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Semiparametric Tests for the Order of Integration in the Possible Presence of Level Breaks

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Abstract

Lobato and Robinson (1998) develop semiparametric tests for the null hypothesis that a series is weakly autocorrelated, or $I(0)$, about a constant level, against fractionally integrated alternatives. These tests have the advantage that the user is not required to specify a parametric model for any weak autocorrelation present in the series. We extend this approach in two distinct ways. First we show that it can be generalised to allow for testing of the null hypothesis that a series is $I(\delta)$ for any δ lying in the usual stationary and invertible region of the parameter space. The second extension is the more substantive and addresses the well known issue in the literature that long memory and level breaks can be mistaken for one another, with unmodelled level breaks rendering fractional integration tests highly unreliable. To deal with this inference problem we extend the Lobato and Robinson (1998) approach to allow for the possibility of changes in level at unknown points in the series. We show that the resulting statistics have standard limiting null distributions, and that the tests based on these statistics attain the same asymptotic local power functions as infeasible tests based on the unobserved errors, and hence there is no loss in asymptotic local power from allowing for level breaks, even where none is present. We report results from a Monte Carlo study into the finite-sample behaviour of our proposed tests, as well as several empirical examples.

Keywords: Fractional integration; level breaks; Lagrange multiplier testing principle; spurious long memory; local Whittle likelihood; conditional heteroskedasticity.

JEL Classification: C22

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1 Introduction

It is well known that if not accounted for, level shifts in a weakly autocorrelated (or short memory) process, denoted $I(0)$, can induce features in the autocorrelation function and the periodogram of a time series that can be mistaken as evidence of long memory; see, *inter alia*, Diebold and Inoue (2001), Gourieroux and Jasiak (2001), Granger and Hyung (2004), Mikosch and Stărică (2004), Qu (2011), and Iacone, Leybourne, and Taylor (2019). To avoid the possibility of spurious inference being made about the memory properties of a time series, it is therefore important to develop tests on the fractional integration (memory) parameter of a time series which are robust to level shifts. As a consequence, Iacone *et al.* (2019) generalise the parametric Lagrange Multiplier [LM] time domain based fractional integration tests of Tanaka (1999) and Nielsen (2004) to allow for the possibility of a single break in the deterministic trend function at an unknown point in the sample. These tests are equivalent to analogous extensions of the frequency domain tests of Robinson (1994) to allow for breaks in the deterministic trend function. Iacone *et al.* (2019) show that this approach delivers an LM test which, regardless of whether a break occurs or not, is a locally most powerful test and has a χ^2_1 limiting null distribution.

However, a significant practical disadvantage of the tests of Iacone *et al.* (2019) is that, like the tests of Robinson (1994), Tanaka (1999) and Nielsen (2004) from which they are derived, they are based on fitting a full parametric model to the data. Crucially, the short run component of this model must be correctly specified under the null hypothesis for the resulting test to be correctly asymptotically sized. This requirement is clearly problematic in practice, and is likely to be further complicated in the case where level breaks are present as this would likely interfere with any preliminary model selection stage used to specify the form used for the short memory component. It therefore seems worth developing long memory tests analogous to those of Iacone *et al.* (2019) but which do not require the user to specify a parametric model for the short memory component of the series.

Our contribution in this paper is therefore to develop semiparametric analogues of the parametric tests of Iacone *et al.* (2019). We will base our approach on an extension of the semiparametric frequency domain based fractional integration tests of Lobato and Robinson (1998). This approach is based on the use of a low frequency approximation provided by the local Whittle [LW] likelihood, which obviates the need to explicitly model any short range dependence present in the data. To account for the possibility of level breaks, the Lobato and Robinson (1998)-type statistics we propose are constructed from data which have been de-trended allowing for the possibility of level breaks, the locations of which are estimated by a standard residual sum of squares estimator applied to the levels data. The tests proposed in Lobato and Robinson (1998), again based on the LM testing principle, are specifically designed for testing the null hypothesis that a time series is $I(0)$. We show that, as conjectured in Lobato and Robinson (1998, p. 478), their approach can

be generalised to provide a valid test for the null hypothesis that the series is integrated of order δ, for any δ lying in the stationary and invertible region of the parameter space ($-0.5 < \delta < 0.5$). It is also possible to test orders of integration outside the stationary and invertible region using data transformations. For example, the null hypothesis of an autoregressive unit root can be obtained by testing for the null hypothesis of short memory in the first differences of the series; as such this is then a test in the levels data for a unit root allowing for the possibility of trend breaks. Because the tests are based on the LM testing principle, no preliminary estimation of the memory parameter is required.

Our focus on the Lobato and Robinson (1998) testing approach is due, at least in part, to results in Shao and Wu (2007a) who show that the standard Lobato and Robinson (1998) tests are, for a suitable choice of the bandwidth parameter m used in the local Whittle loss function, considerably more powerful than other semiparametric tests for testing the null of $I(0)$ against the alternative of fractional integration that are available in the literature. In particular, they show that tests based on the rescaled range and rescaled variance statistics and tests based on the well-known KPSS statistic of Kwiatkowski, Phillips, Schmidt and Shin (1992) have power against local alternatives of order $(\ln(T))^{-1}$, where T denotes the sample size. On the other hand, the Lobato and Robinson (1998) tests have power against local alternatives of order $m^{-1/2}$, where the bandwidth parameter m is typically of the type $m = T^{\alpha}$ for some $0 < \alpha < 4/5$. Moreover, these other approaches have only been developed to test the null hypothesis of $I(0)$ against the alternative of fractional integration, whereas we wish to maintain the flexibility to test a more general $I(\delta)$ null hypothesis. Busetti and Harvey (2001, 2003) develop extensions of the KPSS test that allow for a single level break at an unknown point in the sample, although their approach is based on the assumption that a level break is known to occur.

We establish that, regardless of whether level breaks occur or not, the large sample properties of the tests we propose are identical to those which obtain for the standard Lobato and Robinson (1998) tests for $\delta = 0$ in the case where no level breaks occur. In particular, our proposed LM-type test has a χ^2_1 limiting null distribution and the corresponding t-type test a $N(0,1)$ limiting null distribution, regardless of the value of δ being tested under the null hypothesis, and each attains the same asymptotic local power function as the corresponding infeasible test based on the unobserved errors. Moreover, these asymptotic local power functions do not alter between the break and no break cases and so there is no loss in asymptotic local power from allowing for level breaks, even where no breaks are present. Although based on different and hence not directly comparable models, these large sample properties contrast with those of most popular *unit root* tests, such as that of Dickey and Fuller (1979), and *stationarity* tests, such as that of KPSS. In particular, the limiting null distributions of unit root and stationarity test statistics tend to be non-standard and depend on the functional form of the fitted deterministic, differing between the no break and break cases, and dependent on the locations of the breaks.

Moreover, where breaks are fitted but not actually present in the data, these tests show a considerable decline in asymptotic local power relative to the case where no break is fitted.

The remainder of the paper is organised as follows. Section 2 sets out the fractionally integrated level break model within which we work. Section 3 describes our tests in the infeasible case where the errors are observable. Our proposed semiparametric statistics for the case of unknown level breaks are described in Section 4, where we also establish their large sample properties. Section 5 summarises the results from a Monte Carlo simulation study into the finite sample size and power properties of our proposed tests and compares with the nonparameteric KPSS-type tests of Busetti and Harvey (2001, 2003). Illustrative empirical examples of the methods developed in this paper to bitcoin returns data, VIX market volatility, U.S. CPI inflation, and U.S. real GDP growth are considered in Section 6. Section 7 concludes. Proofs of our main results are provided in a mathematical appendix. A supplementary appendix contains full details of the Monte Carlo design and results.

2 The Fractionally Integrated Model with Level Breaks

Consider the scalar time series process, y_t, satisfying the data generating process (DGP),

$$y_t = \beta_1 + \boldsymbol{\beta}_2' \boldsymbol{DU}_t(\boldsymbol{\tau}^*) + u_t, \quad t = 1, \ldots, T. \tag{1}$$

In (1), $\boldsymbol{\beta} = (\beta_1, \boldsymbol{\beta}_2')'$ is a vector of fixed parameters and $\boldsymbol{DU}_t(\boldsymbol{\tau}^*) := (DU_t(\tau_1^*), \ldots, DU_t(\tau_k^*))'$ is a vector of k level break terms, where $DU_t(\tau)$ is defined for a generic argument τ as $DU_t(\tau) := \mathbb{I}(t \geq \lfloor \tau T \rfloor)$, $\mathbb{I}(\cdot)$ denotes the usual indicator function, $\lfloor \cdot \rfloor$ denotes the integer part of its argument, and where $A := B$ and $B =: A$ is used to denote that A is defined by B. The formulation in (1) therefore allows for up to k level breaks where $\boldsymbol{\tau}^* := (\tau_1^*, \ldots, \tau_k^*)'$ is the vector of (unknown) putative level break fractions and $\boldsymbol{\beta}_2 := (\beta_{2,1}, \ldots, \beta_{2,k})'$ the associated break magnitude parameters, such that a level break occurs at time $\lfloor \tau_i^* T \rfloor$ when $\beta_{2,i} \neq 0$ for $i = 1, \ldots, k$. The true but unknown number of level breaks present, k^* say, is then given by the number of non-zero elements of the vector $\boldsymbol{\beta}_2$. The (putative) level break fractions are assumed to be such that $\tau_i^* \in [\tau_L, \tau_U] =: \Lambda$ for all $i = 1, \ldots, k$, where $\Lambda \subset (0, 1)$ is compact and the quantities τ_L and τ_U are trimming parameters below and above which, respectively, a level break is deemed not to occur. We also make the standard assumption that $|\tau_i^* - \tau_j^*| \geq \lambda > 0$ for all $i \neq j$, such that there are at least $\lfloor \lambda T \rfloor$ observations between breaks. Note that these conditions imply that the number of breaks that can feasibly be calculated, given τ_L, τ_U, λ, satisfies $k \leq 1 + \lfloor (\tau_U - \tau_L)/\lambda \rfloor$.

In the context of (1) the shocks, u_t, are assumed to follow a stationary and invertible process which is fractionally integrated of order δ, denoted $u_t \in I(\delta)$. For our purposes, we define fractional integration for u_t as

$$u_t := \Delta^{-\delta} \eta_t, \tag{2}$$

where η_t is a zero mean $I(0)$ process. We define $I(0)$ to be such that η_t has spectral density $f(\lambda)$ with $f(\lambda) \to G$ for some $G \in (0, \infty)$ as $\lambda \to 0$; formal assumptions on η_t required for our large sample theory results will be delayed until Section 3. The assumption that u_t is stationary and invertible entails that the long memory parameter, δ, is such that $\delta \in (-0.5, 0.5)$. A process satisfying the conditions just stated for u_t is often referred to in the literature as a type I fractionally integrated process.

Our interest focuses on testing the null hypothesis that u_t, and hence y_t, is $I(\delta_0)$ for some $\delta_0 \in (-0.5, 0.5)$; that is, $H_0 : \delta = \delta_0$ in (1). Note that the extension to allow $\delta_0 \neq 0$ is non-trivial, in the sense that testing $\delta = \delta_0$ on y_t, as we do in this paper, is different from testing $\delta = 0$ on $\Delta^{\delta_0} y_t$, as is done for example in Iacone *et al.* (2019), since the latter process has a different unconditional mean thereby changing the model and its interpretation. Based on the familiar LM testing principle we will develop tests against two-sided alternatives of the form $H_1 : \delta \neq \delta_0$ (y_t is not $I(\delta_0)$) together with corresponding t-type tests against one-sided alternatives of the form $H_1 : \delta > \delta_0$ (y_t is more persistent than an $I(\delta_0)$ series) or $H_1 : \delta < \delta_0$ (y_t is less persistent than an $I(\delta_0)$ series).

Next, in Section 3, we discuss the tests proposed in Lobato and Robinson (1998) which were developed for testing the specific null hypothesis that y_t is short memory. These tests apply to the case where either u_t in (1) is observable or where it is *known* that $\boldsymbol{\beta}_2 = \mathbf{0}$ (so that no level breaks are present). We show that this approach can be readily extended to develop tests for the null hypothesis that y_t is $I(\delta_0)$ for some $\delta_0 \in (-0.5, 0.5)$. Then, in Section 4, we show how these tests can be generalised to allow for the possibility that $\boldsymbol{\beta}_2 \neq \mathbf{0}$ in (1), such that a level shift could potentially occur in the data. The testing approach we outline in Section 4 does not assume knowledge of whether a level shift genuinely occurs; that is, we do not assume knowledge of whether $\boldsymbol{\beta}_2 = \mathbf{0}$ or $\boldsymbol{\beta}_2 \neq \mathbf{0}$.

3 Tests of $H_0 : \delta = \delta_0$ when it is known that $\boldsymbol{\beta}_2 = \mathbf{0}$

Suppose for the purposes of this section that it is *known* to be the case that $\boldsymbol{\beta}_2 = \mathbf{0}$ in (1). Under this restriction we can also set $\beta_1 = 0$ with no loss of generality because, as discussed in Lobato and Robinson (1998, p. 477), the statistics we will discuss in this paper are invariant to β_1 in the case where $\boldsymbol{\beta}_2 = \mathbf{0}$. The restriction that $\boldsymbol{\beta}_2 = \mathbf{0}$ is therefore equivalent to the case where β_1, $\boldsymbol{\beta}_2$ and $\boldsymbol{\tau}^*$ are all known, such that u_t in (1) is observable. We may therefore proceed as if u_t were observable. We will discuss the application of the tests to u_t, although in the context of this section they could equally be applied to y_t because no mean-correction is required (provided the mean is constant) due to invariance to β_1.

For observable u_t, semiparametric inference on δ based on the approximation of the Whittle likelihood at low frequencies was proposed by Künsch (1987) and analysed further in Robinson (1995b). This approach is semiparametric as it does not require the specification of a parametric model for $f(\lambda)$ and, within the class of semiparametric methods,

it has the advantage of being based on a (local) likelihood, and it is therefore considerably more efficient than other semiparametric estimates such as the log-periodogram regression of Geweke and Porter-Hudak (1983) and Robinson (1995a).

For a generic series a_t, let $w_a(\lambda) := \frac{1}{\sqrt{2\pi T}} \sum_{t=1}^{T} a_t e^{i\lambda t}$ denote the Fourier transform of a_t, and let $I_a(\lambda) := |w_a(\lambda)|^2$ denote the periodogram. Then, as discussed in Robinson (1995b), for the observable series u_t, the local Whittle estimate of δ is obtained by minimising the loss function $R(d)$ with respect to d, where

$$R(d) := \ln\left(\frac{1}{m}\sum_{j=1}^{m} \lambda_j^{2d} I_u(\lambda_j)\right) - 2d\frac{1}{m}\sum_{j=1}^{m} \ln(\lambda_j) \tag{3}$$

and m denotes the bandwidth, satisfying the rate condition that $1/m + m/T \to 0$ as $T \to \infty$. Recall that $\lambda_j := \frac{2\pi j}{T}$ for integer j are the Fourier frequencies. Applying the LM principle to the objective function in (3) yields the LM-type statistic to test $H_0 : \delta = \delta_0$,

$$LM_m^*(\delta_0) := m\left(\frac{\partial^2 R(d)}{\partial^2 d}\right)^{-1}\left(\frac{\partial R(d)}{\partial d}\right)^2\Bigg|_{d=\delta_0}.$$

Defining the t-type statistic

$$t_m^*(\delta_0) := -\left(\frac{m^{-1/2}\sum_{j=1}^{m} \nu_j \lambda_j^{2\delta_0} I_u(\lambda_j)}{\frac{1}{m}\sum_{j=1}^{m} \lambda_j^{2\delta_0} I_u(\lambda_j)}\right), \text{ where } \nu_j := \ln j - \frac{1}{m}\sum_{j=1}^{m} \ln j, \tag{4}$$

the $LM_m^*(\delta_0)$ statistic in (5) can be equivalently re-written in terms of the Fourier frequencies and the periodogram ordinates at those frequencies as

$$LM_m^*(\delta_0) = t_m^*(\delta_0)^2 = \left(\frac{m^{-1/2}\sum_{j=1}^{m} \nu_j \lambda_j^{2\delta_0} I_u(\lambda_j)}{\frac{1}{m}\sum_{j=1}^{m} \lambda_j^{2\delta_0} I_u(\lambda_j)}\right)^2. \tag{5}$$

The null hypothesis H_0 that u_t is $I(\delta_0)$ can then be rejected for large values of $LM_m^*(\delta_0)$, while a large positive (negative) value of $t_m^*(\delta_0)$ would allow rejection against the one-sided alternative $H_1 : \delta > \delta_0$ ($H_1 : \delta < \delta_0$). It will turn out that standard critical values can be employed in the context of these decision rules.

Lobato and Robinson (1998) analyse the special case of the $t_m^*(0)$ and $LM_m^*(0)$ statistics in (4) and (5), respectively, which obtain setting $\delta_0 = 0$, such that one is testing the null hypothesis of short memory, $H_0 : \delta = 0$. For the purpose of later sections, we need to also define the Lobato and Robinson (1998) t- and LM-type test statistics for the hypothesis $H_0 : \delta = \delta_0$ applied to the observed data, $\{y_t\}$, and which do not account for the possibility of level breaks; we will denote these as $t_m(\delta_0)$ and $LM_m(\delta_0)$, respectively. These differ from the infeasible statistics $t_m^*(\delta_0)$ and $LM_m^*(\delta_0)$ for the hypothesis $H_0 : \delta = \delta_0$ which are applied to the unobserved innovations, $\{u_t\}$. In the context of this section, where it is *known* that $\boldsymbol{\beta}_2 = \mathbf{0}$, then $t_m(\delta_0)$ and $t_m^*(\delta_0)$ coincide, as do $LM_m(\delta_0)$ and $LM_m^*(\delta_0)$. Lobato and Robinson (1998) establish that, under certain regularity con-

ditions (see Assumption 1 below), $t_m^*(0)$ and $LM_m^*(0)$ have $N(0,1)$ and χ_1^2 limiting null distributions, respectively. Shao and Wu (2007a) subsequently demonstrate that under local alternatives of the form $H_c : \delta = cm^{-1/2}$, where c is a constant (such that H_c reduces to $H_0 : \delta = 0$ when $c = 0$), $t_m^*(0) \xrightarrow{d} N(2c, 1)$ and, hence, $LM_m^*(0) \xrightarrow{d} \chi_1^2(4c^2)$, where $\chi_1^2(4c^2)$ denotes a non-central χ_1^2 distribution with non-centrality parameter $4c^2$.

Before progressing to consider the case where u_t is not observable, that is where it is not *known* for sure that $\boldsymbol{\beta}_2 = \mathbf{0}$ in (1), we first show that the properties established for the $LM_m^*(0)$ and $t_m^*(0)$ statistics in Lobato and Robinson (1998) and Shao and Wu (2007a) carry over to the general case of the $LM_m^*(\delta_0)$ statistic in (5) and corresponding $t_m^*(\delta_0)$ statistic in (4) for testing $H_0 : \delta = \delta_0$ for any $\delta_0 \in (-0.5, 0.5)$. To do so we first introduce sufficient conditions for establishing these large sample justifications. We will discuss two sets of possible assumptions under which our large sample results obtain. The first set, given in Assumption 1, coincides with the conditions adopted by Robinson (1995b). The second set, given in Assumption 2, coincides with those employed by Shao and Wu (2007a).

Assumption 1.

(i) $\eta_t := \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$ *and* ε_t *is a martingale difference sequence with* $E(\varepsilon_t|\mathcal{F}_{t-1}) = 0$, $E(\varepsilon_t^2|\mathcal{F}_{t-1}) = 1$, $E(\varepsilon_t^3|\mathcal{F}_{t-1}) = \kappa_3 < \infty$, $E(\varepsilon_t^4|\mathcal{F}_{t-1}) = \kappa_4 < \infty$, *a.s., where* $\mathcal{F}_t$ *is the* σ*-field of events generated by* ε_s, $s \leq t$.

(ii) The weights ψ_j *are such that* $\sum_{j=0}^{\infty} \psi_j^2 < \infty$.

(iii) The spectral density of η_t, $f(\lambda)$, *is twice boundedly differentiable in a neighbourhood of* $\lambda = 0$ *and satisfies, as* $\lambda \to 0^+$, *that* $f(\lambda) = G(1+O(\lambda^2))$ *and* $\frac{\partial}{\partial \lambda} \ln f(\lambda) = O(\lambda^{-1})$ *for some* $G \in (0, \infty)$.

(iv) The bandwidth, m, *is such that* $\frac{1}{m} + \frac{m^5 (\ln m)^2}{T^4} \to 0$ *as* $T \to \infty$.

Remark 3.1. The conditions on η_t detailed in Assumption 1 coincide with those given in Robinson (1995b) and are slightly stronger than those in Lobato and Robinson (1998). A full discussion of these conditions is given in Robinson (1995b, pp. 1634 and 1641) and Lobato and Robinson (1998, p. 478). Assumption 1 includes all stationary and invertible finite-order ARMA models for η_t. Assumption 1 allows for non-linearity via the martingale difference assumption on the innovations, but is otherwise linear. Notice also that Assumption 1 requires $f(\lambda)$ to be smooth only around $\lambda = 0$ and so does not rule out long memory behaviour at frequencies other than $\lambda = 0$ (although this needs to be strengthened in Assumption 3 to obtain results for our feasible tests). ♢

The assumption of conditional homoskedasticity imposed by part (i) of Assumption 1 may be considered unacceptable for many data applications, in particular those involving financial data. Shao and Wu (2007a,b) show that this can be weakened to allow for a wide class of stationary, causal non-linear processes. To that end, suppose that

$$\eta_t = F(\ldots, \varepsilon_{t-1}, \varepsilon_t), \tag{6}$$

where ε_t are independent and identically distributed (IID) random variables and F is a measurable function such that η_t is well defined as a stationary, causal, ergodic process. For a random variable ξ and $p > 0$, write $\xi \in \mathcal{L}^p$ if $\|\xi\|_p := (E(|\xi|^p))^{1/p} < \infty$. Let $\{\varepsilon_t'\}$, $t \in Z$, be an IID copy of $\{\varepsilon_t\}$, $\mathcal{F}_t = (\ldots, \varepsilon_{t-1}, \varepsilon_t)$, $\mathcal{F}_k^* := (\mathcal{F}_{-1}, \varepsilon_0)$, $\eta_k' = F(\mathcal{F}_0^*, \varepsilon_1, \ldots, \varepsilon_t)$ and $\vartheta_q(k) = \|\eta_k - \eta_k'\|_q$.

Assumption 2. *For η_t and F defined as in (6) and for some $q > 4$:*

(i) $\eta_t \in \mathcal{L}^q$ *and* $\sum_{k_1,k_2,k_3} cum(\eta_0, \eta_{k_1}, \eta_{k_2}, \eta_{k_3}) < \infty$, *where $cum(\cdot)$ denotes the joint cumulant of the arguments.*

(ii) $\sum_{k=1}^{\infty} k\vartheta_q(k) < \infty$.

(iii) The spectral density of η_t, $f(\lambda)$, satisfies $f(\lambda) = G(1 + O(\lambda^2))$ as $\lambda \to 0^+$ for some $G \in (0, \infty)$.

(iv) The bandwidth, m, is such that $\frac{(\ln T)^3}{m} + \frac{m}{T^{2/3}} \to 0$ as $T \to \infty$.

Remark 3.2. Assumption 2 includes a number of widely used nonlinear time series models for η_t such as bilinear models, threshold models, GARCH and ARMA-GARCH models; see Shao and Wu (2007a, p. 254) and Shao and Wu (2007b) and the references therein for further discussion of this assumption and further examples of classes of nonlinear processes which satisfy it. While Assumption 2 weakens, *inter alia*, the conditional homoskedasticity restriction of Assumption 1, this comes at the cost of a stronger assumption on the bandwidth, that is restricted to be such that $m = o(T^{2/3})$. Moreover, as discussed in Shao and Wu (2007b, Remark 3.1), Assumption 2(ii) implies continuous differentiability of $f(\lambda)$ for all frequencies, whereas, as discussed in Remark 3.1 and Robinson (1995b), Assumption 1 only imposes conditions on $f(\lambda)$ in a local-to-zero band. There is therefore a clear trade-off between the conditions imposed on η_t by Assumptions 1 and 2. ♢

In Theorem 1 we now derive the large sample properties of the $LM_m^*(\delta_0)$ and $t_m^*(\delta_0)$ statistics, obtained for the case where it is *known* that $\boldsymbol{\beta}_2 = \mathbf{0}$ in (1). To facilitate discussion of asymptotic local power, we consider the local alternative $H_c : \delta = \delta_0 + cm^{-1/2}$.

Theorem 1. *Let y_t be generated according to (1) with $\boldsymbol{\beta}_2 = \mathbf{0}$, and let either Assumption 1 or Assumption 2 hold on η_t. Then, for any $\delta_0 \in (-0.5, 0.5)$, under $H_c : \delta = \delta_0 + cm^{-1/2}$:*

(i) $LM_m^*(\delta_0) \xrightarrow{d} \chi_1^2(4c^2)$; *and*

(ii) $t_m^*(\delta_0) \xrightarrow{d} N(2c, 1)$.

Remark 3.3. Theorem 1 shows that the results obtained for the limiting null distributions of the $LM_m^*(0)$ and $t_m^*(0)$ statistics in Lobato and Robinson (1998) apply more generally to the $LM_m^*(\delta_0)$ and $t_m^*(\delta_0)$ statistics for testing the null hypothesis that u_t is $I(\delta_0)$ for any δ_0 in the usual stationary and invertible region. Theorem 1 also shows that tests based on the $LM_m^*(\delta_0)$ and $t_m^*(\delta_0)$ statistics posses the same local power functions as tests based on

the $LM_m^*(0)$ and $t_m^*(0)$ statistics. Moreover, these results hold regardless of whether u_t is conditionally homoskedastic or conditionally heteroskedastic (satisfying Assumption 2). Finally, note that the result in Theorem 1 was anticipated without proof by Marinucci and Robinson (2001, Section 4), at least under H_0 and Assumption 1. ♢

4 Feasible Tests of $H_0 : \delta = \delta_0$ Allowing for up to k Level Breaks

Recall that the LM- and t-type tests discussed in Section 3 are based on the assumption that $\boldsymbol{\beta}_2 = \mathbf{0}$, such that the $LM_m(\delta_0)$ and $t_m(\delta_0)$ statistics calculated on the observed data $\{y_t\}$ will coincide with the $LM_m^*(\delta_0)$ and $t_m^*(\delta_0)$ statistics based on the shocks, $\{u_t\}$, even if $\beta_1 \neq 0$ such that $\{u_t\}$ are unobservable (because the statistics are invariant to β_1). However, where $\boldsymbol{\beta}_2 \neq \mathbf{0}$ this is no longer the case, and we cannot proceed as if the tests were based on the unobservable shocks, $\{u_t\}$. Moreover, where $\boldsymbol{\beta}_2 \neq \mathbf{0}$ the $LM_m(\delta_0)$ and $t_m(\delta_0)$ statistics constructed from the observed data, $\{y_t\}$, are non-similar tests and will diverge. For example, if $\delta_0 = 0$ it can be shown that the (exact) rates of divergence are $LM_m(0) = O_p(m\ln(m)^2)$ and $t_m(0) = O_p(\sqrt{m}\ln(m))$ under H_0, so that both statistics will diverge with the sample size, even under the null hypothesis. As a consequence, therefore, the Lobato and Robinson (1998) tests will spuriously reject the null with probability tending to one as the sample size diverges. That is, tests based on $LM_m(\delta_0)$ or $t_m(\delta_0)$ are uninformative if it is unknown whether $\boldsymbol{\beta}_2 = \mathbf{0}$ or not. In this section we will therefore discuss how feasible versions of the tests discussed in Section 3 can be derived for the case where it is not known for certain whether $\boldsymbol{\beta}_2 = \mathbf{0}$ or not.

In the context of (1), the disturbances u_t are not observable and so they must be estimated. For a generic vector of (putative) break locations, $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_k)'$, we can use ordinary least squares (OLS) estimators of the parameters β_1 and $\boldsymbol{\beta}_2$ in (1). To that end, let $\boldsymbol{\beta} := (\beta_1, \boldsymbol{\beta}_2')'$, and let $y := (y_1, \ldots, y_T)'$, $x_t(\boldsymbol{\tau}) := (1, DU_t(\tau_1), \ldots, DU_t(\tau_k))'$, and $x(\boldsymbol{\tau}) := (x_1(\boldsymbol{\tau}), \ldots, x_T(\boldsymbol{\tau}))'$. Then the OLS estimate of $\boldsymbol{\beta}$ is given by $\widehat{\boldsymbol{\beta}}(\boldsymbol{\tau}) = (\widehat{\beta}_1(\boldsymbol{\tau}), \widehat{\boldsymbol{\beta}}_2(\boldsymbol{\tau})')' := (x(\boldsymbol{\tau})'x(\boldsymbol{\tau}))^{-1}x(\boldsymbol{\tau})'y$. For a given value of $\boldsymbol{\tau}$ we then have the corresponding estimated residuals $\widehat{u}_t(\boldsymbol{\tau}) := y_t - \widehat{\boldsymbol{\beta}}(\boldsymbol{\tau})'x(\boldsymbol{\tau})$, with associated periodogram $I_{\widehat{u}(\boldsymbol{\tau})}(\lambda_j)$.

Based on $I_{\widehat{u}(\boldsymbol{\tau})}(\lambda_j)$, we can then define analogues of the $LM_m^*(\delta_0)$ statistic of (5) and the corresponding t-type statistic $t_m^*(\delta_0)$ in (4), for testing $H_0 : \delta = \delta_0$ as follows

$$t_m(\delta_0; \boldsymbol{\tau}) := -\left(\frac{m^{-1/2} \sum_{j=1}^{m} \nu_j \lambda_j^{2\delta_0} I_{\widehat{u}(\boldsymbol{\tau})}(\lambda_j)}{\frac{1}{m} \sum_{j=1}^{m} \lambda_j^{2\delta_0} I_{\widehat{u}(\boldsymbol{\tau})}(\lambda_j)} \right) \tag{7}$$

$$LM_m(\delta_0; \boldsymbol{\tau}) := (t_m(\delta_0; \boldsymbol{\tau}))^2. \tag{8}$$

If the true vector of break fractions, $\boldsymbol{\tau}^*$, were known then one would simply evaluate $LM_m(\delta_0; \boldsymbol{\tau})$ and $t_m(\delta_0; \boldsymbol{\tau})$ at $\boldsymbol{\tau} = \boldsymbol{\tau}^*$. Our focus, however, is on the case where $\boldsymbol{\tau}^*$ is unknown and so will need to be estimated from the data. An obvious candidate is the minimum residual sum of squares (RSS) estimator considered in Lavielle and Moulines

(2000, pp. 38–39), which can be written as

$$\widehat{\boldsymbol{\tau}} = (\widehat{\tau}_1, \ldots, \widehat{\tau}_k)' := \arg \min_{\substack{\tau_1,\ldots,\tau_k \in [\tau_L, \tau_U] \\ |\tau_i - \tau_j| \geq \lambda > 0 \forall i \neq j}} \sum_{t=1}^{T} \widehat{u}_t(\boldsymbol{\tau})^2, \tag{9}$$

where it is recalled that τ_L and τ_U are trimming parameters such that $[\tau_L, \tau_U] \subset (0, 1)$.

Given the RSS estimator $\widehat{\boldsymbol{\tau}}$ in (9), tests for $H_0 : \delta = \delta_0$ can then be based on $LM_m(\delta_0; \widehat{\boldsymbol{\tau}})$ and $t_m(\delta_0; \widehat{\boldsymbol{\tau}})$. For these tests to be operational, we will need to establish the large sample behaviour of the $LM_m(\delta_0; \widehat{\boldsymbol{\tau}})$ and $t_m(\delta_0; \widehat{\boldsymbol{\tau}})$ statistics under the null hypothesis, $H_0 : \delta = \delta_0$, and show that unique asymptotic critical values (in the sense that they do not depend on any nuisance parameters) for the tests can be obtained from these distributions. In fact, we will be able to show in what follows that these statistics have the same limiting null distributions as were obtained for their infeasible counterparts $LM_m^*(\delta_0)$ and $t_m^*(\delta_0)$ in Theorem 1. In order to do so, however, we must impose some additional regularity conditions on η_t. In particular, Assumptions 1 and 2 must be strengthened to Assumptions 3 and 4, respectively, as follows:

Assumption 3. *Let Assumption 1 hold. Assume further that:*

(i) $\sup_t E(|\varepsilon_t|^q) < \infty$ *for some* $q > 1/(1 + 2\delta)$.

(ii) The weights ψ_j *are such that* $\sum_{j=0}^{\infty} j|\psi_j| < \infty$.

(iv) For some $\epsilon > 0$*, the bandwidth,* m*, is such that* $\frac{T^\epsilon}{m} \to 0$ *as* $T \to \infty$.

Assumption 4. *Let Assumption 2 hold, and define the projection operator* $\wp_k \xi := E(\xi|\mathcal{F}_k) - E(\xi|\mathcal{F}_{k-1})$*. Then we assume further that:*

(i) $\eta_t \in \mathcal{L}^q$ *and* $\sum_{k=0}^{\infty} \|\wp_0 \eta_k\|_q < \infty$ *for some* $q > 1/(1 + 2\delta)$.

(iv) For some $\epsilon > 0$*, the bandwidth,* m*, is such that* $\frac{T^\epsilon}{m} \to 0$ *as* $T \to \infty$.

Remark 4.1. Both Assumptions 3 and 4 impose the additional moment condition that $q > 1/(1 + 2\delta)$ moments exist. This condition is needed so that we can appeal to the functional central limit theorem [FCLT] for fractional processes for which the moment condition is necessary; see Theorem 2 of Johansen and Nielsen (2012). The fractional FCLT also requires that $q > 2$, but this is implied in Assumptions 1 or 2 so is not stated explicitly here. The condition placed on the weights ψ_j in Assumption 3(ii) is quite standard for the (fractional) FCLT and is met by all stationary and invertible finite-order ARMA models. This condition also implies continuity of the spectral density of η_t and hence rules out long memory at other frequencies, see Remarks 3.1 and 3.2. The condition that $0 < \sum_{j=0}^{\infty} \psi_j < \infty$ (and a similar condition for the non-linear process) is again omitted because it is implied by the assumption $0 < f(0) < \infty$. The additional condition required to hold on the bandwidth in part (iv) of Assumptions 3 and 4 is not restrictive in practice because much larger bandwidths will typically be used. ♢

We are now in a position to state our main result in Theorem 2 which details the large sample behaviour of the feasible statistics $LM_m(\delta_0;\widehat{\boldsymbol{\tau}})$ and $t_m(\delta_0;\widehat{\boldsymbol{\tau}})$ under local alternatives of the form $H_c:\delta=\delta_0+cm^{-1/2}$. We will first state our main result and then provide some discussion around this result. We will also provide further insights into this result through the case of $k=1$ where results are easier to explain.

Theorem 2. *Let y_t be generated according to (1), and let either Assumption 3 or Assumption 4 hold on η_t. Then, for any $\delta_0\in(-0.5,0.5)$, under $H_c:\delta=\delta_0+cm^{-1/2}$, and regardless of whether $\boldsymbol{\beta}_2=\mathbf{0}$ or $\boldsymbol{\beta}_2\neq\mathbf{0}$, it holds that $LM_m(\delta_0,\widehat{\boldsymbol{\tau}})\xrightarrow{d}\chi^2_1(4c^2)$ and $t_m(\delta_0,\widehat{\boldsymbol{\tau}})\xrightarrow{d}N(2c,1)$.*

Remark 4.2. A comparison of the results in Theorem 2 with those given previously in Theorem 1 yields the following immediate consequence. Regardless of whether any particular element $\beta_{2,i}$, $i=1,\ldots,k$, of $\boldsymbol{\beta}_2$ in (1) is zero or non-zero, the tests based on $LM_m(\delta_0;\widehat{\boldsymbol{\tau}})$ and $t_m(\delta_0;\widehat{\boldsymbol{\tau}})$ attain exactly the same asymptotic local power functions as obtained by the infeasible tests based on $LM^*_m(\delta_0)$ and $t^*_m(\delta_0)$, respectively. Moreover, under $H_0:\delta=\delta_0$, $LM_m(\delta_0;\widehat{\boldsymbol{\tau}})\xrightarrow{d}\chi^2_1$ and $t_m(\delta_0;\widehat{\boldsymbol{\tau}})\xrightarrow{d}N(0,1)$, so that standard critical values can be used for both tests, again regardless of whether $\beta_{2,i}=0$ or $\beta_{2,i}\neq 0$, $i=1,\ldots,k$, holds in (1). ♢

A proof of Theorem 2 is provided in the appendix. The proof strategy is to consider the distances between the feasible statistics $LM_m(\delta_0;\widehat{\boldsymbol{\tau}})$ and $t_m(\delta_0;\widehat{\boldsymbol{\tau}})$ with the infeasible $LM^*_m(\delta_0)$ and $t^*_m(\delta_0)$ statistics, respectively, in large samples. Inherent in doing so is to analyse the distance between $\widehat{u}_t$ and $\widehat{u}_t(\widehat{\boldsymbol{\tau}})$, the latter given by $\widehat{u}_t(\boldsymbol{\tau})$ evaluated at $\boldsymbol{\tau}=\widehat{\boldsymbol{\tau}}$, and establish how this affects the distance between the feasible and infeasible statistics. The behaviour of both $LM_m(\delta_0,\widehat{\boldsymbol{\tau}})$ and $t_m(\delta_0,\widehat{\boldsymbol{\tau}})$ clearly depend on the large sample properties of the estimates $\widehat{\boldsymbol{\tau}}$ in (9) and $\widehat{\boldsymbol{\beta}}(\widehat{\boldsymbol{\tau}})$, the latter given by $\widehat{\boldsymbol{\beta}}(\boldsymbol{\tau})$ evaluated at $\boldsymbol{\tau}=\widehat{\boldsymbol{\tau}}$. For the properties of $\widehat{\boldsymbol{\tau}}$ we apply a result of Lavielle and Moulines (2000), and we combine this with a fractional FCLT for u_t to obtain results for $\widehat{\boldsymbol{\beta}}(\widehat{\boldsymbol{\tau}})$.

Remark 4.3. In order to give some insight into the mechanics behind the proof, it is instructive to specialise our discussion to the case where $k=1$. Accordingly, and with an obvious notation, we re-define $\boldsymbol{\tau}$, $\widehat{\boldsymbol{\tau}}$, $\boldsymbol{\beta}_2$, and $\widehat{\boldsymbol{\beta}}_2$ as τ, $\hat{\tau}$, β_2, and $\hat{\beta}_2$, respectively. The proof proceeds by establishing that two key results hold under the conditions of Theorem 2. The first result is that if $\beta_2=0$ (so that no level break occurs), then $t_m(\delta_0;\tau)-t^*_m(\delta_0)=o_p(1)$ and $LM_m(\delta_0;\tau)-LM^*_m(\delta_0)=o_p(1)$, in each case uniformly in τ. This result establishes that when no level break occurs, the differences between the statistics based on $\widehat{u}_t$ and $\widehat{u}_t(\tau)$ are asymptotically negligible, and that this holds uniformly in τ and, hence, holds for $\widehat{\tau}$. To prove this, we first establish uniformly in τ results for $\widehat{\beta}(\tau)$. It is at this stage that the fractional FCLT is used. We can then derive properties of the estimated residuals $\widehat{u}_t(\tau)$ and analyse the distance between the Fourier transforms (and hence the periodograms)

of $\widehat{u}_t(\tau)$ and of u_t. The second result is that if $\beta_2 \neq 0$, then $t_m(\delta_0;\widehat{\tau}) - t_m^*(\delta_0) = o_p(1)$ and $LM_m(\delta_0;\widehat{\tau}) - LM_m^*(\delta_0) = o_p(1)$. That is, when $\beta_2 \neq 0$, such that a level break occurs, the differences between the statistics based on $\widehat{u}_t$ and $\widehat{u}_t(\widehat{\tau})$ are asymptotically negligible. In this case, we first establish the properties of the estimate of the break fraction, $\widehat{\tau}$, using results from Lavielle and Moulines (2000). These properties allow us to bound the distance between $\widehat{\beta}(\widehat{\tau})$ and $\widehat{\beta}(\tau^*)$, and use this to analyse the distance between the Fourier transforms (and the periodograms) of $\widehat{u}_t(\widehat{\tau})$ and of $\widehat{u}_t(\tau^*)$. ♢

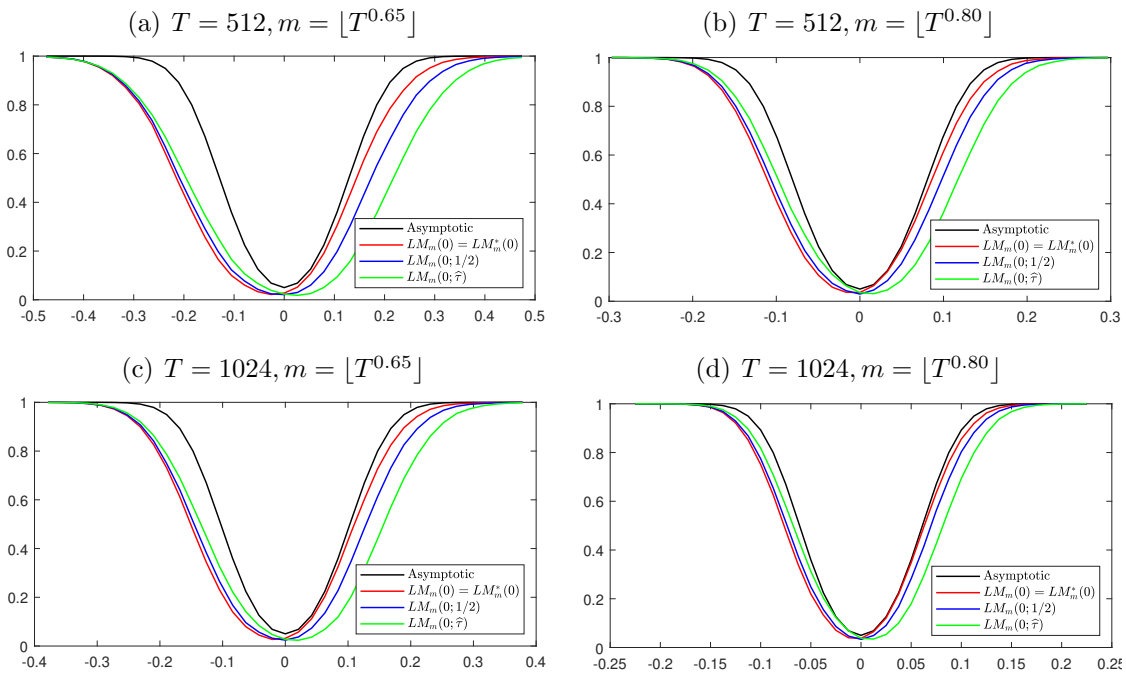
Remark 4.4. As discussed in Remark 4.3, the difference between the feasible and infeasible test statistics is shown to be $o_p(1)$ in Theorem 2. However, these remainder terms are nonetheless functions of δ_0, or equivalently of δ because of the local asymptotic framework; see e.g. (26), (33), and (34) in Appendix A.2. This finite sample dependence on δ_0 can also be observed in the Monte Carlo results; see point (v) in Section 5. ♢

Remark 4.5. The result in Theorem 2 shows that there is no loss in asymptotic local efficiency from allowing for k breaks when the true number of breaks, k^* say, is smaller than k. However, as the simulation results in Section 5 show, the finite sample size and power properties of the feasible LM-type test, $LM_m(\delta_0;\widehat{\tau})$, deteriorate somewhat if k is chosen to be larger than k^*. On the other hand, if k is chosen to be smaller than k^* then we know from the discussion at the start of Section 4 that the $LM_m(\delta_0;\widehat{\tau})$ statistic will diverge, even under the null hypothesis. In practical applications it would therefore seem sensible to select the number breaks used in constructing the $LM_m(\delta_0;\widehat{\tau})$ statistic according to a consistent information criterion. Theorem 9 of Lavielle and Moulines (2000, pp. 49–50) provides the conditions required on the penalty function such that an information criterion-based approach will consistently select the true number of breaks in the context of the DGP in (1) under the conditions of Theorem 2. Their result shows that, provided the maximum number of breaks allowed, k, is at least as large as the true number of breaks, k^*, then the commonly used Bayes information criterion [BIC] of Schwarz (1978) and Hannan-Quinn information criterion [HQIC] of Hannan and Quinn (1979) will both deliver consistent estimates of k^*. We recommend the use of the HQIC as this is less parsimonious than the BIC, and hence constitutes a safer choice in practice, given the severe implications of fitting too few level breaks. We will illustrate the use of the BIC and HQIC in the empirical applications in Section 6. ♢

5 Monte Carlo Evidence

We begin this section by investigating how well the large sample predictions of Theorem 2 hold in finite samples for a DGP that has either zero or one level break and we accordingly set $k = 1$ so that the notation of Remark 4.3 applies. To that end, Figures 1 and 2 graph simulated finite sample power functions of the the feasible LM-type test, $LM_m(\delta_0;\widehat{\tau})$, proposed in Section 4 and the corresponding Lobato and Robinson (1998) test, $LM_m(\delta_0)$,

Figure 1: Rejection frequencies, $\delta_0 = 0$ and $\beta_2 = 0$



that does not allow for the possibility of a level break. In the context of the $LM_m(\delta_0; \widehat{\tau})$ statistic we set the trimming parameters to be $\tau_L = 0.15$ and $\tau_U = 0.85$. Also graphed are the power functions of the corresponding infeasible tests, $LM_m(\delta_0; \tau^*)$, defined just under (8), and $LM_m^*(\delta_0)$ defined in (5). The former assumes knowledge of the true break location, τ^*, but not the innovations, u_t, and the latter assumes knowledge of the innovations.

The simulated data used to construct the power curves in Figures 1 and 2 were generated according to the DGP in (1)–(2) for $T = 512$ and $T = 1024$ setting $k = 1$ and with $\eta_t \sim NIID(0, 1)$, and where β_1 was set equal to zero with no loss of generality. All of the reported tests are for testing $H_0 : \delta = 0$ at the nominal asymptotic 5% level. The graphs depict the simulated power functions of the tests under the local alternative $H_c : \delta = cm^{-1/2}$ for a range of values of c and with the corresponding values of δ shown on the horizontal axes. Results are reported for two bandwidth choices, namely $m = \lfloor T^{0.65} \rfloor$ and $m = \lfloor T^{0.8} \rfloor$. The results in Figure 1 relate to the case considered in part (i) of Theorem 2 with no level break, i.e. $\beta_2 = 0$, while the results in Figure 2 relate to part (ii) of Theorem 2 for the specific case of a level break with $\beta_2 = 2$ at $\tau^* = 0.5$, i.e. a break equal to two standard deviations of the innovation process occurring midway through the sample. The simulated power curves were computed using 10,000 Monte Carlo replications using the `RNDN` function of Gauss 20. As a benchmark, we also include in each graph the corresponding asymptotic local power curves obtained directly from the non-central $\chi^2_1(4c^2)$ distribution, where $c = \delta\sqrt{m}$.

Consider first the results in Figure 1 for the no break case. Here, given knowledge

that no level break was present, the best possible test to use among the three considered would be the basic Lobato and Robinson (1998) test, $LM_m(\delta_0) = LM_m^*(\delta_0)$. Against positive values of δ this test has power closest to the asymptotic local power function and is somewhat more powerful than the infeasible $LM_m(\delta_0;\tau^*)$ test, which in turn is more powerful than the feasible $LM_m(\delta_0;\widehat{\tau})$ test. These differences are, however, reduced for $T = 1024$ *vis-à-vis* $T = 512$ and for $m = \lfloor T^{0.8} \rfloor$ *vis-à-vis* $m = \lfloor T^{0.65} \rfloor$; indeed for $T = 1024$ and $m = \lfloor T^{0.8} \rfloor$ the differences between the three tests are quite small with all three lying close to the asymptotic local power curve. For negative values of δ there are only very slight differences between the three tests. Overall, the large sample predictions from part (i) of Theorem 2 appear to hold reasonably well in finite samples, particularly so for the larger bandwidth considered.

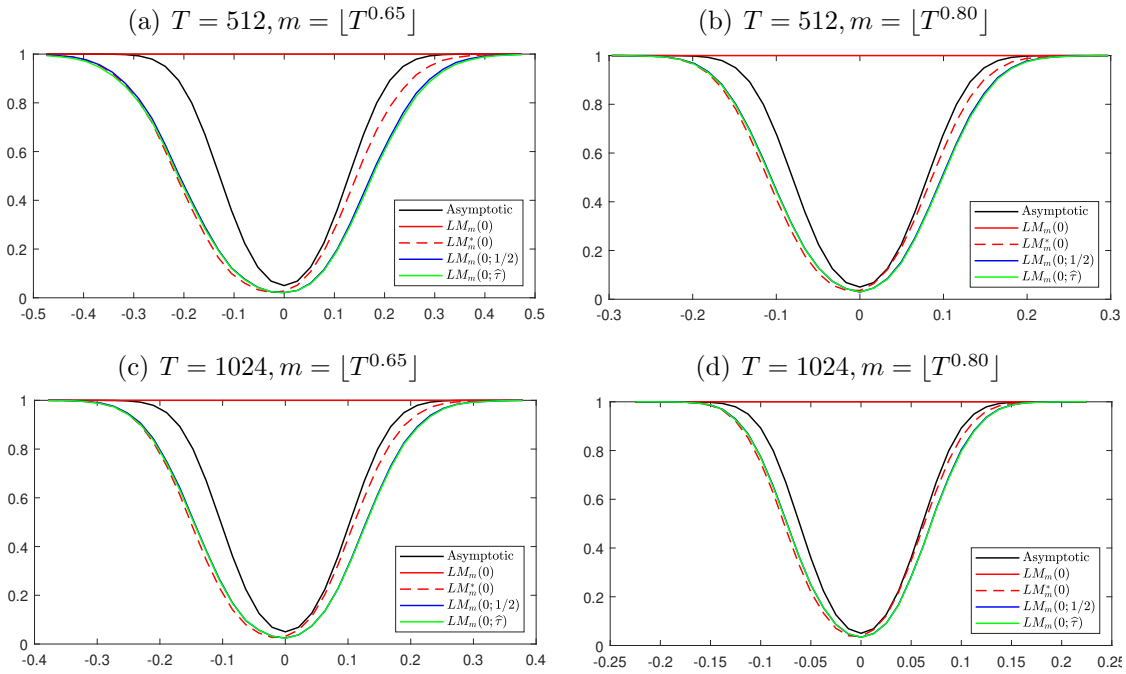
Consider next the the results in Figure 2 for the case where a level break of magnitude $\beta_2 = 2$ occurs. Here the infeasible $LM_m^*(\delta_0)$ test no longer coincides with the feasible Lobato and Robinson (1998) test, $LM_m(\delta_0)$. In this case the divergence of the $LM_m(\delta_0)$ test is clearly seen, regardless of whether the null hypothesis holds or not, with the test rejecting essentially 100% of the time even for the smaller sample size considered. The power functions of the infeasible $LM_m(\delta_0;\tau^*)$ and feasible $LM_m(\delta_0;\widehat{\tau})$ tests essentially coincide regardless of the sample size or bandwidth considered, suggesting that τ^* is very accurately estimated by $\widehat{\tau}$ in this case. As with the results for the no break case in Figure 1, for positive values of δ the power curve of the feasible $LM_m(\delta_0;\widehat{\tau})$ test lies only slightly below that of the infeasible $LM_m^*(\delta_0)$ test, which in turn lies close to the asymptotic local power curve, with the differences between the power curves reducing as T and/or m is increased. For negative values of δ the power curves of the $LM_m(\delta_0;\widehat{\tau})$ and $LM_m^*(\delta_0)$ tests are almost indistinguishable regardless of m or T. Again the large sample predictions from part (ii) of Theorem 2 would appear to hold reasonably well in finite samples.

In the remainder of this section we summarise the results from an large set of Monte Carlo experiments designed to investigate the finite sample size and power properties of the semiparametric long memory tests proposed in Section 4. Specifically, we compare the empirical size and power properties of the $LM_m(\delta_0;\widehat{\boldsymbol{\tau}})$, $LM_m(\delta_0;\boldsymbol{\tau}^*)$ and $LM_m(\delta_0)$ tests along with the corresponding t-type tests, $t_m(\delta_0;\widehat{\boldsymbol{\tau}})$, $t_m(\delta_0;\boldsymbol{\tau}^*)$ and $t_m(\delta_0)$, respectively. Results are reported for DGPs with either zero, one or two level breaks. In the case where a maximum of one level break is allowed, comparison is also made with the KPSS stationarity test, denoted $KPSS$, together with the generalisations thereof proposed in Busetti and Harvey (2001, 2003) which allow for a level break at either a known or unknown location, denoted $KPSS(\tau^*)$ and $KPSS(\widehat{\tau})$, respectively. The full set of results together with details of the experimental design can be found in the supplementary appendix.

We considered models for $\{y_t\}$ of the form given in (1) with either $k = 1$ or $k = 2$:

- For the $k = 1$ (so that up to one level break is allowed) case the DGP had either no level break or a level break at the sample midpoint with magnitude $\beta_2 \in \{0.5, 1, 2\}$.

Figure 2: Rejection frequencies, $\delta_0 = 0$ and $\beta_2 = 2$



Results are reported related to testing $H_0 : \delta = 0$; both where $\delta = 0$ (empirical size) and where $\delta \in \{-0.15, 0.15\}$ (empirical power). The empirical size properties of tests for $H_0 : \delta = 0.3$ and $H_0 : \delta = -0.3$ were also explored. For the empirical size results the error process η_t was allowed to follow either an IID process, an AR(1) process or an ARCH(1) process, while for empirical power IID and ARCH(1) processes were considered.

- For the $k = 2$ (so that up to two level breaks are allowed) case the DGP had either no level break or was such that two level breaks occurred with the level shifting from 0 to β_2 to $2\beta_2$ at 1/3 and 2/3, respectively, of the way through the sample with $\beta_2 \in \{0.5, 1, 2\}$. Results are again reported related to testing $H_0 : \delta = 0$; both where $\delta = 0$ (empirical size) and where $\delta \in \{-0.15, 0.15\}$ (empirical power).

All of the tests were implemented for both a range of fixed bandwidths and using data-driven bandwidth rules. Again we set the search set as $\Lambda = [0.15, 0.85]$. The principal findings of our Monte Carlo results can be summarised as follows, where comments (i)–(vi) relate to results in Tables S.1–S.18 for the single (putative) level break case, and comment (vii) relates to results in Tables S.19–S.24 for the double (putative) break case:

(i) As with the findings in Lobato and Robinson (1998) our results demonstrate that the bandwidth m has a significant impact on the finite sample properties of the tests, with a clear trade-off seen between size and power. In particular, for a given sample size, excluding those tests which are non-similar (i.e., excluding the $LM_m(\delta_0)$ and $t_m(\delta_0)$ tests when $\beta_2 \neq 0$), we observe the following general patterns: (a) for a given

pattern of weak dependence and a given bandwidth, m, the observed distortions from the nominal (asymptotic) significance level are greater the larger is m, and (b) empirical power against a given fixed alternative increases as the bandwidth, m, increases. Generally, a range of bandwidths between $m = \lfloor T^{0.5} \rfloor$ and $m = \lfloor T^{0.65} \rfloor$ provides reasonable finite sample size control across the cases considered.

(ii) Our results suggest that the automatic bandwidth, m_{LR}, of Lobato and Robinson (1998) delivers a reasonable trade-off between finite sample size and power considerations, at least when the data are conditionally homoskedastic. In the conditionally heteroskedastic ARCH(1) case, the empirical size of tests based on m_{LR} do not improve, other things equal, as the sample size is increased. This is perhaps not surprising given that the m_{LR} bandwidth rule is not consistent with the bandwidth rate imposed on m by Assumption 2, and we therefore recommend caution in using the m_{LR} bandwidth rule with data which are suspected to display conditional heteroskedasticity. For the KPSS-type tests, the automatic bandwidth rule recommended in Lobato and Robinson (1998) also appears to deliver a reasonable size-power trade-off.

(iii) Overall, our results suggest that it may be helpful in practice to consider the automatic bandwidth, m_{LR}, together with a range of bandwidths between $m = \lfloor T^{0.5} \rfloor$ and $m = \lfloor T^{0.65} \rfloor$. This is what we will do in the empirical examples in Section 6.

(iv) As expected, where a level break occurs ($\beta_2 \neq 0$), the non-similar $LM_m(\delta_0)$, $t_m(\delta_0)$ and $KPSS$ tests are highly unreliable displaying severe oversize (excepting the left-tailed $t_m(\delta_0)$ test which is correspondingly undersized), and hence spurious evidence of long memory. The observed size distortions seen with these tests are higher, other things equal, the larger is the sample size or the level break magnitude.

(v) Although asymptotically equivalent under both the null and local alternatives (cf. Theorem 2), differences are observed between the finite sample size and power properties of the pairs of tests $LM_m(\delta_0;\widehat{\tau})$ and $LM_m(\delta_0;\tau^*)$, and $t_m(\delta_0;\widehat{\tau})$ and $t_m(\delta_0;\tau^*)$. The $LM_m(\delta_0;\tau^*)$ and $t_m(\delta_0;\tau^*)$ tests are based on knowledge of whether a level break occurs or not (i.e. whether $\beta_2 = 0$ or $\beta_2 \neq 0$) and, where a break occurs, also knowledge of the level break location τ^*, while $LM_m(\delta_0;\widehat{\tau})$ and $t_m(\delta_0;\widehat{\tau})$ do not assume knowledge of either. The differences between the finite sample properties of these pairs of tests are seen to diminish as either the sample size or, in the case where a level break occurs, the break magnitude increases; indeed, for the largest magnitude considered, $\beta_2 = 2$, these differences are largely eliminated even for the smaller of the two sample sizes considered. The observed differences between the empirical power properties of these pairs of tests are seen to be slightly larger, other things equal, in the case where the errors are ARCH(1) *vis-à-vis* the IID case. Moreover, the finite sample differences between the pairs of tests are smallest for the tests of $H_0 : \delta = -0.3$ and largest for the tests of $H_0 : \delta = 0.3$; c.f. Remark 4.4. Where no

level break is present, the finite sample differences between the $LM_m(\delta_0;\widehat{\tau})$ test and $LM_m(\delta_0)$ (which assume no level break is present) are again relatively small, other things equal, particularly for the larger sample size considered. This is also broadly true for a comparison between the $t_m(\delta_0;\widehat{\tau})$ and $t_m(\delta_0)$ tests, although the differences are larger than for the LM-type tests. Overall, the asymptotic theory presented in Theorem 2 appears to provide a reasonable prediction of the finite sample behaviour of the $LM_m(\delta_0;\widehat{\tau})$ and $t_m(\delta_0;\widehat{\tau})$ tests.

(vi) For a given DGP, the one-sided t-tests have more power (in the correct tail) than the corresponding two-sided LM tests, as would be expected. Moreover, and consistent with both the discussion concerning theoretical power rates against local alternatives in Shao and Wu (2007a) and the simulation findings in Lobato and Robinson (1998), the KPSS-type tests have considerably lower power to detect departures from short memory than do the corresponding LM- and t-based fractional integration tests discussed in this paper, at least provided reasonable bandwidths m are chosen.

(vii) The results for the case where two putative breaks are allowed for ($k = 2$) are qualitatively similar to the corresponding results discussed above for the case of a single (putative) level break. However, as might be expected, the patterns seen for $k = 1$ are somewhat magnified for $k = 2$.

6 Empirical Examples

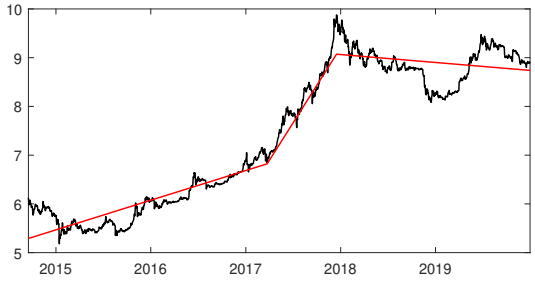
Throughout the empirical examples in this section, we set the trimming parameters equal to the same values as were used in the Monte Carlo experiments in Section 5, that is $\tau_L = 0.15$ and $\tau_U = 0.85$. Where multiple breaks were estimated, we set the minimum spacing parameter λ defined in Section 4 to $\lambda = 0.10$, except for the VIX example where we set $\lambda = 0.05$ to allow larger values of k. For $k \geq 4$, a complete enumeration of all possible break date combinations is infeasible, so the break dates are estimated by numerical (integer) optimization of the RSS function using a Matlab program which is available in the supporting material.

6.1 Bitcoin Returns

We apply the semiparametric long memory tests described in this paper to the daily returns of Bitcoin over the period 17 September 2014 to 31 December 2019, giving a total of $T = 1932$ daily observations. The data were retrieved from Yahoo Finance. The logarithm of the closing price of Bitcoin in USD is graphed in Figure 3 along with the returns series, defined as first differences of the (log) closing price series. A visual inspection of the data suggests the plausibility of changes in slope, implying changes in level at the same point in the returns series, with the most obvious case being at around the beginning of 2018. The red line on the graphs shows the fitted deterministic trend/level of the series allowing for two breaks, the locations of which are estimated by applying the RSS-based estimator

Figure 3: Bitcoin daily data 9/17/2014 to 12/31/2019

(a) Bitcoin/USD log-price



(b) Bitcoin/USD returns

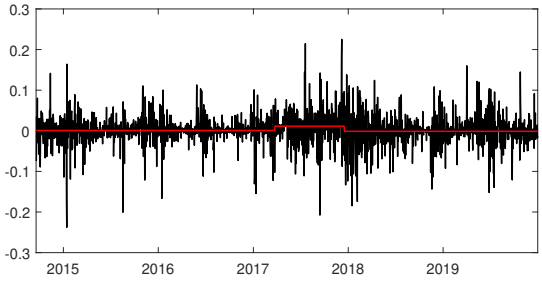


Table 1: Tests of $H_0 : \delta = 0$ versus $H_1 : \delta > 0$ in Bitcoin returns data

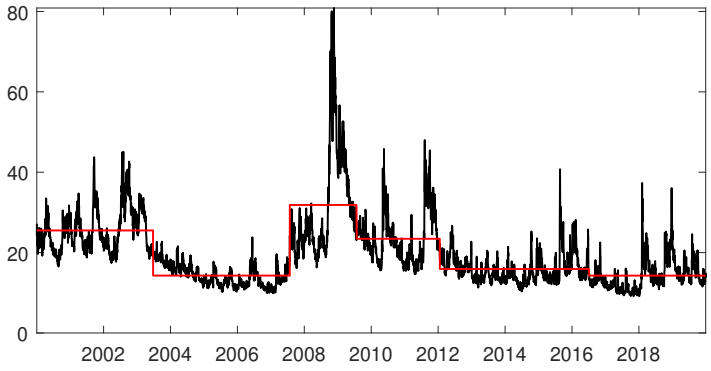
	m	$t_m(0)$ $k = 0^{BIC}$	$t_m(0;\widehat{\tau})$ $k = 1$	$t_m(0;\widehat{\boldsymbol{\tau}})$ $k = 2^{HQ}$	$t_m(0;\widehat{\boldsymbol{\tau}})$ $k = 3$
$\lfloor T^{0.50} \rfloor$	43	1.19	0.80	−0.30	−0.43
	50	1.34*	0.94	−0.21	−0.37
$\lfloor T^{0.55} \rfloor$	64	1.45*	1.02	−0.11	−0.31
	75	1.99**	1.51*	0.31	0.09
$\lfloor T^{0.60} \rfloor$	93	2.02**	1.54*	0.33	0.10
	100	1.38*	0.91	−0.27	−0.50
	125	1.57*	1.08	−0.13	−0.40
$\lfloor T^{0.65} \rfloor$	137	1.09	0.60	−0.59	−0.86
	m_{LR}	1.62* ($m = 510$)	1.37* ($m = 510$)	0.13 ($m = 510$)	−0.16 ($m = 510$)

Note: *, ** and *** denote outcomes which are statistically significant at the 10%, 5% and 1% level, respectively, while ^{HQ} and ^{BIC} indicate the number of breaks chosen by the HQIC and BIC, respectively.

discussed in Section 4 to the returns data setting $k = 2$. The estimated break dates are 24 March 2017 and 16 December 2017.

Evidence of long memory in returns would of course be in strong violation of the efficient market hypothesis, and so it is of interest in the context of the Bitcoin return data to test $H_0 : \delta = 0$ against the alternative $H_1 : \delta > 0$. We do so using both the test based on the $t_m(0)$ statistic of Lobato and Robinson (1998), which does not allow for a level break, and the analogues of this test based on the $t_m(0;\widehat{\tau})$ and $t_m(0;\widehat{\boldsymbol{\tau}})$ statistics allowing for the presence of either one or two level breaks, respectively, in each case occurring at unknown points in the sample. Following the recommendations from our Monte Carlo study we computed the statistics for a range of values of the bandwidth parameter, m, lying between $\lfloor T^{0.5} \rfloor = 43$ and $\lfloor T^{0.65} \rfloor = 137$, inclusive, as well as for the automatic bandwidth rule, m_{LR} of Lobato and Robinson (1998) with the value that this takes reported in parentheses below the outcome of the statistics. The results are summarised in Table 1. Here, and also in Tables 4 and 5, the superscripts *, ** and *** denote outcomes which are statistically significant at the 10%, 5% and 1% level, respectively, while the superscripts ^{HQ} and ^{BIC}

Figure 4: VIX daily data 1/1/2000 to 12/31/2019



indicate the number of breaks chosen by the HQIC and BIC, respectively; cf. Remark 4.5.

Using Lobato and Robinson's $t_m(0)$ test we can reject H_0 at the 10% level when using the data-dependent bandwidth rule, m_{LR}, and for all but the smallest and largest of the other bandwidths considered. The null can also be rejected at the 5% level for $m = 75$ and $m = 93$. On balance we surmise from the results for the standard Lobato and Robinson test that the short memory null hypothesis is rejected in favour of long memory in the Bitcoin returns data. On the other hand, for the test based on $t_m(0; \widehat{\tau})$, which fits a level break to the data, the evidence against the null hypothesis is considerably weaker and, in particular, H_0 can only be rejected at the 10% level for bandwidths $m \in \{75, 93, m_{LR}\}$. Allowing for two breaks, which is the number chosen by our preferred HQIC, no choice of bandwidth results in a rejection at even the 10% level for the $t_m(0; \widehat{\boldsymbol{\tau}})$ test. This suggests that the finding of long memory in Bitcoin returns by the Lobato and Robinson (1998) test is likely attributable to the presence of at least one level break in the returns data.

6.2 VIX Market Volatility

In the next example we consider market volatility, measured by VIX, using daily data from 1 January 2000 to 31 December 2019 for a total of $T = 5031$ observations. The data were downloaded from Yahoo Finance and are graphed in Figure 4. The red step function on the graph shows the fitted deterministic level of the series allowing for 10 level breaks.

It has been argued by several authors that long memory in volatility is an important stylized fact; see e.g. Andersen *et al.* (2001) and references therein. Furthermore, long memory in volatility is relevant in asset pricing. For example, Baillie *et al.* (1996) use asset pricing as motivation for their FIGARCH model, and Christensen and Nielsen (2007) discuss implications of long memory in volatilty in the context of stock pricing. Other authors, however, suggest volatility might be a short memory process with the statistical evidence for long memory disappearing once level shifts in the data are accounted for; see, among others, Granger and Hyung (2004).

To investigate this further, we test the short memory null hypothesis $H_0 : \delta = 0$

Table 2: Tests of $H_0 : \delta = 0$ versus $H_1 : \delta > 0$ in VIX volatility data

	m	$t_m(0)$ $k=0$	$t_m(0;\widehat{\tau})$ $k=1$	$t_m(0;\widehat{\boldsymbol{\tau}})$ $k=2$	$t_m(0;\widehat{\boldsymbol{\tau}})$ $k=3$	$t_m(0;\widehat{\boldsymbol{\tau}})$ $k=4$	$t_m(0;\widehat{\boldsymbol{\tau}})$ $k=5$
$\lfloor T^{0.50} \rfloor$	70	13.39	11.35	10.46	9.86	5.63	3.38
	75	14.39	12.27	11.35	10.65	6.34	4.03
	100	18.98	16.51	15.26	13.91	9.27	6.70
$\lfloor T^{0.55} \rfloor$	108	20.44	17.86	16.50	14.96	10.25	7.57
	125	23.43	20.63	19.14	17.42	12.33	9.44
	150	27.52	24.40	22.57	20.32	14.96	11.85
$\lfloor T^{0.60} \rfloor$	166	30.08	26.79	24.75	22.30	16.67	13.44
	175	31.49	28.10	25.90	23.29	17.60	14.27
	200	35.40	31.76	29.38	26.52	20.48	16.91
	225	39.17	35.30	32.72	29.62	23.25	19.45
	250	42.75	38.65	35.93	32.60	25.90	21.88
$\lfloor T^{0.65} \rfloor$	254	43.30	39.17	36.40	32.98	26.25	22.21
	m_{LR}	10.38 $(m=54)$	8.63 $(m=54)$	8.15 $(m=54)$	7.99 $(m=54)$	4.08 $(m=54)$	2.04^a $(m=54)$

	m	$t_m(0)$ $k=6$	$t_m(0;\widehat{\tau})$ $k=7$	$t_m(0;\widehat{\boldsymbol{\tau}})$ $k=8$	$t_m(0;\widehat{\boldsymbol{\tau}})$ $k=9^{BIC}$	$t_m(0;\widehat{\boldsymbol{\tau}})$ $k=10^{HQ}$
$\lfloor T^{0.50} \rfloor$	70	3.21	3.15	2.48	2.33	2.63
	75	3.86	3.79	3.09	2.91	3.23
	100	6.51	6.41	5.64	5.46	5.81
$\lfloor T^{0.55} \rfloor$	108	7.38	7.28	6.46	6.26	6.63
	125	9.23	9.12	8.29	8.08	8.44
	150	11.62	11.50	10.54	10.34	10.74
$\lfloor T^{0.60} \rfloor$	166	13.19	13.07	12.05	11.82	12.25
	175	14.02	13.90	12.86	12.60	13.03
	200	16.64	16.52	15.42	15.13	15.58
	225	19.17	19.03	17.87	17.54	18.03
	250	21.59	21.44	20.21	19.85	20.38
$\lfloor T^{0.65} \rfloor$	254	21.92	21.77	20.53	20.17	20.70
	m_{LR}	1.89^a $(m=54)$	1.86^a $(m=54)$	1.25^c $(m=54)$	1.11^c $(m=54)$	1.40^b $(m=54)$

Note: All statistics in this table are significant at the 1% level, excepting those with a superscript a which are significant at the 5% level, those with a superscript b which are significant at the 10% level, and those with a superscript c which are not significant at the 10% level. Superscripts ^{HQ} and ^{BIC} indicate the number of breaks chosen by the HQIC and BIC, respectively.

against the long memory alternative $H_1 : \delta > 0$ in the VIX data. We report the outcomes of the $t_m(0)$ statistic, the $t_m(0;\widehat{\tau})$ statistic which allows for the presence of up to one level break, and the $t_m(0;\widehat{\boldsymbol{\tau}})$ statistic which allows for up k level breaks and do this for each of $k = 2, \ldots, 10$. We again computed these statistics for a range of values of the bandwidth parameter, m, between $\lfloor T^{0.5} \rfloor = 70$ and $\lfloor T^{0.65} \rfloor = 254$, inclusive, together with the automatic bandwidth rule, m_{LR}. The results are summarised in Table 2. Following

Andersen *et al.* (2001), we also conducted the analysis using logarithmically transformed VIX data, and the results were nearly identical to those reported in Table 2.

It is seen from the results in Table 2 that the short memory null hypothesis is easily rejected at the 1% significance level for all of the bandwidths considered, other than m_{LR}, regardless of how many level breaks we fit to the data. The tests based on m_{LR} provide weaker evidence of long memory in the VIX data where 5 or more levels breaks are fitted; for example the HQIC selects 10 breaks and here the $t_m(0;\widehat{\boldsymbol{\tau}})$ test is only able to reject at the 10% level when using m_{LR}. In conclusion, though, the results of these tests strongly suggest that long memory is a feature of the VIX data, and that this would not appear to be spurious long memory due to unmodelled level breaks.

Table 3 repeats the analysis of Table 2, but testing null hypothesis $H_0 : \delta = 0.4$ against the two-sided alternative $H_1 : \delta \neq 0.4$. The value $\delta = 0.4$ is very commonly found to characterize volatility data in empirical work, e.g. Andersen *et al.* (2001) and Christensen and Nielsen (2007), and thus seems like a natural null hypothesis. For bandwidths $m \geq 125$, including m_{LR}, the null hypothesis is rejected at the 1% level regardless of the number of breaks allowed for. However, for $m \leq 108$ the evidence against the null hypothesis becomes weaker. Using the number of breaks selected by either HQIC or BIC, the null cannot be rejected at the 10% level for any $m \leq 100$, but can be rejected for larger m. On balance, unless a relatively small bandwidth is used, we conclude that the VIX is more persistent than an $I(0.4)$ series (because rejection is in the right tail). The latter finding is in line with some recent empirical work; e.g. Frederiksen, Nielsen and Nielsen (2012).

6.3 U.S. CPI Inflation

We next consider U.S. CPI inflation, defined as the first differences of the logarithm of the price index. Specifically, we used the series CPIAUCSL from the FRED database, which is the CPI for all items, Urban consumers, seasonally adjusted, base year 1984. We used monthly observations spanning January 1970 to December 2019, for $T = 599$ observations on the first differences. The log-CPI data along with the inflation data, the latter multiplied by 1200 to return a measure that is compatible with the commonly reported inflation rate, are both plotted in Figure 5. U.S. inflation is widely argued to have gone through several different policy regimes over the sample period considered here, most notably the Great Inflation period of the 1970s, the subsequent Volcker-Greenspan era of inflation rate targeting by the U.S. Federal Reserve starting in the early 1980s, and the response to the financial crisis of 2008. Figure 5 is indeed suggestive of the possibility of several level breaks in the inflation data. The red step line on the graphs again shows the fitted deterministic trend/level of the series allowing for up to four breaks. The estimated break dates are August 1977, July 1982, January 1991, and July 2008, broadly consistent with the regimes discussed above.

We again test the short memory null hypothesis, $H_0 : \delta = 0$, against the alternative

Table 3: Tests of $H_0 : \delta = 0.4$ versus $H_1 : \delta \neq 0.4$ in VIX volatility data

	m	$t_m(0)$ $k=0$	$t_m(0;\widehat{\tau})$ $k=1$	$t_m(0;\widehat{\boldsymbol{\tau}})$ $k=2$	$t_m(0;\widehat{\boldsymbol{\tau}})$ $k=3$	$t_m(0;\widehat{\boldsymbol{\tau}})$ $k=4$	$t_m(0;\widehat{\boldsymbol{\tau}})$ $k=5$
$\lfloor T^{0.50} \rfloor$	70	4.34	3.49	1.94^a	0.35^c	0.25^c	-0.53^c
	75	4.92	4.04	2.47^a	0.73^c	0.71^c	-0.05^c
	100	7.08	6.14	4.13	1.66^b	2.00^a	1.37^c
$\lfloor T^{0.55} \rfloor$	108	7.90	6.88	4.72	1.99^a	2.50^a	1.86^b
	125	9.47	8.39	6.08	3.20	3.63	2.94
	150	11.07	9.87	7.05	3.64	4.37	3.79
$\lfloor T^{0.60} \rfloor$	166	12.26	10.99	7.84	4.32	5.01	4.52
	175	12.87	11.57	8.10	4.45	5.30	4.79
	200	14.98	13.60	9.93	6.04	6.99	6.44
	225	16.96	15.52	11.58	7.51	8.53	7.93
	250	18.44	16.94	13.03	8.79	9.84	9.18
$\lfloor T^{0.65} \rfloor$	254	18.60	17.09	13.13	8.74	9.82	9.21
	m_{LR}	11.54 ($m=157$)	10.63 ($m=160$)	8.06 ($m=174$)	4.99 ($m=183$)	5.85 ($m=183$)	5.47 ($m=185$)

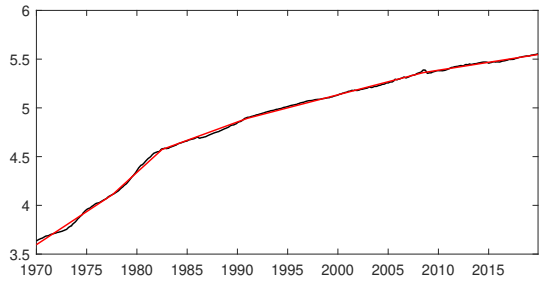
	m	$t_m(0)$ $k=6$	$t_m(0;\widehat{\tau})$ $k=7$	$t_m(0;\widehat{\boldsymbol{\tau}})$ $k=8$	$t_m(0;\widehat{\boldsymbol{\tau}})$ $k=9^{BIC}$	$t_m(0;\widehat{\boldsymbol{\tau}})$ $k=10^{HQ}$
$\lfloor T^{0.50} \rfloor$	70	-0.57^c	-0.65^c	-0.78^c	-0.72^c	-0.67^c
	75	-0.09^c	-0.18^c	-0.32^c	-0.29^c	-0.23^c
	100	1.34^c	1.21^c	1.10^c	1.15^c	1.21^c
$\lfloor T^{0.55} \rfloor$	108	1.83^b	1.71^b	1.55^c	1.59^c	1.65^b
	125	2.91	2.79	2.72	2.76	2.75
	150	3.75	3.64	3.45	3.55	3.56
$\lfloor T^{0.60} \rfloor$	166	4.48	4.37	4.16	4.19	4.22
	175	4.76	4.66	4.46	4.43	4.45
	200	6.41	6.32	6.13	6.07	6.08
	225	7.90	7.78	7.62	7.49	7.50
	250	9.15	9.03	8.83	8.68	8.76
$\lfloor T^{0.65} \rfloor$	254	9.17	9.06	8.87	8.72	8.77
	m_{LR}	5.44 ($m=185$)	5.40 ($m=186$)	5.14 ($m=185$)	5.22 ($m=187$)	5.33 ($m=188$)

Note: All statistics in this table are significant at the 1% level, excepting those with a superscript a which are significant at the 5% level, those with a superscript b which are significant at the 10% level, and those with a superscript c which are not significant at the 10% level. Superscripts ^{HQ} and ^{BIC} indicate the number of breaks chosen by the HQIC and BIC, respectively.

of (positive) long memory in the U.S. inflation data. We consider both the test based on the $t_m(0)$ statistic of Lobato and Robinson (1998), and the corresponding tests based on the $t_m(0;\widehat{\tau})$ and $t_m(0;\widehat{\boldsymbol{\tau}})$ statistics allowing for the presence of up to $k = 1, \ldots, 4$ level breaks, in each case at unknown points in the sample. The results are reported in Table 4 again for a range of values of the bandwidth parameter, m, lying between $\lfloor T^{0.5} \rfloor = 24$ and $\lfloor T^{0.65} \rfloor = 63$, inclusive, and the data-dependent bandwidth rule, m_{LR}.

Figure 5: U.S. CPI monthly data Jan. 1970 to Dec. 12/2019

(a) Log-CPI



(b) Inflation

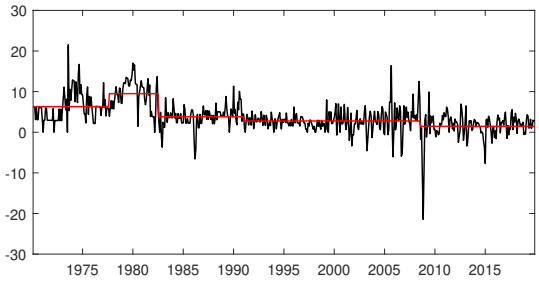


Table 4: Tests of $H_0 : \delta = 0$ versus $H_1 : \delta > 0$ in U.S. monthly CPI inflation data

	m	$t_m(0)$ $k=0$	$t_m(0;\widehat{\tau})$ $k=1$	$t_m(0;\widehat{\boldsymbol{\tau}})$ $k=2$	$t_m(0;\widehat{\boldsymbol{\tau}})$ $k=3^{BIC}$	$t_m(0;\widehat{\boldsymbol{\tau}})$ $k=4^{HQ}$	$t_m(0;\widehat{\boldsymbol{\tau}})$ $k=5$
$\lfloor T^{0.50} \rfloor$	24	6.59***	2.80***	1.56*	0.50	0.15	0.25
	30	8.05***	3.55***	2.18**	1.04	0.67	0.80
$\lfloor T^{0.55} \rfloor$	33	8.76***	4.00***	2.57***	1.41*	1.02	1.15
	40	9.96***	4.32***	2.67***	1.50*	1.08	1.22
$\lfloor T^{0.60} \rfloor$	46	10.96***	4.80***	3.08***	1.89**	1.44*	1.59*
	50	11.72***	5.22***	3.43***	2.22**	1.75**	1.90**
	60	13.04***	5.66***	3.69***	2.41***	1.93**	2.08**
$\lfloor T^{0.65} \rfloor$	63	13.40***	5.78***	3.78***	2.48***	1.98**	2.13**
	m_{LR}	7.66*** $(m=28)$	4.46*** $(m=41)$	2.99*** $(m=45)$	2.05** $(m=47)$	1.62* $(m=48)$	1.77** $(m=48)$

Note: *, ** and *** denote outcomes which are statistically significant at the 10%, 5% and 1% level, respectively, while ^{HQ} and ^{BIC} indicate the number of breaks chosen by the HQIC and BIC, respectively.

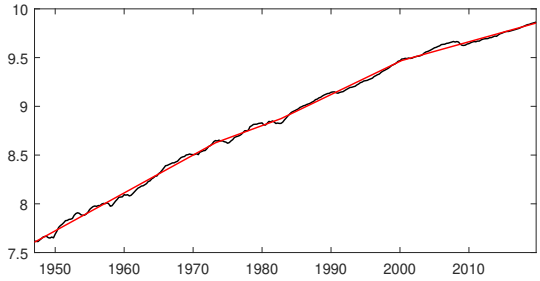
Lobato and Robinson's $t_m(0)$ test overwhelmingly rejects short memory at any conventional significance level for all of the bandwidths considered. Allowing for the presence of level breaks considerably reduces the magnitude of the test statistics. The test outcomes are generally still strongly significant when allowing for one or two level breaks, but when three level breaks are allowed for (the number chosen by BIC), the null cannot be rejected at the 5% level for bandwidths up to $m = 40$. When allowing for four level breaks (the number chosen by HQIC) only the tests based on bandwidths of $m = 50$ and $m = 63$ are significant at the 5% level. Consequently, while the standard Lobato and Robinson (1998) test presents very strong evidence in favour of long memory in the U.S. inflation rate, tests which allows for different policy regimes within the sample period are more suggestive that U.S. inflation is a short memory series.

6.4 Real U.S. GDP Growth Rate

Finally we consider U.S. GDP growth rates obtained as the first difference of the logarithm of real U.S. quarterly GDP (seasonally adjusted) over the period 1947Q1 to 2019Q4

Figure 6: U.S. GDP quarterly data 1947Q1 to 2019Q4

(a) Log-GDP



(b) Growth rate

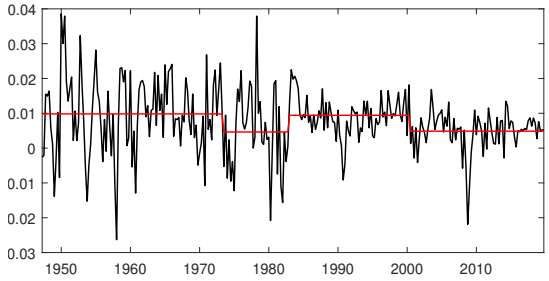


Table 5: Tests of $H_0 : \delta = 0$ versus $H_1 : \delta < 0$ in U.S. quarterly growth rates

	m	$t_m(0)$ $k = 0^{BIC}$	$t_m(0;\widehat{\tau})$ $k = 1^{HQ}$	$t_m(0;\widehat{\boldsymbol{\tau}})$ $k = 2$	$t_m(0;\widehat{\boldsymbol{\tau}})$ $k = 3$
$\lfloor T^{0.50} \rfloor$	17	−0.12	−1.07	−1.47*	−1.10
	20	0.04	−0.90	−1.32*	−0.89
$\lfloor T^{0.60} \rfloor$	22	0.02	−0.89	−1.30*	−0.89
	25	0.30	−0.63	−1.05	−0.66
$\lfloor T^{0.60} \rfloor$	30	0.19	−0.70	−1.09	−0.73
	35	−0.52	−1.33*	−1.66**	−1.43*
$\lfloor T^{0.65} \rfloor$	40	0.02	−0.82	−1.17	−0.93
	m_{LR}	0.10 ($m = 31$)	−0.83 ($m = 32$)	−1.70** ($m = 33$)	−1.47* ($m = 34$)

Note: *, ** and *** denote outcomes which are statistically significant at the 10%, 5% and 1% level, respectively, while ^{HQ} and ^{BIC} indicate the number of breaks chosen by the HQIC and BIC, respectively.

obtained from the FRED database (series GDPC1), for a total of $T = 292$ quarterly observations. The data for U.S. (log) GDP and the GDP growth rates are both graphed in Figure 6. The red line on the graphs again shows the fitted deterministic trend/level of the series allowing for up to three breaks. The estimated break dates are 1973Q2, 1982Q3 and 2000Q2, broadly consistent with the first oil crisis, changes in the Fed policy (discussed in the context of the U.S. CPI data in Section 6.3) and the end of the dot-com bubble.

In particular we will test the null hypothesis that growth rates are short memory, $H_0 : \delta = 0$, such that the log-level of GDP follows an $I(1)$ process, against the alternative of negative long memory (antipersistence) in growth rates, $H_1 : \delta < 0$, such that the log-level of GDP is less persistent than an $I(1)$ process. As in the previous examples, we consider the test of Lobato and Robinson (1998) based on the $t_m(0)$ statistic, and the corresponding tests based on the $t_m(0;\widehat{\tau})$ and $t_m(0;\widehat{\boldsymbol{\tau}})$ statistics allowing for up to $k = 1, 2, 3$ level breaks, in each case at unknown points in the sample. The results are reported in Table 5, again for a range of values of the bandwidth parameter, m, lying between $\lfloor T^{0.5} \rfloor = 17$ and $\lfloor T^{0.65} \rfloor = 40$, inclusive, and the data-dependent bandwidth rule, m_{LR}.

With only a few exceptions, the tests reported are unable to reject the null hypothesis

that GDP growth rates are short memory against $H_1 : \delta < 0$ at conventional significance levels. The results from these tests do not therefore appear to support the conjecture of Perron (1989) that U.S. GDP is $I(0)$ about a broken linear trend, particularly when recalling that our test is of the null hypothesis that U.S. GDP is $I(1)$ around a broken trend.

7 Conclusions

We have developed semiparametric tests, based on the Lagrange multiplier testing principle, for the fractional order of integration of a univariate time series which may be subject to the presence of level breaks. This is of significant practical importance as it is well known that long memory and level breaks can be mistaken for one another, with unmodelled level breaks rendering standard fractional integration tests highly unreliable. Our approach generalises the tests for the null hypothesis of weak dependence ($I(0)$) developed in Lobato and Robinson (1998). These tests are based on the local Whittle approach, and therefore do not require the user to specify a parametric model for any weak autocorrelation present in the data, which is a considerable practical advantage where the confounding effects of long memory and level breaks are present. We also show how, as conjectured in Lobato and Robinson (1998, p. 478), their testing approach can be generalised to develop tests of the null hypothesis that a series is $I(\delta)$ for any δ lying in the usual stationary and invertible region of the parameter space, not just $\delta = 0$. In spite of these generalisations, our tests are shown to attain the same standard asymptotic null distributions and asymptotic local power functions as the corresponding tests in Lobato and Robinson (1998); hence, there is no loss of asymptotic local power from allowing for level breaks, even where no level breaks are present. Monte Carlo simulations suggest that the tests perform well and that the predictions from the asymptotic theory appear to hold reasonably well in finite samples. The practical relevance of our proposed tests was highlighted with a number of empirical examples relating to macroeconomics and finance.

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Appendix A Mathematical Proofs

In this appendix we provide proofs of Theorems 1 and 2.

A.1 Proof of Theorem 1

We use the notation $\delta_c := cm^{-1/2}$, so that, under H_c, we have $\delta = \delta_0 + \delta_c$.

Consider first the proof under Assumption 1. We re-write $t_m^*(\delta_0)$ in (4) as

$$t_m^*(\delta_0) = \frac{-m^{-1/2}G^{-1}\sum_{j=1}^m \nu_j \lambda_j^{2\delta} j^{-2\delta_c} I_u(\lambda_j)}{m^{-1}G^{-1}\sum_{j=1}^m \lambda_j^{2\delta} j^{-2\delta_c} I_u(\lambda_j)}. \tag{10}$$

The numerator of $t_m^*(\delta_0)$ in (10) is

$$\begin{aligned} -m^{-1/2}\sum_{j=1}^m \nu_j G^{-1}\lambda_j^{-2\delta_c}\lambda_j^{2\delta} I_u(\lambda_j) = & -m^{-1/2}\sum_{j=1}^m \nu_j j^{-2\delta_c}\left(G^{-1}\lambda_j^{2\delta} I_u(\lambda_j) - 2\pi I_\varepsilon(\lambda_j)\right) && (11)\\ & -m^{-1/2}\sum_{j=1}^m \nu_j 2\pi I_\varepsilon(\lambda_j) && (12)\\ & -m^{-1/2}\sum_{j=1}^m \nu_j (j^{-2\delta_c}-1) 2\pi I_\varepsilon(\lambda_j). && (13) \end{aligned}$$

Letting $I_\varepsilon(\lambda_j)$ denote the periodogram of ε_t, (4.8) of Robinson (1995b) shows that, for $r \leq m$,

$$\sum_{j=1}^r \left(\frac{I_u(\lambda_j)}{G\lambda_j^{-2\delta}} - 2\pi I_\varepsilon(\lambda_j)\right) = O_p\left(r^{1/3}(\ln(r))^{2/3} + r^3 T^{-2} + r^{1/2}T^{-1/4}\right). \tag{14}$$

Then, letting $b_j := \nu_j j^{-2\delta_c}$ and proceeding as in Robinson (1995b) it follows that the remainder term (11) is $o_p(1)$. This involves using summation by parts, (14), and the bound $|b_j - b_{j+1}| = O(j^{-1})$, which follows by elementary calculations. From (4.11) of Robinson (1995b) it follows directly that (12) converges in distribution to $N(0,1)$.

Next, by a Taylor series expansion and by definition of δ_c, $j^{-2\delta_c} = 1 - 2cm^{-1/2}(\ln j) + 2c^2 m^{-1}(\ln j)^2 j^{-2\delta_{mvt}}$ for $|\delta_{mvt}| \leq |\delta_c|$, so that (13) is

$$2cm^{-1}\sum_{j=1}^m \nu_j (\ln j) 2\pi I_\varepsilon(\lambda_j) + 2c^2 m^{-3/2}\sum_{j=1}^m \nu_j (\ln j)^2 j^{-2\delta_{mvt}} 2\pi I_\varepsilon(\lambda_j). \tag{15}$$

Writing $\ln j = \nu_j + m^{-1}\sum_{k=1}^m \ln k$, the first term of (15) is

$$2cm^{-1}\sum_{j=1}^m \nu_j^2 2\pi I_\varepsilon(\lambda_j) + 2cm^{-1}\left(m^{-1}\sum_{k=1}^m \ln k\right)\sum_{j=1}^m \nu_j 2\pi I_\varepsilon(\lambda_j).$$

Noting that $2cm^{-1}\sum_{j=1}^m \nu_j^2 2\pi E(I_\varepsilon(\lambda_j)) = 2cm^{-1}\sum_{j=1}^m \nu_j^2 \to 2c$, the first term converges in probability to $2c$ by a law of large numbers. Using the result for (12) and the fact that $m^{-1}\sum_{k=1}^m \ln k = O(\ln m)$, the second term is $O_p(m^{-1/2}\ln m) = o_p(1)$. Next, the expectation of the absolute value of the second term of (15) is

$$O\left(m^{-3/2}\sum_{j=1}^m \nu_j (\ln j)^2 j^{2|\delta_c|}\right) = O\left(m^{-1/2}(\ln m)^3 m^{2|\delta_c|}\right) = o(1),$$

where the last equality follows because $m^{1/2} \geq \ln m$, which implies $m^{2|\delta_c|} \leq m^{2|c|/\ln m} =$

$e^{2|c|}$. This shows that the second term of (15) converges to zero in L_1-norm and hence in probability.

The denominator of $t_m^*(\delta_0)$ in (10) may be analyzed in the same way to establish the result that $m^{-1}\sum_{j=1}^m \lambda_j^{2\delta} j^{-2\delta_c} 2\pi I_u(\lambda_j) \to_p G$. The claim of Theorem 1 under Assumption 1 follows by combining these results.

Next, we prove the theorem under Assumption 2. Instead of the bound (14) from (4.8) of Robinson (1995b), we let $\alpha_T(\lambda) := (1-e^{i\lambda})^{-(\delta_0+\delta_c)}$ and use Lemma 4 of Shao and Wu (2007a), where it is shown that, under Assumption 2,

$$\sum_{j=1}^{r}\left(\frac{I_u(\lambda_j)}{|\alpha_T(\lambda_j)|^2 f(\lambda_j)} - \frac{I_\eta(\lambda_j)}{f(\lambda_j)}\right) = O_p\left(r^{1/4}(\ln r)^{1/2} + r^{1/2}T^{-1/4}\right). \tag{16}$$

Denoting $\alpha(\lambda_j) := (1-e^{i\lambda})^{-\delta_0}$ and $\alpha_c(\lambda_j) := (1-e^{i\lambda})^{-\delta_c}$, so that $\alpha_T(\lambda_j) = \alpha(\lambda_j)\alpha_c(\lambda_j)$, the (scaled negative) numerator of $t_m^*(\delta_0)$ in (4) is

$$\begin{aligned} G^{-1}\sum_{j=1}^m \nu_j \lambda_j^{2\delta_0} I_u(\lambda_j) &= \sum_{j=1}^m \nu_j G^{-1}\lambda_j^{-2\delta_c}\lambda_j^{2\delta} I_u(\lambda_j) \\ &= \sum_{j=1}^m \nu_j G^{-1}\lambda_j^{-2\delta_c}|\alpha_c(\lambda_j)|^{-2} f(\lambda_j)^{-1}|\alpha_c(\lambda_j)|^2 f(\lambda_j)|\alpha_T(\lambda_j)|^2|\alpha_T(\lambda_j)|^{-2}\lambda_j^{2\delta} I_u(\lambda_j) \\ &= \sum_{j=1}^m \nu_j |\alpha_c(\lambda_j)|^2 |\alpha_T(\lambda_j)|^{-2} f(\lambda_j)^{-1} I_u(\lambda_j) + o_p(m^{1/2}), \end{aligned} \tag{17}$$

where the last equality follows by using bounds for the low-frequency approximation of the ratio of $f(\lambda_j)$ to G, see Assumption 2(iii), and of $|\alpha_T(\lambda_j)|^2$ to $\lambda_j^{-2\delta}$ as in Robinson (1995b).

For the leading term in (17), we let $b_j := \nu_j|\alpha_c(\lambda_j)|^2$ (with slight abuse of notation), and re-write it as

$$\sum_{j=1}^m b_j\left(|\alpha_T(\lambda_j)|^{-2} f(\lambda_j)^{-1} I_u(\lambda_j) - f(\lambda_j)^{-1} I_\eta(\lambda_j)\right) \tag{18}$$

$$+ \sum_{j=1}^m b_j f(\lambda_j)^{-1} I_\eta(\lambda_j). \tag{19}$$

As in the analysis of (11) it holds that (18) is $o_p(1)$ using (16). The term (19) is asymptotically normal as shown in Shao and Wu (2007a). As in the previous case, the same arguments also give $m^{-1}\sum_{j=1}^m \lambda_j^{2\delta} 2\pi I_u(\lambda_j) \to_p G$, and the claim of Theorem 1 under Assumption 2 follows combining these results.

A.2 Proof of Theorem 2

Let $W(\tau; d)$ denote a type I fractional Brownian motion; that is, with $E(W(s;d)^2) = s^{2(d-1/2)}$, and $W(s;0)$ a standard Brownian motion. Also let $\kappa(d) := \frac{A(d)}{\Gamma(d+1)}$, where $A(d) := \left(\frac{1}{1+2\delta} + \int_0^\infty ((1+s)^d - s^d)^2 ds\right)^{1/2}$. Then we have the following result.

Lemma 1. *Under H_c and either Assumption 3 or Assumption 4, $T^{-(1/2+\delta_0)}\sum_{t=1}^{\lfloor \tau T\rfloor} u_t \Rightarrow \kappa(\delta_0)(2\pi f(0))^{1/2}W(\tau;\delta_0)$ as a càdlàg process indexed by $\tau \in \Lambda$.*

Proof of Lemma 1. Given that $\delta_0 < 1/2$, for m large enough there is $\overline{\delta} = 1/2 - \epsilon$ such that $\delta_0 < \overline{\delta}$ and $\delta < \overline{\delta}$. Using a mean value theorem expansion,

$$u_t = \Delta^{-\delta}\eta_t = \Delta^{-\delta_0}\eta_t + (cm^{-1/2})(-\ln\Delta)\Delta^{-\delta_0}\eta_t + 1/2(cm^{-1/2})^2(-\ln\Delta)^2\Delta^{-\delta_0}\eta_t + \ldots + 1/(k!)(cm^{-1/2})^k(-\ln\Delta)^k\Delta^{-\delta_{mvt}}\eta_t, \quad (20)$$

where $|\delta_{mvt} - \delta_0| < |\delta - \delta_0|$ and k is an integer to be chosen.

From the fractional FCLT, the first term on the right-hand side of (20) satisfies

$$T^{-(1/2+\delta_0)}\sum_{t=1}^{\lfloor \tau T\rfloor}\Delta^{-\delta_0}\eta_t \Rightarrow \kappa(\delta_0)(2\pi f(0))^{1/2}W(\tau;\delta_0)$$

in the Skorohod metric; see, e.g., Hosoya (2005) and Wu and Shao (2006). Moreover, because the jumps in the partial sums take place at fixed points in time, and the limit $W(\tau;\delta)$ is a.s. continuous, the weak convergence also takes place in the uniform metric.

By the same argument, including a slowly varying function, it follows that

$$\frac{1}{\ln(T)}T^{-(1/2+\delta_0)}\sum_{t=1}^{\lfloor \tau T\rfloor}(-\ln\Delta)\Delta^{-\delta_0}\eta_t = O_p(1),$$

and hence $cm^{-1/2}T^{-(1/2+\delta_0)}\sum_{t=1}^{\lfloor \tau T\rfloor}(-\ln\Delta)\Delta^{-\delta_0}\eta_t = o_p(1)$; in both cases uniformly in τ. The $k-2$ remaining terms in the expansion of $\Delta^{-\delta}\eta_t$ in (20) can be analyzed the same way.

For the last term on the right-hand side of (20), notice that

$$\begin{aligned} E((-\ln\Delta)^k\Delta^{-\delta_{mvt}}\eta_t)^2 &= \int_{-\pi}^{\pi}|1-e^{i\lambda}|^{-2\delta_{mvt}}(\ln|1-e^{i\lambda}|)^{2k}f(\lambda)d\lambda \\ &\leq \int_{-\pi}^{\pi}|1-e^{i\lambda}|^{-2\overline{\delta}}(\ln|1-e^{i\lambda}|)^{2k}f(\lambda)d\lambda < \infty, \end{aligned}$$

where we recall that $f(\lambda)$ is the spectral density of η_t, which is bounded, uniformly in λ, under either Assumption 3 or Assumption 4. Then, by the Cauchy-Schwarz inequality,

$$\sum_{t=1}^{\lfloor \tau T\rfloor}(-\ln\Delta)^k\Delta^{-\delta_{mvt}}\eta_t \leq \left(\sum_{t=1}^{T}((-\ln\Delta)^k\Delta^{-\delta_{mvt}}\eta_t)^2\right)^{1/2}T^{1/2} = O_p(T),$$

and note that this is uniform in τ. So, upon choosing k finite but sufficiently large, $T^{-(1/2+\delta_0)}m^{-k/2}T \to 0$ by Assumption 3(iv) or Assumption 4(iv), and consequently

$$T^{-(1/2+\delta_0)}\frac{(cm^{-1/2})^k}{k!}\sum_{t=1}^{\lfloor \tau T\rfloor}(\ln\Delta)^k\Delta^{-\delta_{mvt}}\eta_t = o_p(1).$$

Combining these arguments we obtain the desired result. □

An important consequence of Lemma 1 is that Assumption H1(ϕ) of Lavielle and Moulines (2000) holds with $\phi = 1 + 2\delta_0 < 2$ under the conditions of Lemma 1. In what follows, results for stochastic functionals of τ are to be considered as uniform in τ, unless otherwise specified. We omit the the reference to uniformity in τ for brevity.

We divide the remainder of the proof into three parts for readability. Recall that k^* is the true number of breaks, i.e. the number of non-zero elements of $\boldsymbol{\beta}_2$.

A.2.1 Proof of Theorem 2 When $k > k^* = 0$

In this case $\beta_{2,i} = 0$ for all $i = 1, \ldots, k$. To lighten the notation, we give the proof for the case with $k = 1$ and $k^* = 0$; see also the notation in Remark 4.3. The proof for the general case is the same, but with vectors and matrices replacing scalar quantities. Thus, we prove that, when $\beta_2 = 0$, $t_m(\delta_0; \tau) - t_m^*(\delta_0) = o_p(1)$ uniformly in τ. It is sufficient to show that

$$m^{-1/2} \sum_{j=1}^{m} \nu_j \lambda_j^{2\delta_0} I_{\widehat{u}(\tau)}(\lambda_j) - m^{-1/2} \sum_{j=1}^{m} \nu_j \lambda_j^{2\delta_0} I_u(\lambda_j) = o_p(1), \tag{21}$$

$$m^{-1} \sum_{j=1}^{m} \lambda_j^{2\delta_0} I_{\widehat{u}(\tau)}(\lambda_j) - m^{-1} \sum_{j=1}^{m} \lambda_j^{2\delta_0} I_u(\lambda_j) = o_p(1). \tag{22}$$

We give only the proof of (21). The proof of (22) is almost identical leaving out the factor ν_j and noting the different normalization.

We first note that

$$\begin{aligned} w_{\widehat{u}(\tau)}(\lambda_j) &= (\beta - \widehat{\beta}(\tau))' w_{x(\tau)}(\lambda_j) + w_u(\lambda_j) = -\widehat{\beta}_2(\tau) w_{DU(\tau)}(\lambda_j) + w_u(\lambda_j), \\ I_{\widehat{u}(\tau)}(\lambda_j) &= \widehat{\beta}_2(\tau)^2 I_{DU(\tau)}(\lambda_j) + I_u(\lambda_j) - 2\,\mathrm{Re}\left(w_{DU(\tau)}(\lambda_j) w_u(-\lambda_j)\right) \widehat{\beta}_2(\tau). \end{aligned}$$

The absolute value of the left-hand side of (21) is then

$$\begin{aligned} &\left| m^{-1/2} \sum_{j=1}^{m} \nu_j \lambda_j^{2\delta_0} I_{DU(\tau)}(\lambda_j) \widehat{\beta}_2(\tau)^2 - 2m^{-1/2} \sum_{j=1}^{m} \nu_j \lambda_j^{2\delta_0}\, \mathrm{Re}\left(w_{DU(\tau)}(\lambda_j) w_u(-\lambda_j)\right) \widehat{\beta}_2(\tau) \right| \\ &\quad \le m^{-1/2} \sum_{j=1}^{m} |\nu_j| \lambda_j^{2\delta_0} I_{DU(\tau)}(\lambda_j) \widehat{\beta}_2(\tau)^2 \qquad (23) \\ &\quad + C m^{-1/2} \sum_{j=1}^{m} |\nu_j| \lambda_j^{2\delta_0} \left|\mathrm{Re}\left(w_{DU(\tau)}(\lambda_j) w_u(-\lambda_j)\right)\right| |\widehat{\beta}_2(\tau)|. \qquad (24) \end{aligned}$$

From Iacone (2010) we have the bound

$$I_{DU(\tau)}(\lambda_j) \le C (j/T)^{-1} j^{-1}, \tag{25}$$

and the bound $\widehat{\beta}_2(\tau) = O_p(T^{\delta_0 - 1/2})$ follows from an application of Lemma 1 to the regres-

sion estimate. Applying also the simple bound $|\nu_j| = O(\ln m)$, the term (23) is

$$m^{-1/2}\sum_{j=1}^{m}|\nu_j|\lambda_j^{2\delta_0}I_{DU(\tau)}(\lambda_j)\widehat{\beta}_2(\tau)^2 = O_p\left((\ln m)m^{-1/2}\sum_{j=1}^{m}\left(\frac{j}{T}\right)^{2\delta_0}\left(\frac{j}{T}\right)^{-1}j^{-1}T^{2\delta_0-1}\right)$$
$$= O_p\left((\ln m)m^{-1/2}\sum_{j=1}^{m}j^{2\delta_0-2}\right) = O_p\left((\ln m)m^{-1/2}\right) = o_p(1).$$

For the term (24), first note that $E\left|\lambda_j^{2\delta}I_u(\lambda_j)\right| < C$; see Lemma 3 of Shao and Wu (2007a), the proof of which also applies under Assumption 3 or 4. Thus, (24) is bounded by

$$Cm^{-1/2}\sum_{j=1}^{m}|\nu_j|\lambda_j^{2\delta_0}\left|\mathrm{Re}\left(w_{DU(\tau)}(\lambda_j)w_u(-\lambda_j)\right)\right||\widehat{\beta}_2(\tau)|$$
$$= O_p\left((\ln m)m^{-1/2}\sum_{j=1}^{m}\left(\frac{j}{T}\right)^{2\delta_0}\left(\frac{j}{T}\right)^{-1/2}j^{-1/2}\left(\frac{j}{T}\right)^{-\delta}T^{\delta_0-1/2}\right)$$
$$= O_p\left((\ln m)m^{-1/2}T^{\delta-\delta_0}\sum_{j=1}^{m}j^{(\delta_0-1)+(\delta_0-\delta)}\right).$$

Note that, by mean value expansion, $T^{\delta-\delta_0} = 1 + cm^{(-1/2)}(\ln T)T^{\delta_{mvt}-\delta_0}$, where $|\delta_{mvt} - \delta_0| \leq |\delta - \delta_0|$. Thus, for T large enough, $|T^{\delta_0-\delta} - 1| < cm^{(-1/2)}(\ln T)T^{\epsilon}$ for any $\epsilon > 0$ arbitrarily small. Thus, $T^{\delta_0-\delta} \to 1 = O(1)$. The term $j^{\delta_0-\delta}$ can be discussed in the same way. Therefore, the stochastic order of (24) is

$$O_p\left((\ln m)m^{-1/2}\sum_{j=1}^{m}j^{\delta_0-1}\right) = O_p\left((\ln m)^2m^{-1/2}m^{\max(\delta_0,0)}\right), \tag{26}$$

which is $o_p(1)$ recalling that $\delta_0 < 1/2$.

A.2.2 Proof of Theorem 2 When $k = k^* > 0$

In this case $\beta_{2,i} \neq 0$ for all $i = 1, \ldots, k$. Again, we give the proof for the case with $k = k^* = 1$ to lighten the notation, with the general proof being nearly identical. Thus, we prove that, when $\beta_2 \neq 0$, $t_m(\delta_0; \widehat{\tau}) - t_m^*(\delta_0) = o_p(1)$. Because $\beta_2 \neq 0$, we have to account for the difference between $\widehat{\tau}$ and τ^*. From Theorem 7 of Lavielle and Moulines (2000) it holds that, for any $|\delta| < 1/2$,

$$\widehat{\tau} - \tau^* = O_p(T^{-1}). \tag{27}$$

Proceeding as Bai (1994) and Iacone *et al.* (2019), we also establish that $\widehat{\beta}_2(\widehat{\tau}) - \beta_2 = O_p(T^{\delta_0-1/2})$. Notice here that the key step of the proof in Iacone *et al.* (2019) exploits the rate in (27), the Hájek-Rényi-type inequality in (8) of Lavielle and Moulines (2000), and the fractional FCLT.

We now re-write

$$\begin{aligned}\widehat{u}_t(\widehat{\tau}) &= \beta_1 + \beta_2 DU_t(\tau^*) + u_t - \widehat{\beta}_1(\widehat{\tau}) - \widehat{\beta}_2(\widehat{\tau})DU_t(\widehat{\tau}) \\ &= \beta_1 + \beta_2 DU_t(\tau^*) - \widehat{\beta}_2(\widehat{\tau})DU_t(\tau^*) + \widehat{\beta}_2(\widehat{\tau})DU_t(\tau^*) + u_t - \widehat{\beta}_1(\widehat{\tau}) - \widehat{\beta}_2(\widehat{\tau})DU_t(\widehat{\tau}) \\ &= (\beta_1 - \widehat{\beta}_1(\widehat{\tau})) + (\beta_2 - \widehat{\beta}_2(\widehat{\tau}))DU_t(\tau^*) + u_t + \widehat{\beta}_2(\widehat{\tau})(DU_t(\tau^*) - DU_t(\widehat{\tau})),\end{aligned}$$

and thus, for $j \neq 0$,

$$\begin{aligned}I_{\widehat{u}(\widehat{\tau})}(\lambda_j) = I_u(\lambda_j) &+ (\beta_2 - \widehat{\beta}_2(\widehat{\tau}))^2 I_{DU(\tau^*)}(\lambda_j) + \widehat{\beta}_2(\widehat{\tau})^2 I_{(DU(\tau^*)-DU(\widehat{\tau}))}(\lambda_j) \\ &+ 2\,\mathrm{Re}\left(w_{DU(\tau^*)}(\lambda_j) w_u(-\lambda_j)\right)(\beta_2 - \widehat{\beta}_2(\widehat{\tau})) \\ &+ 2(\beta_2 - \widehat{\beta}_2(\widehat{\tau}))\,\mathrm{Re}\left(w_{DU(\tau^*)}(\lambda_j) w_{DU_t(\tau^*)-DU_t(\widehat{\tau})}(-\lambda_j)\right)\widehat{\beta}_2(\widehat{\tau}) \\ &+ 2\,\mathrm{Re}\left(w_u(\lambda_j) w_{DU_t(\tau^*)-DU_t(\widehat{\tau})}(-\lambda_j)\right)\widehat{\beta}_2(\widehat{\tau}).\end{aligned}$$

Proceeding as in Appendix A.2.1, we need to show that

$$m^{-1/2}(\ln m)\sum_{j=1}^{m}\lambda_j^{2\delta_0}\left|(\beta_2 - \widehat{\beta}_2(\widehat{\tau}))^2 I_{DU(\tau^*)}(\lambda_j)\right| = o_p(1), \tag{28}$$

$$m^{-1/2}(\ln m)\sum_{j=1}^{m}\lambda_j^{2\delta_0}\left|2\,\mathrm{Re}\left(w_{DU(\tau^*)}(\lambda_j) w_u(-\lambda_j)\right)(\beta_2 - \widehat{\beta}_2(\widehat{\tau}))\right| = o_p(1), \tag{29}$$

and

$$m^{-1/2}(\ln m)\left|\sum_{j=1}^{m}\widehat{\beta}_2(\widehat{\tau})^2\lambda_j^{2\delta_0} I_{(DU(\tau^*)-DU(\widehat{\tau}))}(\lambda_j)\right| = o_p(1), \tag{30}$$

$$m^{-1/2}(\ln m)\sum_{j=1}^{m}|\beta_2 - \widehat{\beta}_2(\widehat{\tau})|\lambda_j^{2\delta_0}\left|\mathrm{Re}\left(w_{DU(\tau^*)}(\lambda_j) w_{DU(\tau^*)-DU(\widehat{\tau})}(-\lambda_j)\right)\right||\widehat{\beta}_2(\widehat{\tau})| = o_p(1), \tag{31}$$

$$m^{-1/2}(\ln m)\sum_{j=1}^{m}\lambda_j^{2\delta_0}\left|\mathrm{Re}\left(w_u(\lambda_j) w_{DU(\tau^*)-DU(\widehat{\tau})}(-\lambda_j)\right)\right||\widehat{\beta}_2(\widehat{\tau})| = o_p(1). \tag{32}$$

The bounds in (28) and (29) follow as in the discussion of (23) and (24), respectively.

Next, assuming without loss of generality that $\widehat{\tau} < \tau^*$,

$$\begin{aligned}w_{(DU(\tau^*)-DU(\widehat{\tau}))}(\lambda) &= \frac{1}{\sqrt{2\pi T}}\sum_{t=\lfloor\widehat{\tau}T\rfloor}^{\lfloor\tau^*T\rfloor+1} e^{i\lambda}, \\ \left|w_{(DU(\tau^*)-DU(\widehat{\tau}))}(\lambda)\right| &\leq \frac{1}{\sqrt{2\pi T}}(\lfloor\tau^*T\rfloor - \lfloor\widehat{\tau}T\rfloor) = O_p(T^{-1/2}),\end{aligned}$$

and hence $I_{DU(\tau^*)-DU(\widehat{\tau})}(\lambda_j) = O_p(T^{-1})$. Thus, since $\widehat{\beta}_2(\widehat{\tau}) \to_p \beta_2$, (30) is of order

$$O_p\left(m^{-1/2}(\ln m)\sum_{j=1}^{m} j^{2\delta_0}T^{-2\delta_0}T^{-1}\right) = O_p\left(m^{-1/2}(\ln m)T^{-2\delta_0-1}m^{2\delta_0+1}\right) = o_p(1).$$

The term (31) is, using (25),

$$O_p\left(m^{-1/2}(\ln m)T^{\delta_0-1/2}\sum_{j=1}^{m} j^{2\delta_0}T^{-2\delta_0}(T^{1/2}j^{-1}T^{-1/2})\right)$$
$$= O_p\left(m^{-1/2}(\ln m)T^{-\delta_0-1/2}\sum_{j=1}^{m} j^{2\delta_0-1}\right) = O_p\left(m^{-1/2}(\ln m)^2T^{-\delta_0-1/2}m^{\max(2\delta_0,0)}\right), \tag{33}$$

which is $o_p(1)$. Finally, (32) is of order

$$O_p\left(m^{-1/2}(\ln m)\sum_{j=1}^{m} j^{\delta_0+(\delta_0-\delta)}T^{-\delta_0-(\delta_0-\delta)}T^{-1/2}\right) = O_p\left(m^{\delta_0+1/2}(\ln m)T^{-\delta_0-1/2}\right) = o_p(1). \tag{34}$$

A.2.3 Proof of Theorem 2 When $k > k^* > 0$

In this case we partition the k-vectors $\boldsymbol{\beta}_2, \boldsymbol{\tau}, \boldsymbol{\tau}^*, \widehat{\boldsymbol{\tau}}$, and $\boldsymbol{DU}_t(\boldsymbol{\tau})$ according to whether $\beta_{2,i} = 0$ or $\beta_{2,i} \neq 0$ for $i = 1, \ldots, k$. Once more, we give the proof for the case with $k = 2$ and $k^* = 1$ to lighten the notation. We assume without loss of generality that $\beta_{2,1} \neq 0$ and $\beta_{2,2} = 0$, and partition $\boldsymbol{\beta}_2, \boldsymbol{\tau}, \boldsymbol{\tau}^*, \widehat{\boldsymbol{\tau}}$, and $\boldsymbol{DU}_t(\boldsymbol{\tau})$ accordingly.

We write

$$t_m(\delta_0; (\widehat{\tau}_1, \tau_2)') - t_m^*(\delta_0) = t_m(\delta_0; (\widehat{\tau}_1, \tau_2)') - t_m(\delta_0; (\tau_1^*, \tau_2)') \tag{35}$$
$$+ t_m(\delta_0; (\tau_1^*, \tau_2)') - t_m^*(\delta_0), \tag{36}$$

and prove that each of the two terms on the right-hand side are $o_p(1)$ uniformly in τ_2. The result for (36) follows directly from the result for $k > k^* = 0$ in Appendix A.2.1.

Thus, we prove that the right-hand side of (35) is $o_p(1)$ uniformly in τ_2, and to this end we need to take into account the estimation of τ_1 and $\boldsymbol{\beta}_2$. We have $\widehat{\boldsymbol{\beta}}_2(\boldsymbol{\tau}) = (\widehat{\beta}_{2,1}(\tau_1), \widehat{\beta}_{2,2}(\tau_2))'$. From Lemma 4.1 and (17) of Lavielle and Moulines (2000) we find that, for either $j = 1$ or $j = 2$, it holds that $\widehat{\tau}_j - \tau_1^* = O_p(T^{-1})$; see also (27). Since the labelling of the coefficients does not matter for the proof, we will assume without loss of generality that this property holds for $j = 1$. Then, as in Appendix A.2.2, $\widehat{\beta}_{2,1}(\widehat{\tau}_1) - \beta_{2,1} = O_p(T^{-1})$. Furthermore, as in Appendix A.2.1, $\widehat{\beta}_{2,2}(\tau_2) = O_p(T^{\delta_0-1/2})$ uniformly in τ_2.

With the above notation, the DGP in (1) is

$$y_t = \beta_1 + \beta_{2,1}DU_t(\tau_1^*) + u_t, \quad t = 1, \ldots, T. \tag{37}$$

Then we can write, for generic (τ_1, τ_2),

$$\widehat{u}_t(\tau_1, \tau_2) = \beta_1 + \beta_{2,1}DU_t(\tau_1^*) + u_t - \widehat{\beta}_1(\tau_1, \tau_2) - \widehat{\beta}_{2,1}(\tau_1)DU_t(\tau_1) - \widehat{\beta}_{2,2}(\tau_2)DU_t(\tau_2). \tag{38}$$

Evaluating this expression at $\tau_1 = \widehat{\tau}_1$ and $\tau_1 = \tau_1^*$, respectively, we find that

$$\begin{aligned}\widehat{u}_t(\widehat{\tau}_1, \tau_2) = \widehat{u}_t(\tau_1^*, \tau_2) + \widehat{\beta}_1(\tau_1^*, \tau_2) - \widehat{\beta}_1(\widehat{\tau}_1, \tau_2) + (\widehat{\beta}_{2,1}(\tau_1^*) - \widehat{\beta}_{2,1}(\widehat{\tau}_1))DU_t(\tau_1^*) \\ + \widehat{\beta}_{2,1}(\widehat{\tau}_1)(DU_t(\tau_1^*) - DU_t(\widehat{\tau}_1)),\end{aligned}$$

and consequently, for $j \neq 0$,

$$\begin{aligned}I_{\widehat{u}(\widehat{\tau}_1,\tau_2)}(\lambda_j) = I_{\widehat{u}(\tau_1^*,\tau_2)}(\lambda_j) + (\widehat{\beta}_{2,1}(\tau_1^*) - \widehat{\beta}_{2,1}(\widehat{\tau}_1))^2 I_{DU(\tau_1^*)}(\lambda_j) + \widehat{\beta}_{2,1}(\widehat{\tau}_1)^2 I_{(DU(\tau_1^*)-DU(\widehat{\tau}_1))}(\lambda_j) \\ + 2(\widehat{\beta}_{2,1}(\tau_1^*) - \widehat{\beta}_{2,1}(\widehat{\tau}_1)) \operatorname{Re}\left(w_{DU(\tau_1^*)}(\lambda_j) w_{\widehat{u}(\tau_1^*,\tau_2)}(-\lambda_j)\right) \\ + 2(\widehat{\beta}_{2,1}(\tau_1^*) - \widehat{\beta}_{2,1}(\widehat{\tau}_1)) \operatorname{Re}\left(w_{DU(\tau_1^*)}(\lambda_j) w_{(DU(\tau_1^*)-DU(\widehat{\tau}_1))}(-\lambda_j)\right) \widehat{\beta}_{2,1}(\widehat{\tau}_1) \\ + 2\widehat{\beta}_{2,1}(\widehat{\tau}_1) \operatorname{Re}\left(w_{(DU(\tau_1^*)-DU(\widehat{\tau}_1))}(\lambda_j) w_{\widehat{u}(\tau_1^*,\tau_2)}(-\lambda_j)\right).\end{aligned}$$

Proceeding as in Appendices A.2.1 and A.2.2, we need to show that

$$m^{-1/2}(\ln m) \sum_{j=1}^{m} \lambda_j^{2\delta_0} \left|(\widehat{\beta}_{2,1}(\tau_1^*) - \widehat{\beta}_{2,1}(\widehat{\tau}_1))^2 I_{DU(\tau_1^*)}(\lambda_j)\right| = o_p(1), \tag{39}$$

$$m^{-1/2}(\ln m) \left|\sum_{j=1}^{m} \widehat{\beta}_{2,1}(\widehat{\tau}_1)^2 \lambda_j^{2\delta_0} I_{(DU(\tau_1^*)-DU(\widehat{\tau}_1))}(\lambda_j)\right| = o_p(1), \tag{40}$$

$$m^{-1/2}(\ln m) \sum_{j=1}^{m} \lambda_j^{2\delta_0} \left|2 \operatorname{Re}\left(w_{DU(\tau_1^*)}(\lambda_j) w_{\widehat{u}(\tau_1^*,\tau_2)}(-\lambda_j)\right) (\widehat{\beta}_{2,1}(\tau_1^*) - \widehat{\beta}_{2,1}(\widehat{\tau}_1))\right| = o_p(1), \tag{41}$$

$$m^{-1/2}(\ln m) \sum_{j=1}^{m} |\widehat{\beta}_{2,1}(\tau_1^*) - \widehat{\beta}_{2,1}(\widehat{\tau}_1)| \lambda_j^{2\delta_0} \left|\operatorname{Re}\left(w_{DU(\tau_1^*)}(\lambda_j) w_{DU(\tau_1^*)-DU(\widehat{\tau}_1)}(-\lambda_j)\right)\right| |\widehat{\beta}_{2,1}(\widehat{\tau}_1)| = o_p(1), \tag{42}$$

$$m^{-1/2}(\ln m) \sum_{j=1}^{m} \lambda_j^{2\delta_0} \left|\operatorname{Re}\left(w_{\widehat{u}(\tau_1^*,\tau_2)}(\lambda_j) w_{DU(\tau_1^*)-DU(\widehat{\tau}_1)}(-\lambda_j)\right)\right| |\widehat{\beta}_{2,1}(\widehat{\tau}_1)| = o_p(1). \tag{43}$$

By the same argument as in Appendix A.2.2, we find that $\widehat{\beta}_{2,1}(\tau_1^*) - \widehat{\beta}_{2,1}(\widehat{\tau}_1) = O_p(T^{-1})$, and the proofs for (39) and (42) are then identical to those of (28) and (31), respectively. The proof for (40) is identical to that of (30).

Finally, to prove the results in (41) and (43), we use (38) to find that, for $j \neq 0$,

$$w_{\widehat{u}(\tau_1^*,\tau_2)}(\lambda_j) = w_u(\lambda_j) + (\beta_{2,1} - \widehat{\beta}_{2,1}(\tau_1^*)) w_{DU(\tau_1^*)} - \widehat{\beta}_{2,2}(\tau_2) w_{DU(\tau_2)}.$$

Using this expression, the required result for (41) follows by identical arguments to those for (23), (28), and (29). Similarly, the result for (43) follows by identical arguments to those for (23), (31), and (32).

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