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# CONGRUENCES OF LOCAL ORIGIN AND AUTOMORPHIC INDUCTION

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Abstract. We explore the possibilities for the Galois representation $\rho_g$ attached to a weight-one newform $g$ to be residually reducible, i.e. for the Hecke eigenvalues to be congruent to those of a weight-one Eisenstein series. A special role is played by Eisenstein series $E_1^{1,\eta_K}$ of level $d_K$, where $\eta_K$ is the quadratic character associated with an imaginary quadratic field $K$, of discriminant $d_K$, with respect to which $\rho_g$ is of dihedral type. We prove congruences, where the modulus divides either the class number $h_K$ or $(p-\eta_K(p))$ (for a prime $p$), and $g$ is of level $d_K$ in the first case, level $d_Kp$ or $d_Kp^2$ (according as $\eta_K(p)=1$ or $-1$ respectively) in the second. We also prove analogous congruences where 1 and $\eta_K$ are replaced by a newform $f$ and its twist by $\eta_K$, and $g$ is replaced by a Siegel cusp form of genus 2 and paramodular level, induced in some sense from a Hilbert modular form.

## 1. Introduction

Ramanujan's famous congruence $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$ is a congruence between the Hecke eigenvalues of a cusp form and an Eisenstein series of weight 12, both of level 1, modulo a prime divisor of $\zeta(12)/\pi^{12}$. It is easily generalised to other weights $k$. Moreover, keeping the Eisenstein series at level 1, we may replace the cusp form of level 1 by one of level $p$, and prove a congruence modulo $\ell$, where $\ell > 3$ divides $p^k - 1$. Such congruences "of local origin", anticipated by Harder [Ha], were proved by Billerey and Menares [BM1], and independently in [DF]. Ramanujan's congruence is also easily generalised to the case that the Eisenstein series and the cusp form are both of level $N$, and the modulus divides a certain Dirichlet $L$-value, as in [Du1, Proposition 2.1]. The present paper is based on the 2018 Sheffield Ph.D. thesis of the second-named author [Sp], supervised by the first-named author. The original problem was to prove congruences of local origin where the Eisenstein series is of level $N$ and the cusp form is of level $Np$. The technique involves finding a linear combination of old Eisenstein series at level $Np$ whose constant term at each cusp is divisible by $\ell$, then using a lemma of Deligne and Serre. Billerey and Menares independently obtained a somewhat more complete result [BM2]; see Theorem 2.10 below.

This covers weights $k \geq 2$, so, following an idea of the second-named author, we decided to see what happens for weight 1. The main result on this is Theorem 2.8 in §2, where we use a cusp form of dihedral type with respect to an imaginary quadratic field $K$. In other words, the associated cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ is automorphically induced from one of $\mathrm{GL}_1(\mathbb{A}_K)$. Preceding this we show, in Propositions 2.2 and 2.5 (summarised in Corollary 2.7), that that is essentially

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the only case where we can expect to see a congruence (up to twist–see Remark 2.9 (4)). The congruence is of the form

$$a_q(f) \equiv 1 + \eta_K(q) \pmod{\lambda},$$

for almost all primes $q$, where $a_q(f)$ is the Hecke eigenvalue at $q$ of a certain weight 1 eigenform $f$, and $\lambda \mid \ell$. The condition that $\ell \mid h_K$ or $\ell \mid (p - \eta_K(p))$ leads directly to the construction of a character $\chi$ of $\mathrm{GL}_1(\mathbb{A}_K)$, the finite part of whose conductor is trivial or a divisor of $p$, respectively. The important property of $\chi$ is that, taking values in $\ell^{\text{th}}$ roots of unity, it is congruent mod $\lambda$ to the trivial character (and, unlike the trivial character, not Galois self-conjugate). This congruence is the essential one. That the CM form $f = f_\chi$ should then satisfy the above congruence is an easy consequence.

In §3 we turn to an analogous situation where the cusp form of dihedral type is replaced by a genus 2 Siegel modular form whose associated cuspidal automorphic representation of $\mathrm{GSp}_2(\mathbb{A})$ is automorphically induced from a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_K)$, where now $K$ is a real quadratic field. The main result is Theorem 3.1. The passage from the trivial character to the congruent character $\chi$ is replaced now by the passage from a base-change Hilbert modular form to a congruent non-base-change Hilbert modular form, possibly of higher level.

This is a short paper, and the theorems are set in context by extended remarks, which we shall not repeat here, except to refer to Remark 3.2 (5) for the relationship between the results proved in this paper and a general conjecture on Eisenstein congruences [BD].

## 2. Eisenstein congruences for weight one cusp forms

The following is a theorem of Deligne and Serre [DeSe, Théorème 4.1].

**Theorem 2.1.** *Let $f \in S_1(\Gamma_0(N), \epsilon)$ be a normalised newform, $f = \sum_{n=1}^{\infty} a_n(f) q^n$, where $\epsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ is a Dirichlet character such that $\epsilon(-1) = -1$ (i.e. $\epsilon$ is odd). Then there exists an irreducible, continuous linear representation $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$ such that for all primes $q \nmid N$, $\rho_f$ is unramified at $q$, with $\mathrm{tr}(\rho_f(\mathrm{Frob}_q^{-1})) = a_q(f)$ and $\det(\rho_f(\mathrm{Frob}_q^{-1})) = \epsilon(q)$.*

We have an associated projective representation $\rho'_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{PGL}_2(\mathbb{C})$, whose image is necessarily either dihedral of order $2n$ for some $n \geq 2$, or isomorphic to one of the groups $A_4, S_4$ or $S_5$. Since $\rho_f$ factors through a finite group, it may be defined over a number field, in fact over the field $K_f$ generated over $\mathbb{Q}$ by the $a_n(f)$. For any prime divisor $\lambda$ in $K_f$, dividing a rational prime $\ell$, we may choose an invariant $\mathcal{O}_\lambda$-lattice then reduce modulo $\lambda$ to obtain a residual representation $\overline{\rho}_{f,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_\lambda)$, whose composition factors are independent of the choice of lattice.

**Proposition 2.2.**
1. *For any $\ell$, if $\rho'_f(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \simeq S_4$ or $A_5$ then $\overline{\rho}_{f,\lambda}$ is irreducible.*
2. *Suppose that $\ell > 2$. If $\rho'_f(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \simeq A_4$ then $\overline{\rho}_{f,\lambda}$ is irreducible.*

*Proof.* For background on Schur covers and projective representations, see [HH]. Schur covers of $A_4, S_4$ and $A_5$ may be taken as $\mathrm{SL}_2(\mathbb{F}_3)$, $\mathrm{GL}_2(\mathbb{F}_3)$ and $\mathrm{SL}_2(\mathbb{F}_5)$, respectively. In each case the Schur index is 2, so these are all double covers. The projective representation $\rho'_f$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, viewed as a projective representation of the finite quotient isomorphic to $G = A_4, S_4$ or $A_5$, lifts to a linear representation $\tilde{\rho}_f$ of the Schur cover $C$, and it is not difficult to show that if $\overline{\rho}_{f,\lambda}$ were reducible then so would be a reduction modulo $\lambda$ of this irreducible 2-dimensional $\tilde{\rho}_f$, and we would see a congruence between the character of $\tilde{\rho}_f$ and a sum of two 1-dimensional characters of $C$. (Here, as elsewhere, where necessary we replace $K_f$ by a sufficiently large finite extension, and $\lambda$ by a divisor of the original $\lambda$.) A careful inspection of the character tables of $\mathrm{SL}_2(\mathbb{F}_3)$, $\mathrm{GL}_2(\mathbb{F}_3)$ and $\mathrm{SL}_2(\mathbb{F}_5)$ establishes that for $C = \mathrm{SL}_2(\mathbb{F}_5)$, for which the only 1-dimensional character is the trivial character, there are no such congruences whatsoever, and for $C = \mathrm{SL}_2(\mathbb{F}_3)$ the only such congruence is for $\ell = 2$. For $C = \mathrm{GL}_2(\mathbb{F}_3)$ the only such congruence is for $\ell = 3$, but the corresponding 2-dimensional representation of $C$ factors not only through $S_4$, but through the quotient of $S_4$ by its Klein-4 subgroup, in which case $\rho'_f(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \not\simeq S_4$. □

**Remark 2.3.** (1) It is possible a priori for $\tilde{\rho}_f$ to become reducible modulo $\lambda$, only when $\ell$ divides the order of $C$, thus excluding right away all $\ell$ except $\{2,3\}$ (for $\mathrm{SL}_2(\mathbb{F}_3), \mathrm{GL}_2(\mathbb{F}_3)$) and $\{2,3,5\}$ (for $\mathrm{SL}_2(\mathbb{F}_5)$).

(2) For $\ell > 2$, the composition factors of a 2-dimensional representation, in characteristic $\ell$, of a finite group, are determined by the Brauer character. Hence reducibility of $\overline{\rho}_{f,\lambda}$ would not only imply, but be implied by, the existence of Dirichlet characters $\psi, \phi$, of conductors whose product divides $N$, such that

$$a_q(f) \equiv \psi(q) + \phi(q) \pmod{\lambda} \quad \text{for all primes } q \nmid \ell N,$$

i.e. a congruence of Hecke eigenvalues between $f$ and the Eisenstein series denoted $E_1^{\psi,\phi}$ in [DiSh, §4.8], which belongs to $M_1(\Gamma_0(N), \psi\phi)$.

We turn now to the remaining case, where $\rho'_f(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ is dihedral.

**Proposition 2.4.** *Let $K/\mathbb{Q}$ be a quadratic field, with associated quadratic character $\eta_K$. Let $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathbb{C}^\times$ be a character of order $n \geq 2$, such that*

(1) *$\chi \neq \chi_\sigma$, where $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ represents the non-trivial element of $\mathrm{Gal}(K/\mathbb{Q})$ and $\chi_\sigma(g) := \chi(\sigma g \sigma^{-1})\ \forall g \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$;*

(2) *if $K$ is real then $\chi$ has signature $(+,-)$ at the infinite places.*

*Let $\mathfrak{f}$ be the conductor of $\chi$, and $d_K$ the discriminant of $K/\mathbb{Q}$. Using class field theory to view $\chi$ as a Dirichlet character $\chi : (\mathcal{O}_K/\mathfrak{f})^\times \to \mathbb{C}^\times$ (ignoring the components at infinite places of an idele class character), let $\chi_\mathbb{Q}$ be its restriction to $(\mathbb{Z}/(\mathfrak{f} \cap \mathbb{Z}))^\times$. Then*

(1) *$f_\chi(z) := \sum_{\substack{\mathfrak{a} \in \mathcal{O}_K \\ \mathfrak{a}\ \text{integral}}} \chi(\mathfrak{a}) q^{\mathrm{N}(\mathfrak{a})}$ is a newform in $S_1(\Gamma_0(|d_K| \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{f})), \eta_K \chi_\mathbb{Q})$.*

(2) *$\rho_{f_\chi} \simeq \mathrm{Ind}_{K/\mathbb{Q}}(\chi)$ is irreducible, with $\rho_{f_\chi}|_{\mathrm{Gal}(\overline{\mathbb{Q}}/K)} \simeq \chi \oplus \chi_\sigma$ and $\rho'_{f_\chi}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \simeq D_m$, where $m \mid n$ is the order of $\chi^{-1}\chi_\sigma$.*

For a discussion of this, see [Se, §7]. Note that $\chi \neq \chi_\sigma$ ensures that $\mathrm{Ind}_{K/\mathbb{Q}}(\chi)$ is irreducible, while the condition on $\chi$ when $K$ is real ensures that $\eta_K \chi_\mathbb{Q}$ is odd.

**Proposition 2.5.** *Suppose that $\rho'_f(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ is dihedral. Suppose also that there exist Dirichlet characters $\psi, \phi$ such that*

$$a_q(f) \equiv \psi(q) + \phi(q) \pmod{\lambda} \quad \textit{for all but finitely many primes } q,$$

*with $\lambda$ a prime divisor, in a large enough number field, of a rational prime $\ell > 2$. Then $f = f_\chi$ for some $\chi$ as in Proposition 2.4, with*

$$\psi \equiv \eta_K \phi \pmod{\lambda}.$$

*Furthermore, $K$ is imaginary quadratic.*

*Proof.* By a theorem of Kani [K, Theorem 9(b)], since $\rho'_f(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ is dihedral, there exists some Dirichlet character $\theta$ such that $a_q(f) = 0$ for any prime $q$ such that $\theta(q) \neq 1$. For such a $q$, $\psi(q) + \phi(q) \equiv 0 \pmod{\lambda}$. Define a Dirichlet character $\alpha$ by $\alpha = \psi/\phi$ (where each of $\psi$ and $\phi$ is pulled back from its natural domain to a suitable $(\mathbb{Z}/N\mathbb{Z})^\times$). Then $\alpha(q) \equiv -1 \pmod{\lambda}$ for any prime $q$ such that $\theta(q) \neq 1$. Since $\ell > 2$, the reduction $\overline{\alpha}$ with values in $\overline{\mathbb{F}}_\ell^\times$ is non-trivial, so cannot take the non-identity value $-1$ for a set of $q$ of Dirichlet density greater than $1/2$. It follows that $\theta$ must be a quadratic character. Let's call it $\eta_K$, associated to a quadratic field $K$. We see also that $\overline{\alpha} = \overline{\eta_K}$, so $\psi \equiv \eta_K \phi \pmod{\lambda}$, as required. (Though the various characters may have different conductors, we can view this as a congruence of pullbacks to a suitable $(\mathbb{Z}/N\mathbb{Z})^\times$, or of associated characters of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.) Now by [K, Theorem 9(a)], $\rho_f$ is induced from some character $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathbb{C}^\times$, forcing $f = f_\chi$.

To show that $K$ must be imaginary quadratic, $\det \overline{\rho}_{f,\lambda}$ is odd, but as a Dirichlet character it is $\overline{\psi\phi} = \eta_K(\overline{\psi}^2)$, which implies that $\eta_K$ is odd, as required. This observation and its proof are due to T. Berger. □

**Remark 2.6.** With $\ell > 2$, the congruence is equivalent to reducibility of $\overline{\rho}_{f,\lambda}$.

Summarising Propositions 2.2 and 2.5 we get the following.

**Corollary 2.7.** *Let $f$ be as in Theorem 2.1. If $\overline{\rho}_{f,\lambda}$ is reducible, with $\lambda \mid \ell$ and $\ell > 2$, then $\rho'_f(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ is dihedral, and $f = f_\chi$ for some character $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathbb{C}^\times$, with $K$ imaginary quadratic.*

**Theorem 2.8.** *Let $K$ be an imaginary quadratic field, with associated quadratic character $\eta_K$, discriminant $N = d_K$. Let $h_K$ be the class number of $\mathcal{O}_K$, and $w = 2, 4$ or $6$ the number of units in $\mathcal{O}_K$. Let $\ell$ be a rational prime such that $\ell \nmid w$.*

(1) *If $\ell \mid h_K$ then (for $\lambda \mid \ell$ in a sufficiently large number field) there exists a newform $f \in S_1(\Gamma_0(N), \eta')$, for some character $\eta' \equiv \eta_K \pmod{\lambda}$, such that*

$$a_q(f) \equiv 1 + \eta_K(q) \pmod{\lambda} \text{ for all primes } q \neq \ell.$$

(2) *If $\ell \nmid h_K$ but $\ell \mid (p-1)$, for some prime $p$ split in $K$, then there exists a newform $f \in S_1(\Gamma_0(Np), \eta')$, for some character $\eta' \equiv \eta_K \pmod{\lambda}$, such that*

$$a_q(f) \equiv 1 + \eta_K(q) \pmod{\lambda} \text{ for all primes } q \nmid \ell p.$$

(3) *If $\ell \nmid h_K$ but $\ell \mid (p+1)$, for some prime $p$ inert in $K$, then there exists a newform $f \in S_1(\Gamma_0(Np^2), \eta')$, for some character $\eta' \equiv \eta_K \pmod{\lambda}$, such that*

$$a_q(f) \equiv 1 + \eta_K(q) \pmod{\lambda} \text{ for all primes } q \neq \ell.$$

*Proof.* (1) Since $\ell \mid h_K$, there exists a character $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathbb{C}^\times$ of order $\ell$, factoring through $\mathrm{Gal}(H/K)$, where $H$ is the Hilbert class field of $K$. As noted by Serre [Se, §7.2], the fact that $\mathfrak{a}\sigma(\mathfrak{a})$ is always principal implies that $\chi_\sigma = \chi^{-1}$. Since $\ell$ is odd, it follows that $\chi_\sigma \neq \chi$. Let $f = f_\chi$, which is a newform in $S_1(\Gamma_0(N), \eta_K\chi_{\mathbb{Q}})$. The values of $\chi$ are $\ell^{\text{th}}$ roots of unity, so $\chi(\mathfrak{a}) \equiv 1 \pmod{\lambda}$ for any ideal $\mathfrak{a}$ of $\mathcal{O}_K$. In particular, $\eta_K\chi_{\mathbb{Q}} \equiv \eta_K \pmod{\lambda}$. If $q$ is inert in $K$ (so $\eta_K(q) = -1$) then

$$a_q(f) = 0 = 1 + \eta_K(q),$$

while if $q = \mathfrak{q}\overline{\mathfrak{q}}$ is split in $K$ (so $\eta_K(q) = 1$) then

$$a_q(f) = \chi(\mathfrak{q}) + \chi(\overline{\mathfrak{q}}) \equiv 1 + 1 = 1 + \eta_K(q) \pmod{\lambda}.$$

If $q = \mathfrak{q}^2$ is ramified in $K$ (so $\eta_K(q) = 0$) then

$$a_q(f) = \chi(\mathfrak{q}) \equiv 1 = 1 + \eta_K(q) \pmod{\lambda}.$$

(2) Say $(p) = \mathfrak{p}\overline{\mathfrak{p}}$. Since $\ell \mid (p-1)$, there exists a character $\chi : (\mathcal{O}_K/\mathfrak{p})^\times \to \mathbb{C}^\times$ of exact order $\ell$. Since $\ell \nmid w$, $\chi$ kills the image of the units in all cases, and since $\ell \nmid h_K$ it lifts to a character of the ray class group of conductor $\mathfrak{p}$, and may be considered $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathbb{C}^\times$. Since $\chi_\sigma$ has conductor $\overline{\mathfrak{p}}$, $\chi_\sigma \neq \chi$. Letting $f = f_\chi$, which is a newform in $S_1(\Gamma_0(Np), \eta_K\chi_{\mathbb{Q}})$, we proceed exactly as before, except we must exclude $q = p$, since $\chi(\mathfrak{p}) + \chi(\overline{\mathfrak{p}}) = 0 + 0 \neq 2 = 1 + \eta_K(p)$.

(3) Since $\ell \mid (p^2-1) = \#(\mathcal{O}_K/(p))^\times$, we get a character $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathbb{C}^\times$ as above, of exact order $\ell$ but this time of conductor $(p)$. The automorphism $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ maps to the Frobenius element in $\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$, so $\chi_\sigma = \chi^p = \chi^{-1}$ since $\ell \mid (p+1)$, so again $\chi_\sigma \neq \chi$ because $\ell$ is odd. Letting $f = f_\chi$, which is a newform in $S_1(\Gamma_0(Np^2), \eta_K\chi_{\mathbb{Q}})$, we prove the congruence as above. This time $q = p$ is OK, since $1 + \eta_K(p) = 0$.

□

**Remark 2.9.** (1) The right hand side of the congruence is the Hecke eigenvalue at $q$ of the Eisenstein series $E_1^{1,\eta_K}$, which lives in $M_1(\Gamma_0(N), \eta_K)$.

(2)

$$L(1, \eta_K) = \frac{2\pi h_K}{w\sqrt{|d_K|}}.$$

If $L_{\{p\}}(s, \eta_K)$ is the Dirichlet series with the Euler factor at $p$ omitted, then

$$L_{\{p\}}(1, \eta_K) = \begin{cases} \frac{2\pi h_K(1-p)}{pw\sqrt{|d_K|}} & \text{if } p \text{ is split in } K; \\ \frac{2\pi h_K(1+p)}{pw\sqrt{|d_K|}} & \text{if } p \text{ is inert in } K. \end{cases}$$

So, given that $\ell \nmid w$, the condition on $\ell$ is

$$\begin{cases} \mathrm{ord}_\ell\left(\frac{L(1,\eta_K)}{2\pi/\sqrt{|d_K|}}\right) > 0 & \text{in case (1);} \\ \mathrm{ord}_\ell\left(\frac{L_{\{p\}}(1,\eta_K)}{2\pi/\sqrt{|d_K|}}\right) > 0 \text{ but } \mathrm{ord}_\ell\left(\frac{L(1,\eta_K)}{2\pi/\sqrt{|d_K|}}\right) = 0 & \text{in cases (2),(3).} \end{cases}$$

(3) In the case that $K$ is real quadratic, $L(1, \eta_K)$ is not a critical value.

(4) For a congruence with $\psi(q)+\phi(q)$ (the Hecke eigenvalue at $q$ of $E_1^{\psi,\phi}$) on the right hand side, where $\psi=\eta_K\phi$, we may just twist $f_\chi$ by $\phi$, or equivalently multiply the original $\chi$ by the restriction to $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ of the Galois character associated to $\phi$.

For comparison, we look briefly at the situation for higher weights.

**Theorem 2.10.** *Let $\psi,\phi$ be Dirichlet characters of conductors $u,v$ respectively, and let $N=uv$. Let $k\geq 2$ be an integer. Suppose that $p\nmid N$ is a prime, and that $\lambda\mid\ell$ with $\ell>k+1$ and $\ell\nmid N$. Let $\Sigma_N$ be the set of primes dividing $N$.*

(1) *If*
$$\mathrm{ord}_\lambda\left(\frac{L_{\Sigma_N}(k,\psi\phi^{-1})}{(2\pi)^k}\right)>0,$$
*and $(N,k)\neq(1,2)$, there exists a cuspidal eigenform $f\in S_k(\Gamma_0(N),\chi)$, with $\chi\equiv\psi\phi \pmod{\lambda}$, such that*
$$a_q(f)\equiv\psi(q)+q^{k-1}\phi(q)\pmod{\lambda}\quad\textit{for all primes } q\nmid\ell N.$$

(2) *If*
$$\mathrm{ord}_\lambda\left(\frac{L_{\Sigma_{Np}}(k,\psi\phi^{-1})}{(2\pi)^k}\right)>0,$$
*and $(N,k)\neq(1,2)$, there exists a cuspidal eigenform $f\in S_k(\Gamma_0(Np),\chi)$, with $\chi\equiv\psi\phi \pmod{\lambda}$, such that*
$$a_q(f)\equiv\psi(q)+q^{k-1}\phi(q)\pmod{\lambda}\quad\textit{for all primes } q\nmid\ell pN.$$

This follows from work of Billerey and Menares [BM2, Theorems 1,2]. The right hand side is the Hecke eigenvalue at $q$ of the Eisenstein series $E_k^{\psi,\phi}\in M_k(\Gamma_0(N),\psi\phi)$. Their proof involves finding an element of $M_k(\Gamma_0(Np),\psi\phi)$, in the old space for $E_k^{\psi,\phi}$, with constant terms at all cusps divisible by $\ell$, lifting its reduction modulo $\ell$ to a cusp form, then applying the Deligne-Serre lemma [DeSe, Lemme 6.1]. In the case $(N,k)=(1,2)$, the corresponding condition, which would be $\ell\mid(p^2-1)$, is strengthened to $\ell\mid(p-1)$, following a theorem of Mazur [Maz].

## 3. Congruences between genus one and genus two cusp forms

The paramodular group of level $N$ is given by
$$\Gamma^{\mathrm{para}}(N)=\begin{bmatrix}\mathbb{Z}&N\mathbb{Z}&\mathbb{Z}&\mathbb{Z}\\ \mathbb{Z}&\mathbb{Z}&\mathbb{Z}&\frac{1}{N}\mathbb{Z}\\ \mathbb{Z}&N\mathbb{Z}&\mathbb{Z}&\mathbb{Z}\\ N\mathbb{Z}&N\mathbb{Z}&N\mathbb{Z}&\mathbb{Z}\end{bmatrix}\cap\mathrm{Sp}_2(\mathbb{Q}),$$
where $\mathrm{Sp}_2(\mathbb{Q}):=\{g\in M_4(\mathbb{Q}):{}^tgJg=J\}$, $J=\begin{pmatrix}0_2&-I_2\\ I_2&0_2\end{pmatrix}$. Let $F$ be a cuspidal Hecke eigenform of weight $\rho=\mathrm{Sym}^j\otimes\det^\kappa$ for $\Gamma^{\mathrm{para}}(N)$. Then $F:\mathfrak{H}_2\to V$, where $\mathfrak{H}_2=\{Z\in M_2(\mathbb{C}):{}^tZ=Z,\mathrm{Im}(Z)>0\}$ is Siegel's upper half space of genus 2, $V$ is the space of the representation $\rho=\mathrm{Sym}^j(\mathbb{C}^2)\otimes\det^\kappa$ of $\mathrm{GL}_2(\mathbb{C})$, and
$$F\left((AZ+B)(CZ+D)^{-1}\right)=\rho(CZ+D)(f(Z))\ \text{ for all }\begin{pmatrix}A&B\\ C&D\end{pmatrix}\in\Gamma^{\mathrm{para}}(N).$$

Let the elements $T(q), T(q^2)$ of the genus-2 Hecke algebra be as in [vdG, §16] (with the scaling as following Definition 8). Let $\lambda_F(q), \lambda_F(q^2)$ be the respective eigenvalues for these operators acting on $F$. The spinor $L$-function of $F$ is $L(s, F, \mathrm{Spin}) = \prod_{q \text{ prime}} L_q(s, F, \mathrm{Spin})$, where for primes $q \nmid N$, $L_q(s, F, \mathrm{Spin})^{-1}$

$= 1 - \lambda_F(q)q^{-s} + (\lambda_F(q)^2 - \lambda_F(q^2) - q^{j+2\kappa-4})q^{-2s} - \lambda_F(q)q^{j+2\kappa-3-3s} + q^{2j+4\kappa-6-4s}$.

**Theorem 3.1.** *Let $f \in S_k(\Gamma_0(N))$ be a normalised newform, with $k \geq 2$. Let $K$ be a real quadratic field with discriminant $d_K$, narrow class number $h'_K$ and associated quadratic character $\eta_K$. Suppose that $(d_K, N) = 1$, and that $f$ is not monomial with respect to $K$ (equivalently $a_q(f) \neq 0$ for some prime $q$ inert in $K$).*

(1) *Choose a prime $p$ that splits in $K$, $(p) = \mathfrak{p}\mathfrak{p}^\sigma$. Suppose that $p \nmid \ell N$, $\ell \nmid (p+1)$ and*
$$a_p(f)^2 \equiv p^{k-2}(p+1)^2 \pmod{\lambda},$$
*for some $\lambda \mid \ell$. Further assume that there exists a prime $q$, split in $K$, such that $a_q(f)^{h'_K} \not\equiv \left(q^{\frac{k}{2}} + q^{\frac{k}{2}-1}\right)^{h'_K} \pmod{\lambda}$, and that $\overline{\rho}_{f,\lambda}|_{\mathrm{Gal}(\overline{\mathbb{Q}}/K)}$ is irreducible. Let $\rho = \mathrm{Sym}^{k-2}(\mathbb{C}^2) \otimes \det^2$.*

*Then there exists a Siegel paramodular cusp form $F \in S_\rho(\Gamma^{\mathrm{para}}(N^2 d_K^2 p))$ satisfying*
$$\lambda_F(q) \equiv a_q(f)\,(1 + \eta_K(q)) \pmod{\lambda} \quad \textit{for all primes } q \nmid N d_K p.$$

(2) *Choose a prime $p$ that is inert in $K$. Suppose that $p \nmid \ell N$, $\ell \nmid (p^2+1)$ and*
$$a_p(f)^2 \equiv -p^{k-2}(p-1)^2 \pmod{\lambda},$$
*for some $\lambda \mid \ell$. Further assume that there exists a prime $q$, split in $K$, such that $a_q(f)^{h'_K} \not\equiv \left(q^k + q^{k-2}\right)^{h'_K} \pmod{\lambda}$, and that $\overline{\rho}_{f,\lambda}|_{\mathrm{Gal}(\overline{\mathbb{Q}}/K)}$ is irreducible. Let $\rho = \mathrm{Sym}^{k-2}(\mathbb{C}^2) \otimes \det^2$.*

*Then there exists a Siegel paramodular cusp form $F \in S_\rho(\Gamma^{\mathrm{para}}(N^2 d_K^2 p^2))$ satisfying*
$$\lambda_F(q) \equiv a_q(f)\,(1 + \eta_K(q)) \pmod{\lambda} \quad \textit{for all primes } q \nmid N d_K p.$$

(3) *Suppose that $\mathrm{ord}_\lambda\left(\frac{\sqrt{d_K}\, L(k, \mathrm{Sym}^2 f, \eta_K)}{\pi^{k+1}\Omega_f^+\Omega_f^-}\right) > 0$, where the periods $\Omega_f^\pm$ are as in [TiU, §3] and $\lambda \mid \ell$ with $\ell > k$ and $\ell \nmid 6Nd_K\#(O_K/N\,O_K)^\times$. Suppose also that $\overline{\rho}_{f,\lambda}$ is irreducible, and that for some prime $q \mid\mid N$, there is no newform of level dividing $N/q$ sharing this same residual representation. Then there exists a Siegel paramodular cusp form $F \in S_\rho(\Gamma^{\mathrm{para}}(N^2 d_K^2))$ satisfying*
$$\lambda_F(q) \equiv a_q(f)\,(1 + \eta_K(q)) \pmod{\lambda} \quad \textit{for all primes } q \nmid N d_K.$$

*Proof.* (1) Let $\pi_f$ be the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ associated to $f$. Since $\pi_f$ is not monomial with respect to $K$, the base change $\pi' := \mathrm{BC}_{K/\mathbb{Q}}(\pi_f)$ is a *cuspidal* automorphic representation of $\mathrm{GL}_2(\mathbb{A}_K)$ [L][Ge, §6.1], of weight $(k,k)$, level $(N)$ and trivial central character. The associated $\lambda$-adic Galois representation is $\rho_{f,\lambda}|_{\mathrm{Gal}(\overline{\mathbb{Q}}/K)}$, of conductor $(N)$.

Since $p$ is split, the congruence may be read as
$$a_{\mathfrak{p}}(\pi')^2 \equiv \left(\mathrm{Nm}(\mathfrak{p})^{\frac{k}{2}} + \mathrm{Nm}(\mathfrak{p})^{\frac{k}{2}-1}\right)^2 \pmod{\lambda}.$$

By a level-raising theorem of Taylor for Hilbert modular forms [Ta, Theorem 1], there exists an automorphic representation $\tilde{\pi}$ of $\mathrm{GL}_2(\mathbb{A}_K)$, of weight $(k,k)$, level $(N)\mathfrak{p}$ and trivial central character, such that

$$a_{\mathfrak{q}}(\tilde{\pi}) \equiv a_{\mathfrak{q}}(\pi') \pmod{\lambda} \quad \text{for all primes } \mathfrak{q} \neq \mathfrak{p}.$$

Note that the conditions $a_q(f)^{h'_K} \not\equiv \left(q^{\frac{k}{2}} + q^{\frac{k}{2}-1}\right)^{h'_K} \pmod{\lambda}$ and $\ell \nmid (p+1)$ allow us to ignore the "error term" in Taylor's theorem. Strictly speaking, Taylor's theorem gives us $\tilde{\pi}$ of central character trivial modulo $\lambda$, but we can replace it by one of trivial character using Jarvis's generalisation of Carayol's Lemma to Hilbert modular forms [Ja, Theorem 4.1], which uses the irreducibility of $\overline{\rho}_{f,\lambda}|_{\mathrm{Gal}(\overline{\mathbb{Q}}/K)}$. In his notation, $B = \mathrm{GL}_{2/K}$, $S$ is the set of infinite places together with divisors of $(N)\mathfrak{p}$, $U = U_0((N)\mathfrak{p})$, $U_1 = U_1((N)\mathfrak{p})$, $r$ is essentially the central character, and $\chi$ is its inverse.

Since $\tilde{\pi}$ has level $(N)\mathfrak{p}$, $(\tilde{\pi})^\sigma$ has level $(N)\mathfrak{p}^\sigma$, so $(\tilde{\pi})^\sigma \neq \tilde{\pi}$. We now employ a mild generalisation of a special case of a theorem of Johnson-Leung and Roberts [JR, Main Theorem]. It is likewise an application of a theorem of Roberts [Ro, Theorem 8.6, Introduction]. The analysis at finite places (leading to paramodular level) is exactly as in [JR]. The only difference is at archimedean places. In [JR] they have a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_K)$ whose components at the infinite places are isomorphic to discrete series representations $D_2$ and $D_{2n+2}$, leading to a scalar-valued Siegel cusp form of weight $n+2$. For us it is $D_k$ and $D_k$, so the case $n=0$ for them is the case $k=2$ for us. To make the generalisation, we simply observe that the $L$-packet $\Pi(\phi(\pi_{0,\infty}))$ (in the notation of [JR, §3]) contains the limit of discrete series representation denoted $\pi_\lambda[c]$ in [Mo, p. 207], with $c=0$ and Harish-Chandra parameter $\lambda = (\lambda_1, \lambda_2) = (k-1, 0)$. To cover the fact that $\lambda_2 > 0$ does not hold, see [Mo, p.210, Remark (ii)]. The Blattner parameter is $(\Lambda_1, \Lambda_2) = (\lambda_1, \lambda_2) + (1,2) = (k,2)$. This is $(j+\kappa, \kappa)$, where the lowest $K_\infty$-type is $\mathrm{Sym}^j(\mathbb{C}^2) \otimes \det^\kappa$, so we recover $j = k-2$, $\kappa = 2$.

As in [JR, Main Theorem (iii)], $L(s, F, \mathrm{Spin})$ is the same as the standard $L$-function of $\tilde{\pi}$. Looking at the coefficient of $q^{-s}$, we could write

$$\lambda_F(q) = \begin{cases} a_{\mathfrak{q}}(\tilde{\pi}) + a_{\mathfrak{q}^\sigma}(\tilde{\pi}) & q \text{ split;} \\ 0 & q \text{ inert.} \end{cases}$$

Since $a_{\mathfrak{q}}(\tilde{\pi}) \equiv a_{\mathfrak{q}}(\pi') = a_q(f) \pmod{\lambda}$ and $a_{\mathfrak{q}^\sigma}(\tilde{\pi}) \equiv a_{\mathfrak{q}^\sigma}(\pi') = a_q(f) \pmod{\lambda}$, the required congruence follows.

(2) This time for Taylor's level-raising theorem, since $\mathrm{Nm}(\mathfrak{p}) = p^2$ and since the Hecke eigenvalue for $\mathfrak{p}$ on $\pi'$ is $a_p(f)^2 - 2p^{k_1}$, we need $\lambda$ to divide

$$(a_p(f)^2 - 2p^{k-1})^2 - (p^k + p^{k-2})^2.$$

This factorises as

$$(a_p(f)^2 + p^{k-2}(p-1)^2)(a_p(f)^2 - p^{k-2}(p+1)^2),$$

and we have assumed that $\lambda$ divides the first of these factors, so we get $\tilde{\pi}$ as before. We need to know that $(\tilde{\pi})^\sigma \neq \tilde{\pi}$. If $(\tilde{\pi})^\sigma = \tilde{\pi}$ then by a theorem of Langlands [L], $\tilde{\pi}$ would be in the image of the base change map, say $\tilde{\pi} = \mathrm{BC}_{K/\mathbb{Q}}(\pi_g)$. The level raising congruence implies that $\overline{\rho}_{g,\lambda}|_{\mathrm{Gal}(\overline{\mathbb{Q}}/K)} \simeq$

$\overline{\rho}_{f,\lambda}|_{\mathrm{Gal}(\overline{\mathbb{Q}}/K)}$. Replacing $g$ by a quadratic twist by $\eta$ if necessary (which does not alter its base change), we may assume that $\overline{\rho}_{g,\lambda} \simeq \overline{\rho}_{f,\lambda}$. (We have used the irreducibility of $\overline{\rho}_{f,\lambda}|_{\mathrm{Gal}(\overline{\mathbb{Q}}/K)}$ again.) Since $\tilde{\pi} = \mathrm{BC}_{K/\mathbb{Q}}(\pi_g)$, the level of $g$ is exactly divisible by $p$, whereas the level of $f$ is not divisible by $p$. The (easier) necessary condition for raising the level of $f$ at $p$ implies that $\lambda$ divides the second factor $(a_p(f)^2 - p^{k-2}(p+1)^2)$. This would imply that $\lambda$ divides the difference of the two factors, i.e. $2p^{k-2}(p^2+1)$, which it does not. Hence $(\tilde{\pi})^\sigma \neq \tilde{\pi}$, and we have everything we need to proceed as before.

(3) Note that $L(k, \mathrm{Sym}^2 f, \eta_K) = L(1, \mathrm{ad}^0(f), \eta_K)$, a "twisted adjoint $L$-value". According to a recent preprint of Tilouine and Urban, under the conditions stated there is a congruence modulo $\lambda$ of Hecke eigenvalues between the base change $\pi' := \mathrm{BC}_{K/\mathbb{Q}}(\pi_f)$ and some non-base change cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_K)$ [TiU, Theorem 4.5]. We substitute this for $\tilde{\pi}$ in the above proof. It is crucial for it to be non-base change, to get the required $\mathrm{Gal}(K/\mathbb{Q})$ non-invariance.

□

**Remark 3.2.** (1) We may view (1) and (2) as analogues of (2) and (3) of Theorem 2.8. There the ray-class character $\chi$ of conductor $\mathfrak{p}$ is congruent modulo $\lambda$ to the trivial character of conductor 1, which may be viewed as a "base-change" to $K$ of the trivial character of $\mathbb{A}^\times/\mathbb{Q}^\times$, whose value at $q$ contributes the 1 to the right-hand-side of the congruence. Here $\tilde{\pi}$ of level $(N)\mathfrak{p}$ is congruent modulo $\lambda$ to $\pi'$ of level $(N)$, which is the base-change to $K$ of $\pi_f$, whose Hecke eigenvalue at $q$ contributes to the right-hand-side of the congruence. Note that it was essential to level-raise, indeed to level-raise *after* base-changing rather than before, to ensure the condition of $\mathrm{Gal}(K/\mathbb{Q})$ non-invariance. Just as in Theorem 2.8 the $L$-function attached to $\mathrm{Ind}_K^{\mathbb{Q}}(\chi)$ is $L(s, f_\chi)$, here the $L$-function attached to $\rho_{F,\lambda} := \mathrm{Ind}_K^{\mathbb{Q}}(\rho_{\tilde{\pi},\lambda})$ is $L(s, F, \mathrm{Spin})$.

(2) In (2) of Remark 2.9, we had the $L$-values $L(1, \eta_K)$ and $L_{\{p\}}(1, \eta_K)$, which could just as well have been described as $L_{\Sigma_N}(1, \eta_K)$ and $L_{\Sigma_N \cup \{p\}}(1, \eta_K)$, since the Euler factors at primes dividing $N$ are trivial. The congruence there could be interpreted as reducibility of $\overline{\rho}_{f,\lambda}$, with composition factors $\mathbb{F}_\lambda$, $\mathbb{F}_\lambda(\eta_K)$. Combining this with the irreducibility of $\rho_{f,\lambda}$, a construction of Ribet [Ri] ensures the existence of a non-trivial extension of $\mathbb{F}_\lambda$ by $\mathbb{F}_\lambda(\eta_K)$, hence a non-zero element of $H^1(\mathbb{Q}, \mathrm{Hom}(\mathbb{F}_\lambda, \mathbb{F}_\lambda(\eta_K))) = H^1(\mathbb{Q}, \mathbb{F}_\lambda(\eta_K))$. Since $\rho_{f,\lambda}$ is crystalline at $\ell$ (if $\ell > k$) and unramified at all $q \nmid \ell p N$ (say in cases (2) and (3)), this produces a non-zero element in the $\lambda$ part of a Bloch-Kato Selmer group appearing in the numerator of a conjectural formula for $L_{\Sigma_N \cup \{p\}}(1, \eta_K)$, cf. [DF, Propositions 3.1, 4.2]. This accounts for the involvement of that $L$-value. Note that the Bloch-Kato conjecture for Dirichlet characters is actually proved, in work completed by Huber and Kings [HK, Theorem 5.4.1].

Here, the congruence may be interpreted as reducibility of $\overline{\rho}_{F,\lambda}$, with composition factors $\overline{\rho}_{f,\lambda}$, $\overline{\rho}_{f,\lambda}(\eta_K)$ (assuming their irreducibility). Now

$$\mathrm{Hom}(\overline{\rho}_{f,\lambda}, \overline{\rho}_{f,\lambda}(\eta_K)) \simeq \mathrm{ad}(\overline{\rho}_{f,\lambda})(\eta_K),$$

but since the image of $\rho_{F,\lambda}$ acts by symplectic similitudes, in fact we get a cocycle with values in $\mathrm{ad}^0(\overline{\rho}_{f,\lambda}(\eta_K)$ (trace 0 endomorphisms, twisted by $\eta_K$), cf. [Du2, Lemma 6.5]. Since $\mathrm{ad}^0(\rho_{f,\lambda}) \simeq \mathrm{Sym}^2\rho_{f,\lambda}(k-1)$, we are led to consider $L_{\Sigma_N \cup \{p\}}(k, \mathrm{Sym}^2 f, \eta_K)$. (Note that $L(k, \mathrm{Sym}^2 f, \eta_K)$ is paired with $L(k-1, \mathrm{Sym}^2 f, \eta_K^{-1}) = L(k-1, \mathrm{Sym}^2 f, \eta_K)$ by the functional equation.) Indeed, it is easy to check that

$$L_p(k, \mathrm{Sym}^2(f), \eta_K)^{-1} = \frac{1-p}{p^{k+1}}\left(a_p(f)^2 - \left(p^{\frac{k}{2}} + p^{\frac{k}{2}-1}\right)^2\right),$$

so the level-raising condition (in the case that $p$ is split) implies that $\mathrm{ord}_\lambda(L_p(k, \mathrm{Sym}^2(f), \eta_K)^{-1}) > 0$, and is analogous to the condition $\ell \mid (p-1)$ in Theorem 2.8.

(3) When $p$ is inert, the level-raising condition is that $\lambda$ divides $L_{(p)}(1, \mathrm{ad}^0(\pi'))^{-1}$, which naturally arises from Galois deformation theory. This factors as $L_p(1, \mathrm{ad}^0(f), \eta_K)^{-1} L_p(1, \mathrm{ad}^0(f))^{-1}$, and to get $(\tilde{\pi})^\sigma \neq \tilde{\pi}$ we have imposed the stronger condition that $\lambda$ divides the first factor. This is analogous to what happened with $p^2-1 = (p+1)(p-1)$ (i.e. $(1-p^{-2}) = (1-\eta_K(p)p^{-1})(1-p^{-1})$) in (3) of Theorem 2.8.

(4) We see now that (3) is the analogue of (1) of Theorem 2.8. At least in the case $N=1$, the existence of congruences between base-change and non base-change Hilbert modular forms, modulo divisors of twisted adjoint $L$-values, was originally a conjecture of Doi, Hida, and Ishii [DHI],[Gh1, §7, Conjecture 2], who provided numerical evidence. Subject to a conjectured compatibility of periods (of the kind now proved by Tilouine and Urban), it was supported by a theorem of Ghate [Gh2, Theorem 5, Corollary 2] (which did not restrict to level 1).

(5) In [BD] we give a general conjecture on congruences of Hecke eigenvalues between cuspidal automorphic representations of split connected reductive groups $G$ and representations induced from cuspidal automorphic representations of Levi subgroups $M$ of maximal parabolic subgroups $P$. In §6 we work out the example where $G = \mathrm{GSp}_2$, $P$ is the Klingen parabolic and $M \simeq \mathrm{GL}_2 \times \mathrm{GL}_1$. There we start with the representation $\pi_f \times 1$ of $M(\mathbb{A})$. If instead we start with $\pi_f \times \eta_K$, we arrive at a conjecture that if $j + \kappa = k$, with $j \geq 0$ even and $\kappa \geq 4$, and if $\lambda$ (for $\ell > 2k-1$) divides $L_{\Sigma_{Nd_K}}(k-2+\kappa, \mathrm{Sym}^2 f, \eta_K)/\Omega$ (for a suitable Deligne period $\Omega$) then there exists a genus 2 cuspidal Hecke eigenform $F$ of weight $\mathrm{Sym}^j(\mathbb{C}^2) \otimes \det^\kappa$, associated automorphic representation $\pi_F$ of $\mathrm{GSp}_2(\mathbb{A})$ unramified away from $\Sigma_{Nd_K}$, and

$$\lambda_F(q) \equiv a_q(f)\left(q^{\kappa-2} + \eta_K(q)\right) \pmod{\lambda} \quad \text{for all primes } q \nmid Nd_K.$$

(Likewise with all three occurrences of $Nd_K$ replaced by $Nd_K p$.) Theorem 3.1 proves a degenerate case of such a congruence, with $\kappa = 2$, which is smaller than what is allowed by the conjecture in [BD]. (For $\kappa = 2$ we have $s = 0$ in the notation of [BD], rather than the required $s > 1$.) Similarly Theorem 2.8 is related to the case $G = \mathrm{GL}_2, M \simeq \mathrm{GL}_1 \times \mathrm{GL}_1$ [BD, §5], with $k = 1$ rather than the required $k \geq 3$. Theorem 2.10 supports the conjecture for $G = \mathrm{GL}_2, M \simeq \mathrm{GL}_1 \times \mathrm{GL}_1$, while Propositions 2.2 and 2.5 show that it does not extend to $k = 1$ without severe restrictions.

(6) We have used a form $F$ of weight $\mathrm{Sym}^{k-2}(\mathbb{C}^2)\otimes\det^2$ and level $\Gamma^{\mathrm{para}}(N^2 d_K^2 p)$, following Johnson-Leung and Roberts. We might have tried to use instead a form of level $\Gamma_0^{(2)}(Nd_k p) = \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{Z}) : \ C \equiv 0_2 \pmod{Nd_k p}\right\}$, with the same spinor $L$-function. Such a function was constructed by Yoshida [Y] for $k = 2$, and by Hsieh and Namikawa [HN] in general, as a theta lift from the multiplicative group $D^\times$ of a definite quaternion algebra. However, they require $N$ to be square-free, and there is also a strong restriction on Atkin-Lehner eigenvalues to ensure that the theta lift does not vanish. The construction of Roberts uses a theta lift from $\mathrm{GO}(2, 2)$ instead, and avoids these restrictions.

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