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Review of Incurvati *Conceptions of Set and the Foundations of Mathematics*

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The philosophy of set theory examines the ontological and epistemological implications of the mathematical concept of set, often with a particular focus on how set theory could provide a foundation for (the rest of) mathematics. With this focus in mind, Luca Incurvati's *Conceptions of Set and the Foundations of Mathematics* explores different approaches to the concept of set, comparing and evaluating these approaches in terms of how they might provide such a foundation. Along the way, it offers an excellent, self-contained introduction to a wide range of important topics in the philosophy of set theory, while at the same time proposing a novel framework within which one can evaluate different conceptions of set.

Incurvati's approach focuses on the idea of a *conception*, as distinguished from a *concept*. He begins by describing his understanding of what a conception is. Conceptions differ from concepts in that the former provide a sharpening, or perhaps a precisification, of the latter. Focusing on the concept of set, articulating this concept will settle some questions about its extension, i.e., the range of objects that the concept applies to. Incurvati argues that the concept of set has three core features: (1) a set is a unity, such that it is a single object, over and above its members; (2) a set has a unique decomposition into its members; and (3) a set is extensional. The third condition provides a criterion of identity for sets, such that two sets are identical if and only if they have the same members. While these core features may be essential to the concept of set, they do not settle every question of application, which is where conceptions can help. Conceptions of set aim to sharpen this core concept, and different conceptions sharpen the concept of set in different, perhaps conflicting ways.

With this approach to conception, Incurvati then describes what conceptions are for. One of the primary purposes of a conception of set is to provide a justification for a *theory* of set. The idea here is that a conception can be formalized, usually axiomatically, and that the axioms of a theory of set follow from that formalization. Discussion of theories, which are collections of sentences formulated in a particular language, naturally leads to a discussion of truth. One might argue, for instance, that a particular theory of set, a theory that purports to describe the concept of set, is justified by, or in Incurvati's terms *sanctioned* by, a given conception. But the fact that a set theory is sanctioned by a particular conception of set does not indicate that the theory is true. The truth of a theory of set can then supported by a further argument that the conception that sanctions it is correct.

In this book, Incurvati provides a framework for evaluating different conceptions of set along these lines, with an eye toward arguing for the correctness of a particular conception. The framework, which he calls "inference to the best conception", describes several virtues that a conception of set might exhibit. He focuses, in particular, on two such virtues. The first concerns the extent to which a conception sanctions a theory in which one can carry out a substantial amount of mathematics. If one theory can do this to a greater extent than another, then a conception sanctioning the former should be preferred over a conception sanctioning the latter, at least with respect to this virtue. It is this feature of set theory that is often seen as providing a foundation of mathematics. And it is important for Incurvati's framework that this foundation be autonomous, in the sense that the foundational theory can be motivated or justified without appealing to any other theory.

The second virtue involves a conception's ability to explain the set-theoretic paradoxes, where in most cases this will be an explanation of why the paradoxes do not arise on that conception. The reader will no doubt be aware of set theory's historical connections to several paradoxes, in particular the paradoxes of the naive conception of set. The naive conception of set embraces an unrestricted principle of comprehension, according to which every condition determines the existence of a set whose members are all and only those objects that satisfy the condition. The naive conception is a perfectly natural conception of set, which sanctions a simple theory of set, comprising an axiom of extensionality and an axiom schema that captures unrestricted comprehension. But this theory of set is also inconsistent, as one can derive a contradiction in various ways (for example, by formalizing the paradoxes of Russell, Cantor, or Burali-Forti).

Incurvati articulates a standard diagnosis of the problems with the naive conception, explaining that the paradoxes arise from a combination of two features that one may expect to hold of the concept of set: *universality* and *indefinite extensibility*. A concept is universal when there is a set of all things that fall under that concept; and it is indefinitely extensible when, given a set of things that fall under the concept, one can generate another object that falls under the concept, but which does not belong to the original set. Incurvati argues that we can have confidence in the consistency of a conception of set in so far as it (1) rejects universality or indefinite extensibility, and (2) provides us with an intuitive model of the theory that the conception sanctions. Grasping this intuitive model suggests that there is a coherent picture of the universe of sets, based on the relevant conception, in which no contradiction arises. While consistency is often seen as a virtue, Incurvati does not explicitly require it in his framework of inference to the best conception, as he recognizes that the naive conception is at least a contender.

In addition to the naive conception, Incurvati considers five other conceptions of set, devoting, for the most part, a chapter of the book to each conception. The first conception that he explores in detail is the iterative conception, which most readers will already be familiar with. The iterative conception offers a picture of the set-theoretic universe that is divided into a cumulative hierarchy of levels. The first level is either the empty set or a collection of non-sets (the *Urelemente*), and the universe of sets is built up from there, where each subsequent level collects together all of the subsets of the previous levels. For each level in the hierarchy, there is a "higher" level, and so this picture corresponds to a conception of set that is indefinitely extensible. The iterative conception avoids paradox by rejecting universality: there is no level with a set of all sets. The iterative conception, as captured by the picture of the set-theoretic universe as a cumulative hierarchy, is usually taken to sanction the axioms of first-order Zermelo-Fraenkel set theory with the axiom of choice, either with Urelemente (ZFCU) or without (ZFC). The cumulative hierarchy provides an intuitive model of this theory, and so we have reason to think that the theory is consistent.

Incurvati skillfully takes the reader through well-known philosophical debates involving the iterative conception. For example, though the concept of set is indefinitely extensible, the conception itself does not tell us how far it can be extended. So there are questions about the height of the hierarchy. There are also questions about its width. Each level collects together all of the subsets of the previous level, but the conception does not say what counts as an arbitrary subset. Incurvati explains that various answers to these questions can determine whether the iterative conception sanctions further axioms about the height (e.g., large cardinal axioms) or the width (e.g., $V = L$) of the universe of sets.

In addition to these familiar philosophical questions about the universe of sets, Incurvati also offers his perspective on a different topic in the philosophy of set theory, which addresses the metaphysical nature (if any) of the membership relation. One way to argue in favour of the iterative conception is to focus on the idea that there is a relation of priority or metaphysical dependence between sets and their members. The idea here is that positing such a relation would naturally lead one to the view that the cumulative hierarchy comprises the entire universe of sets. While Incurvati endorses the iterative conception, he rejects this line of argument because the notions of priority and dependence, or any similar relation used for these purposes, must be taken as primitive. And so the view that sets depend on their members offers little explanatory value in terms of why one should prefer the iterative conception. Incurvati then uses this discussion to present his version of the iterative conception, which he calls the 'minimalist' account.

As a whole, the book can be seen as an extended argument for the minimalist approach, and thereby as an argument in favour of the iterative conception of set. Incurvati's argument proceeds by comparing the iterative conception to alternative conceptions, within the framework of making an inference to the best conception. According to Incurvati, the iterative conception "is the most satisfactory conception of set that we have ... it is better than its rivals with regards to certain virtues that a conception of set may have" (p. 65). These virtues include, at least, (1) being able to provide an explanation of the paradoxes, and (2) sanctioning theories within which one can carry out a substantial amount of mathematics. The goal of the book is to convince the reader that the iterative conception scores higher with respect to these virtues than the alternatives.

Incurvati begins this comparison of alternative conceptions of set with a more detailed discussion of the naive conception, exploring three strategies to defend this conception. The naive conception sanctions a theory of sets that is classically inconsistent. In order to avoid triviality, a theory in which every sentence is true, one must deploy a logic that is weaker than classical logic. The first strategy that Incurvati describes uses Graham Priest's logic LP, and the second strategy focuses on a family of relevant logics known as depth-relevant logics. Beyond these proof-theoretic approaches, Incurvati also considers a model-theoretic perspective, looking at the cumulative hierarchy as a consistent substructure in the larger universe of sets, a universe that also contains inconsistent sets. Incurvati argues that, in all three cases, these strategies are either too weak, in the sense that they cannot serve as a foundation for a substantial amount of mathematics, or they are unmotivated when compared to the iterative conception.

Because the naive conception is natural, though inconsistent, one might try to adjust it in order to recover consistency. There is a temptation to focus on maximally consistent sets of instances of naive comprehension. Incurvati quickly dispels this temptation by showing that for any sentence, there is a maximally consistent set of instances of naive comprehension implying it, and a maximally consistent set of instances implying its negation. And unfortunately, there is no principled way to choose one set over another. Incurvati then looks at a pair of conceptions that attempt to adjust the naive conception in different ways, in order to avoid paradox.

The first is based on the 'limitation of size' idea, which modifies the naive comprehension principle by placing restrictions on the properties that can determine a set. According to this conception, a property determines a set as long as the property's extension isn't too big. In order to articulate this idea, Incurvati starts with a proposal from Cantor that a property determines a set when its extension is smaller than the ordinals. He then considers an alternative, inspired by von Neumann, that a property determines a set when it is not as big as the universe (i.e., the extension of the universal property $x : x = x$). Finally, he describes the idea that a property determines a set when it is definite, that is, when it is not indefinitely extensible. In defending the iterative conception against these versions of limitation of size, Incurvati argues that they are ultimately motivated by a desire to avoid paradox. As such, they do not offer an explanation of the paradoxes, which according to Incurvati is something that the iterative conception can do.

Incurvati then looks at the stratified conception of set, due to Quine, which restricts naive comprehension in a syntactic way. On this view, only stratified formulae are permitted in the naive comprehension schema. A formula is stratified when one can assign natural numbers to its variables, so that x and y are assigned the same number in any subformula of the form $x = y$, and the number assigned to y is one more than the number assigned to x in any subformula of the form $x \in y$. (Those familiar with type theory will recognize this idea.) This conception embraces universality, in that there is a universal set, and it avoids paradox by rejecting indefinite extensibility. However, it is often objected that there is no stratified *conception* of set, no picture of the set-theoretic universe on the stratified approach. Incurvati provides an interesting defense against this objection by trying to develop a plausible route by which one could start from type theory and naturally end up with the stratified view.

In comparing the stratified and iterative conceptions, rather than argue against the stratified conception directly, Incurvati proposes an interesting picture according to which the two conceptions can co-exist. This strategy proceeds by appealing to a familiar distinction between logical and combinatorial conceptions of collection. Logical conceptions tie membership in a collection to the satisfaction of a particular condition. The naive conception is probably the most straightforward example of a logical conception of set. In this case, as it is with other logical conceptions, satisfying the relevant condition is of primary importance, and belonging to the correlated set is in some sense derivative. On a combinatorial conception, the focus is instead on the members themselves, and there may not be any condition that all of the members of a given collection satisfy.

To argue that the iterative and stratified conceptions can exist together, Incurvati proposes that the universe of sets is provided by the iterative conception, which is a combinatorial conception of collection. On the other hand, the stratified conception is a logical conception, and it provides us with another kind of collection, which Incurvati describes as an objectified property. An objectified property is an object that is systematically associated with a property.

In order to describe this idea, by way of analogy, Incurvati compares it to the linguistic process of nominalization. One can associate a predicate, e.g., *runs*, with a nominal expression, *running*. While the predicate refers to a property, the nominal expression refers to an object, which Incurvati calls an objectified property. Whether or not this strategy is ultimately convincing, it offers an interesting approach to understanding the distinction between logical and combinatorial conceptions of collection: combinatorial collections are associated with sets and logical collections are associated with objectified properties.

The book finishes with a description of the graph conception of set, most commonly associated with four theories of set developed by Peter Aczel. On this approach, once one is familiar with directed graphs, comprising nodes and arrows between nodes, one can simply take sets to be things depicted by these kinds of graphs. Sets are represented by nodes in a graph, and an arrow from one node to another represents the membership relation, in that the set represented by the first node contains the set represented by the second. Of Aczel's four theories of set, Incurvati argues that the graph conception sanctions the theory ZFA, which is similar to ZFC, but swaps the axiom of foundation for an anti-foundation axiom. According to ZFA, not all sets are well-founded. There can be circular sets (sets that are members of themselves), because there can be circular directed graphs. And there can be infinitely descending chains of membership. Incurvati argues, however, that the graph conception still falls short when compared to the iterative conception, as the latter provides an autonomous foundation of mathematics, while the former can only provide a foundation that must appeal to the theory of graphs.

By carefully elucidating the details of many of the philosophical and technical topics involved in these conceptions of set, Incurvati's book offers an indispensable resource for current debates in the philosophy of set theory, which also includes some of the interesting history of those debates. But the book goes further by introducing a novel framework within which we can compare and evaluate these conceptions, the framework of inference to the best conception.

There are, as one might anticipate, several questions that remain regarding this framework. One concerns the question of pluralism with respect to conceptions of set. Given Incurvati's argument in favour of the iterative conception, there is a natural interpretation that the conclusion of the argument should be that the iterative conception is the objectively correct conception, and that the other conceptions are incorrect. But that raises further questions as to why the particular virtues that Incurvati has focussed on make a conception correct. There is a weaker position that one could take, arguing instead that the iterative conception is best for certain purposes. And one such purpose could be, for example, to provide a foundation for mathematics. Other conceptions might be best for other purposes. Though Incurvati explicitly says his approach is compatible with a moderate form of pluralism, this idea is unfortunately not explored much further.

Other questions that the book raises surround the relationship between inference to the best conception and the more familiar method of inference to the best explanation. It is likely that these methods are not entirely distinct. On Incurvati's approach to conceptions of set, the former appears to include the latter. For one virtue of a conception of set is that a conception is able to provide an explanation of the set-theoretic paradoxes. A better conception will therefore, presumably, offer a better explanation. But it would be interesting to determine how close the connections are between these methodologies. This would be relevant, in particular, to the question of whether inference to the best conception is a legitimate or valid form of inference. Along these lines, one might be able to take advantage of arguments in favour of inference to the best explanation in order to support inference to the best conception.

There are also questions regarding the extent to which the framework can be generalized to conceptions outside of the philosophy of set theory, as the particular virtues described here are very specific to conceptions of set. But one would not expect Incurvati's book to answer all of these questions. Elucidating this framework in detail would most likely require a separate book-length treatment of its own. In proposing this framework, Incurvati has put it to good use in this book by developing a clear, thoughtful and thorough argument in favour of the iterative conception of set. While the reader may not agree with each step of Incurvati's argument, the book will no doubt inspire new and fruitful research in this area, as well as provide an invaluable, self-contained resource for a solid grounding in the philosophy of set theory.

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