha(i)\}$$

We will consider $\mathbb{N}_\infty$ as a poset with the pointwise ordering, i.e. $\alpha \leq \beta$ if for all $i \in \mathbb{N}$, $\alpha(i) \leq \beta(i)$.

Proposition 4.2. *If $\alpha, \beta \in \mathbb{N}_\infty$, then the join $\alpha \vee \beta$ exists and is defined pointwise, i.e. for $i \in \mathbb{N}$*

$$(\alpha \vee \beta)(i) := \alpha(i) \vee \beta(i)$$

Hence, if F is a finitely enumerable subset of $\mathbb{N}_\infty$, then $\bigvee F$ exists and is defined pointwise.

The top element of $\mathbb{N}_\infty$ is the function constantly equal to 1. We'll write this function as 1.

Lemma 4.3. *For all $\alpha \in \mathbb{N}_\infty$, we have $\neg\neg\alpha = 1 \to \alpha = 1$.*

Proof. Suppose $\neg\neg\alpha = 1$. For each $i \in \mathbb{N}$, we have that $\alpha(i)$ is either 0 or 1. But if $\alpha(i) = 0$, then we would have $\neg\alpha = 1$, contradicting $\neg\neg\alpha = 1$. Hence $\alpha(i) = 1$ for all $i \in \mathbb{N}$, and so $\alpha = 1$. □

Lemma 4.4. *Assume Markov's principle. Suppose that $\mathcal{F} \subseteq \mathbb{N}_\infty$ is a finitely enumerable set such that $\bigwedge \mathcal{F} \neq 1$. Then for some $\alpha \in \mathcal{F}$, $\alpha \neq 1$.*

Proof. Suppose $\bigwedge \mathcal{F} \neq 1$. Then by Markov's principle, there is some n such that $\bigwedge \mathcal{F}(n) = 0$. However, we now clearly have $\alpha(n) = 0$ for some $\alpha \in \mathcal{F}$ (since $\{\alpha(n) \mid \alpha \in \mathcal{F}\}$ is a finitely enumerable set of natural numbers), and hence $\alpha \neq 1$. □

Lemma 4.5. *Assume Markov's principle. Suppose that $\mathcal{F} \subseteq \mathbb{N}_\infty$ is a finitely enumerable set such that for each $\alpha \in \mathcal{F}$, $\alpha \neq 1$. Then $\bigvee \mathcal{F} \neq 1$.*

Proof. Since $\mathcal{F}$ is finitely enumerable, we can write $\mathcal{F} = \{\alpha_1, \ldots, \alpha_k\}$. By Markov's principle we have for each i, n_i such that $\alpha_i(n_i) = 0$. Take $N := \max_i n_i$. Then we have that $(\bigvee \mathcal{F})(N) = 0$ and therefore $\bigvee \mathcal{F} \neq 1$. □

Recall that **LLPO** is usually formulated as below.

Definition 4.6. The lesser limited principle of omniscience (**LLPO**) is the following axiom. Let $\alpha : \mathbb{N} \to 2$ be a binary sequence such that for all $i, j \in \mathbb{N}$, if $\alpha(i) = \alpha(j) = 1$ then $i = j$. Then either for all $i \in \mathbb{N}$, $\alpha(2i) = 0$, or for all $i \in \mathbb{N}$ $\alpha(2i + 1) = 0$.

We now obtain the equivalent presentations of **LLPO** below.

Proposition 4.7. *The following are equivalent:*

1. **LLPO**
2. *for all* $\alpha, \beta \in \mathbb{N}_\infty$*, if* $\alpha \vee \beta = 1$*, then* $\alpha = 1$ *or* $\beta = 1$
3. *for all inhabited finitely enumerable sets* $F \subseteq \mathbb{N}_\infty$*, if* $\bigvee F = 1$*, then there exists* $\alpha \in F$ *such that* $\alpha = 1$

Proof. To show $1 \Rightarrow 2$, let $\alpha, \beta \in \mathbb{N}_\infty$ be such that $\alpha \vee \beta = 1$. Then define $\gamma : \mathbb{N} \to 2$ as below.

$$\gamma(i) = \begin{cases} 1 & \text{if } i = 2j, \alpha(j) = 1 \text{ and } \alpha(j+1) = 0 \\ 1 & \text{if } i = 2j+1, \beta(j) = 1 \text{ and } \beta(j+1) = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then by applying **LLPO** to γ, we can show either $\alpha = 1$ or $\beta = 1$.

Now to show $2 \Rightarrow 1$, let $\gamma : \mathbb{N} \to 2$ be such that for all i, j if $\gamma(i) = \gamma(j) = 1$, then $i = j$. Define α and β as follows.

$$\alpha(i) = \begin{cases} 1 & \text{for all } j \leq i, \gamma(2j) = 0 \\ 0 & \text{for some } j \leq i, \gamma(2j) = 1 \end{cases}$$

$$\beta(i) = \begin{cases} 1 & \text{for all } j \leq i, \gamma(2j+1) = 0 \\ 0 & \text{for some } j \leq i, \gamma(2j+1) = 1 \end{cases}$$

Then one can easily check that $\alpha \vee \beta = 1$, and if $\alpha = 1$ then $\gamma(2i) = 0$ for all i, and if $\beta = 1$ then $\gamma(2i+1) = 0$ for all i.

Finally note that 2 is a special case of 3, and that 3 follows from 2 by showing by induction on n that the result holds for all F that admit a surjection $n \twoheadrightarrow F$. □

4.2 Generalising to $\mathbf{LLPO}_n$

In [20], Richman considered for each $n \geq 2$ a variant of **LLPO**, that he denoted $\mathbf{LLPO}_n$. These axioms were also studied by Hendtlass and Lubarsky, who showed (amongst other results) that $\mathbf{LLPO}_{n+1}$ is strictly weaker than $\mathbf{LLPO}_n$. In this section we show that like **LLPO**, $\mathbf{LLPO}_n$ can also be formulated using $\mathbb{N}_\infty$.

Definition 4.8. Let $n \geq 2$. $\mathbf{LLPO}_n$ is the following statement: Let $\alpha : \mathbb{N} \to 2$ be a binary sequence such that for all $i, j \in \mathbb{N}$, $\alpha(i) = \alpha(j) = 1$ implies $i = j$. Then there is some k with $0 \leq k < n$ such that for all i, $\alpha(in + k) = 0$.

Remark 4.9. *In [3] Akama, Hayashi, Berardi and Kohlenbach studied a separate hierarchy of variants of* **LLPO***, denoted* $\Sigma^0_n - \mathbf{LLPO}$*. They show (amongst other results) that for each* n*,* $\Sigma^0_{n+1} - \mathbf{LLPO}$ *is strictly stronger than* $\Sigma^0_n - \mathbf{LLPO}$*. Another variant of Lifschitz realizability (relativised to* Δ^0_n *functions) was used for one of their separation results.*

We now give the equivalent formulation using $\mathbb{N}_\infty$.

Proposition 4.10. *Let $n \geq 2$. The following are equivalent:*

1. $\mathbf{LLPO}_n$
2. *Let $\alpha_1, \ldots, \alpha_n \in \mathbb{N}_\infty$ be such that for all i, j with $1 \leq i \neq j \leq n$, $\alpha_i \vee \alpha_j = 1$. Then there exists $i \in \{1, \ldots, n\}$ such that $\alpha_i = 1$.*

Proof. Similar to the proof of proposition 4.7. $\square$

We now aim towards another characterisation of $\mathbf{LLPO}_n$ analogous to part 3 of proposition 4.7 that will be useful later.

Definition 4.11. For each n, we define the set of *n-trees* by the following recursive definition.

1. There is an n-tree `nil`.
2. If we have a list of n-trees $T_1, \ldots, T_n$ and a list of decreasing sequences $\alpha_1, \ldots, \alpha_n \in \mathbb{N}_\infty$, then $\mathrm{Tr}(T_1, \ldots, T_n; \alpha_1, \ldots, \alpha_n)$ is an n-tree.

Definition 4.12. An n-tree is defined to be *good* according to the following recursive definition.

1. `nil` is good.
2. $\mathrm{Tr}(T_i; \alpha_i)$ is good if for any $1 \leq i \neq j \leq n$, $\alpha_i \vee \alpha_j = 1$, and for any $1 \leq i \leq n$, if $\alpha_i = 1$ then T_i is good.

Definition 4.13. An n-tree is defined to be *very good* according to the following inductive definition.

1. `nil` is very good.
2. $\mathrm{Tr}(T_i; \alpha_i)$ is very good if it is good, and for some $1 \leq i \leq n$, $\alpha_i = 1$ and T_i is very good.

Theorem 4.14. $\mathbf{LLPO}_n$ *is equivalent to the statement that every good n-tree is very good.*

Proof. We first assume that every good n-tree is very good and deduce $\mathbf{LLPO}_n$. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{N}_\infty$ be such that for any $1 \leq i \neq j \leq n$, $\alpha_i \vee \alpha_j = 1$. Then note that we can form a good n-tree $\mathrm{Tr}(\mathtt{nil}; \alpha_i)$. If $\mathrm{Tr}(\mathtt{nil}; \alpha_i)$ is very good, then for some i, $\alpha_i = 1$, as required.

For the converse, we assume $\mathbf{LLPO}_n$ and prove by induction that for every n-tree, T, if T is good then T is very good.

For `nil`, this is clear.

For $T = \mathrm{Tr}(T_i; \alpha_i)$, assume that T is good. Then for $1 \leq i \neq j \leq n$ we have $\alpha_i \vee \alpha_j = 1$. Hence, for some i, $\alpha_i = 1$ by $\mathbf{LLPO}_n$. Since T is good and $\alpha_i = 1$, we have that T_i is good. But by induction we may assume now that T_i is very good. Hence, T is also very good. $\square$

Definition 4.15. $\mathbf{LLPO}_\infty$ is the following statement. Let $(,) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a surjective pairing function, and let $\alpha : \mathbb{N} \to 2$ be a binary sequence such that $\alpha(i) = 1$ for at most one n. Then for some $k \in \mathbb{N}$, and for all $n \in \mathbb{N}$ $\alpha(k, n) = 0$.

Proposition 4.16. $\mathbf{LLPO}_\infty$ *is equivalent to the following statement. Let $(\alpha_i)_{i \in \mathbb{N}}$ be such that $\alpha_i \in \mathbb{N}_\infty$ for each $i \in \mathbb{N}$. Suppose further that for $i \neq j$, $\alpha_i \vee \alpha_j = 1$. Then for some i, $\alpha_i = 1$.*

4.3 Absoluteness Results for n-Trees

We next show how to encode n-trees as functions $\mathbb{N} \to \mathbb{N}$.

Definition 4.17. Let T be an n-tree. We define the *shape* of T, $\mathrm{S}(T) \in \mathbb{N}$ as follows. Assume that we have a standard way of encoding lists of natural numbers as natural numbers such that encoding and decoding can be done in a primitive recursive manner and the code for a list is greater than each of its elements, and write this using brackets $()$.

1. $\mathrm{S}(\texttt{nil})$ is defined to be $()$.
2. $\mathrm{S}(\mathrm{Tr}(T_i; \alpha_i))$ is defined to be $(\mathrm{S}(T_1), \ldots, \mathrm{S}(T_n))$.

We define the data for T, $\mathrm{D}(T) \in 2^{\mathbb{N}}$ as follows.

1. $\mathrm{D}(\texttt{nil})(j) := 0$ for all $j \in \mathbb{N}$.
2. We define $\mathrm{D}(\mathrm{Tr}(T_i; \alpha_i))$ as follows. For any $j \in \mathbb{N}$, j can be written uniquely as either $2nk + 2i$ or $2nk + 2i + 1$ where $0 \leq i < n$. We define

$$\mathrm{D}(\mathrm{Tr}(T_i; \alpha_i))(2nk + 2i) := \alpha_i(k)$$
$$\mathrm{D}(\mathrm{Tr}(T_i; \alpha_i))(2nk + 2i + 1) := \mathrm{D}(T_i)(k)$$

Lemma 4.18. *There are primitive recursive functions b, c, f, g_0 and g_1 such that an n-tree T is good if and only if*

$$\forall l < b(\mathrm{S}(T)) \quad (c(l, \mathrm{S}(T)) = 1 \;\rightarrow\; (\forall i \in \mathbb{N})\ \mathrm{D}(T)(f(l, \mathrm{S}(T), i)) = 1) \;\rightarrow$$
$$(\forall i \in \mathbb{N})\, \neg(\mathrm{D}(T)(g_0(l, \mathrm{S}(T), i)) = 0 \wedge \mathrm{D}(T)(g_1(l, \mathrm{S}(T), i)) = 0) \quad (3)$$

Proof. We define $b(\mathrm{S}(\texttt{nil}))$ to be 0. We can then take c, f, g_0, g_1 to be anything (e.g. constantly equal to 0).

We now deal with the case $T = \mathrm{Tr}(T_i; \alpha_i)$. We define

$$b(\mathrm{S}(\mathrm{Tr}(T_1, \ldots, T_n; \alpha_1, \ldots, \alpha_n)) := \sum_{i=1}^{n} b(\mathrm{S}(T_i)) + n(n-1)$$

Now given $l < b(\mathrm{S}(T_1, \ldots, T_n; \alpha_1, \ldots, \alpha_n))$ we have one of the following two cases (and we can decide which in a primitive recursive manner).

1. For some (unique) $0 \leq l_0 < n$ and $0 \leq l_1 < n - 1$, $l = \sum_{i=1}^{n} b(\mathrm{S}(T_i)) + nl_0 + (l_1 - 1)$.
2. For some $1 \leq k \leq n$ and $0 \leq l' < b(\mathrm{S}(T_k))$, $l = \sum_{i=1}^{k-1} b(\mathrm{S}(T_i)) + l'$, and this is unique when we require furthermore that k is the greatest such value.

For case 1, we take $c(l, S(T)) := 0$. The value of f now makes no difference, so we take it to be constantly 0. Now write l_1' for l_1 if $l_1 < l_0$ and $l_1 + 1$ if $l_1 \geq l_0$ (so that in any case we have $0 \leq l_1' < n$ and $l_0 \neq l_1'$). We define

$$g_0(l, \mathrm{S}(T), i) := 2ni + 2l_0$$
$$g_1(l, \mathrm{S}(T), i) := 2ni + 2l_1'$$

(This corresponds to ensuring that $\alpha_{l_0} \vee \alpha_{l'_1} = 1$)

For case 2, we define $c(l, S(T)) := 1$. Let l' and k be as in the description of case 2. We split into cases on whether or not $c(\mathrm{S}(T_k)) = 1$. If $c(\mathrm{S}(T_k)) = 1$, then define

$$f(l, \mathrm{S}(T), 2i) := 2ni + 2k$$
$$f(l, \mathrm{S}(T), 2i+1) := 2nf(l', \mathrm{S}(T_k), i) + 2k + 1$$

If $c(\mathrm{S}(T_k)) \neq 1$, then define

$$f(l, \mathrm{S}(T), i) := 2ni + k$$

In either case, we define

$$g_0(l, \mathrm{S}(T), i) := 2ng_0(l', \mathrm{S}(T_k), i) + 2k + 1$$
$$g_1(l, \mathrm{S}(T), i) := 2ng_1(l', \mathrm{S}(T_k), i) + 2k + 1$$

(This corresponds to ensuring that if $\alpha_k(j) = 1$ for all j then T_k is good.) □

Theorem 4.19. *Let $\alpha : \mathbb{N} \to \mathbb{N}$. Then the statement "$f$ is the code of a good tree" is absolute in $V^{(\mathcal{S})}$, for any proper formal topology $\mathcal{S}$.*

Proof. Note that if f is a primitive recursive function, then the formula $f(n) = m$ is equivalent to one built from bounded universal quantifiers, conjunctions, $\perp$ and implication, and hence is absolute. Note that formula (3) is built from formulas of this form together with function application, bounded universal quantification implication and negation. Hence it is absolute. We showed in lemma 4.18 that the statement that α codes a good tree is equivalent to this formula and so that is also absolute. □

Lemma 4.20. *There are primitive recursive functions b and f such that for any n-tree T, T is very good if and only if there is $l < b(\mathrm{S}(T))$ such that for all $i \in \mathbb{N}$ $\mathrm{D}(T)(f(l, \mathrm{S}(T), i)) = 1$. Furthermore, assuming Markov's principle, if for all $l < b(\mathrm{S}(T))$, there exists $i \in \mathbb{N}$ such that $\mathrm{D}(T)(f(l, \mathrm{S}(T), i)) = 0$, then T is not good.*

Proof. For $T = \mathtt{nil}$ we define $b(\mathrm{S}(T))$ to be 0, so we can take $f(l, \mathrm{S}(\mathtt{nil}), i)$ to be anything.

For $T = \mathrm{Tr}(T_1, \ldots, T_n; \alpha_1, \ldots, \alpha_n)$, we define

$$b(\mathrm{S}(T)) := \sum_{i=1}^{n} b(\mathrm{S}(T_i))$$

Then, note that for $l < b(\mathrm{S}(T))$, l can be written as

$$l = \sum_{i=1}^{k} b(\mathrm{S}(T_i)) + l'$$

where $0 \leq k < n$ and $0 \leq l' < b(\mathrm{S}(T_k))$ and this is unique if we require the greatest such k.

Then splitting into cases depending on whether the input to f is odd or even, we define

$$f(l, \mathrm{S}(T), 2i) := 2ni + k$$
$$f(l, \mathrm{S}(T), 2i+1) := 2nf(l', \mathrm{S}(T_k), i) + 2k + 1$$

□

Corollary 4.21 (CZF + MP). *For any n-tree T, and any list of n-trees $T_1, \ldots, T_k$, we have*

1. *If T is good, then the double negation of "T is very good" is true.*
2. *Suppose the following statement is false: T_i is very good for every $1 \leq i \leq k$. Then for some $1 \leq i \leq k$, T_i is not good.*

Proof. Note that part 1 follows directly from lemma 4.20.

We now show part 2.

Suppose that it is false that T_i is very good for every $1 \leq i \leq k$. We define a finite sequence $\alpha_{0,1}, \ldots, \alpha_{0,k} \in \mathbb{N}_\infty$ using f from lemma 4.20 by,

$$\alpha_{0,i}(j) := f(0, \mathrm{S}(T_i), j)$$

Note that we cannot have $\alpha_{0,i} = 1$ for all i, since then each T_i would be very good. Hence by Markov's principle, there is some i_0 such that $\alpha_{0,i_0} \neq 1$. We then define $\alpha_{1,i}$ by

$$\alpha_{1,i}(j) := \begin{cases} \alpha_{0,i}(j) & i \neq i_0 \\ f(1, \mathrm{S}(T_i), j) & \text{otherwise} \end{cases}$$

Then, repeating the same argument as before, we find i_1 such that $\alpha_{1,i_1} \neq 1$. We continue this process until reach n such that $i_n = b(\mathrm{S}(T_{i_n}) - 1$. At this point, we have found j such that $f(l, \mathrm{S}(T_{i_n}), j) \neq 1$ for every $l < b(\mathrm{S}(T_{i_n}))$ and hence can apply lemma 4.20 to show that T_{i_n} is not good. □

5 Some Special Cases of Independence of Premisses

In this section we define a family of variants of independence of premisses (**IP**). The motivation for this it that it allows us to easily state some special cases of **IP** that hold in certain realizability models and are needed to construct the formal topologies we will use later.

Definition 5.1. Let $\Phi(x, y)$ be a formula with only x and y free variables and $\Psi(z)$ a formula with only z as a free variable. We will think of Ψ as a class, and write $z \in \Psi$ to mean $\Psi(z)$. We think of $\Phi(x, y)$ as a class of pairs and write $\langle x, y \rangle \in \Phi$ to mean $\Phi(x, y)$.

Write $\mathbf{IP}_{\Phi,\Psi}$ for the following axiom schema. For any formula ϕ,

$$\langle x, y \rangle \in \Phi \quad \to \quad ((\forall u \in y)(\exists v \in \Psi)\, \phi) \;\to\; ((\forall u \in x)(\exists v \in \Psi)\, (u \in y \to \phi))$$

Lemma 5.2. *Let X and Y be definable sets. By viewing them as classes in the usual way, we can define $\mathbf{IP}_{\Phi,X}$ and $\mathbf{IP}_{\Phi,Y}$. If there are (provably and definably) functions $f : X \to Y$ and $g : Y \to X$ such that $f \circ g = 1_Y$, then $\mathbf{IP}_{\Phi,X}$ implies $\mathbf{IP}_{\Phi,Y}$.*

Proof. We want to show

$$\langle x, y\rangle \in \Phi \quad \to \quad ((\forall u \in y)(\exists v \in Y)\,\phi) \;\to\; ((\forall u \in x)(\exists v \in Y)\,(u \in y \to \phi))$$

So assume that $x, y \in \Phi$ and $((\forall u \in y)(\exists v \in Y)\,\phi)$. Note that we can define a formula $\phi'(u, w)$ equivalent to $\phi(u, f(w))$ and show

$$(\forall u \in y)(\exists w \in X)\,\phi'$$

This is because for every $u \in y$, we have some $v \in Y$ such that $\phi(v)$, but we can then take w to be $g(v)$. Then since $f(w) = f(g(v)) = v$, we have $\phi(u, f(w))$.

Now applying $\mathbf{IP}_{\Phi,X}$, we have

$$(\forall u \in x)(\exists w \in X)\,(u \in y \to \phi')$$

Taking v to be $f(w)$, we have

$$(\forall u \in x)(\exists v \in Y)\,(u \in y \to \phi)$$

But we have now proved $\mathbf{IP}_{\Phi,Y}$, as required. □

5.1 The Schema $\mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^\mathbb{N}}$

We now come to the special cases, $\mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^\mathbb{N}}$, of $\mathbf{IP}_{\Phi,\Psi}$ that we will need to construct the formal topologies later.

Definition 5.3. Let $n \in \mathbb{N}$. Define $\mathcal{F}_n$ to be the class of pairs $\langle x, y\rangle$ where x is of the form $\{\alpha_1, \ldots, \alpha_n\}$ where $\alpha_1, \ldots, \alpha_n \in \mathbb{N}_\infty$ such that for any $1 \le i \ne j \le n$, $\alpha_i \vee \alpha_j = 1$ and $y = x \cap \{1\}$.

Then viewing $\mathbb{N}^\mathbb{N}$ as a class, we define $\mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^\mathbb{N}}$ according to definition 5.1.

It is important to note that $\mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^\mathbb{N}}$ implies several variants, that will also be used throughout this paper. Where it is clear from context, we will write that we invoke $\mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^\mathbb{N}}$ when we actually mean one of the variants listed below.

Proposition 5.4. $\mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^\mathbb{N}}$ *implies* $\mathbf{IP}_{\mathcal{F}_n,\mathbb{N}}$, $\mathbf{IP}_{\mathcal{F}_n,\mathrm{List}(\mathbb{N}^\mathbb{N})}$, $\mathbf{IP}_{\mathcal{F}_n,\mathcal{T}_n}$ *and* $\mathbf{IP}_{\mathcal{F}_n,\mathrm{List}(\mathcal{T}_n)}$ *where we write* $\mathrm{List}(X)$ *for the set of finite lists of elements of X and $\mathcal{T}_n$ to mean the set of n-trees.*

Proof. One can easily define suitable functions to apply lemma 5.2. For n-trees we use the "shape and data" encoding from definition 4.17. □

Lemma 5.5. $\mathbf{LLPO}_n$ *implies* $\mathbf{IP}_{\mathcal{F}_n,\Psi}$ *for any* Ψ *(and in particular* $\mathbf{LLPO}_n$ *implies* $\mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^\mathbb{N}}$*).*

Proof. Suppose that $x = \{\alpha_1, \ldots, \alpha_n\}$ where $\alpha_i \vee \alpha_j = 1$ for $i \ne j$ and such that for all $u \in x \cap \{1\}$ there exists $v \in \Psi$ such that $\phi(u, v)$.

By $\mathbf{LLPO}_n$, we know that $\alpha_i = 1$ for some i. However, this implies that $1 \in u \cap \{1\}$, so there must exist $v \in \Psi$ such that $\phi(1, v)$. Note that we trivially have that $u = 1$ implies $\phi(u, v)$, and so we have now proved this instance of $\mathbf{IP}_{\mathcal{F}_n,\Psi}$. □

5.2 $\mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^\mathbb{N}}$ in $V(\mathcal{K}_1)$

We now check that $\mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^\mathbb{N}}$ actually holds in the most basic realizability model for set theory, $V(\mathcal{K}_1)$, developed by McCarty in [14]. The proof uses a key idea that is already implicit in Lifschitz's original presentation of Lifschitz realizability [12] and also appears the newer versions by Van Oosten [22].

Lemma 5.6 (**CZF** + **MP**). *$\mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^\mathbb{N}}$ holds in $V(\mathcal{K}_1)$. In fact, a more general version holds. Let Φ be the class of pairs $\langle x, y\rangle$ with x any subset of $\mathbb{N}^\mathbb{N}$ and $y = x \cap \{1\}$ (writing 1 for the function constantly equal to 1). Then $\mathbf{IP}_{\Phi,\mathbb{N}^\mathbb{N}}$ holds in $V(\mathcal{K}_1)$.*

Proof. Note firstly that we can show in **CZF** that for any $f \in \mathbb{N}^\mathbb{N}$, $\neg\neg f = 1$ implies $f = 1$. Hence, we can replace y by $\{f \in x \mid \neg\neg f = 1\}$.

We are given $a_0, a_1 \in \mathcal{K}_1$ such that

$$a_0 \Vdash (\forall u \in x)\, u \in \mathbb{N}^\mathbb{N}$$
$$a_1 \Vdash (\forall u \in x)\, \neg\neg u = 1 \to (\exists v \in \mathbb{N}^\mathbb{N})\, \phi$$

and need to construct computably $b \in \mathcal{K}_1$ such that

$$b \Vdash (\forall u \in x)(\exists v \in \mathbb{N}^\mathbb{N})\, \neg\neg u = 1 \to \phi$$

Note that for any formula ψ, we have $c \Vdash \neg\psi$ for some $c \in \mathcal{K}_1$ if and only if $c \Vdash \neg\psi$ for *every* $c \in \mathcal{K}_1$. Hence, if $c \Vdash \neg\neg u = 1$ for some $c \in \mathcal{K}_1$, then $0 \Vdash u = 1$.

Now let $\langle d, u\rangle \in x$. Note that $(a_0 d)_0$ is a code for a total computable function. We define a new computable function as follows. Given input n, in parallel, run the following two algorithms.

First algorithm: For each m in turn, evaluate $(a_0 d)_0 m$. If $(a_0 d)_0 m \neq 1$, then halt and return 0. Otherwise, continue running.

Second algorithm: Try to evaluate $a_1 d 0$. If this is successful, then try to evaluate $(a_1 d 0)_0 n$. If this is successful, then halt and return $(a_1 d 0)_0 n$.

Let $n \in \mathcal{K}_1$. Suppose that neither of these algorithms halts. Then in particular, for all m, $(a_0 d)_0 m = 1$. However, we would then have $0 \Vdash \neg\neg u = 1$ and so $a_1 d 0$ must be defined, with $(a_1 d 0)_0$ a total computable function. This implies that the second algorithm halts successfully, giving a contradiction. Hence by **MP** one of the algorithms must halt, and so we get a total computable function. Note that we did this uniformly in d, so in fact we have $b_0 \in \mathcal{K}_1$ such that for each $\langle d, u\rangle \in x$, $b_0 d$ denotes and is a total computable function defined as above.

Now define b such that for every $d \in \mathcal{K}_1$,

$$bd = \mathbf{p}(b_0 d)(\lambda z.(a_1 d 0)_1)$$

Note first that for any $\langle d, u\rangle \in x$, $bd\downarrow$, since $b_0 d \downarrow$ and for any term t, $\lambda z.t$ denotes (even if t does not). Furthermore, as shown above, $(bd)_0$ is always a total computable function. In particular, we have $\langle (bd)_0, \overline{(bd)_0}\rangle \in \overline{\mathbb{N}^\mathbb{N}}$, where $\overline{(bd)_0}$ is the function in $V(\mathcal{K}_1)$ represented by $(bd)_0$, and $\overline{\mathbb{N}^\mathbb{N}}$ is the standard implementation of $\mathbb{N}^\mathbb{N}$ in $V(\mathcal{K}_1)$.

Now suppose that for some $c \in \mathcal{K}_1$, $c \Vdash \neg\neg u = 1$. In particular, this implies that for every m, $(a_0 d)_0 m = 1$. Then the first algorithm above never halts. Hence we must have that for every n, $b_0 dn = (a_0 d0)_0 n$, and so $\overline{b_0 d} = \overline{(a_0 d0)_0}$. But, we also have $(a_1 d0)_1 \Vdash \phi[v/\overline{(a_0 d0)_0}]$. Therefore we have established that

$$(bd)_1 c \Vdash \phi[v/\overline{b_0 d}]$$

and so

$$b \Vdash (\forall u \in x)(\exists v \in \mathbb{N}^{\mathbb{N}})\, \neg\neg u = 1 \to \phi$$

as required. Finally, note that we constructed b uniformly in a, so we do indeed have a realizer for the implication

$$((\forall u \in y)(\exists v \in \mathbb{N}^{\mathbb{N}})\, \phi) \;\to\; ((\forall u \in x)(\exists v \in \mathbb{N}^{\mathbb{N}})\, (u \in y \to \phi))$$

□

5.3 $\mathbf{IP}_{\mathcal{F}_n, \mathbb{N}^{\mathbb{N}}}$ in Realizability with Truth

We now do the same thing for realizability with truth. For this to work we this time need to assume that $\mathbf{IP}_{\mathcal{F}_n, \mathbb{N}^{\mathbb{N}}}$ holds already in the background universe (which was not needed for $V(\mathcal{K}_1)$).

Lemma 5.7 ($\mathbf{CZF} + \mathbf{MP} + \mathbf{IP}_{\mathcal{F}_n, \mathbb{N}^{\mathbb{N}}}$). $\mathbf{IP}_{\mathcal{F}_n, \mathbb{N}^{\mathbb{N}}}$ *holds in the realizability with truth model V^* studied in [17].*

Proof. Let V^* be the realizability with truth model from [17]. We will construct, for each instance ψ of $\mathbf{IP}_{\mathcal{F}_n, \mathbb{N}^{\mathbb{N}}}$ a closed application term t_ψ such that $t_\psi \Vdash_{tr} \psi$.

Recall from the proof of lemma 5.6, that each instance of $\mathbf{IP}_{\mathcal{F}_n, \mathbb{N}^{\mathbb{N}}}$ is equivalent to a formula of the following form.

$$(\forall x \in \mathcal{F}_n)((\forall u \in x)\neg\neg u = 1 \to (\exists v \in \mathbb{N}^{\mathbb{N}})\, \phi) \;\to\; ((\forall u \in x)(\exists v \in \mathbb{N}^{\mathbb{N}})\, \neg\neg u = 1 \to \phi) \quad (4)$$

Finding a realizer for this formula amounts to

1. Showing that the implication is true
2. Constructing a such that whenever

$$b \Vdash_{tr} (\forall x \in \mathcal{F}_n)((\forall u \in x)\neg\neg u = 1 \to (\exists v \in \mathbb{N}^{\mathbb{N}})\, \phi) \quad (5)$$

ab is defined, and

$$ab \Vdash_{tr} (\forall u \in x)(\exists v \in \mathbb{N}^{\mathbb{N}})\, \neg\neg u = 1 \to \phi \quad (6)$$

To show 1, we simply apply $\mathbf{IP}_{\mathcal{F}_n, \mathbb{N}^{\mathbb{N}}}$ in the background.

For 2, let b be as in (5). We need to construct a realizer as in (6). Since the formula is of the form $(\forall u \in x)\, \psi$, we need to show $(\forall u \in x^\circ)(\exists v \in \mathbb{N}^{\mathbb{N}})\, \neg\neg u = 1 \to \phi^\circ$ and construct ab such that for any $\langle d, u \rangle \in x$,

$$abd \Vdash_{tr} (\exists v \in \mathbb{N}^{\mathbb{N}})\, \neg\neg u = 1 \to \phi$$

For the truth part, we once again apply $\mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^\mathbb{N}}$ in the background. For the realizability part, we follow the same proof as for lemma 5.6 to construct a total computable function f.

Finally, we need to construct a realizer for

$$\neg\neg u = 1 \to \phi[v/\overline{f}]$$

Since, this is an implication, it once again consists of both a realizability part and a truth part. However, by [17, Lemma 5.10] we have that if $\neg\neg u^\circ = 1$ is true, then $0 \Vdash_{tr} \neg\neg u = 1$. Hence, we can apply the proof used in lemma 5.6 for both parts, and therefore the same realizer constructed there still works for this case. □

Theorem 5.8. *Let T be one of the theories* **CZF**, **CZF + REA**, **IZF**, **IZF + REA**. *Then $T + \mathbf{MP} + \mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^\mathbb{N}}$ has the numerical existence property and is closed under Church's rule.*

Proof. Using lemma 5.7, the proof of [17, Theorem 1.2] now applies here. □

5.4 $\mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^\mathbb{N}}$ in Function Realizability Models

We now check that the same axioms, $\mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^\mathbb{N}}$, also hold in function realizability models.

Lemma 5.9 (**CZF + MP**)**.** *There is $\alpha \in \mathcal{K}_2$ such that the following holds. Suppose that $\beta \in \mathcal{K}_2$ is such that for all $\gamma \in \mathcal{K}_2$ if $\gamma(n) = 1$ for all $n \in \mathbb{N}$, then $\beta\gamma\downarrow$. Then,*

1. *$\alpha\beta\downarrow$.*
2. *For all $\gamma \in \mathcal{K}_2$, $\alpha\beta\gamma\downarrow$.*
3. *If $\gamma(n) = 1$ for all $n \in \mathbb{N}$, then ($\beta\gamma\downarrow$ by assumption and) $\alpha\beta\gamma = \beta\gamma$.*

Proof. We define α so that for each β, $\alpha\beta$ is as follows.

$$\alpha\beta(\langle n, m_1, \ldots, m_k\rangle) = \begin{cases} 1 & \text{if } m_i \neq 1 \text{ for some } i \leq k \\ \beta(\langle n, m_1, \ldots, m_k\rangle) & \text{otherwise} \end{cases}$$

Note that there is such an α since this is clearly continuous in β and any continuous function is representable in $\mathcal{K}_2$. Also, note that by unfolding the definition of application in $\mathcal{K}_2$ and applying **MP** one can show that α is as required. □

Lemma 5.10 (**CZF + MP**)**.** *Let $V^\mathbb{P}$ be the function realizability model from [16]. Let Φ be the class of pairs $\langle x, y\rangle$ with x any subset of $\mathbb{N}^\mathbb{N}$ and $y = x \cap \{1\}$ (writing 1 for the function constantly equal to 1). Then $\mathbf{IP}_{\Phi,\mathbb{N}^\mathbb{N}}$ (and hence also $\mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^\mathbb{N}}$ for each n) holds in $V^\mathbb{P}$.*

Proof. One can easily use lemma 5.9 to adapt the proof of lemma 5.6 to work over $\mathcal{K}_2$. □

6 The Topological Models $V^{(\mathcal{L}_n)}$

We now define the topological models.

In this section, we will assume a fixed n throughout, and refer to n-trees simply as trees.

6.1 Definition of $\mathcal{L}_n$

In this section we define the formal topologies that we will use for the topological models and check that they are in fact formal topologies. The basic idea is to use the formulation of $\mathbf{LLPO}_n$ in terms of trees to produce the simplest formal topology where $\mathbf{LLPO}_n$ holds in the respective topological model, even when it does not hold in the background universe. This is based on the observation of Van Oosten in [23] that the Lifschitz realizability topos is the largest subtopos of the effective topos where an axiom equivalent to **LLPO** in the presence of Church's thesis holds.

Definition 6.1. Let T be a tree. Then we define the *cover from* T, $\mathrm{Cover}(T) \subseteq \{0\}$, inductively as follows.

1. $\mathrm{Cover}(\mathtt{nil}) = \{0\}$
2. $\mathrm{Cover}(\mathrm{Tr}(T_i; \alpha_i)) = \bigcup_{i=1}^{n} \{0 \in \mathrm{Cover}(T_i) \mid \alpha_i = 1\}$

Lemma 6.2. *Let T be a good tree. Then $0 \in \mathrm{Cover}(T)$ if and only if T is very good.*

Proof. We show this by induction on trees.

For $T = \mathtt{nil}$, we have both $0 \in \mathrm{Cover}(T)$ and T is very good, so the result is clear.

Now suppose that $T = \mathrm{Tr}(T_i; \alpha_i)$. If T is very good then for some i, $\alpha_i = 1$ and T_i is very good. However, if T_i is very good, then $0 \in \mathrm{Cover}(T_i)$ by the induction hypothesis, and so, we have $0 \in \mathrm{Cover}(T)$. We have shown that if T is very good then $0 \in \mathrm{Cover}(T)$. Now suppose that $0 \in \mathrm{Cover}(T)$. Then for some i, $\alpha_i = 1$ and $0 \in \mathrm{Cover}(T_i)$. The latter implies T_i is very good by the induction hypothesis, and so by the former T is very good, as required. $\square$

Proposition 6.3 ($\mathbf{CZF} + \mathbf{MP}$)**.** *Let T be a good tree. Then we have $\neg\neg 0 \in \mathrm{Cover}\, T$.*

Proof. Suppose that T is a good tree and that $0 \notin \mathrm{Cover}(T)$. Since $0 \notin \mathrm{Cover}(T)$, we have by lemma 6.2 that T is not very good. Then by corollary 4.21 we have that T is not good, giving us a contradiction. Hence we have $\neg\neg 0 \in \mathrm{Cover}\, T$ as required. $\square$

Definition 6.4. Let $S, \leq$ be the poset with $S = \{0\}$. Define the relation $\lhd$ as follows. $0 \lhd p$ precisely if $\mathrm{Cover}(T) \subseteq p$ for some good tree, T. Write $\mathcal{L}_n$ for the tuple $\langle S, \leq, \lhd \rangle$ (we will show that this is a formal topology).

Lemma 6.5 ($\mathbf{CZF} + \mathbf{MP}$)**.** *$\mathcal{L}_n$ satisfies axioms 1, 2 and 4 in the definition of formal topology.*

Proof. 1 and 2 are clear. It remains to prove 4, that is, that whenever $0 \lhd p$ and $0 \lhd q$, we have $0 \lhd p \cap q$.

Fix a good tree, T. We will show by induction that for any tree S, there is a tree R such that $\mathrm{Cover}(R) \subseteq \mathrm{Cover}(T) \cap \mathrm{Cover}(S)$, and that if S is good then R is also good.

For $S = \mathtt{nil}$, we just take R to be T.

Now suppose that $S = \mathrm{Tr}(S_i; \alpha_i)$. Then we have for each i, a tree R_i such that $\mathrm{Cover}(R_i) \subseteq \mathrm{Cover}(T) \cap \mathrm{Cover}(S_i)$ and R_i is good if S_i is good. Define R to be the tree $\mathrm{Tr}(R_i; \alpha_i)$. Suppose that $0 \in \mathrm{Cover}(R)$. Then for some $1 \leq i \leq n$ we must have $\alpha_i = 1$ and $0 \in \mathrm{Cover}(R_i)$. Since $\mathrm{Cover}(R_i) \subseteq \mathrm{Cover}(T) \cap \mathrm{Cover}(S_i)$, we also have $0 \in \mathrm{Cover}(S_i)$ and $0 \in \mathrm{Cover}(T)$. But, now recalling that $\alpha_i = 1$, the former implies $0 \in \mathrm{Cover}(S)$. Hence, $\mathrm{Cover}(R) \subseteq \mathrm{Cover}(T) \cap \mathrm{Cover}(S)$.

Now suppose that S is good. Then we have that for any $1 \leq i \neq j \leq n$, $\alpha_i \vee \alpha_j = 1$. Also, for any i, if $\alpha_i = 1$, then S_i is good. But this then implies that R_i is good. Hence R is also good.

We can now easily deduce axiom 4. $\square$

Theorem 6.6 ($\mathbf{CZF} + \mathbf{MP} + \mathbf{IP}_{\mathcal{F}_n, \mathbb{N}^\mathbb{N}}$)**.** *$\mathcal{L}_n$ is a formal topology.*

Proof. We have already shown in lemma 6.5 that axioms 1, 2 and 4 hold. It remains to show that axiom 3 holds. That is, whenever $0 \lhd p$ and $p \lhd q$, we have $0 \lhd q$.

Fix $q \subseteq \{0\}$. We show the following by induction on trees. Let T be a tree. Suppose that T is good and whenever $0 \in \mathrm{Cover}(T)$ we have $0 \lhd q$. Then there is a good tree S such that $\mathrm{Cover}(S) \subseteq q$.

First assume $T = \mathtt{nil}$. Then $0 \in \mathrm{Cover}(T)$, and so we have $0 \lhd q$. Let S be any good tree such that $\mathrm{Cover}(S) \subseteq q$.

Now assume that $T = \mathrm{Tr}(T_i; \alpha_i)$. Assume that T is good and whenever $0 \in \mathrm{Cover}(T)$ we have $0 \lhd q$. Since T is good, we have that for any $1 \leq i \neq j \leq n$, $\alpha_i \vee \alpha_j = 1$. Let $1 \leq i \leq n$ be such that $\alpha_i = 1$. Then T_i is good, and $\mathrm{Cover}(T_i) \subseteq \mathrm{Cover}(T)$. The latter implies that whenever $0 \in \mathrm{Cover}(T_i)$ we have $0 \lhd q$ and so we may apply the induction hypothesis, to show there exists S such that $\mathrm{Cover}(S) \subseteq q$.

However, we can now apply $\mathbf{IP}_{\mathcal{F}_n, \mathbb{N}^\mathbb{N}}$ to find for each $1 \leq i \leq n$, a tree S_i such that if $\alpha_i = 1$ then S_i is good and $\mathrm{Cover}(S_i) \subseteq q$. Define S to be $\mathrm{Tr}(S_i; \alpha_i)$. Then whenever i is such that $\alpha_i = 1$, we have that S_i is good, and so S must be good. Suppose that $0 \in \mathrm{Cover}(S)$. Then for some i we have $\alpha_i = 1$ and $0 \in \mathrm{Cover}(S_i)$. Hence also $0 \in q$. But we have now shown $\mathrm{Cover}(S) \subseteq q$ as required. $\square$

6.2 Some Basic Properties of $\mathcal{L}_n$ and $V^{(\mathcal{L}_n)}$

Lemma 6.7 ($\mathbf{CZF} + \mathbf{MP}$)**.** *If $\mathbf{LLPO}_n$ is true, then we have*

1. *$V^{(\mathcal{L}_n)}$ is isomorphic to the class of all sets, V.*
2. *$V^{(\mathcal{L}_n)} \models \phi$ if and only if ϕ is true.*

Proof. By $\mathbf{LLPO}_n$, we know that every good n-tree is very good. Hence, in this case $\mathcal{L}_n$ reduces to the trivial formal topology, where for every $p \subseteq \{0\}$, $0 \lhd p$ if and only if $0 \in p$. The result clearly follows. $\square$

Lemma 6.8 ($\mathbf{CZF} + \mathbf{MP} + \mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^\mathbb{N}}$)**.** *For each $j \in \mathbb{N}$, let p_j be a subset of $\{0\}$. Suppose that $0 \lhd \bigcup_{j\in\mathbb{N}} p_j$. Then there is some finite set $J \subseteq \mathbb{N}$ such that $0 \lhd \bigcup_{j\in J} p_j$. (That is, $\mathcal{L}_n$ is* countably compact*.)*

Proof. We show by induction on trees, that for every tree T, if T is good and $\mathrm{Cover}(T) \subseteq \bigcup_{j\in\mathbb{N}} p_j$ then there exists a finite set $J \subseteq \mathbb{N}$ and another good tree S such that $\mathrm{Cover}(S) \subseteq \bigcup_{j\in J} p_j$.

For $T = \mathtt{nil}$, we have $0 \in \bigcup_{j\in\mathbb{N}} p_j$ and so for some $j \in \mathbb{N}$, $0 \in p_j$. Hence we can just take $J := \{j\}$ and $S = \mathtt{nil}$.

Now suppose $T = \mathrm{Tr}(T_i; \alpha_i)$. Note that if $1 \le i \le n$ is such that $\alpha_i = 1$, then T_i is good and $\mathrm{Cover}(T_i) \subseteq \mathrm{Cover}(T) \subseteq \bigcup_{j\in\mathbb{N}} p_j$. So by the induction hypothesis, there is a finite set J and a good tree S such that $\mathrm{Cover}(S) \subseteq \bigcup_{j\in J} p_j$. Hence we can apply $\mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^\mathbb{N}}$ to find for each $1 \le i \le n$, a finite set $J_i \subseteq \mathbb{N}$ and a tree S_i such that if $\alpha_i = 1$ then S_i is good and $\mathrm{Cover}(S_i) \subseteq \bigcup_{j\in J_i} p_j$. We then take $J := \bigcup_{i=1}^n J_i$ and $S := \mathrm{Tr}(S_i; \alpha_i)$ and note these are as required. □

Lemma 6.9 ($\mathbf{CZF} + \mathbf{MP} + \mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^\mathbb{N}}$)**.** *Suppose that $V^{(\mathcal{L}_n)} \models (\exists j \in \mathbb{N})\, \phi(j)$. Then there is some finite $J \subseteq \mathbb{N}$ such that $V^{(\mathcal{L}_n)} \models (\exists j \in \hat{J})\, \phi(j)$.*

Proof. Apply lemma 6.8 with $p_j := [\![\phi(\hat{j})]\!]$ for $j \in \mathbb{N}$. □

The following lemma will be key to showing later that certain choice axioms and existence properties hold. It appears to be related to the constructions developed by Lee and Van Oosten in [11, Sections 4 and 5]. We will return to this point in section 8.1.

Lemma 6.10 ($\mathbf{CZF} + \mathbf{MP} + \mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^\mathbb{N}}$)**.** *Let $1 \le k < n$ and for each $j \in \mathbb{N}$, let p_j be a subset of $\{0\}$. Suppose that $0 \lhd \bigcup_{j\in\mathbb{N}} p_j$ (relative to $\mathcal{L}_n$). Suppose further that for every $J \subseteq \mathbb{N}$ such that J is finite and $|J| > k$ we have $\bigcap_{j\in J} p_j = \emptyset$.*

Then for some $j \in \mathbb{N}$ there exists a good $\lceil \frac{n}{k} \rceil$-tree, S such that $\mathrm{Cover}(S) \subseteq p_j$ (where $\lceil \frac{n}{k} \rceil$ means round up $\frac{n}{k}$ to the next integer).

Proof. We show by induction on trees that for every n-tree, T, if T is good and $\mathrm{Cover}(T) \subseteq \bigcup_j p_j$, then there exists $j \in \mathbb{N}$ and an $\lceil \frac{n}{k} \rceil$-tree S such that $\mathrm{Cover}(S) \subseteq p_j$.

For $T = \mathtt{nil}$, we have $0 \in \bigcup_{j\in\mathbb{N}} p_j$. Hence for some $j \in \mathbb{N}$ we in fact have $0 \in p_j$. We can then take S to be $\mathtt{nil}$.

Now suppose that $T = \mathrm{Tr}(T_i; \alpha_i)$.

Suppose that $\alpha_i = 1$. Then T_i is good and $\mathrm{Cover}(T_i) \subseteq \bigcup_i p_i$. So there exist $j \in \mathbb{N}$ and S a good $\lceil \frac{n}{k} \rceil$-tree such that $\mathrm{Cover}(S) \subseteq p_j$.

Hence we can apply $\mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^\mathbb{N}}$ to find for each $1 \le i \le n$, $j_i \in \mathbb{N}$ and an $\lceil \frac{n}{k} \rceil$-tree S_i such that if $\alpha_i = 1$ then S_i is good and $\mathrm{Cover}(S_i) \subseteq p_{j_i}$.

Now suppose that $|\{j_i \mid 1 \le i \le n\}| > k$. Let $I \subseteq \{1, \ldots, n\}$ be such that $|I| = |\{j_i \mid 1 \le i \le n\}| = |\{j_i \mid i \in I\}|$ (which exists by finite choice and decidability of equality for $\mathbb{N}$). By assumption, $\bigcap_{i\in I} p_i = \emptyset$. Suppose that for all $i \in I$, $\alpha_i = 1$. Then we would have that each S_i is good but $\bigcap_{i\in I} \mathrm{Cover}(S_i) = \emptyset$, giving a contradiction by corollary 4.21 and lemma 6.2. Hence by lemma 4.4, for some i, $\alpha_i \ne 1$. Let $i' \in I \setminus \{i\}$. Since $\alpha_i \ne 1$, we vacuously have $\alpha_i = 1$ implies that $\mathrm{Cover}(S_i) \subseteq p_{j_{i'}}$. Hence we may "replace" j_i with $j_{i'}$.

By repeating the above argument we may assume without loss of generality that in fact

$$|\{j_i \mid 1 \leq i \leq n\}| \leq k$$

Write J for the set $\{j_i \mid 1 \leq i \leq n\}$.

Now note that we have

$$\sum_{j \in J} |\{i \mid j_i = j\}| \;=\; n$$

Note that if $l \in \mathbb{N}$ is such that $l < \lceil \frac{n}{k} \rceil$, then $l < \frac{n}{k}$. To show this, see that we can find $p, q \in \mathbb{N}$ with $0 \leq q < k$ such that $n = pk + q$ by Euclid's algorithm. We can then split into cases depending on whether or not $q = 0$, by decidability of equality for $\mathbb{N}$. If $q = 0$, then $l < \lceil \frac{n}{k} \rceil = \frac{n}{k}$. If $q > 0$, then $l \leq \lceil \frac{n}{k} \rceil - 1 < \frac{n}{k}$. So in either case $l < \frac{n}{k}$.

Hence, if we had $|\{i \mid j_i = j\}| < \lceil \frac{n}{k} \rceil$ for all $j \in J$, this would imply $\sum_{j \in J} |\{i \mid j_i = j\}| < \frac{n}{k}.k = n$, giving a contradiction. Hence, for some $j \in J$ we must have $|\{i \mid j_i = j\}| \geq \lceil \frac{n}{k} \rceil$. Choose such a j, and $I \subseteq \{i \mid j_i = j\}$ with $|I| = \lceil \frac{n}{k} \rceil$ and an enumeration of I. Then let S be the $\lceil \frac{n}{k} \rceil$-tree $\mathrm{Tr}((\alpha_i)_{i \in I}; (S_i)_{i \in I})$. Since T is good and S_i is good when $\alpha_i = 1$, S must also be good. Now suppose $0 \in \mathrm{Cover}(S)$. This implies that for some $i \in I$, $\alpha_i = 1$ and $0 \in S_i$. But then also $0 \in p_j$. So $\mathrm{Cover}(S) \subseteq p_j$ as required. □

Remark 6.11. *Note that in the above lemma we do not have $0 \lhd p_j$ relative to $\mathcal{L}_n$, because we require a good n-tree S, such that $\mathrm{Cover}(S) \subseteq p_j$, but have only a good $\lceil \frac{n}{k} \rceil$-tree. We do however have $\neg\neg 0 \in p_j$.*

Lemma 6.12 ($\mathbf{CZF} + \mathbf{MP} + \mathbf{IP}_{\mathcal{F}_n, \mathbb{N}^\mathbb{N}}$)**.** *Suppose that for each $j \in \mathbb{N}$, p_j is a subset of $\{0\}$ such that $0 \lhd \bigcup_{j \in \mathbb{N}} p_j$ and that for all $j \neq j' \in \mathbb{N}$ we have $p_j \cap p_{j'} = \emptyset$. Then for some (necessarily unique) $j \in \mathbb{N}$ we have $0 \in p_j$.*

Proof. This is a special case of lemma 6.10 with $k = 1$. □

Lemma 6.13 ($\mathbf{CZF} + \mathbf{MP} + \mathbf{IP}_{\mathcal{F}_n, \mathbb{N}^\mathbb{N}}$)**.** *Suppose that $V^{(\mathcal{L}_n)} \models f \in \mathbb{N}^\mathbb{N}$. Then for some $g : \mathbb{N} \to \mathbb{N}$, $V^{(\mathcal{L}_n)} \models f = \hat{g}$.*

Proof. We define $g : \mathbb{N} \to \mathbb{N}$ as follows. Let $l \in \mathbb{N}$. For each $m \in \mathbb{N}$, set $p_m := [\![f(\hat{l}) = \hat{m}]\!]$. Note that for $m \neq m'$, we have $p_m \cap p_{m'} = \emptyset$, so we can apply lemma 6.12 to find m such that $0 \in [\![f(\hat{l}) = \hat{m}]\!]$. We take $g(l)$ to be this m.

Note that by construction we have $V^{(\mathcal{L}_n)} \models (\forall l \in \mathbb{N})\, \hat{g}(l) = f(l)$, and so $V^{(\mathcal{L}_n)} \models \hat{g} = f$. □

Lemma 6.14 ($\mathbf{CZF} + \mathbf{MP} + \mathbf{IP}_{\mathcal{F}_n, \mathbb{N}^\mathbb{N}}$)**.** *Suppose that $V^{(\mathcal{L}_n)} \models F : \mathbb{N}^\mathbb{N} \to \mathbb{N}$. Then for some $G : \mathbb{N}^\mathbb{N} \to \mathbb{N}$, $V^{(\mathcal{L}_n)} \models F = \hat{G}$.*

Proof. First note that by lemma 6.13 we can show that $\mathbb{N}^\mathbb{N}$ is absolute, in the sense that in $V^{(\mathcal{L}_n)}$ we can show that $\widehat{\mathbb{N}^\mathbb{N}}$ is the set of functions $\mathbb{N} \to \mathbb{N}$. However, we can now apply the same proof as in lemma 6.13 to get the result. □

Lemma 6.15 ($\mathbf{CZF} + \mathbf{MP} + \mathbf{IP}_{\mathcal{F}_n, \mathbb{N}^\mathbb{N}}$)**.**

$$V^{(\mathcal{L}_n)} \models \mathbf{MP}$$

Proof. Suppose that $f \in V^{(\mathcal{L}_n)}$ is such that $V^{(\mathcal{L}_n)} \models f \in 2^{\mathbb{N}} \ \wedge \ \neg\neg(\exists x \in \mathbb{N})\, f(x) = 1$. Then by lemma 6.13 there is $g : \mathbb{N} \to 2$ such that $V^{(\mathcal{L}_n)} \models \hat{g} = f$. Note that $\neg\neg(\exists x \in \mathbb{N})\, \hat{g}(x) = 1$ is equivalent to $\neg(\forall x \in \mathbb{N})\, \hat{g}(x) = 0$ and so is absolute. Hence we can apply **MP** in the background to find $m \in \mathbb{N}$ such that $g(m) = 1$. But then $V^{(\mathcal{L}_n)} \models (\exists x \in \mathbb{N})\, f(x) = 1$. Therefore **MP** holds in $V^{(\mathcal{L}_n)}$. $\square$

6.3 $\mathbf{LLPO}_n$ in $V^{(\mathcal{L}_n)}$

The motivation for the definition of $\mathcal{L}_n$ was to try to write down the simplest topology where $\mathbf{LLPO}_n$ holds in the topological model. We now check that in fact it really is the case that $\mathbf{LLPO}_n$ holds in $V^{(\mathcal{L}_n)}$. Note that we don't need to assume $\mathbf{LLPO}_n$ holds in the background for this to work, although we did need $\mathbf{IP}_{\mathcal{F}_n, \mathbb{N}^{\mathbb{N}}}$, even just to construct the topological model.

Lemma 6.16 ($\mathbf{CZF} + \mathbf{MP} + \mathbf{IP}_{\mathcal{F}_n, \mathbb{N}^{\mathbb{N}}}$)**.**

$$V^{(\mathcal{L}_n)} \models \mathbf{LLPO}_n$$

Proof. Suppose that $f \in V^{(\mathcal{L}_n)}$ is such that internally in $V^{(\mathcal{L}_n)}$, f is a function $\mathbb{N} \to 2$ such that $f(i) = 1$ for at most one i. Then by lemma 6.13 there must be some (unique) $g : \mathbb{N} \to 2$ such that $V^{(\mathcal{L}_n)} \models \hat{g} = f$. Then by lemma 3.8 we must have that also $g(i) = 1$ for at most one i. We now define a tree by setting for $1 \leq k \leq n$,

$$\alpha_k(i) := 1 - \max_{i' \leq i}(g(ni' + (k-1)))$$

and then define

$$T := \mathrm{Tr}(\mathtt{nil}, \ldots, \mathtt{nil}; \alpha_1, \ldots, \alpha_n)$$

We clearly have that T is a good tree and by lemma 3.8 we know

$$\mathrm{Cover}(T) \subseteq \bigcup_{1 \leq k \leq n} [\![(\forall x \in \mathbb{N})\, f(xn + (\hat{k} - 1)) = 0]\!]$$

Hence

$$V^{(\mathcal{L}_n)} \models \bigvee_{1 \leq k \leq n} (\forall x \in \mathbb{N})\, f(xn + (k-1)) = 0$$

But we now have that $V^{(\mathcal{L}_n)} \models \mathbf{LLPO}_n$ as required. $\square$

6.4 Bounded Existential Formulas and Countable Choice in $V^{(\mathcal{L}_n)}$

Although countable choice fails in each $V^{(\mathcal{L}_n)}$, there are weaker variants that we define below that do hold. To formulate them, we first define some notation for certain bounded existential formulas.

Definition 6.17. Let ϕ be a formula. We write $(\exists^{\leq n} x)\, \phi$ as shorthand for the following formula.

$$(\exists x \in \mathbb{N})\, \phi \quad \wedge \quad (\forall x_1, \ldots, x_{n+1} \in \mathbb{N}) \left(\bigwedge_{i \neq j} (x_i \neq x_j) \ \to \neg \bigwedge_{i} \phi(x_i) \right)$$

Informally, this says that there exists a witness of $\phi(x)$ in $\mathbb{N}$, but given any $X \subseteq \mathbb{N}$ with $|X| = n+1$ it is false that every element of X is a witness of $\phi(x)$. In other words $\phi(x)$ has at least one, but at most n witnesses.

Definition 6.18. We define the following variants of the axiom of choice. Let X be any set.

1. Write $\mathbf{AC}_{X,k}$ for the following principle. Let $\phi(x, y)$ be a bounded formula (that may have parameters). Suppose that we have $(\forall x \in X)(\exists^{\leq k} y)\, \phi(x, y)$. Then there is a function $f : X \to \mathbb{N}$ such that for every $x \in X$, $\phi(x, f(x))$.
2. Write $\mathbf{AC}^m_{X,k}$ for the following principle. Let $\phi(x, y)$ be a bounded formula (that may have parameters). Suppose that we have $(\forall x \in X)(\exists^{\leq k} y)\, \phi(x, y)$. Then there is a function $f : X \to \mathbb{N}$ such that for every $x \in X$, there is a good m-tree, T such that if T is very good then $\phi(x, f(x))$.
3. Write $\mathbf{AC}^{\neg\neg}_{X,k}$ for the following principle. Let $\phi(x, y)$ be a bounded formula. Suppose that we have $(\forall x \in X)(\exists^{\leq k} y)\, \phi(x, y)$. Then there is a function $f : X \to \mathbb{N}$ such that for all $x \in X$, $\neg\neg\phi(x, f(x))$.

Proposition 6.19 ($\mathbf{CZF} + \mathbf{MP}$)**.** *Let X be any set. For all $m, k \in \mathbb{N}$ with $m, k \geq 2$, and all $m' \leq m$,*

$$\mathbf{AC}_{X,k} \;\Rightarrow\; \mathbf{AC}^m_{X,k} \;\Rightarrow\; \mathbf{AC}^{m'}_{X,k} \;\Rightarrow\; \mathbf{AC}^{\neg\neg}_{X,k}$$

Proof. For ($\mathbf{AC}_{X,k} \Rightarrow \mathbf{AC}^m_{X,k}$), note that $\mathbf{AC}^m_{X,k}$ is easily a special case of $\mathbf{AC}_{X,k}$.

For ($\mathbf{AC}^m_{X,k} \Rightarrow \mathbf{AC}^{m'}_{X,k}$), given any good m-tree T, we can generate a good m'-tree by "choosing m' branches at each level."

For ($\mathbf{AC}^{m'}_{X,k} \Rightarrow \mathbf{AC}^{\neg\neg}_{X,k}$), we just apply corollary 4.21. $\square$

Lemma 6.20 ($\mathbf{CZF} + \mathbf{MP} + \mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^\mathbb{N}} + \mathbf{AC}_{\mathbb{N},\mathbb{N}}$)**.** *Let $n, k \in \mathbb{N}$ and $2 \leq k < n$. Then*

$$V^{(\mathcal{L}_n)} \models \mathbf{AC}^{\lceil \frac{n}{k} \rceil}_{\mathbb{N},k}$$

Proof. Let $x \in \mathbb{N}$ and suppose that $0 \in [\![(\exists^{\leq k} y)\, \phi(\hat{x}, y)]\!]$. Then we have by unfolding the interpretation of formulas in $V^{(\mathcal{L}_n)}$ and the definition of $\exists^{\leq k}$ that,

$$0 \lhd \bigcup_{i \in \mathbb{N}} [\![\phi(\hat{x}, \hat{i})]\!]$$

and for every list $i_1, \ldots, i_{k+1}$

$$\bigcap_{1 \leq j \leq k+1} [\![\phi(\hat{x}, \hat{i_j})]\!] \;=\; \emptyset$$

Hence, applying lemma 6.10 with $p_i := [\![\phi(\hat{x}, \hat{i})]\!]$, we have that for every $x \in \mathbb{N}$ there exists $y \in \mathbb{N}$ and a good $\lceil \frac{n}{k} \rceil$-tree S such that if S is very good then $0 \in [\![\phi(\hat{x}, \hat{y})]\!]$.

Now applying $\mathbf{AC}_{\mathbb{N},\mathbb{N}}$ we get a choice function $f : \mathbb{N} \to \mathbb{N}$. That is, for every $x \in \mathbb{N}$, there exists a good $\lceil \frac{n}{k} \rceil$-tree S such that if S is very good then $0 \in [\![\phi(\hat{x}, \widehat{f(x)})]\!]$. For each $x \in \mathbb{N}$, let $g \in \mathbb{N}^\mathbb{N}$ be a code for the tree S as above.

Then the statement that g codes a good tree is absolute by theorem 4.19, so also holds internally.

Also, the statement that g codes a very good tree is equivalent to a formula of the form $(\exists x \in \mathbb{N})\, \psi(x)$, where ψ is negative by lemma 4.20. Hence by lemma 3.9 the statement "$\hat{g}$ codes a very good tree implies $\phi(\hat{x}, \widehat{f(x)})$" must also hold internally. □

Finally, we define another variant of choice that will also hold in our model. This will be denoted *Herbrand choice*, since it also holds in the Herbrand topos developed by Van den Berg in [21].

Definition 6.21. We refer to the following principle as $\mathbf{HAC}_{X,\mathbb{N}}$ or *Herbrand choice*. Let $\phi(x, y)$ be a bounded formula (that may have parameters). Suppose that we have $(\forall x \in X)(\exists y \in \mathbb{N})\phi(x, y)$. Then there exists a function f from X to the set of finite subsets of $\mathbb{N}$, $\mathcal{P}_{\text{fin}}(\mathbb{N})$, such that for all $x \in X$ there exists $m \in f(x)$ such that $\phi(x, m)$.

One can easily show $\mathbf{HAC}_{X,\mathbb{N}}$ can be alternatively formulated as follows.

Proposition 6.22 (CZF). $\mathbf{HAC}_{X,\mathbb{N}}$ *is true if and only if the following holds. Suppose that we have* $(\forall x \in X)(\exists y \in \mathbb{N})\phi(x, y)$*. Then there exists a function* $f \colon X \to \mathbb{N}$ *such that for all* $x \in X$ *there exists* $m < f(x)$ *such that* $\phi(x, m)$*.*

Lemma 6.23 $(\mathbf{CZF} + \mathbf{MP} + \mathbf{IP}_{\mathcal{F}_n, \mathbb{N}^\mathbb{N}} + \mathbf{AC}_{\mathbb{N},\mathbb{N}})$**.**

$$V^{(\mathcal{L}_n)} \models \mathbf{HAC}_{\mathbb{N},\mathbb{N}}$$

Proof. Suppose that $V^{(\mathcal{L}_n)} \models (\forall x \in \mathbb{N})(\exists y \in \mathbb{N})\phi(x, y)$. Then for every $n \in \mathbb{N}$, we have $V^{(\mathcal{L}_n)} \models (\exists y \in \mathbb{N})\, \phi(\hat{n}, y)$. By lemma 6.9 there exists a finite set $J \subseteq \mathbb{N}$ such that $V^{(\mathcal{L}_n)} \models (\exists y \in \hat{J})\, \phi(\hat{n}, y)$. Hence also there exists $N \in \mathbb{N}$ such that $V^{(\mathcal{L}_n)} \models (\exists y < \hat{N})\, \phi(\hat{n}, y)$. By $\mathbf{AC}_{\mathbb{N},\mathbb{N}}$, we deduce that there is a function $f \colon \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, $V^{(\mathcal{L}_n)} \models (\exists y < \widehat{f(n)})\, \phi(\hat{n}, y)$. Finally by absoluteness, we deduce $V^{(\mathcal{L}_n)} \models (\forall x \in \mathbb{N})(\exists y < \hat{f}(x))\, \phi(x, y)$, and thereby $V^{(\mathcal{L}_n)} \models \mathbf{HAC}_{\mathbb{N},\mathbb{N}}$. □

7 Applications

7.1 Consistency of Church's Thesis with $\mathbf{LLPO}_n$

A hallmark of Lifschitz realizability, from Lifschitz's original model for arithmetic in [12] onwards is that it satisfies both Church's thesis and **LLPO**. We will recover the result from [5] that Church's thesis and **LLPO** are compatible over **IZF**. Moreover, we will show something even stronger. Certain variants of the axiom of countable choice are compatible with Church's thesis and **LLPO**, and as n increases, we can show that successively stronger forms of countable choice are compatible with Church's thesis and $\mathbf{LLPO}_n$.

Lemma 7.1 $(\mathbf{CZF} + \mathbf{MP} + \mathbf{CT}_!)$**.**

$$V^{(\mathcal{L}_n)} \models \mathbf{CT}_!$$

Proof. By lemma 6.13 it suffices to show that for every $f \in \mathbb{N}^{\mathbb{N}}$, the statement that f is computable holds in $V^{(\mathcal{L}_n)}$. For any f, we have by applying $\mathbf{CT}_!$ in the background that there exists $e \in \mathbb{N}$ such that $f = \{e\}$. For every $i \in \mathbb{N}$, the statement that $f(i) = \{e\}(i)$ is of the form $(\exists x \in \mathbb{N})\, \phi(x)$ where ϕ is primitive recursive. Since this holds in the background universe we must also have for each i, $V^{(\mathcal{L}_n)} \models \hat{f}(\hat{i}) = \{\hat{e}\}(\hat{i})$. Therefore $V^{(\mathcal{L}_n)} \models (\forall x \in \mathbb{N})\, \hat{f}(x) = \{\hat{e}\}(x)$. Therefore $V^{(\mathcal{L}_n)} \models \mathbf{CT}_!$ as required. □

Theorem 7.2. *Assume that* **CZF** *is consistent. Then for each $n \in \mathbb{N}$, the following theory is consistent.*

$$\mathbf{CZF} + \mathbf{MP} + \mathbf{LLPO}_n + \bigwedge_{2 \leq k < n} \mathbf{AC}_{\mathbb{N},k}^{\lceil \frac{n}{k} \rceil} + \mathbf{HAC}_{\mathbb{N},\mathbb{N}} + \mathbf{CT}_!$$

Assume that **IZF** *is consistent. Then for each $n \in \mathbb{N}$, the following theory is consistent.*

$$\mathbf{IZF} + \mathbf{MP} + \mathbf{LLPO}_n + \bigwedge_{2 \leq k < n} \mathbf{AC}_{\mathbb{N},k}^{\lceil \frac{n}{k} \rceil} + \mathbf{HAC}_{\mathbb{N},\mathbb{N}} + \mathbf{CT}_!$$

Proof. Let T be either **CZF** or **IZF** and assume that T is consistent. It is already known that in both cases **MP** does not change the consistency strength. (**IZF** is the same consistency strength as **ZF** by the main result in [6] and **CZF** is the same consistency strength as **CZF** + **LPO** by [19])

So we have that $T + \mathbf{MP}$ is consistent. Then so is the theory $T + \mathbf{MP} + \mathbf{CT}_0 + \mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^{\mathbb{N}}}$ by working in the McCarty realizability model $V(\mathcal{K}_1)$ and using the main results in [14] and [18] together with lemma 5.6.

However we now get the result by building the model $V^{(\mathcal{L}_n)}$ in $T + \mathbf{MP} + \mathbf{CT}_0 + \mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^{\mathbb{N}}}$ and applying lemmas 6.15, 6.16, 6.20, 6.23 and 7.1. □

In [20], Richman gave a proof in Bishop style constructive mathematics that for each n, $\mathbf{LLPO}_n$ is inconsistent with the statement that all functions are computable (that in fact this is even true for $\mathbf{LLPO}_\infty$). Richman's argument does not hold in **CZF** or even **IZF**, as is already clear from the earlier Lifschitz realizability model in [5]. However, it turns out that the only obstacle is an implicit use of countable choice, and one can use $\mathbf{AC}_{\mathbb{N},n}^{\neg\neg}$ to carry out Richman's argument, as follows.

Theorem 7.3. *For each $n \in \mathbb{N}$, the following theory is inconsistent.*

$$\mathbf{CZF} + \mathbf{LLPO}_n + \mathbf{CT}_! + \mathbf{AC}_{\mathbb{N},n}^{\neg\neg}$$

Proof. For each $i, j \in \mathbb{N}$ with $j < n$, we define $\alpha_{i,j} \in \mathbb{N}_\infty$ as follows. $\alpha_{i,j}(k)$ is equal to 0 if the ith Turing machine with input i has halted by stage k with output j, and $\alpha_{i,j}(k)$ is equal to 1 otherwise.

Note that for any i and for any $j, j' < n$ with $j \neq j'$ we have $\alpha_{i,j} \vee \alpha_{i,j'} = 1$ (since the ith Turing machine on input i can have at most 1 output). Hence we can apply $\mathbf{LLPO}_n$ to show that for some $j < n$, $\alpha_{i,j} = 1$.

Now we can apply $\mathbf{AC}_{\mathbb{N},n}^{\neg\neg}$ to find a function $f : \mathbb{N} \to n$ such that for each i, $\neg\neg\alpha_{i,f(i)} = 1$. (In fact this implies that $\alpha_{i,f(i)} = 1$, but we don't need this.)

Now apply $\mathbf{CT}_!$ to find $e \in \mathbb{N}$ such that for all i, $\{e\}(i) = f(i)$. In particular, the eth Turing machine with input e halts with output $f(e)$. Hence, for sufficiently large k we have $\alpha_{e,f(e)}(k) = 0$ and so $\alpha_{e,f(e)} \neq 1$. However, $f(e)$ was chosen so that $\neg\neg\alpha_{e,f(e)} = 1$. Therefore we get a contradiction, as required. □

Hendtlass and Lubarsky showed in [9] that $\mathbf{LLPO}_{n+1}$ is independent of $\mathbf{LLPO}_n$ over $\mathbf{IZF} + \mathbf{DC}$ using topological models. We obtain here a similar separation result.

Corollary 7.4. *For each n $\mathbf{LLPO}_{n+1}$ does not imply $\mathbf{LLPO}_n$ over $\mathbf{IZF} + \mathbf{MP} + \mathbf{CT}_! + \mathbf{AC}^{\neg\neg}_{\mathbb{N},n} + \mathbf{HAC}_{\mathbb{N},\mathbb{N}}$.*

Proof. $\mathbf{IZF} + \mathbf{MP} + \mathbf{CT}_! + \mathbf{AC}^{\neg\neg}_{\mathbb{N},n} + \mathbf{HAC}_{\mathbb{N},\mathbb{N}} + \mathbf{LLPO}_{n+1}$ is consistent by theorem 7.2 and proposition 6.19 but $\mathbf{IZF} + \mathbf{MP} + \mathbf{CT}_! + \mathbf{AC}^{\neg\neg}_{\mathbb{N},n} + \mathbf{LLPO}_n$ is not by theorem 7.3. □

In addition we get the following corollary by the same argument.

Corollary 7.5. *$\mathbf{AC}^{\neg\neg}_{\mathbb{N},n}$ does not imply $\mathbf{AC}^{\neg\neg}_{\mathbb{N},n+1}$ over $\mathbf{IZF} + \mathbf{MP} + \mathbf{CT}_! + \mathbf{LLPO}_{n+1} + \mathbf{HAC}_{\mathbb{N},\mathbb{N}}$.*

Proof. $\mathbf{IZF} + \mathbf{MP} + \mathbf{CT}_! + \mathbf{LLPO}_{n+1} + \mathbf{HAC}_{\mathbb{N},\mathbb{N}} + \mathbf{AC}^{\neg\neg}_{\mathbb{N},n}$ is consistent by theorem 7.2 and proposition 6.19 but $\mathbf{IZF} + \mathbf{MP} + \mathbf{CT}_! + \mathbf{LLPO}_{n+1} + \mathbf{AC}^{\neg\neg}_{\mathbb{N},n+1}$ is not by theorem 7.3. □

7.2 Existence Properties

Theorem 7.6. *Let T be one of $\mathbf{CZF}$ or $\mathbf{IZF}$. Let $\phi(x)$ be a formula with one free variable, x. Suppose that*

$$T + \mathbf{MP} + \mathbf{LLPO}_n \quad \vdash \quad (\exists j \in \mathbb{N})\, \phi(j)$$

Then there is a finite set $J \subseteq \mathbb{N}$ such that

$$T + \mathbf{MP} + \mathbf{LLPO}_n \quad \vdash \quad \bigvee_{j \in J} \phi(\underline{j})$$

Proof. Suppose that

$$T + \mathbf{MP} + \mathbf{LLPO}_n \quad \vdash \quad (\exists j \in \mathbb{N})\, \phi(j)$$

Then we have by lemma 6.16 that

$$T + \mathbf{MP} + \mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^{\mathbb{N}}} \quad \vdash \quad V^{(\mathcal{L}_n)} \models (\exists j \in \mathbb{N})\, \phi(j)$$

Fix a primitive recursive encoding of finite sets of naturals as naturals. Then by lemma 6.9, working in $T + \mathbf{MP} + \mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^{\mathbb{N}}}$ we can prove that there exists a natural number encoding a finite set J such that $V^{(\mathcal{L}_n)} \models (\exists j \in \hat{J})\, \phi(j)$. Now applying theorem 5.8 and absoluteness for primitive recursive formulas we have a finite set $J \subseteq \mathbb{N}$ such that

$$T + \mathbf{MP} + \mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^{\mathbb{N}}} \quad \vdash \quad V^{(\mathcal{L}_n)} \models \bigvee_{j \in J} \phi(\underline{\hat{j}})$$

By lemma 5.5 we have in particular that,

$$T + \mathbf{MP} + \mathbf{LLPO}_n \quad \vdash \quad V^{(\mathcal{L}_n)} \models \bigvee_{j \in J} \phi(\underline{\hat{j}})$$

Finally we apply lemma 6.7 to get

$$T + \mathbf{MP} + \mathbf{LLPO}_n \quad \vdash \quad \bigvee_{j \in J} \phi(\underline{j})$$

□

Theorem 7.7. *Let T be one of* **CZF** *or* **IZF***. Let $n, k \in \mathbb{N}$ and $k < n$, and let $\phi(x)$ be a formula with one free variable, x. Suppose that*

$$T + \mathbf{MP} + \mathbf{LLPO}_n \vdash (\exists^{\leq k} x)\, \phi(x)$$

Then for some $j \in \mathbb{N}$ we have

$$T + \mathbf{MP} + \mathbf{LLPO}_n \vdash \neg\neg\phi(\underline{j}) \tag{7}$$

$$T + \mathbf{MP} + \mathbf{LLPO}_{\lceil \frac{n}{k} \rceil} \vdash \phi(\underline{j}) \tag{8}$$

Proof. Suppose that $T + \mathbf{MP} + \mathbf{LLPO}_n \vdash (\exists^{\leq k} x)\, \phi(x)$. Then we have by lemma 6.16 that

$$T + \mathbf{MP} + \mathbf{IP}_{\mathcal{F}_n, \mathbb{N}^{\mathbb{N}}} \quad \vdash \quad V^{(\mathcal{L}_n)} \models (\exists^{\leq k} x)\, \phi(x)$$

Hence, applying lemma 6.10 with $p_j := [\![\phi(\hat{j})]\!]$, and writing Good$(T)$ to mean T is a good $\lceil \frac{n}{k} \rceil$ tree and VeryGood(T) to mean T is a very good tree,

$$T + \mathbf{MP} + \mathbf{IP}_{\mathcal{F}_n, \mathbb{N}^{\mathbb{N}}} \quad \vdash \quad (\exists j \in \mathbb{N})(\exists T)\ \mathrm{Good}(T) \wedge \mathrm{VeryGood}(T) \to \left(V^{(\mathcal{L}_n)} \models \phi(\hat{j})\right) \tag{9}$$

We now apply lemma 5.8 to find $j \in \mathbb{N}$ such that

$$T + \mathbf{MP} + \mathbf{IP}_{\mathcal{F}_n, \mathbb{N}^{\mathbb{N}}} \quad \vdash \quad (\exists T)\ \mathrm{Good}(T) \wedge \mathrm{VeryGood}(T) \to \left(V^{(\mathcal{L}_n)} \models \phi(\underline{\hat{j}})\right) \tag{10}$$

By lemma 5.5 we have in particular that,

$$T + \mathbf{MP} + \mathbf{LLPO}_n \quad \vdash \quad (\exists T)\ \mathrm{Good}(T) \wedge \mathrm{VeryGood}(T) \to \left(V^{(\mathcal{L}_n)} \models \phi(\underline{\hat{j}})\right) \tag{11}$$

However, we also have by lemma 6.7 that

$$T + \mathbf{MP} + \mathbf{LLPO}_n \quad \vdash \quad (\forall j \in \mathbb{N})\ \left(V^{(\mathcal{L}_n)} \models \phi(j)\right) \to \phi(j)$$

Finally, we deduce (7) by corollary 4.21 and deduce (8) by theorem 4.14. $\square$

Corollary 7.8. *Let T be one of* **CZF** *or* **IZF***. Let $n, k \in \mathbb{N}$ and $k < n$, and let $\phi_1, \ldots, \phi_k$ be sentences. Suppose that*

$$T + \mathbf{MP} + \mathbf{LLPO}_n \vdash \bigvee_{i=1}^{k} \phi_i$$

Then for some $1 \leq i \leq k$ we have

$$T + \mathbf{MP} + \mathbf{LLPO}_n \vdash \neg\neg\phi_i$$
$$T + \mathbf{MP} + \mathbf{LLPO}_{\lceil \frac{n}{k} \rceil} \vdash \phi_i$$

Corollary 7.9. *Let T be one of* **CZF** *or* **IZF**. *Let $n \in \mathbb{N}$ and let $\phi(x)$ be a formula with one free variable, x. Suppose that*

$$T + \mathbf{MP} + \mathbf{LLPO}_n \vdash (\exists! x \in \mathbb{N})\, \phi(x)$$

Then for some $j \in \mathbb{N}$ we have

$$T + \mathbf{MP} + \mathbf{LLPO}_n \vdash \phi(\underline{j})$$

Proof. This is a special case of (8) in theorem 7.7 taking $k := 1$. □

By contrast, we see below that none of these theories can have the full numerical existence property.

Theorem 7.10. *The numerical existence property does not hold for any consistent, recursively axiomatisable extension of* $\mathbf{CZF} + \mathbf{LLPO}_\infty$.

Proof. Let T be a consistent recursively axiomatisable extension of $\mathbf{CZF} + \mathbf{LLPO}_\infty$. In fact, a similar proof works for any theory T that interprets enough first order arithmetic to state $\mathbf{LLPO}_\infty$ and carry out the constructions used in Gödel's incompleteness theorem. However, for convenience we will use definitions and notation from set theory.

Assume that we are given a bijective pairing on $\mathbb{N}$ with primitive recursive pairing and projection functions, which we write as $(,)$, $()_0$ and $()_1$ respectively, and let Pr be a primitive recursive provability predicate.

Construct by diagonalisation a formula $\phi(n)$, where n is the only free variable and such that

$$T \vdash (\forall n \in \mathbb{N})\, (\phi(n) \;\leftrightarrow\; ((\forall m \in \mathbb{N})\, ((m)_0 = n \wedge \mathrm{Pr}((m)_1, \ulcorner \phi(\underline{n}) \urcorner) \quad \to \\ (\exists m' < m)\ \mathrm{Pr}((m')_1, \ulcorner \phi(\underline{(m')_0}) \urcorner)))) \tag{12}$$

Write $\psi(n, m)$ for the formula

$$\psi(n, m) := ((m)_0 = n \wedge \mathrm{Pr}((m)_1, \ulcorner \phi(\underline{n}) \urcorner) \quad \to \\ (\exists m' < m)\ \mathrm{Pr}((m')_1, \ulcorner \phi(\underline{(m')_0}) \urcorner))) \tag{13}$$

Now define for each $n \in \mathbb{N}$, $\alpha_n \in \mathbb{N}_\infty$ as follows

$$\alpha_n(l) := \begin{cases} 1 & \text{for all } m \leq l, \psi(n, m) \\ 0 & \text{otherwise} \end{cases}$$

So that we can apply $\mathbf{LLPO}_\infty$, we first show that for all $n \neq n'$ we have $\alpha_n \vee \alpha_{n'} = 1$. For any $l \in \mathbb{N}$, assume for a contradiction that $\alpha_n \vee \alpha_{n'}(l) = 0$. Without loss of generality we may assume l is the least such number (since $\psi(n, m)$ is primitive recursive and so decidable). By the minimality of l we must have either $\neg\psi(n, l)$ or $\neg\psi(n', l)$. However, we cannot have both of these since this would imply $(l)_0 = n$ and $(l)_0 = n'$. Hence we have without loss of generality $(l)_0 = n$ and since $\alpha_{n'}(l) = 0$ and $\psi(n', l)$, there must be some $l' < l$ such that $\neg\psi(n', l')$. In particular we have $\mathrm{Pr}((l')_1, \ulcorner \phi(\underline{(l')_0}) \urcorner)$ but also for all $m < l$, $\neg\,\mathrm{Pr}((m)_1, \ulcorner \phi(\underline{(m)_0}) \urcorner)$, giving us a contradiction. Therefore, $\alpha_n \vee \alpha_{n'} = 1$ as required.

We can now apply $\mathbf{LLPO}_\infty$ to show that $T \vdash (\exists n \in \mathbb{N})\, \alpha_n = 1$. Note that this implies $T \vdash (\exists n \in \mathbb{N})\, \phi(n)$.

Now if we assume that the numerical existence property holds for T then there must be some $n \in \mathbb{N}$ such that $T \vdash \phi(\underline{n})$. So there must be m such that $(m)_1$ codes a proof for $\phi(\underline{(m)_0})$ (by taking $(m)_0 = n$). Since the provability predicate is decidable, without loss of generality we can take m to be the least number such that $(m)_1$ codes a proof for $\phi(\underline{(m)_0})$. By the minimality of m we have that for all $m' < m$, $\neg\,\mathrm{Pr}((m')_1, \phi(\underline{(m')_0}))$. But this is a Δ_0 sentence, so by absoluteness for Δ_0 sentences we have

$$T \vdash \neg(\exists m' < \underline{m})\ \mathrm{Pr}((m')_1, \phi(\underline{(m')_0}))$$

Again by absoluteness of Δ_0 sentences, we also have

$$T \vdash (\underline{m})_0 = \underline{(m)_0} \wedge \mathrm{Pr}((\underline{m})_1, \ulcorner\phi(\underline{(m)_0})\urcorner)$$

Hence we have $T \vdash \neg\phi(\underline{(m)_0})$, contradicting that $T \vdash \phi(\underline{(m)_0})$ and the consistency of T. Therefore the numerical existence property must fail for T. □

Corollary 7.11. *For every n, there is a formula with one free variable, $\phi(x)$, such that* $\mathbf{IZF} + \mathbf{LLPO}_n + \mathbf{MP} \vdash (\exists x \in \mathbb{N})\, \phi(x)$ *but for every formula $\psi(x)$,* $\mathbf{IZF} + \mathbf{LLPO}_n + \mathbf{MP} \nvdash (\exists! x \in \mathbb{N})\, \phi(x) \wedge \psi(x)$.

Proof. Let $\phi(x)$ be the formula from the proof of theorem 7.10. If $\mathbf{IZF} + \mathbf{LLPO}_n + \mathbf{MP} \vdash (\exists! x \in \mathbb{N})\, \phi(x) \wedge \psi(x)$ was provable, then by corollary 7.9 there would be some j such that $\mathbf{IZF} + \mathbf{LLPO}_n + \mathbf{MP} \vdash \phi(\underline{j}) \wedge \psi(\underline{j})$. But in particular this gives $\mathbf{IZF} + \mathbf{LLPO}_n + \mathbf{MP} \vdash \phi(\underline{j})$ contradicting theorem 7.10. □

In [7], Friedman showed that for every recursively axiomatisable extension of Heyting arithmetic the disjunction property implies the numerical existence property. He further remarks, without proof, that there is a Δ^0_2 extension that satisfies the disjunction property but not the numerical existence property. As a corollary of the above results, we obtain a reasonably natural example of a Π^0_2 theory with the disjunction property but not the numerical existence property.

Corollary 7.12. *Assume classical logic in the meta theory. The theory* $T := \bigcap_n \mathbf{IZF} + \mathbf{MP} + \mathbf{LLPO}_n$ *(i.e. the set of formulas provable in* $\mathbf{IZF} + \mathbf{MP} + \mathbf{LLPO}_n$ *for every n) has the disjunction property.*

Proof. Suppose that $T \vdash \phi \vee \psi$. Then, for each n, $\mathbf{IZF} + \mathbf{MP} + \mathbf{LLPO}_{2n} \vdash \phi \vee \psi$. Hence either $\mathbf{IZF} + \mathbf{MP} + \mathbf{LLPO}_n \vdash \phi$ or $\mathbf{IZF} + \mathbf{MP} + \mathbf{LLPO}_n \vdash \psi$. Let $X := \{n \in \mathbb{N} \mid \mathbf{IZF} + \mathbf{MP} + \mathbf{LLPO}_n \vdash \phi\}$ and $Y := \{n \in \mathbb{N} \mid \mathbf{IZF} + \mathbf{MP} + \mathbf{LLPO}_n \vdash \phi\}$. X and Y are downwards closed subsets of $\mathbb{N}$ such that $X \cup Y = \mathbb{N}$. By classical logic we therefore have either $X = \mathbb{N}$ or $Y = \mathbb{N}$. Without loss of generality, say $X = \mathbb{N}$. Then we have that for every n,

$$\mathbf{IZF} + \mathbf{MP} + \mathbf{LLPO}_n \vdash \phi$$

But we have now shown the disjunction property for this theory. □

Theorem 7.13. *The theory* $T := \bigcap_n \mathbf{IZF} + \mathbf{MP} + \mathbf{LLPO}_n$ *(i.e. the set of formulas provable in* $\mathbf{IZF} + \mathbf{MP} + \mathbf{LLPO}_n$ *for every n) does not have the numerical existence property.*

Proof. Note that the statement $(\exists n \in \mathbb{N})\,\mathbf{LLPO}_n$ can be formalised in set theory and holds in each $\mathbf{IZF} + \mathbf{MP} + \mathbf{LLPO}_n$ for each n. However, for each n, we have seen that $\mathbf{IZF} + \mathbf{MP} + \mathbf{LLPO}_{n+1}$ does not prove $\mathbf{LLPO}_n$, so it is not provable in T. Hence T proves $(\exists n \in \mathbb{N})\,\mathbf{LLPO}_n$ but does not prove $\mathbf{LLPO}_n$ for any n, so the numerical existence property fails. ☐

7.3 Consistency of Brouwerian Continuity Principles

Recall that the fan theorem and bar induction are defined as below.

Definition 7.14. Write 2^* for the set of finite binary sequences. If $\alpha : \mathbb{N} \to 2$ is an infinite binary sequence, write $\bar{\alpha}(n)$ for the finite binary sequence of length n obtained by restricting α.

A subset R of 2^* is a *bar* if for every $\alpha : \mathbb{N} \to 2$, there exists some $n \in \mathbb{N}$ such that $\bar{\alpha(n)} \in R$.

A bar, R, is *uniform* if there exists $n \in \mathbb{N}$ such that for all $\alpha : \mathbb{N} \to 2$, there exists $m \leq n$ such that $\bar{\alpha}(m) \in R$.

The *fan theorem*, **Fan** is the axiom that every bar is uniform.

A subset R of $\mathbb{N}^*$ is a *bar* if for every $\alpha : \mathbb{N} \to \mathbb{N}$, there exists some $n \in \mathbb{N}$ such that $\bar{\alpha(n)} \in R$.

A bar, R, is *monotone* if whenever $s \in R$ and s' is a finite binary sequence extending s, then also $s' \in R$.

If s and t are finite binary sequences, write $s * t$ for the concatenation of s and t.

Monotone bar induction, $\mathbf{BI}_M$, is the following axiom. Let $Q \subset \mathbb{N}^*$ be such that there is a monotone bar R with $R \subseteq Q$ and Q has the property that whenever $s * \langle n \rangle \in Q$ for all n also $s \in Q$. Then $\langle\rangle \in Q$.

Proposition 7.15 $(\mathbf{CZF} + \mathbf{MP})$**.** *Let $V^{\mathbb{P}}$ be the function realizability model from [16]. Then* $\mathbf{MP}$ *holds in $V^{\mathbb{P}}$.*

Proof. This can easily be checked by applying **MP** in the background and noting that there is a continuous functional that takes as input $\alpha : \mathbb{N} \to 2$ such that there exists n with $\alpha(n) = 1$ and returns the first n such that $\alpha(n) = 1$. ☐

Lemma 7.16 $(\mathbf{CZF} + \mathbf{MP} + \mathbf{IP}_{\mathcal{F}_n, \mathbb{N}^{\mathbb{N}}} + \mathbf{Fan})$**.**

$$V^{(\mathcal{L}_n)} \models \mathbf{Fan}$$

Proof. Let $R \in V^{(\mathcal{L}_n)}$ be such that the statement that R is a bar holds in $V^{(\mathcal{L}_n)}$. We first construct a set R' in the background universe and check that R' is a bar. Let R' be the set of $\sigma \in 2^*$ such that $V^{(\mathcal{L}_n)} \models (\exists \sigma' \in R)\, \sigma' \leq \hat{\sigma}$.

To show that R' is a bar, let $\alpha \in 2^{\mathbb{N}}$. Then $V^{(\mathcal{L}_n)} \models (\exists j \in \mathbb{N})\, \bar{\hat{\alpha}}(j) \in R$, since R is internally a bar in $V^{(\mathcal{L}_n)}$. Hence by lemma 6.9, there is a finite set $J \subseteq N$ such that $V^{(\mathcal{L}_n)} \models (\exists j \in J)\, \bar{\hat{\alpha}}(j) \in R$. Then set $N := \max J$. We clearly have $\bar{\alpha}(N) \in R'$, and so R' is a bar.

We can now apply **Fan** in the background universe to find m such that for every $\alpha \in 2^{\mathbb{N}}$ there exists $l \leq m$ such that $\bar{\alpha}(l) \in R'$. But we now have $V^{(\mathcal{L}_n)} \models (\exists x \leq \hat{m})\, \bar{\alpha}(x) \in R$ as required. ☐

Lemma 7.17 $(\mathbf{CZF} + \mathbf{MP} + \mathbf{IP}_{\mathcal{F}_n, \mathbb{N}^{\mathbb{N}}} + \mathbf{BI}_M)$**.**

$$V^{(\mathcal{L}_n)} \models \mathbf{BI}_M$$

Proof. Suppose that $R, Q \in V^{(\mathcal{L}_n)}$ are such that in $V^{(\mathcal{L}_n)}$ the following holds: $R \subseteq Q \subseteq \mathbb{N}^*$, R is a monotone bar and whenever Q contains every immediate successor of $\sigma \in \mathbb{N}^*$, it also contains σ. We first define external versions of R and Q as follows:

$$R' := \{\sigma \in \mathbb{N}^* \mid V^{(\mathcal{L}_n)} \models \hat{\sigma} \in R\}$$
$$Q' := \{\sigma \in \mathbb{N}^* \mid V^{(\mathcal{L}_n)} \models \hat{\sigma} \in Q\}$$

Note that we can easily show $R' \subseteq Q' \subseteq \mathbb{N}^*$ and that R' is monotone. To apply $\mathbf{BI}_M$ in the background, it only remains to check that R' is a bar and that for any $\sigma \in \mathbb{N}^*$ if Q' contains every immediate successor of σ it also contains σ.

To check that R' is a bar, let $f : \mathbb{N} \to \mathbb{N}$. Then $V^{(\mathcal{L}_n)} \models (\exists x \in \mathbb{N})\, \hat{\bar{f}}(x) \in R$. Hence by lemma 6.9, there is a finite set $J \subseteq N$ such that $V^{(\mathcal{L}_n)} \models \bigvee_{j \in J} \hat{\bar{f}}(j) \in R$. Then set $N := \max J$. By monotonicity we have that for each $j \in J$, $V^{(\mathcal{L}_n)} \models \hat{\bar{f}}(j) \in R \to \hat{\bar{f}}(N) \in R$. So we deduce that $V^{(\mathcal{L}_n)} \models \hat{\bar{f}}(N) \in R$ and so $\bar{f}(N) \in R'$. Therefore R' is a bar as required.

Now let $\sigma \in \mathbb{N}^*$ be such that for all $m \in \mathbb{N}$, $\sigma * \langle m \rangle \in Q'$. Then by absoluteness, we have $V^{(\mathcal{L}_n)} \models (\forall x \in \mathbb{N})\, \hat{\sigma} * \langle x \rangle \in Q$. Therefore, $V^{(\mathcal{L}_n)} \models \hat{\sigma} \in Q$ and so $\sigma \in Q'$.

We can now apply $\mathbf{BI}_M$ in the background to deduce that $\langle\rangle \in Q'$. Therefore $V^{(\mathcal{L}_n)} \models \langle\rangle \in Q$. So we have confirmed $\mathbf{BI}_M$ holds in $V^{(\mathcal{L}_n)}$ as required. ☐

Lemma 7.18 ($\mathbf{CZF} + \mathbf{MP} + \mathbf{IP}_{\mathcal{F}_n, \mathbb{N}^\mathbb{N}} + \mathbf{Cont}(\mathbb{N}^\mathbb{N}, \mathbb{N})$)**.**

$$V^{(\mathcal{L}_n)} \models \mathbf{Cont}(\mathbb{N}^\mathbb{N}, \mathbb{N})$$

Proof. Suppose $V^{(\mathcal{L}_n)} \models F : \mathbb{N}^\mathbb{N} \to \mathbb{N}$. Then by lemma 6.14 there is $G : \mathbb{N}^\mathbb{N} \to \mathbb{N}$ such that $V^{(\mathcal{L}_n)} \models F = \hat{G}$. Let $\alpha \in \mathbb{N}^\mathbb{N}$. By $\mathbf{Cont}(\mathbb{N}^\mathbb{N}, \mathbb{N})$ in the background, there exists j such that for any $\beta \in \mathbb{N}^\mathbb{N}$, $\bar{\alpha}(j) = \bar{\beta}(j)$ implies $G(\alpha) = G(\beta)$. However, by absoluteness we then have $V^{(\mathcal{L}_n)} \models (\forall \beta \in \mathbb{N}^\mathbb{N})\, \bar{\alpha}(j) = \bar{\beta}(j) \to G(\alpha) = G(\beta)$. But we now have that in $V^{(\mathcal{L}_n)}$, $\hat{G}$ and so also F are continuous. We deduce $\mathbf{Cont}(\mathbb{N}^\mathbb{N}, \mathbb{N})$ in $V^{(\mathcal{L}_n)}$. ☐

Lemma 7.19 ($\mathbf{CZF} + \mathbf{MP} + \mathbf{IP}_{\mathcal{F}_n, \mathbb{N}^\mathbb{N}} + \mathbf{AC}_2$)**.** *Let $n, k \in \mathbb{N}$ and $2 \leq k < n$. Then*

$$V^{(\mathcal{L}_n)} \models \mathbf{AC}^{\lceil \frac{n}{k} \rceil}_{\mathbb{N}^\mathbb{N}, k}$$

Proof. By adapting the proof of lemma 6.20 and applying $\mathbf{AC}_2$ in the background. ☐

Lemma 7.20 ($\mathbf{CZF} + \mathbf{MP} + \mathbf{IP}_{\mathcal{F}_n, \mathbb{N}^\mathbb{N}} + \mathbf{AC}_2$)**.**

$$V^{(\mathcal{L}_n)} \models \mathbf{HAC}_{\mathbb{N}^\mathbb{N}, \mathbb{N}}$$

Proof. By adapting the proof of lemma 6.23 and applying $\mathbf{AC}_2$ in the background. ☐

Theorem 7.21. *Assume* **CZF** *is consistent. Then for each n, so is the following theory.*

$$\mathbf{CZF} + \mathbf{CC} + \mathbf{Fan} + \mathbf{AC}_2 + \mathbf{RDC} + \mathbf{MP} + \mathbf{IP}_{\mathcal{F}_n, \mathbb{N}^\mathbb{N}} \tag{14}$$

Assume **CZF**+**MP**+**REA** *is consistent. Then for each n, so is the following theory.*

$$\mathbf{CZF} + \mathbf{REA} + \mathbf{CC} + \mathbf{BI}_M + \mathbf{AC}_2 + \mathbf{RDC} + \mathbf{MP} + \mathbf{IP}_{\mathcal{F}_n,\mathbb{N}^\mathbb{N}} \tag{15}$$

Proof. Using proposition 7.15 and lemma 5.10 one can easily adapt the proof of [16, Theorem 9.10] to show this. ☐

Theorem 7.22. *If* **CZF** *is consistent then for each n, the following theory is also consistent.*

$$\mathbf{CZF}+\mathbf{MP}+\bigwedge_{2\leq k<n}\mathbf{AC}^{\lceil\frac{n}{k}\rceil}_{\mathbb{N}^\mathbb{N},k}+\mathbf{HAC}_{\mathbb{N}^\mathbb{N},\mathbb{N}}+\mathbf{LLPO}_n+\mathbf{Cont}(\mathbb{N}^\mathbb{N},\mathbb{N})+\mathbf{Fan} \tag{16}$$

If **CZF** + **REA** *is consistent then for each n, the following theory is also consistent.*

$$\mathbf{CZF}+\mathbf{MP}+\bigwedge_{2\leq k<n}\mathbf{AC}^{\lceil\frac{n}{k}\rceil}_{\mathbb{N}^\mathbb{N},k}+\mathbf{HAC}_{\mathbb{N}^\mathbb{N},\mathbb{N}}+\mathbf{LLPO}_n+\mathbf{Cont}(\mathbb{N}^\mathbb{N},\mathbb{N})+\mathbf{BI}_M \tag{17}$$

Proof. We build $V^{(\mathcal{L}_n)}$ in the theory (14), which is consistent by theorem 7.21. We then have that $V^{(\mathcal{L}_n)}$ models (16) by lemmas 6.15, 7.19, 7.20, 6.16, 7.18 and 7.16. To add monotone bar induction we also apply lemma 7.17. ☐

(There is already a similar result for **LLPO** over second order arithmetic due to Van Oosten in [22, Section 5].)

Corollary 7.23. $\mathbf{CZF}+\mathbf{MP}+\bigwedge_{2\leq k<n}\mathbf{AC}^{\lceil\frac{n}{k}\rceil}_{\mathbb{N}^\mathbb{N},k}+\mathbf{HAC}_{\mathbb{N}^\mathbb{N},\mathbb{N}}+\mathbf{LLPO}_n+\mathbf{Cont}(\mathbb{N}^\mathbb{N},\mathbb{N})+\mathbf{BI}_M$ *does not prove* **LCP** *or* $\mathbf{AC}_{\mathbb{N}^\mathbb{N},2}$.

Proof. $\mathbf{CZF}+\mathbf{MP}+\bigwedge_{2\leq k<n}\mathbf{AC}^{\lceil\frac{n}{k}\rceil}_{\mathbb{N}^\mathbb{N},k}+\mathbf{HAC}_{\mathbb{N}^\mathbb{N},\mathbb{N}}+\mathbf{LLPO}_n+\mathbf{Cont}(\mathbb{N}^\mathbb{N},\mathbb{N})+\mathbf{BI}_M$ is consistent, so it suffices to show $\mathbf{CZF}+\mathbf{LLPO}_n+\mathbf{LCP}$ and $\mathbf{CZF}+\mathbf{LLPO}_n+\mathbf{Cont}(\mathbb{N}^\mathbb{N},\mathbb{N})+\mathbf{AC}_{\mathbb{N}^\mathbb{N},2}$ are not.

In both cases, we show the theories are inconsistent by first noting that there is a surjection $F : \mathbb{N}^\mathbb{N} \twoheadrightarrow \{\langle\alpha_1,\ldots,\alpha_n\rangle \in \mathbb{N}^n_\infty \mid \alpha_i \vee \alpha_j = 1,\ \text{for } i \neq j\}$, defined as follows.

$$(F(\alpha))_i(k) = \begin{cases} 0 & \alpha(k') \equiv i \mod n+1 \text{ where } k' \leq k \text{ least s.t. } \alpha(k') \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

By $\mathbf{LLPO}_n$, there is $1 \leq i \leq n$ for each $\alpha \in \mathbb{N}^\mathbb{N}$ such that $(F(\alpha))_i = 1$. Let α be such that $(F(\alpha))_i = 1$ for all i. By **LCP** there is some $i, k \in \mathbb{N}$ such that whenever $\bar{\beta}(k) = \bar{\alpha}(k)$, $(F(\beta))_i = 1$. However, we can now easily find β such that $\bar{\beta}(k) = \bar{\alpha}(k)$ but $(F(\beta))_i \neq 1$ to get a contradiction. Similarly, we can use $\mathbf{AC}_{\mathbb{N}^\mathbb{N},2}$ to get a function $G : \mathbb{N}^\mathbb{N} \to \mathbb{N}$ such that for all α, $(F(\alpha))_{G(\alpha)} = 1$, contradicting $\mathbf{Cont}(\mathbb{N}^\mathbb{N},\mathbb{N})$. ☐

8 Connections to Other Formal Systems

8.1 Connections to Topos Theory

The $\mathcal{L}_n$ considered in this paper appear to be strongly related to the local operators in the effective topos previously considered by Lee and Van Oosten in [11], specifically to the local operators corresponding to finitary sights. We expect that in fact these local operators can be obtained by carrying out the construction of $\mathcal{L}_n$ in the effective topos. The realizability model $V^{\mathbb{P}}$ corresponds to the topos $\mathsf{RT}(\mathcal{K}_2)$ (as described, for example, in [24, Section 4.3]). Since we only require computable functions, one might expect our constructions to work also in the relative realizability topos $\mathsf{RT}(\mathcal{K}_2^{\mathrm{REC}}, \mathcal{K}_2)$ (see [24, Section 4.5]). The realizability with truth model is related to the topos $(\mathsf{Eff} \downarrow \Delta)$ obtained by gluing along the inclusion functor from Set to Eff. Putting this all together, we make the following conjecture.

Conjecture 8.1. *Some of the local operators in* Eff *considered in [11] have counterparts in the toposes* $\mathsf{RT}(\mathcal{K}_2)$*,* $\mathsf{RT}(\mathcal{K}_2^{\mathrm{REC}}, \mathcal{K}_2)$ *and* $(\mathsf{Eff} \downarrow \Delta)$*.*

(We again point out that Van Oosten has already shown that the original Lifschitz realizability model has a counterpart in $\mathsf{RT}(\mathcal{K}_2)$ (see [24, Section 4.3]) and for **q**-realizability (an ancestor of realizability with truth) (see [22, Proposition 3.5])).

8.2 Connections to Type Theory

A form of Lifschitz realizability for homotopy type theory has been developed by Koutsoulis [10] based on a preprint of this paper together with other techniques.

We also expect that some of the results of section 7.2 manifest in extensional type theory as follows.

Definition 8.2. Let Γ be a context in type theory. We say that Γ has *propositional canonicity for* $\mathbb{N}$ if whenever $\Gamma \vdash t : \mathbb{N}$, there is some $n \in \mathbb{N}$ and a term p such that $\Gamma \vdash p : \mathrm{Id}_{\mathbb{N}}(t, \underline{n})$.

Suppose we are working in a variant of type theory that has a propositional truncation operator (such as type theory with brackets, as in [4]). In such theories there are two different ways of formalising **LLPO** depending on whether or not we use the propositional truncation operator $\| - \|$. We call these $\mathbf{LLPO}_+$ and $\mathbf{LLPO}_\vee$ and define them as follows.

$$\begin{aligned}
\mathbf{LLPO}_+ :=& \prod_{\alpha:\mathbb{N}\to 2} \left(\prod_{m,n:\mathbb{N}} (\alpha(m) = 1 + \alpha(n) = 1 \ \to \ m = n) \right) \quad \to \\
& \left(\left(\prod_{n:\mathbb{N}} \alpha(2n) = 0 \right) + \left(\prod_{n:\mathbb{N}} \alpha(2n+1) = 0 \right) \right) \\
\mathbf{LLPO}_\vee :=& \prod_{\alpha:\mathbb{N}\to 2} \left(\prod_{m,n:\mathbb{N}} (\alpha(m) = 1 + \alpha(n) = 1 \ \to \ m = n) \right) \quad \to \\
& \left\| \left(\prod_{n:\mathbb{N}} \alpha(2n) = 0 \right) + \left(\prod_{n:\mathbb{N}} \alpha(2n+1) = 0 \right) \right\|
\end{aligned}$$

By adapting the proof of theorem 7.10, we have,

Theorem 8.3. *The context* $(x : \mathbf{LLPO}_+)$ *does not have propositional canonicity for* $\mathbb{N}$ *over any variant of type theory for which it is consistent (that is,* *there is no term of type $\bot$ in context $(x : \mathbf{LLPO}_+)$) and such that the set of judgements is computably enumerable.*

However, we expect by analogy with the results in this paper that the following holds.

Conjecture 8.4. *The context $(x : \mathbf{LLPO}_\vee)$ has propositional canonicity for $\mathbb{N}$ over type theory with bracket types, as studied by Awodey and Bauer in [4], or similar systems studied by Maietti in [13].*

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