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## STRICT NASH EQUILIBRIA IN A DUOPOLISTIC MARKET SHARE MODEL

Jane Binner

*Birmingham Business School University of Birmingham  
Edgbaston Park Rd, Birmingham B15 2TY (UK) j.m.binner@bham.ac.uk*

Francesco Ciardiello \*

*Management School, University of Sheffield  
Conduit Rd, Sheffield B15 2TY (UK) f.ciardiello@shef.ac.uk*

Leslie Fletcher

*School of Computing and Mathematical Sciences,  
Liverpool John Moores University  
Byrom St, Liverpool L3 3AF (UK)*

Vassili Kolokoltsov

*Department of Statistics, University of Warwick Coventry CV4 7AL (UK).  
Faculty Applied Mathematics and Control Processes, St. Petersburg State University (Russia)  
FRC CSC Russian Academy of Science  
v.kolokoltsov@warwick.ac.uk*

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This paper develops a duopolistic discounted marketing model with linear advertising costs and advertised prices for mature markets still in expansion. Generic and predatory advertising effects are combined together in the model. We characterise a class of advertising models with some lowered production costs. For such a class of models, advertising investments have a no-free-riding strict Nash equilibrium in pure strategies if discount rates are small. We discuss the entity of this efficiency at varying of parameters of our advertising model. We provide a computational framework in which market shares can be computed at *equilibrium*, too. We analyse market share dynamics for an asymmetrical numerical scenario where one of the two firms is more effective in generic and predatory advertising. Several numerical insights on market share dynamics are obtained. Our computational framework allows for different scenarios in practical applications and it is developed thanks to *Mathematica* software.

\*Corresponding author

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## 1. Introduction

Advertising expenditures for companies may be generally viewed as a form of investment and the main thrust of the advertising literature is to examine optimal strategies which maximise the net present value of future cash flows. When advertising is aimed at increasing product sales, then it is called generic or informative advertising, whilst when advertising is aimed at gaining market shares, it is called brand advertising. When brand advertising is devoted to stealing customers from competitors, it is called predatory advertising. Many advertising models have been built to provide solutions in monopolistic, duopolistic and oligopolistic environments (Vidale and Wolfe, 1957; Sethi, 1973; Deal, 1979; Little, 1979; Sethi, 1983) for many decision variables.<sup>a</sup> Distinct dynamic advertising strategies were never developed prior to the model developed in (Bass et al., 2005, Table 1) for brand and generic advertising respectively. Even the deep dynamic analysis provided in (Bass et al., 2005) suggests that generic advertising expenditures must be highly resolved separately from its brand advertising ones in order to halt suboptimal advertising. However externalities from simple generic advertising may become significant and may modify brand preferences, as market demand becomes more informed (Kinnucan, 1996; Norman et al., 2008; Rutz and Bucklin, 2011; Brahim et al., 2014). The main controversial *economic* issue is that generic advertising may redistribute market shares, especially in markets that have become strongly differentiated (Chakravarti and Janiszewski, 2004; Brady, 2009; Espinosa and Mariel, 2001; Piga, 1998; Friedman, 1983). In the current paper, predatory and generic advertising expenditures are combined together.

The analysis provided in (Espinosa and Mariel, 2001) deals with a duopoly where generic and predatory advertising are present but separately analyzed. The authors argue that static strategies do not internalise any variations of market shares if firms play a game with time-invariant market shares. Therefore it is expected that Nash equilibria are neither efficient if the advertising is predatory (too high expenditures), nor efficient if advertising is generic (too low expenditures). In the current paper we face a more complex scenario than the static one analysed in (Espinosa and Mariel, 2001). In fact, generic and brand/predatory advertising are unified and their effect on market shares cannot be distinguished at any time for an infinite time

<sup>a</sup>For a complete review of the literature on advertising models before 1995 see (Erickson, 1995). The more recent classification of advertising models consists of six categories (Huang et al., 2012); the Nerlove-Arrow model and its extensions, the Vidale-Wolfe model and its extensions, the Lanchester model and its extensions, the diffusion models, dynamic advertising-competition models with other attributes, and (vi) empirical studies for dynamic advertising problems.

horizon. Strategic generic advertising interactions between firms may cause short-run bubble effects but similar predatory effects may cause long-run lowering effects on market shares, especially for asymmetrical market conditions. For instance, different investment strategies may generate different market share dynamics, which yield the same discounted profit to one of the firms. To some extent, the efficiency of such equilibria is not so intuitive. For instance, in (Espinosa and Mariel, 2001) closed-loop equilibria are more efficient than open-loop equilibria if the advertising is only generic while open-loop are more efficient than closed-loop if the advertising is only predatory. Very little attention has been devoted to the efficiency and stability of Nash equilibria in the game theory of advertising. The simple concept of strong Nash equilibria, which incorporates a strong form of efficiency, may be useful to be analysed in a duopolistic model such as this. By being complementary to the dynamic analysis in (Espinosa and Mariel, 2001), we restrict optimal analysis to static expenditure strategies. The stability we require is the following: if a firm unilaterally changes its strategy from Nash equilibrium strategies, then such a firm has to be *strictly* worse. We fill the gap in this literature and we investigate the existence of strict Nash equilibria in a context of generic and brand advertising models.

## 2. Results

In the current paper, we initially develop a duopolistic discounted advertising model for mature markets in expansion. Generic and predatory advertising effects are combined together. We model generic advertising by adopting a diffusion model and we model predatory advertising by adopting the Lanchester model of combat. For the sake of simplicity we allow firms to advertise prices even if we do not model the latter as a decision variable. In fact, prices may be announced before marketing campaigns (Jiang et al., 2014; Grewal et al., 1998; Tenn and Wendling, 2014; Lu et al., 2016).<sup>b</sup> We define a class of advertising models where production costs are lowered by marketing parameters, being inspired by an economic analysis between advertising and quality of products suggested in (Bagwell, 2007).

We obtain payoffs in a closed-form using tools from the theory of linear differential systems. Significantly, payoffs are continuous but they are not quasi concave on non-compact subsets. Our method provides for an algebraic analysis of best reply correspondences. We find out that a strict Nash equilibrium in pure strategies exists for the class of advertising games with low costs. Interestingly, our equilibria are not free-riding and they cannot be conceptually compared to the ones found in (Krishnamurthy, 2000). The existence of equilibria is not surprising for small discount

<sup>b</sup>We prefer to characterise our model in terms of advertised prices to make our results comparable to another literature stream, i.e. optimal pricing in markets with sticky prices (Gorodnichenko and Weber, 2016; Piga, 2000). Our model is comparable to a limit case in this literature, i.e. the rate of price readjustment is null. Recently, static optimal solutions are also provided in advertising models subject to interferences (Baggio and Viscolani, 2014; Viscolani, 2012).

rates by taking into account Folk theorem in game theory. However, the efficiency of such equilibria is not so intuitive. Our strict equilibria are not obtained for any level of production costs, as it has been found in the static model provided in (Espinosa and Mariel, 2001). Unfortunately, due to the lack of a closed form expression of pure strategy Nash equilibria we cannot analytically discuss the influence of the parameters on optimal investments strategies. However our algebraic method provides a computational framework in order to integrate such optimal investment strategies (Deal, 1979; Erickson, 1985).

As a further investigation, we focus on market share dynamics generated by our strict Nash equilibria in pure strategies. Since generic advertising supports the general standard of the product category, it brings advantages to firms in the market regardless of whether or not they contributed to advertising campaigns (Han et al., 2017; Shapiro, 2018). In the duopolistic model analysed in (Bass et al., 2005), when the asymmetries between the firms increase, there is a larger difference between their generic advertising contributions. However, the weaker firm always invests a not null amount of money and, then, cheaply but not freely rides the market (Krishnamurthy, 2000). In our numerical scenario we assume the similar scenario adopted in (Bass et al., 2005). One firm is *stronger* if it is endowed with more favourable competitive and generic advertising parameters. Interestingly, we find out that the *weaker* firm is not a cheap rider. We believe that this does not happen because our advertising expenditure are combined together for brand and generic advertising. As an expected result, the weaker firm enlarges its market share due to generic advertising while its market share is affected by the long run effect of predatory advertising from its competitor. Our results support those obtained in (Bass et al., 2005). Thanks to our numerical framework we are numerically able to integrate the time at which the weaker firm achieves its maximum market share, if the market in itself is not initially saturated. Similar insights cannot be replicated from the results in (Bass et al., 2005) because the two models are similar but different.

The rest of the paper is organised as follows. In Section 3 we provide modelling background from the literature of advertising models. In Section 4, we provide details of our advertising model. In Section 5 we prove the existence of proper investment equilibria in pure strategies. In Section 6 we provide an asymmetric numerical scenario for our model and we illustrate market share dynamics if pure strategy strict Nash equilibria investments are implemented. Our proofs are provided in the Appendix.

### 3. Modelling background

Firms sell their heterogenous goods in a specific product category. Each firm satisfies a portion of demand, i.e. its market share  $x_i(t) \geq 0$  for time  $t \geq 0$ . The sum of market shares is  $x(t) = x_1(t) + x_2(t)$ . The market is still in expansion and the whole demand is finite and unitary. The sum of market shares  $x(t) = x_1(t) + x_2(t)$  and the remaining portion of the demand is  $1 - x(t)$ , i.e. the potential demand. Potential de-

mand is the demand which has unrevealed wish for the product category. However, the potential demand is not already contained within market shares of any firm and it is *uninformed* on the product category. Generic advertising is only directed to the potential demand through diffusion models. Diffusion models are used to capture the life cycle dynamics of new products or to forecast the demand on markets (Fisher and Pry, 1971). Originally, diffusion models do not incorporate any advertising effort and one of the main challenge is to add exogenous influences, most importantly the influence of advertising efforts (Bass et al., 1994). A generic functional form for diffusion advertising models for firm  $i$  is  $\dot{x}_i(t) = f(x_i(t), u_i(t), t)$  where  $u_i(t)$  is the advertising effort and  $\dot{x}_i(t)$  is the change in market share for firm  $i$  (Dockner and Jørgensen, 1988). For instance, a diffusion model is  $\dot{x}_i(t) = \alpha u_i(t)[1 - x_i(t)] + \gamma x_i(t)[1 - x(t)]$  for a monopolistic firm  $i$  where  $\alpha > 0$  is the advertising effectiveness of the firm and  $\gamma$  represents the effectiveness of word-of-mouth advertising in (Jørgensen et al., 2006). With respect to generic advertising, we symmetrically adopt this diffusion model with  $\gamma = 0$  and  $\alpha = 1$  for both firms in our duopoly. The Lanchester model has been often used to model competitive advertising (Fruchter and Kalish, 1997). The Lanchester dynamics capture the competitive market shares' shifts of firms  $i, j$  due to investments in advertising by the two market rivals, (Chintagunta and Vilcassim, 1992). The Lanchester model is  $\dot{x}_i(t) = \rho_i x_j(t) u_i(t) - \rho_j x_i(t) u_j(t)$  where  $\rho_i, \rho_j \in [0, 1]$  are the effectiveness of brand advertising for firms  $i, j$  (Little, 1979). In particular, here we assume that  $\rho_i, \rho_j > 0$  for both firms.

#### 4. Our advertising model

We adopt some modifications to the models in the previous Section. We assume that the advertising strategy variable  $u_i \in [0, \infty[$  is time-invariant. Moreover we do not adopt a linear form for investment variable  $u_i$ . In spite of mathematical simplifications we add a level of complexity and we shape advertising returns as a functional form  $a_i, b_i : [0, \infty[ \rightarrow [0, \infty[$  for generic and brand advertising expenditures respectively. We assume these functions satisfy a law of diminishing returns on the investment for both generic and brand advertising (Hanssens et al., 2003; Freimer and Horsky, 2012). It thus follows that

- $a_i(u_i) = \frac{u_i}{u_i + \alpha_i} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is the generic advertising return for  $i$ . The coefficient  $\alpha_i \in \mathbb{R}^+ \setminus \{0\}$  is called ineffectiveness of generic advertising return.
- $b_i(u_i) = \frac{u_i}{u_i + \beta_i} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is the brand advertising return for  $i$ . The coefficient  $\beta_i \in \mathbb{R}^+ \setminus \{0\}$  is called ineffectiveness of brand advertising return.

Functions  $a_i(\cdot), b_i(\cdot)$  are increasing and concave and approximate 1 for large investments. The lower  $\alpha_i$  and  $\beta_i$ 's values are, the more profitable advertising investments are. Summing up all these features of the diffusion model and of the Lanchester model, the derivative in primary demand  $\dot{x}_i$  is modelled by the following differential system with initial conditions

$$\begin{cases} \dot{x}_i(t) = [a_i(u_i) + a_j(u_j)][1 - (x_i(t) + x_j(t))] + \rho_i x_j(t) b_i(u_i) - \rho_j x_i(t) b_j(u_j) \\ x_i(0) = x_i^0 \end{cases} \quad (1)$$

where  $u_i \in \mathbb{R}^+$  is the advertising expenditure of firm  $i$ ,  $x_i(t) : \mathbb{R}^+ \rightarrow [0, 1]$  is the market share of firm  $i$  at time  $t$  and  $x_i^0 = x_i(0)$  are the initial market shares. We assume linear advertising costs that are, traditionally, used in the Nerlove-Arrow model (Gould, 1976). In the literature, it is widely accepted that firms have different discount rates. Here firms adopt a unique positive discount rate  $\rho \neq 0$  (Jørgensen, 1982). Therefore, the discounted flow of profits is

$$\pi^i(u_i, u_j) = \int_0^\infty e^{-\rho t} (r_i x_i(t) - u_i) dt. \quad (2)$$

where  $r_i = p_i - c_i$ . The quantity  $p_i > 0$  is the advertised price  $p_i$  for the good  $i$ . The quantity  $c_i > 0$  is the marginal cost for good  $i$ . We assume that marginal profits  $r_i$  are positive. We say that  $G = ([0, +\infty]^2, \pi^i)$  is a non-cooperative advertising game. Classical solution concepts are defined below.

**Definition 1 (Nash equilibria, market shares at equilibrium).** *Let  $(\hat{u}_1, \hat{u}_2)$  be a strict Nash equilibrium in pure strategies for  $G$ . Let us substitute  $(\hat{u}_1, \hat{u}_2)$  in (1). A solution  $\hat{x}_i(t)$  of the first order differential system (1) is the associated market share of firm  $i$  at equilibrium. A pure strategy profile is free-riding if one strategy is null and the remaining strategy is non-null. A pure strategy profile is null when both strategies are null. We say that a pure strategy profile is proper if it is neither null nor free-riding.*

#### 4.1. Advertising games with low costs

We characterise advertising games which have lower costs (or higher profits) by considering that prices are advertised in our marketing model.

**Definition 2.** *The following inequality is satisfied*

$$c_i + c_i^{sh} < p_i \quad (3)$$

where

$$c_i^{sh} = \frac{(\rho_j + \rho_i)}{2\rho_i} \alpha_i + \frac{\rho_j}{\rho_i} \beta_i. \quad (4)$$

*We say that  $G = ([0, +\infty]^2, \pi^i)$  is the advertising game with low costs if the above conditions are satisfied. We say that  $c_i^{sh} > 0$  is the marketing incentive for firm  $i$ .*

Marketing incentives need to be smaller than the advertised price in order to satisfy inequality (3). For instance, if firm  $i$  has a brand advertising effectiveness extremely lower than its competitor's counterpart  $\rho_j$ , inequality (3) is not satisfied. The marketing incentive  $c_i^{sh}$  is increasing with respect to  $\rho_j, \alpha_i, \beta_i$ . The latter means that firm  $i$  has lower costs if the effectiveness of the brand advertising of the competitor

increases or if its advertising returns are marginally worse. The marketing incentive  $c_i^{sh}$  is decreasing with respect to  $\rho_i$ . The latter means that firm  $i$  may have higher costs if the effectiveness of own brand advertising increases. If  $\rho_1 = \rho_2$ , then  $c_i^{sh} = \alpha_i + \beta_i$ . If  $\alpha_i = \beta_i$ , then  $c_i^{sh} = \alpha_i \frac{(3\rho_j + \rho_i)}{2\rho_i}$ .

### 5. Existence of a strict Nash equilibrium in pure strategies

In this section, we find sufficient conditions which guarantee the existence of static advertising expenditures equilibria for our advertising model. Let  $\widehat{U}_i : [0, +\infty[ \rightarrow ]2^{[0, +\infty[}$  be the best reply correspondence for firm  $i$ . The set  $\widehat{U}_i(u_j)$  collects firm  $i$ 's best replies to a strategy  $u_j$ . We transform system (1) through a change of variables into

$$\begin{cases} \dot{x} = 2 \{a_1(u_1) + a_2(u_2)\} (1 - x) \\ \dot{w} = \rho_1 b_1(u_1) (x - w) - \rho_2 b_2(u_2) (x + w) \end{cases} \quad (5)$$

where  $x = x_1 + x_2$  and  $w = x_1 - x_2$ .

**Proposition 1.** *The solution for system (5) is given by the following formulae*

$$\begin{cases} x(t) = 1 - (1 - x_1^0 - x_2^0) e^{-2At} & t \geq 0 \\ w(t) = \frac{B^-}{B^+} + \frac{B^-}{(B^+ - 2A)} (x_1^0 + x_2^0 - 1) (e^{-2At} - e^{-B^+t}) + e^{-B^+t} C_u \end{cases}$$

where  $C_u$  is a constant depending on

$$A = a_1(u_1) + a_2(u_2), B^+ = \rho_1 b_1(u_1) + \rho_2 b_2(u_2), B^- = \rho_1 b_1(u_1) - \rho_2 b_2(u_2).$$

Here, we provide the following result in which market shares and payoffs are obtained in closed formulas.

**Proposition 2.** *Assume  $\rho \neq 0$ . Then market shares and payoffs are*

$$\begin{aligned} x_i(t) &= \frac{\rho_i b_i(u_i)}{B^+} (1 - e^{-B^+t}) + \frac{(x_i^0 + x_j^0 - 1) (\rho_i b_i(u_i) - A)}{(B^+ - 2A)} (e^{-2At} - e^{-B^+t}) \\ &+ x_i^0 e^{-B^+t}. \end{aligned} \quad (7)$$

$$\begin{aligned} \pi^i(u_i, u_j) &= \frac{r_i}{(B^+ + \rho)} \left( \frac{\rho_i b_i(u_i)}{\rho} + \frac{(x_i^0 + x_j^0 - 1) (\rho_i b_i(u_i) - A)}{(2A + \rho)} + x_i^0 \right) \\ &- \frac{u_i}{\rho}. \end{aligned} \quad (8)$$

for  $i, j = 1, 2$ , respectively.

Each payoff on its own variable is defined on a non-compact set  $[0, \infty[$  and may lack quasi-concavity. Therefore, we cannot apply classical results for the existence of pure strategy Nash equilibria. We prefer to follow a different approach to the problem of whether a strict Nash equilibrium exists in pure strategies. First we identify an

algebraic structure for firms' best reply strategies, i.e. best reply strategies are zeros of a polynomial equation.

**Proposition 3.** *Assume  $\rho \neq 0$ . The payoff  $\pi^1(u_1, u_2)$  is rational in  $u_1$  and can be calculated as follows*

$$\pi^1(u_1, u_2) = -\frac{hu_1^3 + (b - r_1d)u_1^2 + (c - r_1f)u_1 - r_1g}{\rho(hu_1^2 + bu_1 + c)}$$

where

$$b = (\alpha_1 + \beta_1)(\rho^2 + (\theta + 2\vartheta)\rho + 2\theta\vartheta) + 2\beta_1(\rho + \theta) + \rho_1\alpha_1(\rho + 2\vartheta)$$

$$c = \alpha_1\beta_1(\rho^2 + (\theta + 2\vartheta)\rho + 2\theta\vartheta)$$

$$d = x_1^0\rho^2 + ((\vartheta + 1)(1 + x_1^0 - x_2^0) + \rho_1(x_1^0 + x_2^0))\rho + 2\rho_1(\vartheta + 1)$$

$$f = x_1^0(\alpha_1 + \beta_1)\rho^2 + (\alpha_1\rho_1(x_1^0 + x_2^0) + ((\alpha_1 + \beta_1)\vartheta + \beta_1)(1 + x_1^0 - x_2^0))\rho + 2\rho_1\alpha_1\vartheta$$

$$g = \alpha_1\beta_1\rho(x_1^0\rho + \vartheta(1 + x_1^0 - x_2^0))$$

$$h = (\rho + 2\vartheta + 2)(\rho + \rho_1 + \theta)$$

$$\theta = \rho_2 \frac{u_2}{u_2 + \beta_2}, \quad \vartheta = \frac{u_2}{u_2 + \alpha_2}.$$

**Proof.** The proof follows from a direct calculation.  $\square$

**Proposition 4.** *Assume  $\rho \neq 0$ . Let  $u_2 \geq 0$  be a strategy of firm 2. Assume that a best reply to strategy  $u_2$  exists, i.e.  $u_1 > 0$ . Then,  $u_1$  is a root of*

$$\begin{aligned} & h^2u_1^4 + 2hbu_1^3 + (r_1fh - r_1db + 2hc + b^2)u_1^2 + \\ & + (2bc - 2r_1dc + 2r_1gh)u_1 + (r_1gb - r_1fc + c^2) = 0 \end{aligned} \quad (11)$$

**Proof.** Let  $\hat{u}_2$  be an investment of firm 2. By simple calculation, we have

$$\begin{aligned} \frac{\partial \pi^1(u_1, \hat{u}_2)}{\partial u_1} &= -\frac{h^2u_1^4 + 2hbu_1^3 + (r_1fh - r_1db + 2hc + b^2)u_1^2}{\rho(hu_1^2 + bu_1 + c)^2} + \\ &+ \frac{(2bc - 2r_1dc + 2r_1gh)u_1 + (r_1gb - r_1fc + c^2)}{\rho(hu_1^2 + bu_1 + c)^2}. \end{aligned} \quad (12)$$

If there exists a best reply in  $u_1$ , trivially  $u_1$  is a root of above equation.  $\square$

We conventionally rank the coefficients on the left side of equation (11) by decreasing order. The first two Lemmas are preliminary to the fundamental Lemma 3.

**Lemma 1.** *If  $\rho \neq 0, \rho \approx 0$  and  $r_1 > \frac{(\rho_2 + \rho_1)}{2\rho_1}\alpha_1 + \beta_1 \frac{\rho_2}{\rho_1}$  then the fourth coefficient is strictly negative, for any  $u_2 \geq 0$ .*

**Lemma 2.** *If  $\rho \neq 0, \rho \approx 0$  and  $r_1 > \beta_1 \frac{\rho_2}{\rho_1}$ , then the fifth coefficient is strictly negative for any  $u_2 \geq 0$ .*

**Lemma 3.** *If  $\rho \neq 0$ ,  $\rho \approx 0$  and  $r_1 > \frac{(\rho_2 + \rho_1)}{2\rho_1}\alpha_1 + \beta_1 \frac{\rho_2}{\rho_1}$  then there exists a unique best reply  $u_1 > 0$  for any strategy  $u_2 \geq 0$ .*

A fundamental property of best reply correspondences is presented below.

**Lemma 4.** *Under the hypothesis of Lemma 3, best reply correspondences  $\widehat{U}_i$  are continuous functions. In addition,  $\lim_{u_j \rightarrow \infty} \widehat{U}_i(u_j) < \infty$ .*

By a classical fixed-point argument, we prove the existence of a strict Nash equilibrium in pure strategies.

**Theorem 1 (Main Existence Result).** *Assume that  $G = ([0, +\infty[^2, \pi^i)$  is an advertising game with low costs. Assume that the discount rate satisfies the following properties:  $\rho \neq 0$  and  $\rho \approx 0$ . Then, a strict Nash equilibrium in proper pure strategies exists for  $G$ .*

**Proof.** The thesis of Lemma 4 is satisfied by hypotheses. By coercivity conditions the image of  $\widehat{U}_2$  is a bounded subset in  $]0, +\infty[$ . Let  $K_1 = Cl(\widehat{U}_2([0, +\infty[)) \subset ]0, +\infty[$  be the closure of the image values of  $\widehat{U}_2$ . Let  $\widehat{U}_1|_{K_1} : K_1 \rightarrow ]0, +\infty[$  be the restricted function to the compact subset  $K_1$ . In addition  $\widehat{U}_1|_{K_1}(K_1)$  is a compact subset in  $]0, +\infty[$  because  $K_1$  is compact and  $\widehat{U}_1|_{K_1}$  is continuous. By definition of  $K_1$ , it follows that  $\widehat{U}_1 \circ \widehat{U}_2|_{K_1} : K_1 \rightarrow K_1$ . In addition, the above function is continuous because it is a composition of continuous functions and  $K_1$  is a compact in  $]0, +\infty[$ . By Brower's fixed point  $\widehat{U}_1 \circ \widehat{U}_2|_{K_1}$  admits a fixed point. Then there exists a strategy  $\widehat{u}_1 \in K_1 \subset ]0, +\infty[$  such that  $\widehat{U}_1(\widehat{U}_2(\widehat{u}_1)) = \widehat{u}_1$ . We define  $\widehat{u}_2 := \widehat{U}_2(\widehat{u}_1)$ . By Lemma 3, we have that  $\widehat{u}_2 \neq 0$ . It is straightforward to prove that  $(\widehat{U}_2(\widehat{U}_1))(\widehat{u}_2) = \widehat{u}_2$ . By definition,  $\widehat{u} = (\widehat{u}_1, \widehat{u}_2)$  is a Nash equilibrium in pure strategies. By construction we have that  $\widehat{u}$  is a strategy profile with  $\widehat{u}_i \neq 0$ . Therefore  $\widehat{u}$  is a proper strategy profile. By Lemma 3 again,  $\widehat{u}$  is a strict Nash equilibrium in pure strategies.  $\square$

## 6. Market shares at equilibrium for an asymmetric advertising situation

In this section we provide a numerical example under some asymmetric conditions for two firms. We assume that firm 1 has higher investment returns for both generic and predatory advertising. Further, we assume that firm 1 is more effective at capturing demand belonging to the market share of the competitor, *ceteribus paribus* their investments. Table 1 contains the numerical parameters for the simulation of our model. Values of parameters of our model are listed in Table 1. It is straightforward to verify that our game is an advertising game with low costs. Using best reply equation (11), we obtain implicit formulae for best reply functions. We numerically integrate their intersection points, i.e. strict Nash equilibria in proper pure strategies. Interestingly, we find that a strict Nash equilibrium is unique and equal to (5.23, 7.24) which is not free-riding. Moreover, the weaker firm invests more than

ineffectiveness of generic advertising return	$\alpha_1 = 5, \alpha_2 = 10$
ineffectiveness of brand advertising return	$\beta_1 = 5, \beta_2 = 15$
effectiveness of brand advertising	$\rho_1 = 1/50, \rho_2 = 1/100$
marketing incentive	$c_1^{sh} = 25/4, c_1^{sh} = 45$
advertised marginal prices	$p_1 = 75, p_2 = 60$
marginal cost	$c_1 = 25, c_2 = 10$
discount rate	$\rho = 0.000001$

Table 1. Parameters of the model are listed. Then we compute marketing incentives. We set advertised marginal prices and marginal costs for both firms. In addition, the discount factor is non-null and it is extremely small.

its opponent. Therefore, the weaker firm does not *cheaply* ride the market (Krishnamurthy, 2000).<sup>c</sup> By replacing Nash equilibrium strategies in formula (7), we compute firms' market shares. Figures 1–5 describe market shares dynamics. The red, blue and green lines represent the dynamics of market shares of the stronger firm (firm 1) and of weaker firm (firm 2) and of the whole market demand, respectively. If the market is initially not saturated, firms invest in marketing advertising

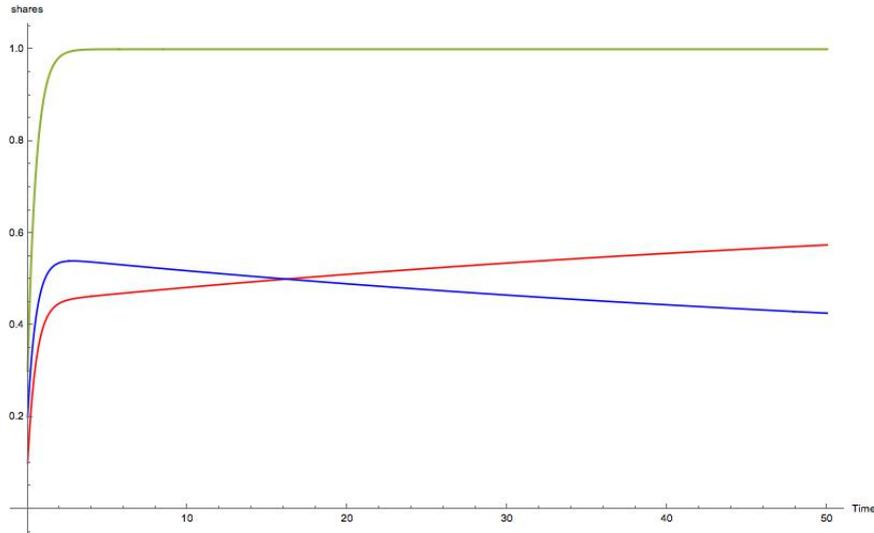


Fig. 1.  $x_1^0 = \frac{1}{10}$ ,  $x_2^0 = \frac{1}{5}$ . Firm 1 is initially weaker in the market. The market is initially not saturated.

<sup>c</sup>We consider the same values of  $\rho_i, \beta_i, \alpha_i, r_i$  in Table 1 and we choose higher discount rates. From our numerical implementation, it follows that pure strategy Nash equilibria fail to exist if  $\rho$  approximatively becomes higher than a threshold value equal to 0.040821.

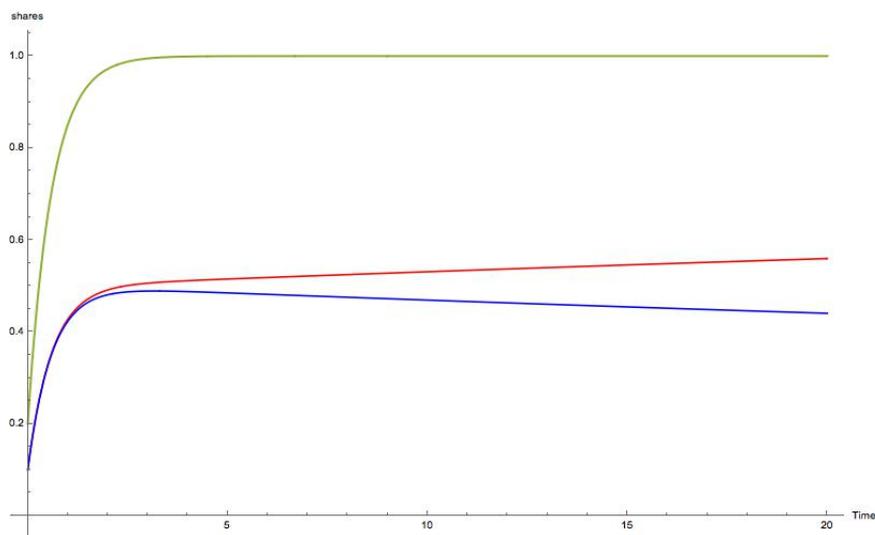


Fig. 2.  $x_1^0 = x_2^0 = \frac{1}{10}$ . Firm 1,2 equally share the market. The market is initially not saturated.

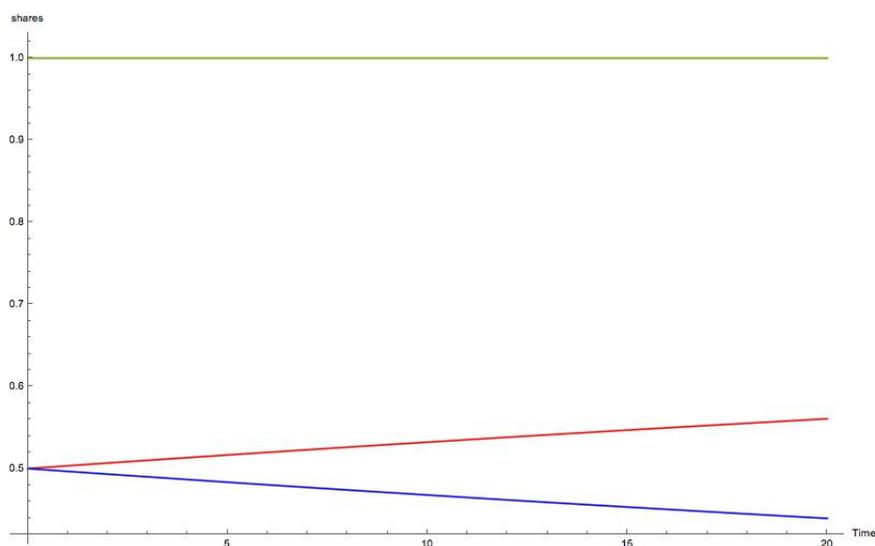


Fig. 3.  $x_1^0 = x_2^0 = \frac{1}{2}$ . Firm 1,2 equally share the market. The market is initially saturated.

and they improve their market positions. The *weaker* firm, i.e.firm 2, initially increases its market position (blue line) and its market share reaches its maximum

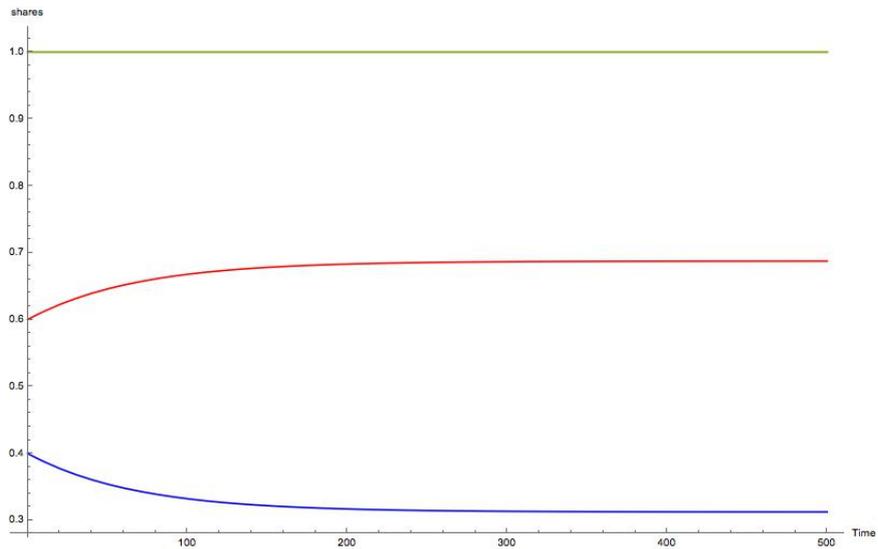


Fig. 4.  $x_1^0 = \frac{9}{10}$ ,  $x_2^0 = \frac{1}{10}$ . Firm 1 is initially much stronger on the market. The market is initially saturated.

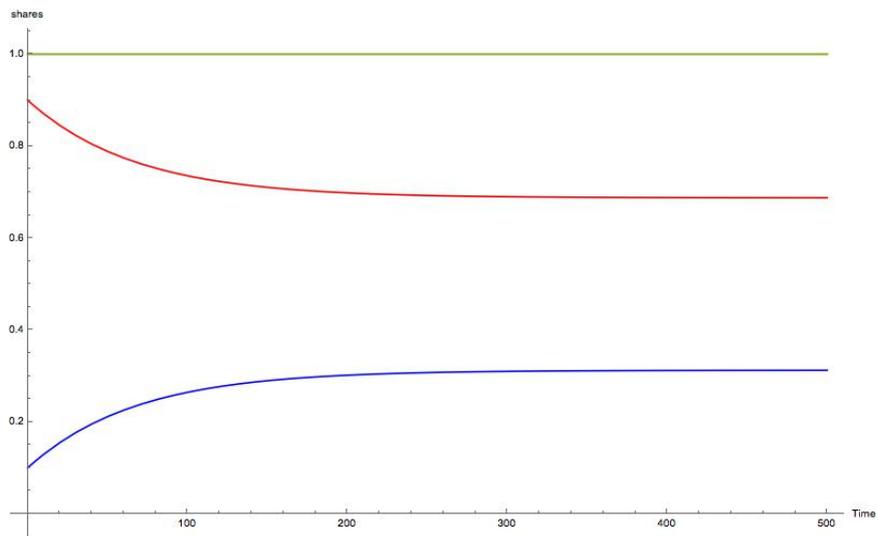


Fig. 5.  $x_1^0 = \frac{3}{5}$ ,  $x_2^0 = \frac{2}{5}$ . Firm 1 is initially stronger on the market. The market is initially saturated.

at time  $t \approx 2$  and, then it decreases because of predatory effects. In Figure 1 the *stronger* firm's market share is initially smaller than the opponent's one and market shares become equal at  $t \approx 16$ . If the market size is saturated, then any generic

marketing effort is uninfluential in extending the market. Although firm 1 is more effective in marketing campaigns and its initial market position is very dominant, its market share decreases (Figure 5). The latter happens when the difference between two initial market shares is high. If this difference is not high enough, then firm 1 keeps on improving its market share by making the opponent's market position weaker (Figures 3–4).

## 7. Conclusions, limitations and future work

We develop a novel duopolistic advertising model describing changes to the market shares in a duopoly caused by strategic investments in generic and predatory advertising. We provide the existence of optimal investment strategies for firms with lower production costs due to marketing incentives. We discuss marketing incentives due to advertising by varying the parameters of the model. We provide numerical insights on market share optima in an asymmetrical marketing scenario. We provide rational insights on how competing firms might ultimately reduce the quality of manufactured goods when they publish the prices at the beginning of marketing campaigns. Our work has some limitations. Optimal advertising investments are not in a closed-form formula. Due to the lack of a closed form expression of these equilibria, we cannot study the sensitivity of optimal strategies to a change in marketing parameters. Our main existence result lies in the assumption that the unique discount rate is small. A natural question arising from this paper is, if firms are given different discount rates, can we extend our main existence result? Future research should consider parameterising this model using advanced econometric models in order to estimate marketing incentives in our model. The latter will help to measure the quality of products provided by firms competing during informative marketing campaigns.

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**Appendix: Proofs**

**Proof of Proposition 1.** Analysing the first equation of system (5) we can see that it is possible to separate the variables and integrate it as follows

$$\begin{aligned}\frac{dx}{dt} &= 2A(1-x) \\ x(t) &= 1 - C_x e^{-2At}.\end{aligned}$$

If at  $t = 0$  we have  $x(0) = x_1^0 + x_2^0$ , then  $C_x = 1 - x_1^0 - x_2^0$ , and hence

$$x(t) = 1 + (x_1^0 + x_2^0 - 1) e^{-2At}.$$

For the function  $w(t)$  we have

$$\frac{dw}{dt} = \rho_1 b_1(u_1)(x-w) - \rho_2 b_2(u_2)(x+w) = B^- x(t) - B^+ w(t).$$

It is a linear equation of the first order. We integrate it using the standard approach and obtain that the exact solution is given as

$$w(t) = e^{-B^+ t} \int e^{B^+ t} B^- x(t) dt + e^{-B^+ t} C_u.$$

Therefore, for the function  $w(t)$  we have

$$\begin{aligned}w(t) &= e^{-B^+ t} \int e^{B^+ t} B^- x(t) dt + e^{-B^+ t} C_u \\ &= e^{-B^+ t} \int e^{B^+ t} B^- ((1 + (x_1^0 + x_2^0 - 1) e^{-2At})) dt + e^{-B^+ t} C_u \\ &= e^{-B^+ t} \left( B^- \int e^{B^+ t} dt + B^- (x_1^0 + x_2^0 - 1) \int e^{(B^+ - 2A)t} dt \right) + e^{-B^+ t} C_u \\ &= e^{-B^+ t} \left( \frac{B^-}{B^+} e^{B^+ t} + \frac{B^- (x_1^0 + x_2^0 - 1)}{(B^+ - 2A)} (e^{(B^+ - 2A)t} - 1) \right) + e^{-B^+ t} C_u \\ &= \frac{B^-}{B^+} + \frac{B^- (x_1^0 + x_2^0 - 1)}{(B^+ - 2A)} (e^{-2At} - e^{-B^+ t}) + e^{-B^+ t} C_u. \quad \square\end{aligned}$$

**Proof of Proposition 2.** Since  $x_1 = \frac{x+w}{2}$  and  $x_2 = \frac{x-w}{2}$  we obtain

$$\begin{aligned}x_1(t) &= \frac{x(t) + w(t)}{2} = \\ &= \frac{1}{2} \left( 1 + (x_1^0 + x_2^0 - 1) e^{-2At} + \frac{B^-}{B^+} + \frac{B^- (x_1^0 + x_2^0 - 1)}{(B^+ - 2A)} (e^{-2At} - e^{-B^+ t}) + e^{-B^+ t} C_u \right) \\ &= \frac{1}{2} \left( \frac{B^+ + B^-}{B^+} + \frac{(x_1^0 + x_2^0 - 1)(B^+ + B^- - 2A)}{(B^+ - 2A)} e^{-2At} + (C_u - \frac{(x_1^0 + x_2^0 - 1)(B^-)}{(B^+ - 2A)}) e^{-B^+ t} \right) \\ &= \frac{\rho_1 b_1(u_1)}{B^+} + \frac{(x_1^0 + x_2^0 - 1)(\rho_1 b_1(u_1) - A)}{(B^+ - 2A)} e^{-2At} + \frac{C_u}{2} e^{-B^+ t}.\end{aligned}$$

Let us now substitute the initial condition and find the constant  $C_u$ . If  $x_1(0) = x_1^0$  then

$$C_u/2 = 2 \left( x_1^0 - \frac{\rho_1 b_1(u_1)}{B^+} - \frac{(x_1^0 + x_2^0 - 1)(-2\rho_1 b_1(u_1) + 2A + B^-)}{2(B^+ - 2A)} \right)$$

Therefore,

$$\begin{aligned} x_1(t) &= \frac{\rho_1 b_1(u_1)}{B^+} + \frac{(x_1^0 + x_2^0 - 1)(\rho_1 b_1(u_1) - A)}{(B^+ - 2A)} e^{-2At} \\ &+ \left( x_1^0 - \frac{\rho_1 b_1(u_1)}{B^+} - \frac{(x_1^0 + x_2^0 - 1)(2\rho_1 b_1(u_1) - 2A)}{2(B^+ - 2A)} \right) e^{-B^+t} = \\ &= \frac{\rho_1 b_1(u_1)}{B^+} (1 - e^{-B^+t}) + \frac{(x_1^0 + x_2^0 - 1)(\rho_1 b_1(u_1) - A)}{(B^+ - 2A)} (e^{-2At} - e^{-B^+t}) + x_1^0 e^{-B^+t}. \end{aligned}$$

For  $x_2(t)$  we have a similar result.

$$\begin{aligned} x_2(t) &= \frac{\rho_2 b_2(u_2)}{B^+} + \frac{(x_1^0 + x_2^0 - 1)(\rho_2 b_2(u_2) - A)}{(B^+ - 2A)} e^{-2At} \\ &- \left( -x_2^0 + \frac{\rho_2 b_2(u_2)}{B^+} + \frac{(x_1^0 + x_2^0 - 1)(\rho_2 b_2(u_2) - A)}{(B^+ - 2A)} \right) e^{-B^+t} \\ &= \frac{\rho_2 b_2(u_2)}{B^+} (1 - e^{-B^+t}) + \frac{(x_1^0 + x_2^0 - 1)(\rho_2 b_2(u_2) - A)}{(B^+ - 2A)} (e^{-2At} - e^{-B^+t}) + x_2^0 e^{-B^+t}. \end{aligned}$$

We therefore have obtained the formulae for the evolution of market shares given the investment rates of firms 1,2. We have

$$\begin{aligned} \pi^1(u_1, u_2) &= \lim_{T \rightarrow \infty} \int_0^T e^{-\rho t} (r_1 x_1(t) - u_1) dt \\ &= \lim_{T \rightarrow \infty} \int_0^T \left( \frac{r_1 \rho_1 b_1(u_1)}{B^+} - u_1 \right) e^{-\rho t} dt \\ &+ \int_0^T \frac{r_1 (x_1^0 + x_2^0 - 1)(\rho_1 b_1(u_1) - A)}{(B^+ - 2A)} e^{-\rho t} e^{-2At} dt \\ &- \int_0^T r_1 \left( \frac{\rho_1 b_1(u_1)}{B^+} + \frac{(x_1^0 + x_2^0 - 1)(\rho_1 b_1(u_1) - A)}{(B^+ - 2A)} - x_1^0 \right) e^{-\rho t} e^{-B^+t} dt \\ &= \frac{1}{\rho} \left( \frac{r_1 \rho_1 b_1(u_1)}{B^+} - u_1 \right) + \frac{r_1 (x_1^0 + x_2^0 - 1)(\rho_1 b_1(u_1) - A)}{(2A + \rho)(B^+ - 2A)} \\ &- \frac{r_1}{(B^+ + \rho)} \left( \frac{\rho_1 b_1(u_1)}{B^+} + \frac{(x_1^0 + x_2^0 - 1)(\rho_1 b_1(u_1) - A)}{(B^+ - 2A)} - x_1^0 \right) \\ &= \frac{r_1}{(B^+ + \rho)} \left( -\frac{\rho_1 b_1(u_1)}{\ln \beta} + \frac{(x_1^0 + x_2^0 - 1)(\rho_1 b_1(u_1) - A)}{(2A + \rho)} + x_1^0 \right) - \frac{u_1}{\rho}. \end{aligned}$$

## 18 REFERENCES

The expression of  $\pi^2(u_1, u_2)$  can be obtained in the same way. So we have derived explicit formulae for payoffs.  $\square$

**Proof of Lemma 1.** We assume that  $\rho \neq 0$ . Consider the exact form of the fourth coefficient and substitute the values for

$$\theta = \rho_2 \frac{u_2}{u_2 + \beta_2} \quad \text{and} \quad \vartheta = \frac{u_2}{u_2 + \alpha_2}.$$

Multiplying the fourth coefficient by  $\frac{(u_2 + \alpha_2)^2 (u_2 + \beta_2)^2}{\alpha_1 \beta_1}$  and representing the expression  $P(u_2) = \frac{(u_2 + \alpha_2)^2 (u_2 + \beta_2)^2}{\alpha_1 \beta_1} (2bc - 2r_1 dc + 2r_1 gh)$  as a polynomial in  $u_2$ , we find that the sign of  $P(u_2)$  is equal to the sign of the fourth coefficient. We have the following polynomial expression

$$\begin{aligned} P(u_2) &= 2\alpha_2^2 \rho^2 \beta_2^2 \left( -2r_1 \rho_1 (1 - x_1^0) + \bar{o}(\rho) \right) \\ &\quad + u_2 \left( 2\alpha_2 \rho \beta_2 \left( -2\rho_1 r_1 (\beta_2 (1 - x_1^0 + x_2^0) + \alpha_2 \rho_2) + \bar{o}(\rho) \right) \right) \\ &\quad + u_2^2 \left( -8r_1 \rho_1 \alpha_2 \rho_2 \beta_2 + \bar{o}(\rho) \right) \\ &\quad + u_2^3 \left( 8\rho_2 \left( \beta_1 \alpha_2 \rho_2 - r_1 \rho_1 \alpha_2 + \rho_1 \alpha_1 \beta_2 - 2r_1 \rho_1 \beta_2 \right) + \bar{o}(\rho) \right) \\ &\quad + u_2^4 \left( 8\rho_2 \left( 2\rho_2 \beta_1 + \alpha_1 \rho_2 + \rho_1 \alpha_1 - 2r_1 \rho_1 \right) + \bar{o}(\rho) \right) \end{aligned}$$

where  $f(\rho) = \bar{o}(\rho)$  is such that  $\lim_{\rho \rightarrow 0} \frac{f(\rho)}{\rho} = 0$ . By hypothesis we know that  $\rho \approx 0$ , then  $\bar{o}(\rho)$  does not contribute to the signs of coefficients. If the following system of equations

$$\begin{cases} 8\rho_2 (\beta_1 \alpha_2 \rho_2 - r_1 \rho_1 \alpha_2 + \rho_1 \alpha_1 \beta_2 - 2r_1 \rho_1 \beta_2) < 0 \\ 8\rho_2 (2\rho_2 \beta_1 + \alpha_1 \rho_2 + \rho_1 \alpha_1 - 2r_1 \rho_1) < 0 \end{cases}$$

or, in equivalent way,

$$\begin{cases} r_1 > \frac{\beta_1 \alpha_2 \rho_2 + \rho_1 \alpha_1 \beta_2}{\rho_1 (\alpha_2 + 2\beta_2)} \\ r_1 > \frac{\rho_2}{\rho_1} \beta_1 + \frac{\rho_2 + \rho_1}{2\rho_1} \alpha_1. \end{cases} \quad (19)$$

is satisfied, then the number of sign alterations of polynomial equation  $P(u_2) = 0$  is zero. Since

$$\frac{\rho_2}{\rho_1} \beta_1 + \frac{\rho_2 + \rho_1}{2\rho_1} \alpha_1 - \frac{\beta_1 \alpha_2 \rho_2 + \rho_1 \alpha_1 \beta_2}{\rho_1 (\alpha_2 + 2\beta_2)} = \frac{1}{2} \frac{4\beta_2 \rho_2 \beta_1 + \alpha_2 \rho_1 \alpha_1 + \alpha_2 \alpha_1 \rho_2 + 2\beta_2 \alpha_1 \rho_2}{\rho_1 (\alpha_2 + 2\beta_2)} > 0$$

then we conclude that the second condition is stronger than the first in (19). By hypothesis the second condition in (19) is satisfied. Then the first condition in (19)

is satisfied. It thus follows there are no positive roots for the equation  $P(u_2) = 0$ . It is straightforward to verify that  $P(0) < 0$ . By continuity argument, it thus follows that  $P(u_2) < 0$  for any  $u_2 \geq 0$ . Then the third coefficient is strictly negative for any  $u_2 \geq 0$ .  $\square$

**Proof of Lemma 2.** Arguing as in the proof of Lemma 1, we consider

$$P(u_2) = \frac{(u_2 + \alpha_2)^2 (u_2 + \beta_2)^2}{\alpha_1 \beta_1} (r_1 g b - r_1 f c + c^2).$$

Here

$$\begin{aligned} P(u_2) &= \rho^3 \alpha_2^2 \beta_2^2 (-r_1 (\alpha_1 \rho_1 x_2^0 - \beta_1 x_2^0 + \beta_1 - \beta_1 x_1^0) + \bar{o}(\rho)) + u_2 \rho^2 \beta_2 \alpha_2 \\ &\quad + [-r_1 [\alpha_2 \beta_1 \rho_2 (1 - x_1^0 - x_2^0) + \beta_2 \rho_1 \alpha_1 (1 - x_1^0 + 3x_2^0) + \alpha_1 \alpha_2 \rho_2 \rho_1 (x_1^0 + x_2^0)] + \bar{o}(\rho)] \\ &\quad + u_2^2 \rho (-2\rho_1 \alpha_1 r_1 \beta_2 (\beta_2 (1 - x_1^0 + x_2^0) + \alpha_2 \rho_2 (1 + x_1^0 + x_2^0)) + \bar{o}(\rho)) \\ &\quad + u_2^3 (-4r_1 \rho_2 \alpha_1 \rho_1 \beta_2 + \bar{o}(\rho)) \\ &\quad + u_2^4 (-4\alpha_1 \rho_2 (r_1 \rho_1 - \beta_1 \rho_2) + \bar{o}(\rho)). \end{aligned}$$

Therefore we impose the set of following conditions

$$\begin{cases} -r_1 (\alpha_1 \rho_1 x_2^0 + \beta_1 (1 - x_1^0 - x_2^0)) < 0 \\ -r_1 (\alpha_2 \beta_1 \rho_2 (1 - x_1^0 - x_2^0) + \beta_2 \rho_1 \alpha_1 (1 - x_1^0 + 3x_2^0) + \alpha_1 \alpha_2 \rho_2 \rho_1 (x_1^0 + x_2^0)) < 0 \\ -2\rho_1 \alpha_1 r_1 \beta_2 (\beta_2 (1 - x_1^0 + x_2^0) + \alpha_2 \rho_2 (1 + x_1^0 + x_2^0)) < 0 \\ -4r_1 \rho_2 \alpha_1 \rho_1 \beta_2 < 0 \\ -4\alpha_1 \rho_2 (r_1 \rho_1 - \beta_1 \rho_2) < 0 \end{cases}$$

which is trivially equivalent to  $r_1 \rho_1 - \beta_1 \rho_2 > 0$ . If  $r_1 > \beta_1 \frac{\rho_2}{\rho_1}$  then the fifth coefficient is negative for any non negative value of  $\hat{u}_2$ .  $\square$

**Proof of Lemma 3.** Let  $\hat{u}_1$  be a best reply to  $\hat{u}_2$ . Then  $\hat{u}_1 > 0$  is a positive root of equation (11). According to the Descartes's rule of sign alterations, if the number of sign alterations in the sequence of the coefficients of equations of a polynomial equation is equal to 1, then there exists exactly one positive root of the equation (Korn and Korn, 1968). Since the first and the second coefficients in left-side of equation (11) are positive, we obtain the different cases in Table 2. Cases are reduced by assuming that the third coefficient is not null in Table 2. If this happens the number of sign alterations may be just lower to the previous case, i.e. the third coefficient is not null. This happens since the first two coefficients of (11) are not strictly negative. We clarify this ambiguity for the first four rows of Table 2. Let us suppose that the third coefficient is null then, the second coefficient is positive. If it is strictly positive the number of alterations does not decrease. If the second coefficient is null, then the first coefficient of (11) is strictly positive, i.e.  $h^2 > 0$ , since  $\rho_1 > 0$ . Then, one of the first three coefficients is at least not null.

## 20 REFERENCES

Therefore The four rows of Table 2 do not present any change of the number of sign alterations if the third coefficient is null. For the remaining last four rows of Table 2, it is straightforward to prove that the number of sign alteration may be lower if the third coefficient is null. In particular the number of sign alterations is 0 in the row 5 and it is 1 in the row 6. Table 2 fully represents the number of sign alterations of (11).

Third coefficient	Fourth coefficient	Fifth coefficient	Sign alterations
$> 0$	$> 0$	$> 0$	0
$> 0$	$> 0$	$< 0$	1
$> 0$	$< 0$	$> 0$	2
$> 0$	$< 0$	$< 0$	1
$< 0(= 0)$	$> 0$	$> 0$	2 (0)
$< 0(= 0)$	$> 0$	$< 0$	3 (1)
$< 0$	$< 0$	$> 0$	2
$< 0$	$< 0$	$< 0$	1

Table 2. Number of sign alterations in equation (11)

Taking into account our hypothesis, theses of Lemmas 1, 2 hold. Therefore the fourth and the fifth coefficients are strictly negative. It follows from Table 2 that  $\hat{u}_1 > 0$  is the unique best reply to  $\hat{u}_2$ . If there is exactly one positive root of the numerator of (12), and since the denominator is positive for any  $u_1 > 0$ , the derivative of payoffs takes negative values for  $u_1 > \hat{u}_1$  and positive values for  $u_1 < \hat{u}_1$ . This proves that  $\hat{u}_1$  is a maximum point of  $\pi^1(u_1, \hat{u}_2)$ .  $\square$

**Proof of Lemma 4.** By hypothesis, the thesis of Lemmas 3 is satisfied. Therefore, best reply multifunctions  $\hat{U}_i$  are functions. In addition,  $\hat{U}_1$  are continuous since payoffs in (8) are continuous. The rest of the proof is just technical and we leave it to the reader. The proof is simply based on convergence properties of  $a_i, b_i$  when  $u$  converges to  $\infty$ . In fact, we have  $\lim_{u_2 \rightarrow \infty} \theta(u_2) = \rho_2$  and  $\lim_{u_2 \rightarrow \infty} \vartheta(u_2) = 1$ .  $\square$