Article:

Bury, RT and Mikhailov, AV (2021) Automorphic Lie algebras and corresponding integrable systems. Differential Geometry and its Applications, 74. 101710. ISSN 0926-2245

https://doi.org/10.1016/j.difgeo.2020.101710

Automorphic Lie algebras and corresponding integrable systems.

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December 11, 2020

Abstract

We study automorphic Lie algebras and their applications to integrable systems. Automorphic Lie algebras are a natural generalisation of celebrated Kac-Moody algebras to the case when the group of automorphisms is not cyclic. They are infinite dimensional and almost graded. We formulate the concept of a graded isomorphism and classify $sl(2,\mathbb{C})$ based automorphic Lie algebras corresponding to all finite reduction groups. We show that hierarchies of integrable systems, their Lax representations and master symmetries can be naturally formulated in terms of automorphic Lie algebras.

1 Introduction

The integrability of a nonlinear partial differential or a differential difference equation can often be related to the existence of a corresponding Lax representation. Having a Lax operator we can construct an infinite hierarchy of commuting symmetries, local conservation laws and find exact multi-soliton solutions. It enables us to find a recursion operator and a multi-Hamiltonian structure for the corresponding equation. Symmetries, local conservation laws, recursion operators and multi-Hamiltonian structures are fundamental properties of integrable equations [1], [2]. Integration of such equations can be reduced to a direct and inverse spectral transform associated with the Lax operator.

Symmetries of the Lax operator play a key role in the spectral transform and are reflected in all structures associated with the corresponding integrable equation. Discrete groups of automorphisms of Lax operators, the *reduction groups*, were introduced in [3], [4], [5]. Reduction groups have been extensively applied for the construction of new integrable systems, recursion operators, R matrices, for the classification of soliton solutions, and the spectral theory of Lax operators (see for example [4], [5], [6], [7], [8], [9], [10]).

Often the structure of Lax operators have a natural Lie algebraic interpretation in terms of Kac-Moody algebras [11]. A new class of Lie algebras over rings of automorphic functions, which can be also regarded as infinite dimensional Lie algebras over $\mathbb{C}$, was proposed in [7]. These algebras have been further studied in [12] where they acquired the name *automorphic Lie algebras* (see also [13]). Automorphic Lie algebras are a natural generalisation of Kac-Moody algebras. While a Kac-Moody algebra can be seen as a subalgebra of a loop algebra, which is invariant with respect to a cyclic group of a finite order automorphism (the Coxeter automorphism [11]), an automorphic Lie algebra is a subalgebra of a generalised loop algebra which is invariant with respect to a reduction group (the reduction group can be non-cyclic, noncommutative, and it can be infinite).

Automorphic Lie algebras are infinite dimensional (over the field $\mathbb{C}$), they are almost graded and can be characterised by a finite set of structure constants. They have a structure of a finitely generated $\mathbb{C}[J]$–Lie module, where J is a primitive automorphic function. The classification of automorphic Lie algebras is part of the programme of classification of Lax operators and hence of integrable systems. The problem of classification of automorphic Lie algebras corresponding to finite reduction groups had been extensively studied in [14] and independently in [15]. An alternative approach to automorphic

Lie algebras and further development can be found in [16], [17]. Automorphic Lie algebras have found further applications to construction of differential-difference and partial-difference integrable systems and Yang-Baxter maps [18], [19].

In this paper we define the concept of a graded isomorphism of almost graded algebras. It is stronger than isomorphism and can be effectively verified. We also study automorphic Lie algebras related to the simple Lie algebra A_1 and finite reduction groups. We show that there are five types of non-isomorphic algebras which include the polynomial part of the A_1 loop algebra, the polynomial part of the Kac-Moody algebra A_1^1 and three others. Explicit realisation of these algebras in terms of finitely generated $\mathbb{C}[J]$–Lie modules is presented in Section 3.3. We discuss the construction of Lax operators, corresponding integrable hierarchies and master symmetries in terms of automorphic Lie algebras and illustrate it with examples.

2 Kac-Moody and automorphic Lie algebras

The construction of automorphic Lie algebras is similar to the construction used in the theory of Kac-Moody Lie algebras. While a Kac-Moody algebra can be realised as a subalgebra of a loop algebra, which is invariant with respect to a cyclic group generated by an automorphism of a finite order, an automorphic Lie algebra can be viewed as a subalgebra of a simple Lie algebra over the field $\mathbb{C}(\lambda)$, which is invariant with respect to a finite group of automorphisms. Automorphic Lie algebras can also be defined for infinite groups, but in this paper we focus on the case of finite groups.

Let $\Gamma = \{\mu_k \in \mathbb{C}\}$ denote a finite set of points and $\mathcal{R}_\lambda(\Gamma)$ denote a ring of rational functions of the variable λ with poles at $\lambda = \mu_k,\ \mu_k \in \Gamma$ and regular elsewhere. In this notation the ring of polynomials $\mathbb{C}[\lambda] = \mathcal{R}_\lambda(\infty)$ and the ring of Laurent polynomials $\mathbb{C}[\lambda^{-1}, \lambda] = \mathcal{R}_\lambda(0, \infty)$.

Let $\mathfrak{A}$ be a simple Lie algebra over $\mathbb{C}$ and

$$\mathfrak{A}_\lambda(\Gamma) = \mathcal{R}_\lambda(\Gamma) \otimes_{\mathbb{C}} \mathfrak{A}\,. \tag{1}$$

Then $\mathfrak{A}_\lambda(\Gamma)$ may be made into a Lie algebra in a unique way satisfying

$$[p \otimes a, q \otimes b] = pq \otimes [a, b]$$

for $p, q \in \mathcal{R}_\lambda(\Gamma),\ a, b \in \mathfrak{A}$. In particular, the algebra $\mathfrak{A}_\lambda(0, \infty) = \mathcal{R}_\lambda(0, \infty) \otimes_{\mathbb{C}} \mathfrak{A}$ is called the *loop algebra* [20]. Elements $a(\lambda) \in \mathfrak{A}_\lambda(0, \infty)$ are Laurent polynomials $\sum_{n\in\mathbb{Z}} \lambda^n a_n$ where $a_n \in \mathfrak{A}$ with finitely many $a_n \neq 0$. We shall call the algebra $\mathfrak{A}_\lambda(\Gamma)$ a generalised loop algebra.

2.1 Kac-Moody algebras

Let $\phi_1 : \mathfrak{A} \to \mathfrak{A}$ be an automorphism of a finite order n, then $\Phi_1 : \mathfrak{A}_\lambda(0, \infty) \to \mathfrak{A}_\lambda(0, \infty)$, defined for any $a(\lambda) \in \mathfrak{A}_\lambda(0, \infty)$ as

$$\Phi_1(a(\lambda)) = \phi_1(a(\omega^{-1}\lambda))\,, \qquad \omega = \exp(\frac{2\pi i}{n}) \tag{2}$$

is an automorphism of $\mathfrak{A}_\lambda(0, \infty)$. The automorphism Φ_1 is of order n and thus it generates a cyclic group of automorphisms $\mathcal{G} = \langle \Phi_1\,;\, \Phi_1^n = \mathrm{id}\,\rangle \simeq \mathbb{Z}/n\mathbb{Z}$.

A Kac-Moody algebra $L(\mathfrak{A}, \phi_1)$ can be defined[1] as a subalgebra of $\mathfrak{A}_\lambda(0, \infty)$ invariant with respect to the cyclic group of automorphisms $\mathcal{G}$

$$L(\mathfrak{A}, \phi_1) = \{a(\lambda) \in \mathfrak{A}_\lambda(0, \infty) \,|\, a(\lambda) = \phi_1(a(\omega^{-1}\lambda))\}\ . \tag{3}$$

[1]There are many comprehensive monographs and textbooks presenting the theory of Kac-Moody algebras (see for example [21], [20]). We shall adopt definitions and some notations from [11] (Section 5) where the most convenient and useful (for our purposes) exposition of Kac-Moody algebras is given. In [11] and in this paper Kac-Moody algebras are assumed to be centreless, i.e. are quotients of the corresponding affine Lie algebras over their centres.

We have $L(\mathfrak{A},\phi_1)=\sum_{k\in\mathbb{Z}}\lambda^k\mathfrak{A}_k$ where $\mathfrak{A}_k=\{a\in\mathfrak{A}\,|\,\phi_1(a)=\omega^k a\}$ and define $L^k(\mathfrak{A},\phi_1)=\lambda^k\mathfrak{A}_k$. It is a graded Lie algebra

$$L(\mathfrak{A},\phi_1)=\bigoplus_{k\in\mathbb{Z}}L^k(\mathfrak{A},\phi_1),\qquad [L^k(\mathfrak{A},\phi_1),L^m(\mathfrak{A},\phi_1)]\subset L^{k+m}(\mathfrak{A},\phi_1)\,.$$

We can also consider two subalgebras $L_\pm(\mathfrak{A},\phi_1)\subset L(\mathfrak{A},\phi_1)$ of polynomials in λ and λ^{-1}:

$$L_+(\mathfrak{A},\phi_1)=\{a\in\mathfrak{A}_\lambda(\infty)\,|\,a=\Phi_1(a)\},\quad L_-(\mathfrak{A},\phi_1)=\{a\in\mathfrak{A}_\lambda(0)\,|\,a=\Phi_1(a)\}\,.$$

The subalgebras $L_\pm(\mathfrak{A},\phi_1)$ are isomorphic and they cover $L(\mathfrak{A},\phi_1)$:

$$L_-(\mathfrak{A},\phi_1)\bigcup L_+(\mathfrak{A},\phi_1)=L(\mathfrak{A},\phi_1)\,,\quad L_-(\mathfrak{A},\phi_1)\bigcap L_+(\mathfrak{A},\phi_1)=\mathfrak{A}_0\,. \tag{4}$$

Example: In $\mathfrak{A}=sl(2,\mathbb{C})$ we take the standard (Cartan-Weyl) basis $\mathbf{e},\mathbf{f},\mathbf{h}$

$$\mathbf{e}=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \mathbf{f}=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad \mathbf{h}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

with commutation relations

$$[\mathbf{e},\mathbf{f}]=\mathbf{h},\quad [\mathbf{h},\mathbf{e}]=2\mathbf{e},\quad [\mathbf{h},\mathbf{f}]=-2\mathbf{f}\,. \tag{5}$$

We define the automorphism Φ_1 of order 2 as

$$\Phi_1(a(\lambda))=\begin{pmatrix}1&0\\0&-1\end{pmatrix}a(-\lambda)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\,. \tag{6}$$

Then $\mathfrak{A}_{2k-1}=\mathrm{Span}_{\mathbb{C}}\,(\mathbf{e},\mathbf{f}),\ \mathfrak{A}_{2k}=\mathrm{Span}_{\mathbb{C}}(\mathbf{h}),\ k\in\mathbb{Z}$ and

$$L(\mathfrak{A},\phi_1)=\bigoplus_{k\in\mathbb{Z}}\mathfrak{A}_k\lambda^k,\quad L_+(\mathfrak{A},\phi_1)=\bigoplus_{k\geqslant 0}\mathfrak{A}_k\lambda^k,\quad L_-(\mathfrak{A},\phi_1)=\bigoplus_{k\leqslant 0}\mathfrak{A}_k\lambda^k\,. \tag{7}$$

The algebra $L(\mathfrak{A},\phi_1)$ is isomorphic to the loop algebra $\mathfrak{A}_\lambda(0,\infty)$. Indeed, the set $\{e_k=\lambda^k\mathbf{e},f_k=\lambda^k\mathbf{f},h_k=\lambda^k\mathbf{h}\}_{k\in\mathbb{Z}}$ is a basis in $\mathfrak{A}_\lambda(0,\infty)$ with non-vanishing commutation relations

$$[e_k,f_p]=h_{k+p},\quad [h_k,e_p]=2e_{k+p},\quad [h_k,f_p]=-2f_{k+p},\qquad k,p\in\mathbb{Z}. \tag{8}$$

In $L(\mathfrak{A},\phi_1)$ one can take the basis $\{e^k=\lambda^{2k+1}\mathbf{e},f^k=\lambda^{2k-1}\mathbf{f},h^k=\lambda^{2k}\mathbf{h}\}_{k\in\mathbb{Z}}$ and verify that the commutators of its elements are exactly the same as in (8).

This is an illustration of a general Theorem (V.Kac [21]) that for any simple Lie algebra $\mathfrak{A}$ and a finite order inner automorphism ϕ_1 the corresponding Kac-Moody algebra $L(\mathfrak{A},\phi_1)$ is isomorphic to the loop algebra $\mathfrak{A}_\lambda(0,\infty)$. Here we would like to stress the fact that the subalgebras $L_\pm(\mathfrak{A},\phi_1)$ and $\mathfrak{A}_0$ in the coverage (4) depend on the choice of the automorphism and that is of importance to our applications to integrable systems.

2.2 Automorphic Lie algebras

The map

$$g_1:\mathcal{R}_\lambda(0,\infty)\to\mathcal{R}_\lambda(0,\infty),\quad g_1(\alpha(\lambda))=\alpha(\omega^{-1}\lambda),\quad \alpha(\lambda)\in\mathcal{R}_\lambda(0,\infty),\quad \omega=\exp\left(\frac{2\pi i}{n}\right) \tag{9}$$

is an automorphism of order n of the ring $\mathcal{R}_\lambda(0,\infty)$. The ring $\mathcal{R}_\lambda(0,\infty)$ has another automorphism g_2 of order 2

$$g_2:\mathcal{R}_\lambda(0,\infty)\to\mathcal{R}_\lambda(0,\infty),\quad g_2(\alpha(\lambda))=\alpha(\lambda^{-1}),\quad \alpha(\lambda)\in\mathcal{R}_\lambda(0,\infty)\,. \tag{10}$$

The automorphisms g_1 and g_2 generate a subgroup $G \subset \operatorname{Aut} \mathcal{R}_\lambda(0,\infty)$ which is isomorphic to $\mathbb{D}_n$ - the group of a dihedron with n vertices. Indeed, we have $g_1^n = g_2^2 = \mathrm{id}$ and it is easy to verify that $g_1 g_2 g_1 g_2 = \mathrm{id}$, thus

$$G = \langle g_1, g_2\, ;\, g_1^n = g_2^2 = g_1 g_2 g_1 g_2 = \mathrm{id}\,\rangle \simeq \mathbb{D}_n\,. \tag{11}$$

The order $|\mathbb{D}_n| = 2n$. In the case $n = 2$ the group is commutative and $\mathbb{D}_2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ (the group of Klein).

The subring of all G–invariant (or automorphic) Laurent polynomials is given by

$$\mathcal{R}_\lambda^G(0,\infty) = \{\alpha \in \mathcal{R}_\lambda(0,\infty)\,|\, g_1(\alpha) = g_2(\alpha) = \alpha\}.$$

The ring $\mathcal{R}_\lambda^G(0,\infty) = \mathbb{C}[J]$, where $J = \frac{1}{2}(\lambda^n + \lambda^{-n}) \in \mathcal{R}_\lambda(0,\infty)$ is an automorphic Laurent polynomial. Moreover J is a primitive automorphic function of the group G, in the sense that any automorphic rational function of λ is a rational function of J [12].

Let ϕ_1, ϕ_2 be two automorphisms of $\mathfrak{A}$ satisfying the conditions $\phi_1^n = \phi_2^2 = \phi_1\phi_2\phi_1\phi_2 = \mathrm{id}$. Then Φ_1 (defined in (2)) and $\Phi_2 : \mathfrak{A}_\lambda(0,\infty) \to \mathfrak{A}_\lambda(0,\infty)$

$$\Phi_2(a(\lambda)) = \phi_2(a(\lambda^{-1}))\,, \qquad a(\lambda) \in \mathfrak{A}_\lambda(0,\infty) \tag{12}$$

generate a subgroup $\mathcal{G} = \langle \Phi_1, \Phi_2\,;\, \Phi_1^n = \Phi_2^2 = \Phi_1\Phi_2\Phi_1\Phi_2 = \mathrm{id}\,\rangle \subset \operatorname{Aut}\mathfrak{A}_\lambda(0,\infty)$ (*a reduction group* [3]-[7]), which is isomorphic to the dihedral group $\mathcal{G} \simeq \mathbb{D}_n$. The subalgebra of $\mathfrak{A}_\lambda(0,\infty)$ invariant with respect to the group of automorphisms $\mathcal{G}$

$$\mathfrak{A}_\lambda^{\mathcal{G}}(0,\infty) = \{a(\lambda) \in \mathfrak{A}_\lambda(0,\infty)\,|\, a = \Phi_1(a) = \Phi_2(a)\} \tag{13}$$

is an example of an automorphic Lie algebra. In this example $\mathfrak{A}_\lambda^{\mathcal{G}}(0,\infty)$ is a subalgebra of the Kac-Moody Lie algebra $L(\mathfrak{A},\phi_1)$.

In order to formulate a general definition of automorphic Lie algebras we need to fix some notations. We will consider the groups G whose elements are Möbius (linear-fractional) transformations

$$g_k(\lambda) = \frac{\alpha_k\lambda + \beta_k}{\gamma_k\lambda + \delta_k}, \qquad \alpha_k\delta_k - \beta_k\gamma_k \neq 0, \quad \alpha_k, \beta_k, \gamma_k, \delta_k \in \mathbb{C}.$$

of the extended complex plane $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The group of all Möbius transformations is called the Möbius group which is isomorphic to $PSL(2,\mathbb{C}) \simeq SL(2,\mathbb{C})/\pm I$, where $SL(2,\mathbb{C})$ is a group of 2×2 matrices whose determinants are equal to 1, and I is the unit matrix. Indeed, if we associate a matrix

$$S_k = \begin{pmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{pmatrix} \tag{14}$$

with the Möbius transformation g_k, then the composition of transformations $g_p \cdot g_k$ corresponds to the product of the matrices S_pS_k. Matrices S_k and θS_k, $\theta \neq 0$, $\theta \in \mathbb{C}$ result in the same Möbius transformation.

In this paper we are interested in finite subgroups of the Möbius group. According to F.Klein [22], all finite subgroups of $PSL(2,\mathbb{C})$ are in the following list:

1. the additive group of integers modulo N, $\mathbb{Z}/N\mathbb{Z}$
2. the symmetry group of the dihedron with N vertices, $\mathbb{D}_N$
3. the symmetry group of the tetrahedron, $\mathbb{T}$
4. the symmetry group of the octahedron, $\mathbb{O}$
5. the symmetry group of the icosahedron, $\mathbb{I}$

In what follows we assume that G is a finite group of Möbius transformations. For any $\gamma_0 \in \bar{\mathbb{C}}$ we denote the *orbit* $G(\gamma_0) = \{g(\gamma_0)\,|\, g \in G\}$ and the *isotropy subgroup* $G_{\gamma_0} = \{g \in G\,|\, g(\gamma_0) = \gamma_0\}$. If the group G_{γ_0} is nontrivial, i.e. $|G_{\gamma_0}| > 1$, then the point γ_0 is called a fixed point of the group G of order $|G_{\gamma_0}|$. Points which are not fixed are called generic. Obviously, the number of points $|G(\gamma_0)| = |G|/|G_{\gamma_0}|$. If γ_0 is a fixed point of order n, then the corresponding orbit is called a degenerate orbit of degree n. We call orbits corresponding to generic points generic. Two points $\gamma_0, \gamma_1 \in \bar{\mathbb{C}}$ are said to be equivalent $\gamma_0 \sim \gamma_1$ if they belong to the same orbit (for non equivalent points $\gamma_0, \gamma_1 \in \bar{\mathbb{C}}$ we shall use the notation $\gamma_0 \not\sim \gamma_1$).

Möbius transformations induce automorphisms of the field of rational functions $\mathbb{C}(\lambda)$ defined as

$$g:\ f(\lambda) \to f(g^{-1}(\lambda)), \qquad f(\lambda) \in \mathbb{C}(\lambda). \tag{15}$$

If G is a finite group of Möbius transformations, then there exists a subfield $\mathbb{C}^G(\lambda)$ of G-invariant rational functions

$$\mathbb{C}^G(\lambda) = \{f \in \mathbb{C}(\lambda)\,|\, g(f) = f,\ \forall g \in G\}$$

Non constant elements of $\mathbb{C}^G(\lambda)$ are called rational automorphic functions of the group G. Moreover, there exists a *primitive* automorphic function $J \in \mathbb{C}^G(\lambda)$, such that any rational automorphic function is a rational function of J, or $\mathbb{C}^G(\lambda) = \mathbb{C}(J)$ (see [12]). The primitive automorphic function J is not uniquely defined - any non constant fractional linear function of J is a primitive automorphic function.

For finite groups automorphic functions can be easily constructed using the group average

$$\langle f(\lambda)\rangle_G = \frac{1}{|G|}\sum_{g\in G} f(g^{-1}(\lambda)).$$

If an automorphic function has a pole (or a zero) in λ at a point $\lambda = \mu$, then its order is divisible by $|G_\mu|$.

Let $\gamma_0 \in \mathbb{C}$, then

$$J_G(\lambda, \gamma_0) = \left\langle \frac{1}{(\lambda - \gamma_0)^{|G_{\gamma_0}|}} \right\rangle_G$$

is a primitive automorphic function which has poles of multiplicity $|G_{\gamma_0}|$ at the points of the orbit $G(\gamma_0)$. If $\gamma_0 = \infty$, then $J_G(\lambda, \infty) = \langle \lambda^{|G_\infty|}\rangle_G$.

Assuming $\gamma_0 \not\sim \gamma_1$ we define a primitive automorphic function

$$J_G(\lambda, \gamma_0, \gamma_1) = J_G(\lambda, \gamma_0) - J_G(\gamma_1, \gamma_0) \tag{16}$$

with poles of multiplicity $|G_{\gamma_0}|$ at points of the orbit $G(\gamma_0)$, zeros of multiplicity $|G_{\gamma_1}|$ at points of $G(\gamma_1)$ and no other poles or zeros. Any primitive automorphic function with a pole at γ_0 and a zero at γ_1 is proportional to $J_G(\lambda, \gamma_0, \gamma_1)$. The following Lemma summarises some useful properties of the function $J_G(\lambda, \gamma_0, \gamma_1)$.

Lemma 1. *Let G be a finite group, $\beta \not\sim \alpha$, $\beta \not\sim \gamma$ and $\beta \not\sim \delta$, then*

$$J_G(\alpha, \beta, \gamma) + J_G(\gamma, \beta, \alpha) = 0 \tag{17}$$

$$J_G(\alpha, \beta, \gamma) - J_G(\delta, \beta, \gamma) = J_G(\alpha, \beta, \delta) \tag{18}$$

$$J_G(\alpha, \beta, \gamma) J_G(\alpha, \gamma, \beta) = C(\beta, \gamma) \tag{19}$$

$$J_G(\alpha, \beta, \gamma) J_G(\alpha, \gamma, \delta) = J_G(\alpha, \beta, \delta) J_G(\beta, \gamma, \delta) \tag{20}$$

where $C(\beta, \gamma) = C(\gamma, \beta) \neq 0$ and $C(\beta, \gamma)$ does not depend on α.

Proof. Identity (17) follows from (18) if we take $\gamma \sim \alpha$, and (18) immediately follows from (16). The left hand side of (19) is a product of two rational functions of α. Poles of $J_G(\alpha, \beta, \gamma)$ are all at $\alpha \sim \beta$ and are canceled by the corresponding zeros of $J_G(\alpha, \gamma, \beta)$. Similarly poles of $J_G(\alpha, \gamma, \beta)$ are all at

$\alpha \sim \gamma$ and they are canceled by the corresponding zeros of $J_G(\alpha, \beta, \gamma)$. Thus the product is a rational automorphic function of α which does not have any poles. Therefore it is a constant function of α. The property $C(\beta, \gamma) = C(\gamma, \beta)$ is obvious from the symmetry. Identity (20) follows from (17)-(19):

$$J_G(\alpha,\beta,\gamma)J_G(\alpha,\gamma,\delta) = J_G(\alpha,\beta,\gamma)(J_G(\alpha,\gamma,\beta) - J_G(\delta,\gamma,\beta)) =$$
$$C(\beta,\gamma) - (J_G(\alpha,\beta,\delta) + J_G(\delta,\beta,\gamma))J_G(\delta,\gamma,\beta) = C(\beta,\gamma) + J_G(\alpha,\beta,\delta)J_G(\beta,\gamma,\delta) - C(\beta,\gamma)\,. \blacksquare$$

If the group G is clearly specified or the result is general and does not depend on the choice of a finite group G, we shall use a simplified notation by omitting the subscript G in $J_G(\alpha, \beta, \gamma)$.

Let $\Gamma = G(\gamma_0)$ be an orbit of a finite subgroup $G \subset PSL(2, \mathbb{C})$ and $\mathcal{R}_\lambda(\Gamma)$ the corresponding ring of rational functions with poles at Γ only. Then G is a group of automorphisms of $\mathcal{R}_\lambda(\Gamma)$. Indeed, the transformations (15) map $\mathcal{R}_\lambda(\Gamma) \to \mathcal{R}_\lambda(\Gamma)$ and respect the ring structure. The G-invariant subring

$$\mathcal{R}_\lambda^G(\Gamma) = \{a \in \mathcal{R}_\lambda(\Gamma) \,|\, g(a) = a, \forall g \in G\}$$

is the ring of polynomials $\mathbb{C}[J]$ of a primitive automorphic function $J = J_G(\lambda, \gamma_0)$.

Let us have a simple Lie algebra $\mathfrak{A}$, the corresponding generalised loop algebra $\mathfrak{A}_\lambda(\Gamma)$ (1) and a homomorphism $\Psi : G \to \operatorname{Aut}\mathfrak{A}_\lambda(\Gamma)$. We denote by $\mathcal{G}$ the image $\Psi(G)$ in $\operatorname{Aut}\mathfrak{A}_\lambda(\Gamma)$ and call it the *reduction group*. An element $\Phi_k \in \mathcal{G}$ can be viewed as a pair $\Phi_k = (g_k, \phi_k)$ consisting of a Möbius transformation g_k and an automorphism $\phi_k \in \operatorname{Aut}\mathfrak{A}$, which could depend on λ. The action of Φ_k on the elements of $\mathfrak{A}_\lambda(\Gamma)$ is similar to (12):

$$\Phi_k(a(\lambda)) = \phi_k(a(g_k^{-1}(\lambda)))\,, \qquad a(\lambda) \in \mathfrak{A}_\lambda(\Gamma).$$

When the elements $\phi_k \in \operatorname{Aut}\mathfrak{A}$ do not depend on λ we can (without loss of generality - see [12]) construct the reduction group as follows. Suppose we have a λ-independent homomorphism $\psi : G \to \operatorname{Aut}\mathfrak{A}$. For every element $g_k \in G$ we define $\Phi_k = (g_k, \psi_{g_k})$ and the reduction group $\mathcal{G}(G, \psi) = \{\Phi = (g, \psi_g) \,|\, g \in G\}$, which is a subgroup of the direct product $G \times \operatorname{Aut}\mathfrak{A}$ and is isomorphic to G.

The *automorphic Lie algebra* corresponding to the group $\mathcal{G}$ and the orbit Γ is the $\mathcal{G}$-invariant subalgebra $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma) = \{a \in \mathfrak{A}_\lambda(\Gamma) \,|\, \Phi(a) = a, \forall \Phi \in \mathcal{G}\}$.

More generally, let $\mathbf{\Gamma}$ be a G orbit of a finite set of points $\{\mu_1, \ldots, \mu_M \,|\, \mu_i \not\sim \mu_j\}$, which is then a union of M simple orbits $\mathbf{\Gamma} = \bigcup_{k=1}^M \Gamma_k$, i.e. orbits of a single point $\Gamma_k = G(\mu_k)$. The group G is still a group of automorphisms of the corresponding ring $\mathcal{R}_\lambda(\mathbf{\Gamma})$ and $\mathcal{G} = \Psi(G)$ is a group of automorphisms of the Lie algebra $\mathfrak{A}_\lambda(\mathbf{\Gamma})$. We shall call the $\mathcal{G}$–invariant subalgebra $\mathfrak{A}_\lambda^{\mathcal{G}}(\mathbf{\Gamma})$ the *automorphic Lie algebra corresponding to the reduction group* $\mathcal{G}$ *and simple orbits* $\Gamma_1, \ldots, \Gamma_M$. The subalgebras $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_k) \subset \mathfrak{A}_\lambda^{\mathcal{G}}(\mathbf{\Gamma})$, $k = 1, \ldots, M$ form a coverage of $\mathfrak{A}_\lambda^{\mathcal{G}}(\mathbf{\Gamma})$ in the following sense:

$$\mathfrak{A}_\lambda^{\mathcal{G}}(\mathbf{\Gamma}) = \bigcup_{k=1}^{M} \mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_k)\,, \qquad \mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_k) \bigcap \mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_n) = \mathfrak{A}^{\mathcal{G}}\,,\ k \neq n, \tag{21}$$

where $\mathfrak{A}^{\mathcal{G}} = \{a \in \mathfrak{A} \,|\, \Phi(a) = a,\ \forall \Phi \in \mathcal{G}\}$ is a $\mathcal{G}$–invariant subalgebra of the $\mathbb{C}$–algebra $\mathfrak{A}$.

In this sense, the Kac-Moody Lie algebra $L(\mathfrak{A}, \phi_1)$ is the automorphic Lie algebra corresponding to the cyclic group $\mathbb{Z}/n\mathbb{Z}$ and $\mathbf{\Gamma} = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \{\infty\}$ and $\Gamma_2 = \{0\}$ (of the Möbius transformation $g_1(\lambda) = \omega\lambda$). Its subalgebra $L_+(\mathfrak{A}, \phi_1)$ is a $\mathbb{Z}_n$–automorphic Lie algebra corresponding to one orbit Γ_1. Similarly, $L_-(\mathfrak{A}, \phi_1)$ corresponds to the orbit Γ_2. The algebra $\mathfrak{A}_\lambda^{\mathcal{G}}(0, \infty)$ (13) is the automorphic Lie algebra corresponding to the group $\mathcal{G} \simeq \mathbb{D}_n$ and a single degenerate orbit $\Gamma = \{0, \infty\}$ of degree n.

There is a natural projection $\mathcal{P}_{\mathcal{G}}$ of the linear space $\mathfrak{A}_\lambda(\mathbf{\Gamma})$ onto $\mathfrak{A}_\lambda^{\mathcal{G}}(\mathbf{\Gamma})$ given by the group average. For $a \in \mathfrak{A}_\lambda(\mathbf{\Gamma})$ we define $\mathcal{P}_{\mathcal{G}}(a) \in \mathfrak{A}_\lambda^{\mathcal{G}}(\mathbf{\Gamma})$ as

$$\mathcal{P}_{\mathcal{G}}(a) = \langle a \rangle_{\mathcal{G}} = \frac{1}{|\mathcal{G}|} \sum_{\Phi \in \mathcal{G}} \Phi(a)\,. \tag{22}$$

Obviously $\mathcal{P}_{\mathcal{G}}^2 = \mathcal{P}_{\mathcal{G}}$. The projection $\mathcal{P}_{\mathcal{G}} : \mathfrak{A}_\lambda(\mathbf{\Gamma}) \to \mathfrak{A}_\lambda^{\mathcal{G}}(\mathbf{\Gamma})$ is a surjective linear map, but it is *not* a Lie algebra homomorphism.

3 Automorphic Lie algebras in the case $\mathfrak{A} = sl(2,\mathbb{C})$

As above, G denotes a finite group of Möbius transformations. In the case $\mathfrak{A} = sl(2,\mathbb{C})$ it is well known ([23], [21], [20]) that all automorphisms are inner and can be represented in the form $a \to UaU^{-1}$ where $U \in GL(2,\mathbb{C})$. We shall denote such an automorphism as ϕ_U, where $\phi_U(a) = UaU^{-1}$. Thus $\mathrm{Aut}\,\mathfrak{A} \simeq PSL(2,\mathbb{C})$. Let us take any *injective*[2] homomorphism $\rho : G \to PSL(2,\mathbb{C})$ (which can be regarded as a faithful projective representation $\rho : G \mapsto \mathrm{End}\mathbb{C}^2$) and define a homomorphism $\psi_\rho : G \to \mathrm{Aut}(sl(2,\mathbb{C}))$ by its action on the Möbius transformations $g \in G:\ \psi_\rho(g) = \phi_{\rho(g)}$. Thus with any Möbius group G and a projective representation ρ we associate a reduction group $\mathcal{G} = \{\Phi_k = (g, \phi_{\rho(g)})\,|\, g \in G\}$.

In the case $\mathfrak{A} = sl(2,\mathbb{C})$ there is a natural homomorphism $\psi : G \to \mathrm{Aut}(sl(2,\mathbb{C}))$, namely $\psi(g_k) = \phi_{S_k}$, where $g_k \in G$ and S_k is the matrix (14) associated to the Möbius transformation g_k. We call the reduction group $\mathcal{G} = \{(g_k, \phi_{S_k})\,|\, g_k \in G\}$ the natural reduction group.

3.1 The case $G = \mathbb{D}_2$ and $\mathfrak{A} = sl(2,\mathbb{C})$

Without loss of generality we can represent the generators of the group G by the Möbius transformations

$$g_1(\lambda) = -\lambda, \qquad g_2(\lambda) = \lambda^{-1}.$$

Thus the (natural) reduction group $\mathcal{G} \sim \mathbb{D}_2$ is generated by the transformations

$$\Phi_1(a(\lambda)) = \mathbf{s}_3 a(-\lambda)\mathbf{s}_3, \qquad \Phi_2(a(\lambda)) = \mathbf{s}_1 a(\lambda^{-1})\mathbf{s}_1. \tag{23}$$

Here we use the notation

$$\mathbf{s}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{s}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{s}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{s}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The set $\{\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ is a two-dimensional irreducible projective representation of the group $\mathbb{D}_2$. Since the group $\mathbb{D}_2$ is commutative, all its linear irreducible representations are one-dimensional.

In order to construct a corresponding automorphic Lie algebra we need to choose an orbit Γ of the group G or a finite union of orbits. There are three degenerate orbits Γ_0, Γ_1 and Γ_i of degree 2

$$\Gamma_0 = \{0, \infty\}, \quad \Gamma_1 = \{\pm 1\}, \quad \Gamma_i = \{\pm i\}$$

and a generic orbit

$$\Gamma_\mu = \{\pm\mu, \pm\mu^{-1}\}, \qquad \mu \notin \{0, \infty, \pm 1, \pm i\}.$$

Elements of a basis of the automorphic Lie algebra $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_0)$ can be constructed using the group average (22). We define $\mathbf{e}^1 = 2\langle\lambda\mathbf{e}\rangle_{\mathcal{G}},\ \mathbf{f}^1 = 2\langle\lambda\mathbf{f}\rangle_{\mathcal{G}},\ \mathbf{h}^2 = 2\langle\lambda^2\mathbf{h}\rangle_{\mathcal{G}}$. Evaluating the group average we get:

$$\mathbf{e}^1 = \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix}, \quad \mathbf{f}^1 = \begin{pmatrix} 0 & \lambda^{-1} \\ \lambda & 0 \end{pmatrix}, \quad \mathbf{h}^2 = (\lambda^2 - \lambda^{-2})\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{24}$$

Their commutators are

$$[\mathbf{e}^1, \mathbf{f}^1] = \mathbf{h}^2, \quad [\mathbf{h}^2, \mathbf{e}^1] = 2(\lambda^2 + \lambda^{-2})\mathbf{e}^1 - 4\mathbf{f}^1, \quad [\mathbf{h}^2, \mathbf{f}^1] = -2(\lambda^2 + \lambda^{-2})\mathbf{f}^1 + 4\mathbf{e}^1. \tag{25}$$

For $\nu \not\sim 0$ we define a primitive automorphic function $J_G(\lambda, 0, \nu) = J_G(\lambda, 0) - J_G(\nu, 0)$ where $J_G(\lambda, 0) = \langle\lambda^{-2}\rangle_G = \frac{1}{2}(\lambda^2 + \lambda^{-2})$ (see (16)). The set

$$B = \bigcup_{n\in\mathbb{N}} B_n, \qquad B_n = \{\mathbf{e}^{2n-1} = J^{n-1}\mathbf{e}^1, \quad \mathbf{f}^{2n-1} = J^{n-1}\mathbf{f}^1, \quad \mathbf{h}^{2n} = J^{n-1}\mathbf{h}^2\}, \tag{26}$$

[2]If the homomorphism ρ is not injective and therefore the corresponding projective representation is not faithful, then its kernel $\ker\rho \subset G$ is a normal subgroup in G and the problem can be effectively reduced to the quotient group $G/\ker\rho$ (see [12]).

where $J = 2J_G(\lambda, 0, \nu)$ is a basis of $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_0)$ (see [12]).

It follows from the commutation relations (25) that

$$\begin{aligned} [\mathbf{e}^n, \mathbf{f}^m] &= \mathbf{h}^{n+m}, \\ [\mathbf{h}^k, \mathbf{e}^n] &= 2\mathbf{e}^{n+k} - 4\mathbf{f}^{n+k-2} + 4J_G(\nu, 0)\mathbf{e}^{n+k-2}, \\ [\mathbf{h}^k, \mathbf{f}^n] &= -2\mathbf{f}^{n+k} + 4\mathbf{e}^{n+k-2} - 4J_G(\nu, 0)\mathbf{f}^{n+k-2}, \end{aligned} \tag{27}$$

where $n, m \in 2\mathbb{N} - 1$ and $k \in 2\mathbb{N}$. Thus the algebra $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_0)$ is almost graded

$$\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_0) = \bigoplus_{k=1}^{\infty} \mathcal{B}^k, \quad [\mathcal{B}^p, \mathcal{B}^q] \subset \mathcal{B}^{p+q} \bigoplus \mathcal{B}^{p+q-1}$$

where homogeneous subspaces are $\mathcal{B}^p = \mathrm{Span}_{\mathbb{C}}(B_p)$. If we set $\nu = \exp \frac{i\pi}{4}$ then $J_G(\nu, 0) = 0$ and the commutation relations (27) take a rather simple form. A choice of the point ν (which controls zeros of the automorphic function $J_G(\lambda, 0, \nu)$) corresponds to a choice of the basis in $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_0)$. The grading structure depends on the choice of ν (see [12]). It follows from the commutation relations (27) that the algebra $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_0)$ is generated by its first homogeneous space $\mathcal{B}^1$ (and actually, in this particular case, by two elements $\mathbf{e}^1$ and $\mathbf{f}^1$).

Remark: Almost graded algebras can be seen as deformations of the corresponding graded algebras. For example the Automorphic Lie algebra $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_0)$ is a deformation of the graded algebra $L_{>0}(\mathfrak{A}, \phi_1) = \bigoplus_{k>0} \mathfrak{A}_k \subset L_+(\mathfrak{A}, \phi_1)$ (7). Indeed, after the re-scaling (which is a graded isomorphism)

$$\hat{\mathbf{e}}^n = \epsilon^n \mathbf{e}^n, \quad \hat{\mathbf{f}}^n = \epsilon^n \mathbf{f}^n, \quad \hat{\mathbf{h}}^n = \epsilon^n \mathbf{h}^n$$

the commutation relations (27) take the form

$$\begin{aligned} [\hat{\mathbf{e}}^n, \hat{\mathbf{f}}^m] &= \hat{\mathbf{h}}^{n+m}, \\ [\hat{\mathbf{h}}^k, \hat{\mathbf{e}}^n] &= 2\hat{\mathbf{e}}^{n+k} - 4\epsilon^2\hat{\mathbf{f}}^{n+k-2} + 4\epsilon^2 J_G(\nu, 0)\hat{\mathbf{e}}^{n+k-2}, \\ [\hat{\mathbf{h}}^k, \hat{\mathbf{f}}^n] &= -2\hat{\mathbf{f}}^{n+k} + 4\epsilon^2\hat{\mathbf{e}}^{n+k-2} - 4\epsilon^2 J_G(\nu, 0)\hat{\mathbf{f}}^{n+k-2}. \end{aligned}$$

Setting (formally) $\epsilon = 0$, we obtain the commutation relations for the algebra $L_{>0}(\mathfrak{A}, \phi_1)$.

Similarly one can construct a basis for the algebra $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma)$, for any orbit $\Gamma = G(\kappa)$ and compute the corresponding structure constants. For example a basis for $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_1)$ can be chosen as:

$$\{\hat{J}^{n-1}\mathcal{P}_{\mathcal{G}}\left(\frac{\mathbf{e}}{\lambda - 1}\right), \; \hat{J}^{n-1}\mathcal{P}_{\mathcal{G}}\left(\frac{\mathbf{f}}{(\lambda - 1)^2}\right), \; \hat{J}^{n-1}\mathcal{P}_{\mathcal{G}}\left(\frac{\mathbf{h}}{\lambda - 1}\right) \mid n \in \mathbb{N}\},$$

where $\hat{J} = J_G(\lambda, 1, \nu), \; \nu \not\sim 1$.

There is, however, a more elegant way to give a description of the automorphic Lie algebra $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma)$ for any orbit $\Gamma = G(\kappa), \; \kappa \not\sim 0$.

Proposition 1. *Let $\mathcal{G} \simeq \mathbb{D}_2$ be the reduction group generated by the automorphisms (23), $\kappa \in \mathbb{C} \setminus \{0, \infty\}$, $\Gamma_\kappa = G(\kappa)$ and $J_\kappa = J_G(\lambda, \kappa, 0)$. Then*

(i) the automorphic Lie algebra $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_\kappa)$ is generated by

$$\mathbf{a}_1 = J_\kappa \mathbf{e}^1, \quad \mathbf{a}_2 = J_\kappa \mathbf{f}^1, \quad \mathbf{a}_3 = J_\kappa \mathbf{h}^2.$$

(ii) The commutation relations between the generators are

$$[\mathbf{a}_1, \mathbf{a}_2] = J_\kappa \mathbf{a}_3, \tag{28}$$

$$[\mathbf{a}_3, \mathbf{a}_1] = 2(\kappa^2 + \kappa^{-2})J_\kappa \mathbf{a}_1 - 4J_\kappa \mathbf{a}_2 + 4C(0, \kappa)\mathbf{a}_1, \tag{29}$$

$$[\mathbf{a}_3, \mathbf{a}_2] = -2(\kappa^2 + \kappa^{-2})J_\kappa \mathbf{a}_2 + 4J_\kappa \mathbf{a}_1 - 4C(0, \kappa)\mathbf{a}_2. \tag{30}$$

(iii) The set

$$B = \bigcup_{n\in\mathbb{N}} B_n\,,\ \ B_n = \{J_\kappa^{n-1}\mathbf{a}_1,\ J_\kappa^{n-1}\mathbf{a}_2,\ J_\kappa^{n-1}\mathbf{a}_3\} \tag{31}$$

is a basis of $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_\kappa)$.

(iv) The algebra $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_\kappa)$ *is almost graded*

$$\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_\kappa) = \bigoplus_{k=1}^{\infty} \mathcal{B}^k, \quad [\mathcal{B}^p, \mathcal{B}^q] \subset \mathcal{B}^{p+q} \bigoplus \mathcal{B}^{p+q-1}$$

where $\mathcal{B}^k = \mathrm{Span}_{\mathbb{C}}(B_k)$.

Proof. (ii): The commutation relation (28) immediately follows from (27):

$$[\mathbf{a}_1, \mathbf{a}_2] = J_\kappa^2[\mathbf{e}^1, \mathbf{f}^1] = J_\kappa^2\mathbf{h}^2 = J_\kappa\mathbf{a}_3,$$

To show (29),(30) we use (27) and (16),(19). To demonstrate (29) we note:

$$[\mathbf{a}_3, \mathbf{a}_1] = J_\kappa^2[\mathbf{h}^2, \mathbf{e}^1] = 2(\lambda^2 + \lambda^{-2})J_\kappa\mathbf{a}_1 - 4J_\kappa\mathbf{a}_2$$

We recall that $(\lambda^2 + \lambda^{-2})J_\kappa = 2J_G(\lambda, 0)J_G(\lambda, \kappa, 0)$. It follows from Lemma 1 (16),(19) that

$$2J_G(\lambda, 0)J_G(\lambda, \kappa, 0) = 2J_G(\kappa, 0)J_G(\lambda, \kappa, 0) + 2J_G(\lambda, 0, \kappa)J_G(\lambda, \kappa, 0) = (\kappa^2 + \kappa^{-2})J_\kappa + 2C(0, \kappa)$$

and thus $[\mathbf{a}_3, \mathbf{a}_1] = 2(\kappa^2 + \kappa^{-2})J_\kappa\mathbf{a}_1 - 4J_\kappa\mathbf{a}_2 + 4C(0, \kappa)\mathbf{a}_1$. The proof of (30) is similar.

(iii): The elements $J_\kappa^{n-1}\mathbf{a}_i$ are $\mathcal{G}$ invariant and have poles at points of Γ_κ only, thus $J_\kappa^{n-1}\mathbf{a}_i \in \mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_\kappa)$. It is easy to show that any element of $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_\kappa)$ can be represented as a finite linear combination of elements of B. For a generic point κ, the proof of the latter statement is given in [12] (Proposition 3.1). In the case of degenerate orbits Γ_1, Γ_i the proof is similar (or can be deduced from the generic orbit case). Thus B is a basis of $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_\kappa)$.

(i): It follows from (ii) that all elements of B can be generated by the set $B_1 = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

(iv): Since B is a basis of $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_\kappa)$, we have $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_\kappa) = \bigoplus_{k=1}^{\infty} \mathcal{B}^k$ where $\mathcal{B}^k = \mathrm{Span}_{\mathbb{C}}(B_n)$. It follows from (ii) that $[\mathcal{B}^p, \mathcal{B}^q] \subset \mathcal{B}^{p+q} \bigoplus \mathcal{B}^{p+q-1}$. ■

It follows from the above Proposition and the preceding discussion that for any orbit Γ an almost graded basis of the algebra $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma)$ can be characterised by a set of generators $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and a primitive automorphic function J with poles at Γ and thus it is convenient to introduce the notation

$$\langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\,;\, J\rangle = \{J^{n-1}\mathbf{a}_i \,|\, n \in \mathbb{N}, i = 1, 2, 3\}.$$

In this notation $B = \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\,;\, J_\kappa\rangle$ (31).

An almost graded $\mathbb{C}$–algebra with the basis $\langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\,;\, J\rangle$ is infinite dimensional and can be viewed as a $\mathbb{C}[J]$–Lie module with three generators. It can be completely characterised by a finite number of structure constants $C_{ijk}^1, C_{ijk}^0 \in \mathbb{C}$:

$$[\mathbf{a}_i, \mathbf{a}_j] = \sum_k C_{ijk}^1 J\mathbf{a}_k + C_{ijk}^0\mathbf{a}_k\,. \tag{32}$$

Two algebras are isomorphic iff there exist bases such that the corresponding structure constants coincide.

Definition 1. *Given two almost graded Lie algebras* $\mathcal{A}$ *and* $\mathcal{B}$ *with bases defined by* $\langle \mathbf{a}_1, \ldots, \mathbf{a}_N\,;\, J_a\rangle$ *and* $\langle \mathbf{b}_1, \ldots, \mathbf{b}_N\,;\, J_b\rangle$ *respectively, we say that the algebras are* graded isomorphic *if there exists a linear transformation of the form*

$$\hat{\mathbf{b}}_i = \sum_{k=1}^{N} W_{ik}\mathbf{b}_k, \quad \hat{J}_b = \Delta J_b + \delta, \qquad \Delta, \delta, W_{ij} \in \mathbb{C}, \quad \det W \neq 0, \tag{33}$$

such that the structure constants of the algebras $\mathcal{A}$ *and* $\mathcal{B}$ *in the bases* $\langle \mathbf{a}_1, \ldots, \mathbf{a}_N\,;\, J_a\rangle$ *and* $\langle \hat{\mathbf{b}}_1, \ldots, \hat{\mathbf{b}}_N\,;\, \hat{J}_b\rangle$ *coincide.*

Graded–isomorphic algebras are of course isomorphic. The advantage of the graded isomorphism is that it can be effectively verified for infinite dimensional almost graded $\mathbb{C}$–algebras. Suppose we are given two almost graded algebras, one with the basis $\langle \mathbf{a}_1, \dots, \mathbf{a}_N\,;\, J_a\rangle$, commutation relations (32) and thus with structure constants C^1_{ijk}, C^0_{ijk} and another one $\langle \mathbf{b}_1, \dots, \mathbf{b}_N\,;\, J_b\rangle$ with commutation relations

$$[\mathbf{b}_i, \mathbf{b}_j] = \sum_{k=1}^{N} S^1_{ijk} J_b \mathbf{b}_k + S^0_{ijk}\mathbf{b}_k$$

and corresponding structure constants S^1_{ijk} and S^0_{ijk}. Then for the transformed elements $\hat{\mathbf{b}}_p = \sum_{i=1}^{N} W_{pi}\mathbf{b}_i$ and $J_b = \Delta^{-1}(\hat{J}_b - \delta)$ we have:

$$[\hat{\mathbf{b}}_p, \hat{\mathbf{b}}_q] = \sum_{i,j,k,s=1}^{N} \frac{\hat{J}_b}{\Delta} W_{pi}W_{qj}S^1_{ijk}W^{-1}_{ks}\hat{\mathbf{b}}_s - \frac{\delta}{\Delta} W_{pi}W_{qj}S^1_{ijk}W^{-1}_{ks}\hat{\mathbf{b}}_s + W_{pi}W_{qj}S^0_{ijk}W^{-1}_{ks}\hat{\mathbf{b}}_s$$

Equating the transformed structure constants with C^1_{ijk}, C^0_{ijk} we obtain the following overdetermined system of $N^2(N-1)$ polynomial equations

$$P^0_{pqk} = 0, \quad P^1_{pqk} = 0, \qquad p, q, k \in \{1, \dots, N\}, \quad p > q, \tag{34}$$

where

$$P^1_{pqk} = \sum_{i,j=1}^{N} W_{pi}W_{qj}S^1_{ijk} - \sum_{s=1}^{N} \Delta C^1_{pqs} W_{sk}, \tag{35}$$

$$P^0_{pqk} = \sum_{i,j=1}^{N} W_{pi}W_{qj}S^0_{ijk} - \sum_{s=1}^{N} (C^0_{pqs} + \delta C^1_{pqs}) W_{sk} \tag{36}$$

for $N^2 + 2$ unknowns W_{ij}, Δ, δ.

Proposition 2. *Let $\mathfrak{A} = sl(2, \mathbb{C})$ and $\mathcal{G} \simeq \mathbb{D}_2$ be the reduction group generated by the automorphisms (23). The automorphic Lie algebras $\mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma_0)$, $\mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma_1)$ and $\mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma_i)$, corresponding to degenerate orbits, are graded isomorphic.*

Proof: The algebra $\mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma_0)$ is almost graded in the basis $\langle \mathbf{e}^1, \mathbf{f}^1, \mathbf{h}^2\,;\, J_0\rangle$ with structure constants defined by (25)

$$[\mathbf{e}^1, \mathbf{f}^1] = \mathbf{h}^2\,, \quad [\mathbf{h}^2, \mathbf{e}^1] = 4J_0\mathbf{e}^1 - 4\mathbf{f}^1\,, \quad [\mathbf{h}^2, \mathbf{f}^1] = -4J_0\mathbf{f}^1 + 4\mathbf{e}^1\,. \tag{37}$$

It follows from Proposition 1 that the algebra $\mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma_1)$ is almost graded in the basis $\langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\,;\, J_1\rangle$ with structure constants defined by (28),(29) and (30), ($\kappa = 1,\ C(0,1) = 1$):

$$[\mathbf{a}_1, \mathbf{a}_2] = J_1\mathbf{a}_3,\ [\mathbf{a}_3, \mathbf{a}_1] = 4J_1\mathbf{a}_1 - 4J_1\mathbf{a}_2 + 4\mathbf{a}_1,\ [\mathbf{a}_3, \mathbf{a}_2] = -4J_1\mathbf{a}_2 + 4J_1\mathbf{a}_1 - 4\mathbf{a}_2.$$

It is easy to verify that the following invertible linear map $\mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma_1) \mapsto \mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma_0)$

$$\mathbf{e}_1 = \mathbf{a}_1 - \mathbf{a}_2 - \frac{1}{2}\mathbf{a}_3, \quad \mathbf{f}_1 = -\mathbf{a}_1 + \mathbf{a}_2 - \frac{1}{2}\mathbf{a}_3, \quad \mathbf{h}_2 = 4\mathbf{a}_1 + 4\mathbf{a}_2, \quad J_0 = 8J_1 + 2$$

is the graded isomorphism. If we denote by $\langle \hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{a}}_3\,;\, J_i\rangle$ the basis of $\mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma_i)$ with structure constants defined by (28),(29) and (30) and $\kappa = i,\ C(0,i) = 1$, then the linear map $\mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma_1) \mapsto \mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma_i)$ given by $\hat{\mathbf{a}}_1 = -\mathbf{a}_1,\ \hat{\mathbf{a}}_2 = \mathbf{a}_2,\ \hat{\mathbf{a}}_3 = \mathbf{a}_3, J_i = -J_1$ is a graded isomorphism. ■

Proposition 3. *Let $\mathfrak{A} = sl(2, \mathbb{C})$ and $\mathcal{G} \simeq \mathbb{D}_2$ be the reduction group generated by the automorphisms (23). The automorphic Lie algebras $\mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma_\mu)$, $\mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma_\nu)$ are graded isomorphic if and only if*

$$\nu \in G(\mu) \cup G(i\mu) \cup G(\frac{\mu-1}{\mu+1}) \cup G(i\frac{\mu-1}{\mu+1}) \cup G(\frac{i\mu-1}{i\mu+1}) \cup G(i\frac{i\mu-1}{i\mu+1})\,. \tag{38}$$

In the proof of the proposition we shall use the following Lemma.

Lemma 2. *Consider two almost graded algebras $\mathcal{A}$ and $\mathcal{B}$ with bases $\langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\,;\, J_a\rangle$ and $\langle \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\,;\, J_b\rangle$ and the commutation relations*

$$\begin{array}{ll} [\mathbf{a}_1,\mathbf{a}_2] = J_a\mathbf{a}_3, & [\mathbf{b}_1,\mathbf{b}_2] = J_b\mathbf{b}_3,\\ [\mathbf{a}_3,\mathbf{a}_1] = 4\alpha J_a\mathbf{a}_1 - 4J_a\mathbf{a}_2 + 4\mathbf{a}_1, & [\mathbf{b}_3,\mathbf{b}_1] = 4\beta J_b\mathbf{b}_1 - 4J_b\mathbf{b}_2 + 4\mathbf{b}_1,\\ [\mathbf{a}_3,\mathbf{a}_2] = -4\alpha J_a\mathbf{a}_2 + 4J_a\mathbf{a}_1 - 4\mathbf{a}_2, & [\mathbf{b}_3,\mathbf{b}_2] = -4\beta J_b\mathbf{b}_2 + 4J_b\mathbf{b}_1 - 4\mathbf{b}_2, \end{array} \tag{39}$$

respectively. If algebras $\mathcal{A}$ and $\mathcal{B}$ are graded isomorphic, then

$$(\alpha^2-\beta^2)((\alpha+3)^2-\beta^2(\alpha-1)^2)((\alpha-3)^2-\beta^2(\alpha+1)^2)=0. \tag{40}$$

Proof: Algebras $\mathcal{A}$ and $\mathcal{B}$ are graded isomorphic and thus there exists a solution of equations (34) for unknowns W_{ij},Δ,δ with the condition $\det W=\gamma\neq 0$. Obviously $P^0_{pqk}, P^1_{pqk},\ p,q,s\in\{1,2,3\}$ (35), (36) are polynomials in $W_{ij},\alpha,\beta,\Delta,\delta$. In the polynomial ring $\mathbb{C}[W_{ij},\alpha,\beta,\gamma,\Delta,\delta]$ we consider the ideal $\mathcal{J}=\langle P^0_{pqk},P^1_{pqk},\det W-\gamma\rangle$ generated by all polynomials $P^0_{pqk},P^1_{pqk},\ p,q,s\in\{1,2,3\}$ and the polynomial $\det W-\gamma$. It can be duly shown that the polynomial

$$\pi=\gamma(\alpha^2-\beta^2)((\alpha+3)^2-\beta^2(\alpha-1)^2)((\alpha-3)^2-\beta^2(\alpha+1)^2)\in\mathcal{J} \tag{41}$$

belongs to the ideal $\mathcal{J}$. Thus $\pi=0$ for every solution of the system (34) and the equation $\det W=\gamma$. Since $\gamma\neq 0$ we get (40). ■

Proof of Proposition 3: First we show that the isomorphism $\mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma_\mu)\simeq\mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma_\nu)$ follows from (38). It is sufficient to show that $\mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma_\mu)\simeq\mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma_\nu)$ for $\nu=i\mu$ and $\nu=\frac{\mu-1}{\mu+1}$. Indeed, if $\nu\in G(\mu)$ then $\Gamma_\nu=\Gamma_\mu$ and the algebras coincide, while the remaining cases can be reduced to the above two cases by compositions. In the case of a degenerate orbit Γ_μ condition (38) means that $\nu\in\Gamma_0\bigcup\Gamma_1\bigcup\Gamma_i$ and the statement follows from Proposition 2. Thus we shall assume that point μ is generic. Since $C(0,\mu)\neq 0$, by the re-scaling $\mathbf{a}_i\mapsto C(0,\mu)\mathbf{a}_i,\ J\mapsto C(0,\mu)J_a$ we can reduce the commutation relations (28),(29) and (30) for the algebra $\mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma_\mu)$ to (39) for the algebra $\mathcal{A}$ with $\alpha=\frac12(\mu^2+\mu^{-2})$. For a generic point μ we have $\alpha\neq\pm1$. Similarly for the algebra $\mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma_\nu)$ we get structure constants for the algebra $\mathcal{B}$ (39) with $\beta=\frac12(\nu^2+\nu^{-2})$.

In the case $\nu=i\mu$ we have $\beta=-\alpha$ and it is easy to verify that the linear map $\mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma_{i\mu})\mapsto\mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma_\mu)$ of the form (33)

$$\mathbf{a}_1=-\mathbf{b}_1,\quad \mathbf{a}_2=\mathbf{b}_2,\quad \mathbf{a}_3=\mathbf{b}_3,\quad J_a=-J_b$$

is the algebra homomorphism. Similarly,

$$\mathbf{a}_1=\frac{1}{1-\alpha}(\hat{\mathbf{a}}_1-\hat{\mathbf{a}}_2-\frac12\hat{\mathbf{a}}_3),\quad \mathbf{a}_2=\frac{1}{1-\alpha}(-\hat{\mathbf{a}}_1+\hat{\mathbf{a}}_2-\frac12\hat{\mathbf{a}}_3),\quad \mathbf{a}_2=\frac{1}{1-\alpha}(4\hat{\mathbf{a}}_1+4\hat{\mathbf{a}}_2)$$

and $J_a=-\frac{4}{(\alpha-1)^2}J_{\hat a}-\frac{1}{\alpha-1}$ maps the basis of $\mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma_\mu)$ into the basis of $\mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma_\nu)$ with $\nu=\frac{\mu-1}{\mu+1}$.

The necessity follows from the statement (40) of Lemma 2. If the algebras $\mathcal{A}$ and $\mathcal{B}$ are graded isomorphic, then $\beta\in\{\pm\alpha,\pm\frac{\alpha+3}{\alpha-1},\pm\frac{\alpha-3}{\alpha+1}\}$. The case $\beta=\alpha$ corresponds to $\nu\in G(\mu)$, the cases $\beta=-\alpha,\frac{\alpha+3}{\alpha-1},-\frac{\alpha+3}{\alpha-1},\frac{\alpha-3}{\alpha+1}$ and $-\frac{\alpha-3}{\alpha+1}$ correspond to $\nu\in G(i\mu),G(\frac{\mu-1}{\mu+1}),G(i\frac{\mu-1}{\mu+1}),G(\frac{i\mu-1}{i\mu+1})$ and $\nu\in G(i\frac{i\mu-1}{i\mu+1})$ respectively. ■

The group of automorphisms Aut $\mathcal{G}$ of the reduction group $\mathcal{G}\simeq\mathbb{D}_2$ (23) is isomorphic to the dihedral group $\mathbb{D}_3$, and it has six elements. Elements of Aut $\mathcal{G}$ act by permutations on the set of the orbits $G(\mu),G(i\mu),G(\frac{\mu-1}{\mu+1}),G(i\frac{\mu-1}{\mu+1}),G(\frac{i\mu-1}{i\mu+1}),G(i\frac{i\mu-1}{i\mu+1})$. That explains the number of solutions to the equation (41).

In this Section we have shown that for $\mathfrak{A}=sl(2,\mathbb{C})$ and $\mathcal{G}\simeq\mathbb{D}_2$ (23) there are two essentially different types of automorphic Lie algebra. The first one corresponds to degenerate orbits of the Möbius group, and the algebras corresponding to different degenerate orbits are all isomorphic. The second type is the automorphic Lie algebras $\mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma_\mu)$ corresponding to generic orbits of the Möbius group G. If μ and ν are two generic points, then the corresponding automorphic Lie algebras are graded isomorphic if and only if the condition (38) is satisfied.

Proposition 4. *Let $\mathfrak{A} = sl(2,\mathbb{C})$ and $\mathcal{G} \simeq \mathbb{D}_2$ be the reduction group generated by the automorphisms (23). Automorphic Lie algebras corresponding to generic and degenerate orbits are not isomorphic.*

Proof: Assuming μ to be a generic point, it follows from (28),(29),(30) and (25) that

$$\dim\left(\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_\mu)\big/\langle[\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_\mu),\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_\mu)]\rangle\right) = 3\,, \qquad \dim\left(\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_0)\big/\langle[\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_0),\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_0)]\rangle\right) = 2.$$

where $\langle[\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_\nu),\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_\nu)]\rangle$ denotes the ideal generated by the commutator of the algebra $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_\nu)$ with itself. ■

3.2 Automorphic Lie algebras corresponding to finite reduction groups, $\mathfrak{A} = sl(2,\mathbb{C})$

In this section we consider automorphic Lie algebras $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma)$ where $\mathfrak{A} = sl(2,\mathbb{C})$, Γ is an orbit (with respect to the Möbius group associated with a finite reduction group $\mathcal{G}$) of a point, which may be either degenerate or generic. With every finite Möbius group and every 2-dimensional faithful projective representation of the group we can associate a reduction group. Thus one could expect that the number of automorphic Lie algebras is rather big. However, it turns out that there exist many graded isomorphisms between these automorphic Lie algebras.

Proposition 5. *Automorphic Lie algebras corresponding to groups $\mathbb{Z}_N$, $N \geq 2$ and the degenerate orbit $\Gamma = \{\infty\}$ are isomorphic.*

Proof. For a given $N \geq 2$ we choose the following generators:

$$\mathbf{a}_1 = \lambda\mathbf{e}, \quad \mathbf{a}_2 = \lambda^{N-1}\mathbf{f}, \quad \mathbf{a}_3 = \mathbf{h}, \quad J = \lambda^N$$

then the commutation relations are

$$[\mathbf{a}_1,\mathbf{a}_2] = J\mathbf{a}_3, \qquad [\mathbf{a}_3,\mathbf{a}_1] = 2\mathbf{a}_1, \qquad [\mathbf{a}_3,\mathbf{a}_2] = -2\mathbf{a}_2$$

and they do not depend on the choice of N. These are the commutation relations of the Kac-Moody subalgebra $L_+(\mathfrak{A},\phi)$, $\phi^2 = id$. ■

Furthermore, automorphic Lie algebras corresponding to the dihedral groups $\mathbb{D}_N$ for any $N \geq 2$ and any irreducible projective representations are graded isomorphic to the cases considered in Section 3.1 for the group $\mathbb{D}_2$ - with the distinction remaining between when we take Γ degenerate or generic (for generic orbits this has been shown in [12]).

There also exist graded isomorphisms between all algebras associated with non-cyclic reduction groups and degenerate orbits.

Theorem 1. *Let $\mathfrak{A} = sl(2,\mathbb{C})$, $\mathcal{G}$ be any finite non-cyclic reduction group and Γ be a degenerate orbit of the corresponding Möbius group. Then the automorphic Lie algebra $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma)$ is graded isomorphic to the algebra with $\mathcal{G} \simeq \mathbb{D}_2$ and the degenerate orbit $\Gamma = \{0,\infty\}$.*

Sketch of the proof. Our proof is elementary, but long[3]. We consider all finite non-cyclic Möbius groups, namely the groups $\mathbb{D}_N, \mathbb{T}, \mathbb{O}$ and $\mathbb{I}$. We take (in turn) each one of the groups. We take (in turn) each one of the three possible degenerate orbits (these orbits are listed in [12], Appendix A). We consider (in turn) all faithful 2-dimensional projective representations of the chosen group and

[3]The conjecture that the algebras mentioned in the Theorem are isomorphic had been formulated by one of the authors (AVM) in 2008. When our proof of the Theorem was completed and announced at a number of seminars and the conference 'Symmetry in Nonlinear Mathematical Physics - 2009", Kiev, we were informed that a short and elegant proof of the conjecture has been done by S.Lombardo and J.Sanders (now published in [15]). Their proof is based on the classical theory of invariants. In [15] the authors also introduced a canonical basis for automorphic Lie algebras, which is analogous to the Cartan-Weyl basis.

construct the corresponding reduction groups. Taking each one of the reduction groups and each orbit we evaluate the reduction group average to find a basis of the associated automorphic Lie algebra $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma)$ and compute the corresponding structure constants. Finally we find a linear transformation (33) which transforms the structure constants obtained to the structure constants (25) of the algebra with $\mathcal{G} \simeq \mathbb{D}_2$ and the degenerate orbit $\Gamma = \{0, \infty\}$.

For example, let us take the icosahedral group $G = \mathbb{I}$. As a Möbius group it can be generated by two linear-fractional transformations

$$g_1(\lambda) = \varepsilon\lambda, \qquad g_2(\lambda) = \frac{(\varepsilon^2 + \varepsilon^3)\lambda + 1}{\lambda - \varepsilon^2 - \varepsilon^3}, \quad \varepsilon = \exp\left(\frac{2\pi i}{5}\right).$$

The group $\mathbb{I}$ has two 2-dimensional irreducible projective representations. Let us take the natural representation generated by

$$U_1 = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} \qquad U_2 = \begin{pmatrix} \varepsilon^2 + \varepsilon^3 & 1 \\ 1 & -\varepsilon^2 - \varepsilon^3 \end{pmatrix}.$$

Thus the reduction group $\mathcal{G}$ is generated by two automorphisms

$$\Phi_1(a(\lambda)) = U_1 a(g_1^{-1}(\lambda))U_1^{-1}, \qquad \Phi_2(a(\lambda)) = U_2 a(g_2^{-1}(\lambda))U_2^{-1}.$$

Let us choose the degenerate orbit of order 5

$$\Gamma = \{0, \infty, \varepsilon^{k+1} + \varepsilon^{k-1}, \varepsilon^{k+2} + \varepsilon^{k-2} \,|\, k = 0, 1, 2, 3, 4\}.$$

The corresponding automorphic Lie algebra $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma)$ has generators

$$\mathbf{b}_1 = \langle \lambda \mathbf{e} \rangle_{\mathbb{I}}, \quad \mathbf{b}_2 = \langle \lambda^4 \mathbf{f} \rangle_{\mathbb{I}}, \quad \mathbf{b}_3 = \langle \lambda^5 \mathbf{h} \rangle_{\mathbb{I}}, \quad J_{\mathbb{I}} = \langle \lambda^5 \rangle_{\mathbb{I}}$$

with the following commutation relations

$$[\mathbf{b}_1, \mathbf{b}_2] = \frac{5}{2}\mathbf{b}_1 - \frac{5}{6}\mathbf{b}_2 + \frac{1}{12}\mathbf{b}_3$$

$$[\mathbf{b}_1, \mathbf{b}_3] = -5\mathbf{b}_1 + \frac{11}{3}\mathbf{b}_2 + \frac{5}{6}\mathbf{b}_3 - 2J_{\mathbb{I}}\mathbf{b}_1$$

$$[\mathbf{b}_2, \mathbf{b}_3] = -1653\mathbf{b}_1 + 5\mathbf{b}_2 + \frac{5}{2}\mathbf{b}_3 + 2J_{\mathbb{I}}\mathbf{b}_2$$

After an invertible transformation of the form (33)

$$\hat{\mathbf{a}}_1 = 2\mathbf{b}_1, \qquad \hat{\mathbf{a}}_2 = \frac{1}{6}\mathbf{b}_2, \qquad \hat{\mathbf{a}}_3 = \frac{5}{6}\mathbf{b}_1 - \frac{5}{18}\mathbf{b}_2 + \frac{1}{36}\mathbf{b}_3, \qquad \hat{J} = \frac{1}{36}J_{\mathbb{I}} + \frac{5}{12}$$

one can easily verify that the commutation relations for $\hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{a}}_3$ (structure constants) coincide with the ones for the $\mathbb{D}_2$ group (25), and thus the two algebras are graded isomorphic. We treated all other cases similarly. ■

This covers the situation for all groups where we choose degenerate orbits, but we can also consider the case where we choose generic orbits (we did this for $\mathbb{D}_2$ in Section 3.1). For all groups $\mathcal{G}$, we take the generators

$$\mathbf{a}_1 = \left\langle \frac{1}{\lambda - \mu}\sigma_1 \right\rangle_{\mathcal{G}}, \quad \mathbf{a}_2 = \left\langle \frac{1}{\lambda - \mu}\sigma_2 \right\rangle_{\mathcal{G}}, \quad \mathbf{a}_3 = \left\langle \frac{1}{\lambda - \mu}\sigma_3 \right\rangle_{\mathcal{G}}$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

being a basis of $sl(2, \mathbb{C})$. If G is the Möbius group associated with the reduction group $\mathcal{G}$, the automorphic function J is given by

$$J = \left\langle \frac{1}{\lambda - \mu} \right\rangle_G$$

With all finite reduction groups the commutation relations take the form

$$\begin{aligned}[\mathbf{a}_1,\mathbf{a}_2] &= p\mathbf{a}_1 + q\mathbf{a}_2 + r_3\mathbf{a}_3 + 2J\mathbf{a}_3,\\ [\mathbf{a}_1,\mathbf{a}_3] &= s\mathbf{a}_1 + r_2\mathbf{a}_2 - q\mathbf{a}_3 + 2J\mathbf{a}_2,\\ [\mathbf{a}_2,\mathbf{a}_3] &= r_1\mathbf{a}_1 - s\mathbf{a}_2 + p\mathbf{a}_3 + 2J\mathbf{a}_1\end{aligned} \tag{42}$$

where s, p, q, r_i are functions of a generic point μ. For example, for the trivial group all coefficients s, p, q, r_i are equal to zero, for the group $\mathbb{Z}_N$ we have:

$$r_3 = -\frac{2}{\mu}, \quad p = q = s = r_1 = r_2 = 0$$

and for $\mathbb{D}_N$ we have:

$$r_2 = \frac{(\mu^N-1)^2}{\mu^{N+1}}, \quad r_1 = \frac{\mu^{2N}-1}{\mu^{N+1}}, \quad s = p = q = r_3 = 0.$$

In fact using transformations (33) we can reduce the relations (42) to one of the above listed cases.

Here we assert that the list of automorphic Lie algebras corresponding to all finite reduction groups and $\mathfrak{A} = sl(2,\mathbb{C})$ is rather short and up to graded isomorphism can be represented by algebras of the following types:

- $\mathcal{A}^0$ the polynomial part of the Loop algebra ($\mathfrak{A}_\lambda(\infty) = \mathbb{C}[\lambda] \otimes_{\mathbb{C}} sl(2,\mathbb{C})$), when the reduction group is trivial;
- $\mathcal{A}^1$ the subalgebra $L_+(\mathfrak{A},\phi)$, $\phi^2 = id$ of the Kac-Moody algebra, which corresponds to $\mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma)$ with $\mathcal{G} \simeq \mathbb{Z}_2$ and a degenerate orbit $\Gamma = \{\infty\}$;
- $\mathcal{A}^1_1$ the algebra $\mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma_1)$ with $\mathcal{G} \simeq \mathbb{Z}_2$ and a generic orbit $\Gamma_1 = \{\pm 1\}$;
- $\mathcal{A}^2$ the algebra $\mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma)$ with $\mathcal{G} \simeq \mathbb{D}_2$ and a degenerate orbit $\Gamma = \{0,\infty\}$;
- $\mathcal{A}^2_\mu$ the algebra $\mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma_\mu)$ with $\mathcal{G} \simeq \mathbb{D}_2$ and a generic orbit Γ_μ.

Proposition 6. *The algebras of the types $\mathcal{A}^0$, $\mathcal{A}^1$, $\mathcal{A}^1_1$, $\mathcal{A}^2$ and $\mathcal{A}^2_\mu$ are not graded isomorphic.*

Proof: For any algebra $\mathcal{A}$ we set $\hat{\mathcal{A}} = \mathcal{A}/\langle[\mathcal{A},\mathcal{A}]\rangle$, where $\langle[\mathcal{A},\mathcal{A}]\rangle$ denotes the ideal generated by the commutator of the algebra $\mathcal{A}$ with itself. It follows from the commutation relations of these algebras, given earlier, that

$$\dim\hat{\mathcal{A}}^0 = 0, \quad \dim\hat{\mathcal{A}}^1 = 1, \quad \dim\hat{\mathcal{A}}^1_1 = 2, \quad \dim\hat{\mathcal{A}}^2 = 2, \quad \dim\hat{\mathcal{A}}^2_\mu = 3.$$

Our analysis of equations (34), (35) and (36), these being the equations determining the mapping between the bases of two algebras to establish a graded isomorphism between them, shows that the algebras $\mathcal{A}^1_\mu$ and $\mathcal{A}^2$ are not graded isomorphic, since in this case the equations (34), (35) and (36) have no solution. ■

3.3 Explicit realisations of $sl(2,\mathbb{C})$ automorphic Lie algebras as finitely generated $\mathbb{C}[J]$–Lie modules

Automorphic Lie algebras $\mathcal{A}^0$, $\mathcal{A}^1$, $\mathcal{A}^1_1$, $\mathcal{A}^2$ and $\mathcal{A}^2_\mu$ are almost graded infinite dimensional Lie algebras over $\mathbb{C}$. They also can be viewed as $\mathbb{C}[J]$–Lie modules with three generators $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, where J is the corresponding automorphic function of the parameter λ [7]. Each of these algebras and the corresponding $\mathbb{C}[J]$–Lie module can be extended by a derivation $\mathcal{D}$. In this Section we give explicit realisations for all automorphic Lie algebras listed above, their derivations and commutation relations.

$\mathcal{A}^0$: $\quad J = \lambda, \quad \mathcal{D} = \frac{d}{d\lambda}, \quad \mathbf{a}_1 = \mathbf{e}, \quad \mathbf{a}_2 = \mathbf{f}, \quad \mathbf{a}_3 = \mathbf{h},$

with commutation relations (5), $[\mathcal{D}, \mathbf{a}_i] = 0,\ [\mathcal{D}, J] = 1$.

$\mathcal{A}^1$: $\quad J = \lambda^2, \quad \mathcal{D} = \lambda\frac{d}{d\lambda}, \quad \mathbf{a}_1 = \lambda\mathbf{e}, \quad \mathbf{a}_2 = \lambda\mathbf{f}, \quad \mathbf{a}_3 = \mathbf{h},$

and commutation relations: $[\mathcal{D}, J] = 2J$

$$\begin{array}{lll} [\mathbf{a}_1, \mathbf{a}_2] = J\mathbf{a}_3, & [\mathbf{a}_3, \mathbf{a}_1] = 2\mathbf{a}_1, & [\mathbf{a}_3, \mathbf{a}_2] = -2\mathbf{a}_2, \\ [\mathcal{D}, \mathbf{a}_1] = \mathbf{a}_1, & [\mathcal{D}, \mathbf{a}_2] = \mathbf{a}_2, & [\mathcal{D}, \mathbf{a}_3] = 0. \end{array}$$

$\mathcal{A}^1_1$: $\quad J = \frac{1}{\lambda^2 - 1}, \quad \mathcal{D} = \lambda\frac{d}{d\lambda}, \quad \mathbf{a}_1 = \frac{\lambda}{\lambda^2-1}\mathbf{e}, \quad \mathbf{a}_2 = \frac{\lambda}{\lambda^2-1}\mathbf{f}, \quad \mathbf{a}_3 = \mathbf{h},$

with commutation relations: $[\mathcal{D}, J] = -2J - 2J^2$,

$$\begin{array}{lll} [\mathbf{a}_1, \mathbf{a}_2] = (J + J^2)\mathbf{a}_3, & [\mathbf{a}_3, \mathbf{a}_1] = 2\mathbf{a}_1, & [\mathbf{a}_3, \mathbf{a}_2] = -2\mathbf{a}_2, \\ [\mathcal{D}, \mathbf{a}_1] = -(1 + 2J)\mathbf{a}_1, & [\mathcal{D}, \mathbf{a}_2] = -(1 + 2J)\mathbf{a}_2, & [\mathcal{D}, \mathbf{a}_3] = 0. \end{array}$$

$\mathcal{A}^2$: $\quad J = \lambda^2 + \lambda^{-2}, \quad \mathcal{D} = \lambda(\lambda^2 - \lambda^{-2})\frac{d}{d\lambda},$

$$\mathbf{a}_1 = \lambda\mathbf{e} + \lambda^{-1}\mathbf{f}, \quad \mathbf{a}_2 = \lambda^{-1}\mathbf{e} + \lambda\mathbf{f}, \quad \mathbf{a}_3 = 2(\lambda^2 - \lambda^{-2})\mathbf{h},$$

with commutation relations: $[\mathcal{D}, J] = 2J^2 - 8$,

$$\begin{array}{lll} [\mathbf{a}_1, \mathbf{a}_2] = \mathbf{a}_3, & [\mathbf{a}_3, \mathbf{a}_1] = 2J\mathbf{a}_1 - 4\mathbf{a}_2, & [\mathbf{a}_3, \mathbf{a}_2] = -2J\mathbf{a}_2 + 4\mathbf{a}_1, \\ [\mathcal{D}, \mathbf{a}_1] = J\mathbf{a}_1 - 2\mathbf{a}_2, & [\mathcal{D}, \mathbf{a}_2] = J\mathbf{a}_2 - 2\mathbf{a}_1, & [\mathcal{D}, \mathbf{a}_3] = 2J\mathbf{a}_3. \end{array}$$

$\mathcal{A}^2_\mu$: $J = \frac{(\mu^4 - 1)\lambda^2}{(\lambda^2 - \mu^2)(1 - \lambda^2\mu^2)}, \quad \mathcal{D} = \frac{\mu^2\lambda(1 - \lambda^4)}{2(\lambda^2 - \mu^2)(1 - \lambda^2\mu^2)}\frac{d}{d\lambda}, \quad \mu \neq 0,\ \mu^4 \neq 1, \quad \mu \in \mathbb{C},$

$$\mathbf{a}_1 = \frac{\lambda\mu}{\lambda^2 - \mu^2}\mathbf{e} + \frac{\lambda\mu}{1 - \lambda^2\mu^2}\mathbf{f}, \quad \mathbf{a}_2 = \frac{\lambda\mu}{1 - \lambda^2\mu^2}\mathbf{e} + \frac{\lambda\mu}{\lambda^2 - \mu^2}\mathbf{f}, \quad \mathbf{a}_3 = \frac{\mu^2(\lambda^4 - 1)}{(\lambda^2 - \mu^2)(1 - \lambda^2\mu^2)}\mathbf{h},$$

with commutation relations: $[\mathcal{D}, J] = J^3 + 2\alpha J^2 + J$,

$$\begin{array}{ll} [\mathbf{a}_1, \mathbf{a}_2] = J\mathbf{a}_3, & [\mathcal{D}, \mathbf{a}_3] = (J^2 + \alpha J)\mathbf{a}_3, \\ [\mathbf{a}_3, \mathbf{a}_1] = 2(J + \alpha)\mathbf{a}_1 + 4\beta\mathbf{a}_2, & [\mathcal{D}, \mathbf{a}_1] = \frac{1}{2}(2J^2 + 3\alpha J + 1)\mathbf{a}_1 + \beta J\mathbf{a}_2, \\ [\mathbf{a}_3, \mathbf{a}_2] = -2(J + \alpha)\mathbf{a}_2 - 4\beta\mathbf{a}_1, & [\mathcal{D}, \mathbf{a}_2] = \frac{1}{2}(2J^2 + 3\alpha J + 1)\mathbf{a}_2 + \beta J\mathbf{a}_1, \end{array}$$

where $\alpha = \frac{\mu^4 + 1}{\mu^4 - 1}$ and $\beta = \frac{\mu^2}{\mu^4 - 1}$.

4 Integrable systems corresponding to finite reduction groups

Automorphic Lie algebras can be used to find systems of integrable equations, by using them to construct an *automorphic* Lax pair (L, A):

$$L = \partial_x + \mathbf{U}(x, t, \lambda), \qquad A = \partial_t + \mathbf{V}(x, t, \lambda) \tag{43}$$

where $\mathbf{U}, \mathbf{V} \in \mathfrak{A}^{\mathcal{G}}_\lambda(\Gamma)$.

A Lax pair defines the linear differential system

$$L\psi = 0, \quad A\psi = 0 \tag{44}$$

where ψ is a fundamental solution matrix to this linear problem. In order for this linear system to be consistent, the following *compatibility condition* must hold

$$\mathbf{V}_x - \mathbf{U}_t + [\mathbf{U}, \mathbf{V}] = 0, \tag{45}$$

which means that operators L and A commute $[L, A] = 0$.

Let us denote as $\mathfrak{A}^{\mathcal{G}} = \{a \in \mathfrak{A} \,|\, \Phi(a) = a, \forall \Phi \in \mathcal{G}\}$ a $\mathcal{G}$-invariant subalgebra of a simple finite dimensional Lie algebra $\mathfrak{A}$. Let $\mathbf{G}^{\mathcal{G}}$ be a Lie group corresponding to $\mathfrak{A}^{\mathcal{G}}$ and $\mathbf{g} \in \mathbf{G}^{\mathcal{G}}$ be a differentiable function of x, t with values in $\mathbf{G}^{\mathcal{G}}$.

Definition 2. *The map*

$$L \to \hat{L} = \mathbf{g}^{-1} L \mathbf{g}, \qquad A \to \hat{A} = \mathbf{g}^{-1} A \mathbf{g} \tag{46}$$

is called a gauge transformation of the Lax pair.

Obviously

$$\hat{L} = \partial_x + \hat{\mathbf{U}}(x, t, \lambda), \qquad \hat{A} = \partial_t + \hat{\mathbf{V}}(x, t, \lambda)$$

where

$$\hat{\mathbf{U}} = \mathbf{g}^{-1}\mathbf{g}_x + \mathbf{g}^{-1}\mathbf{U}\mathbf{g}, \qquad \hat{\mathbf{V}} = \mathbf{g}^{-1}\mathbf{g}_t + \mathbf{g}^{-1}\mathbf{V}\mathbf{g}$$

and $[\hat{L}, \hat{A}] = 0$. If ψ is a fundamental solution of the problem (44), then $\chi = \mathbf{g}^{-1}\psi$ is a fundamental solution of the problem $\hat{L}\chi = 0$, $\hat{A}\chi = 0$.

Lax pairs (L, A) and $(\hat{L}, \hat{A})$ related by a gauge transformation are called *gauge equivalent*. Choosing an appropriate gauge we can transform a Lax pair to a convenient form.

Definition 3. *We say that a lax pair (43) is in the canonical gauge if* $\mathbf{U} \cap \mathfrak{A}^{\mathcal{G}} = 0$ *and* $\mathbf{V} \cap \mathfrak{A}^{\mathcal{G}} = 0$.

The canonical gauge is almost unique. The remaining gauge freedom is due to constant (x, t-independent) elements $\mathbf{g} \in \mathbf{G}^{\mathcal{G}}$, which are point symmetries of the resulting integrable non-linear system.

There is also a freedom in the choice of independent variables x, t. Suppose

$$x = X(\xi, \eta), \qquad t = T(\xi, \eta) \tag{47}$$

is an invertible change of variables, then

$$\psi_\xi = \frac{\partial X}{\partial \xi}\psi_x + \frac{\partial T}{\partial \xi}\psi_t = -\frac{\partial X}{\partial \xi}\mathbf{U}\psi - \frac{\partial T}{\partial \xi}\mathbf{V}\psi,$$

$$\psi_\eta = \frac{\partial X}{\partial \eta}\psi_x + \frac{\partial T}{\partial \eta}\psi_t = -\frac{\partial X}{\partial \eta}\mathbf{U}\psi - \frac{\partial T}{\partial \eta}\mathbf{V}\psi$$

and thus in new variables the Lax pair corresponding to (43) can be written in the form

$$\tilde{L} = \partial_\xi + \tilde{\mathbf{U}}, \qquad \tilde{A} = \partial_\eta + \tilde{\mathbf{V}},$$

where

$$\tilde{\mathbf{U}} = \frac{\partial X}{\partial \xi}\mathbf{U} + \frac{\partial T}{\partial \xi}\mathbf{V}, \quad \tilde{\mathbf{V}} = \frac{\partial X}{\partial \eta}\mathbf{U} + \frac{\partial T}{\partial \eta}\mathbf{V}.$$

In the following subsections we consider Lax pairs and corresponding second-order systems of two equations using the automorphic Lie algebras constructed above.

4.1 Equations corresponding to algebra $\mathcal{A}^2$

Theorem 1 states that automorphic Lie algebras corresponding to the groups $\mathbb{D}_N$, $\mathbb{T}$, $\mathbb{O}$ and $\mathbb{I}$ and degenerate orbits are all graded isomorphic and can be represented by algebra $\mathcal{A}^2$. For this algebra we choose a basis $\langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, J\rangle$ with commutation relations of the form (25)

$$[\mathbf{a}_1, \mathbf{a}_2] = \mathbf{a}_3 \quad [\mathbf{a}_1, \mathbf{a}_3] = 4\mathbf{a}_2 - 2J\mathbf{a}_1 \quad [\mathbf{a}_2, \mathbf{a}_3] = -4\mathbf{a}_1 + 2J\mathbf{a}_2.$$

In order to obtain a system of two second order equations we choose the following Lax pair

$$L = \partial_x - \sum_{i=1}^{3} u_i(x,t)\mathbf{a}_i \tag{48}$$

$$A = \partial_t - \sum_{i=1}^{3} v_i(x,t)\mathbf{a}_i - \sum_{i=1}^{3} w_i(x,t)J\mathbf{a}_i \tag{49}$$

In this case the subalgebra $\mathfrak{A}^{\mathcal{G}}$ is trivial and therefore the Lax pair is already in the canonical gauge.

Decomposing the compatibility condition (45) over the basis we obtain a system of eight equations

$$\mathbf{a}_1: \quad u_{1t} = v_{1x} - 4(u_3v_2 - u_2v_3) \tag{50}$$
$$\mathbf{a}_2: \quad u_{2t} = v_{2x} - 4(u_1v_3 - u_3v_1) \tag{51}$$
$$\mathbf{a}_3: \quad u_{3t} = v_{3x} - u_1v_2 + u_2v_1 \tag{52}$$
$$J\mathbf{a}_1: \quad 4(u_3w_2 - u_2w_3) + 2(u_3v_1 - u_1v_3) - w_{1,x} = 0 \tag{53}$$
$$J\mathbf{a}_2: \quad 4(u_1w_3 - u_3w_1) + 2(u_2v_3 - u_3v_2) - w_{2,x} = 0 \tag{54}$$
$$J\mathbf{a}_3: \quad u_1w_2 - u_2w_1 - w_{3,x} = 0 \tag{55}$$
$$J^2\mathbf{a}_1: \quad 2(u_3w_1 - u_1w_3) = 0 \tag{56}$$
$$J^2\mathbf{a}_2: \quad 2(u_2w_3 - u_3w_2) = 0. \tag{57}$$

for nine functions u_i, v_i, w_i. This system is underdetermined. In order to make it well determined we use transformation (47) of the form $x \to X(x', t),\ t \to t$ to make $u_3 = 1$. Then it follows from (55)-(57) that $w_{3,x} = 0$ and thus $w_3 = w_3(t)$ is a function of t only. By an appropriate change of the variable t, $t \to T(t')$, we can fix $w_3 = 2$. We shall omit primes and use the notations x and t for the new independent variables. Then from equations (52)-(57) it follows that

$$w_1 = 2u_1,\ w_2 = 2u_2,\ v_3 = -u_1u_2 + \alpha,\ v_1 = u_{1,x} - u_1^2u_2 + \alpha u_1,\ v_2 = -u_{2,x} - u_1u_2^2 + \alpha u_2,$$

where $\alpha = \alpha(t)$ is an arbitrary function of t.

Finally, the equations at $\mathbf{a}_1$ and $\mathbf{a}_2$ give us our nonlinear integrable system:

$$\begin{aligned} u_{1,t} &= u_{1,xx} - (u_1^2u_2)_x + 4u_{2,x} + \alpha u_{1,x}, \\ -u_{2,t} &= u_{2,xx} + (u_1u_2^2)_x - 4u_{1,x} - \alpha u_{2,x}. \end{aligned} \tag{58}$$

The arbitrary function $\alpha(t)$ can be removed by a Galilean transformation $x \to x + \int \alpha(t)\,dt$.

System (58) possesses an infinite hierarchy of symmetries. They can be found using the same L operator (48) and $A_k,\ k \in \mathbb{N}$ operators of the form

$$A_k = \partial_{t_k} - \sum_{s=1}^{k}\sum_{i=1}^{3} v_i^{(s)} J^{s-1}\mathbf{a}_i.$$

In particular (58) corresponds to $k = 2$

$$t = t_2, \qquad A = A_2 = \partial_{t_2} - \sum_{i=1}^{3}(v_i\mathbf{a}_i + 2u_iJ\mathbf{a}_i)$$

and in the case of $k = 1$ the system is linear

$$(u_1)_{t_1} = u_{1,x}, \qquad (u_2)_{t_1} = u_{2,x}.$$

The coefficients $v_i^{(s)}$ can be found from the compatibility condition $[L, A_k] = 0$, or using the method proposed in [11], [27].

The existence of the derivation

$$\mathcal{D} = \lambda(\lambda^2 - \lambda^{-2})\frac{d}{d\lambda}$$

of the automorphic Lie algebra $\mathcal{A}^2$ enables us to construct a Lax representation for the master symmetry. Let us consider the same Lax operator L (48) and define the second operator M, which includes the $\mathcal{D}$ derivation

$$M = \frac{\partial}{\partial\tau} + \mathcal{D} - \sum_{i=1}^{3}(V_i\mathbf{a}_i + W_i J\mathbf{a}_i).$$

It follows from the compatibility conditions $[L, M] = 0$ that $W_i = 2xu_i + \gamma u_i$, where γ is an arbitrary function of τ. We choose $\gamma = 0$ without loss of generality. Then we can establish that

$$V_1 = \frac{1}{2}u_1 + u_1V_3 + xu_{1,x}, \qquad V_2 = -\frac{1}{2}u_2 + u_2V_3 - xu_{2,x}, \qquad V_3 = -xu_1u_2 + \alpha.$$

We shall omit the inessential constant of integration α (an arbitrary function of τ). The resulting system

$$\begin{aligned} u_{1,\tau} &= 4u_2 - u_1^2u_2 + \frac{3}{2}u_{1,x} + x(u_{1,x} - u_1^2u_2 + 4u_2)_x, \\ u_{2,\tau} &= 4u_1 - u_1u_2^2 - \frac{3}{2}u_{2,x} - x(u_{2,x} + u_2^2u_1 - 4u_1)_x \end{aligned} \tag{59}$$

is a master symmetry of the the system (58). Indeed, ∂_t and ∂_τ do not commute, but their commutator commutes with ∂_t and defines a symmetry of (58):

$$\begin{aligned} \partial_{t_3}u_1 = [\partial_\tau, \partial_t]u_1 = 2u_{1,xxx} + (16u_1 - 4u_1^3 - 12u_1u_2^2 + 3u_1^3u_2^2 - 6u_1u_2u_{1,x})_x, \\ \partial_{t_3}u_2 = [\partial_\tau, \partial_t]u_2 = 2u_{2,xxx} + (16u_2 - 4u_2^3 - 12u_1^2u_2 + 3u_1^2u_2^3 + 6u_1u_2u_{2,x})_x. \end{aligned} \tag{60}$$

An infinite hierarchy of commuting local symmetries of equation (58) can be constructed recursively

$$\partial_{t_{n+1}} = [\partial_\tau, \partial_{t_n}], \qquad [\partial_{t_n}\partial_{t_m}] = 0.$$

Moreover, the operators A_k can also be found recursively $A_{k+1} = [M, A_k]$.

For example, taking

$$\begin{aligned} A_2 &= \partial_{t_2} - \mathcal{V}, & \mathcal{V} &= \textstyle\sum_{i=1}^{3}(v_i + 2u_iJ)\mathbf{a}_i, \\ M &= \partial_\tau + \mathcal{D} - \mathcal{W}, & \mathcal{W} &= \textstyle\sum_{i=1}^{3}(W_i + 2xu_iJ)\mathbf{a}_i, \end{aligned}$$

we obtain

$$[M, A_2] = [\partial_\tau, \partial_{t_2}] - \mathcal{V}_\tau - \mathcal{D}(\mathcal{V}) + \mathcal{W}_{t_2} + [\mathcal{W}, \mathcal{V}] = \partial_{t_3} - \sum_{i=1}^{3}(z_i + 4v_iJ + 8u_iJ^2)\mathbf{a}_i$$

where

$$\begin{aligned} z_1 &= 2u_{1,xx} - 6u_2u_1u_{1,x} + 8u_{2,x} + 3u_2^2u_1^3 - 4u_1^3 - 4u_2^2u_1 - 16u_1, \\ z_2 &= 2u_{2,xx} + 6u_1u_2u_{2,x} - 8u_{1,x} + 3u_1^2u_2^3 - 4u_2^3 - 4u_1^2u_2 - 16u_2, \\ z_3 &= 2u_1u_{2,x} - 2u_2u_{1,x} + 3u_2^2u_1^2 - 4u_1^2 - 4u_2^2 - 16. \end{aligned}$$

The Lax pair (L, A_3) yields equations (60).

There is a complete classification of second-order two component integrable systems [24], [25]. Equation (58) corresponds to the equation (D) in the list of integrable systems provided in [25]. This system is often called the deformed derivative nonlinear Schrödinger equation.

The system (I) in [25]

$$\begin{aligned} u_{1,t} &= u_{1,xx} - (u_1^2 u_2)_x, \\ -u_{2,t} &= u_{2,xx} + (u_1 u_2^2)_x \end{aligned}$$

corresponds to the algebra $\mathcal{A}^1$ in a similar way. It can be transformed into the well known derivative nonlinear Schrödinger equation [26] by the change of independent variables $t \mapsto it,\ x \mapsto ix$ and the additional assumption $u := u_1 = u_2^*$

$$iu_t + u_{xx} + i(|u|^2 u)_x = 0. \tag{61}$$

The condition $u_1 = u_2^*$ corresponds to a choice of the real form of $\mathcal{A}^1$ and requires a $\mathbb{Z}_2$ extension of the reduction group, which includes complex and hermitian conjugations $a(\lambda) = a^+(\lambda^*)$. The Lax representations for the famous nonlinear Schrödinger equation and the Heisenberg model originate from the algebra $\mathcal{A}^0$. They correspond to different choices of the gauge of the Lax operator and a real form of $\mathcal{A}^0$.

4.2 Equations associated with generic orbits of $sl(2,\mathbb{C})$ automorphic Lie algebras

In the case of $sl(2,\mathbb{C})$ the automorphic Lie algebras $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_\mu)$ corresponding to a generic orbit of a finite group have a basis $\langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, J\rangle$ such that the commutation relations take the form

$$[\mathbf{a}_1, \mathbf{a}_2] = p\mathbf{a}_1 + q\mathbf{a}_2 + r_3\mathbf{a}_3 + 2J\mathbf{a}_3$$

$$[\mathbf{a}_1, \mathbf{a}_3] = s\mathbf{a}_1 + r_2\mathbf{a}_2 - q\mathbf{a}_3 + 2J\mathbf{a}_2$$

$$[\mathbf{a}_2, \mathbf{a}_3] = r_1\mathbf{a}_1 - s\mathbf{a}_2 + p\mathbf{a}_3 + 2J\mathbf{a}_1$$

where s, p, q, r_i are constants depending on a generic point μ and the choice of representation of a reduction group $\mathcal{G}$. The algebra $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_\mu)$ is almost graded

$$\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_\mu) = \bigoplus_{k=1}^{\infty} \mathfrak{A}^k, \qquad [\mathfrak{A}^p, \mathfrak{A}^q] \subset \mathfrak{A}^{p+q} \oplus \mathfrak{A}^{p+q+1}$$

where $\mathfrak{A}^1 = \mathrm{span}_{\mathbb{C}}\langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\rangle$ and $\mathfrak{A}^k = J^{k-1}\mathfrak{A}^1$.

Let us take a Lax pair (L, A) where the operator L is spanned by the first homogeneous space $\mathfrak{A}^1$ of the automorphic Lie algebra, while the operator A is spanned by the first and second homogeneous spaces:

$$L = \partial_x + \sum_{i=1}^{3} S_i(x,t)\mathbf{a}_i$$

$$A = \partial_t + \sum_{i=1}^{3} v_i(x,t)\mathbf{a}_i + \sum_{i=1}^{3} w_i(x,t)J\mathbf{a}_i\ .$$

The compatibility condition $[L, A] = 0$ results in a system of equations in the first three homogeneous spaces. In $\mathfrak{A}^3$ we get the equations (vanishing the coefficients at $J^2\mathbf{a}_1, J^2\mathbf{a}_2, J^2\mathbf{a}_3$ respectively):

$$2(S_2w_3 - S_3w_2) = 0, \quad 2(S_1w_3 - S_3w_1) = 0, \quad 2(S_1w_2 - S_2w_1) = 0.$$

and thus $w_i = \gamma(x,t)S_i$. Taking this into account we see that the coefficients at $J\mathbf{a}_1, J\mathbf{a}_2, J\mathbf{a}_3$ vanish if

$$w_{1,x} + 2(S_2v_3 - S_3v_2) = 0, \quad w_{2,x} + 2(S_1v_3 - S_3v_1) = 0, \quad w_{3,x} + 2(S_1v_2 - S_2v_1) = 0. \tag{62}$$

Equations (62) are compatible and enable us to express the functions v_i in terms of S_i and their x–derivatives if

$$S_1w_{1,x} + S_2w_{2,x} + S_3w_{3,x} = \gamma_x(S_1^2 - S_2^2 + S_3^2) + \frac{1}{2}\gamma(S_1^2 - S_2^2 + S_3^2)_x = 0.$$

Assuming that $S_1^2 - S_2^2 + S_3^2 \not\equiv 0$ we can make a change of variables $x \to \hat{x} = \alpha(x,t),\ t \to \hat{t} = \beta(t)$ (and thus $S_i \to \hat{S}_i = S_i/\alpha_x$), such that $\hat{S}_1^2 - \hat{S}_2^2 + \hat{S}_3^2 = 1$ and $\gamma = 2$. Thus, without a loss of generality we shall assume that

$$S_1^2 - S_2^2 + S_3^2 = 1, \qquad w_i = 2S_i.$$

Taking this into account we represent a general solution of eq (62) in the form

$$v_1 = S_{2,x}S_3 - S_{3,x}S_2 + \Phi S_1, \quad v_2 = S_{1,x}S_3 - S_{3,x}S_1 + \Phi S_2, \quad v_3 = S_{1,x}S_2 - S_{2,x}S_1 + \Phi S_3,$$

where $\Phi = \Phi(x,t)$ is an as yet undetermined function. In $\mathfrak{A}^1$ the coefficients at $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ vanish if

$$\begin{aligned}
S_{1,t} &= v_{1,x} + p(S_1v_2 - S_2v_1) + s(S_1v_3 - S_3v_1) + r_1(S_2v_3 - S_3v_2),\\
S_{2,t} &= v_{2,x} + q(S_1v_2 - S_2v_1) + r_2(S_1v_3 - S_3v_1) - s(S_2v_3 - S_3v_2),\\
S_{3,t} &= v_{3,x} + r_3(S_1v_2 - S_2v_1) - q(S_1v_3 - S_3v_1) + p(S_2v_3 - S_3v_2).
\end{aligned}$$

It follows from the equation $(S_1^2 - S_2^2 + S_3^2)_t = 0$ that

$$\Phi = pS_1S_3 + sS_1S_2 - qS_2S_3 + \frac{1}{2}(r_1S_1^2 - r_2S_2^2 + r_3S_3^2) + \theta(t)$$

where $\theta(t)$ is an arbitrary function. Finally we obtain the following integrable system of equations

$$\begin{aligned}
S_{1,t} &= S_3S_{2,xx} - S_2S_{3,xx} + [S_1(pS_1S_3 + sS_1S_2 - qS_2S_3)]_x\\
&+ \frac{1}{2}[S_1(r_1S_1^2 - r_2S_2^2 + r_3S_3^2)]_x - pS_{3,x} - sS_{2,x} - r_1S_{1,x} + \theta(t)S_{1,x},\\
S_{2,t} &= S_3S_{1,xx} - S_1S_{3,xx} + [S_2(pS_1S_3 + sS_1S_2 - qS_2S_3)]_x\\
&+ \frac{1}{2}[S_2(r_1S_1^2 - r_2S_2^2 + r_3S_3^2)]_x - qS_{3,x} - r_2S_{2,x} + sS_{1,x} + \theta(t)S_{2,x},\\
S_{3,t} &= S_2S_{1,xx} - S_1S_{2,xx} + [S_3(pS_1S_3 + sS_1S_2 - qS_2S_3)]_x\\
&+ \frac{1}{2}[S_3(r_1S_1^2 - r_2S_2^2 + r_3S_3^2)]_x - r_3S_{3,x} + qS_{2,x} - pS_{1,x} + \theta(t)S_{3,x}.
\end{aligned}$$

The functions $S_1(x,t), S_2(x,t), S_3(x,t)$ satisfy the condition $S_1^2 - S_2^2 + S_3^2 = 1$ and can be parametrised by two functions $u = u(x,t)$ and $v = v(x,t)$:

$$S_1 = \frac{1-uv}{u-v}, \quad S_2 = \frac{1+uv}{u-v}, \quad S_3 = \frac{u+v}{u-v}.$$

In these new variables the above system takes the form

$$\begin{aligned}
u_t &= u_{xx} - \frac{2u_x^2}{u-v} - \frac{2}{(u-v)^2}[2P(u,v)u_x - P(u,u)v_x] + \eta(t)u_x\\
-v_t &= v_{xx} - \frac{2v_x^2}{u-v} + \frac{2}{(u-v)^2}[2P(u,v)v_x - P(v,v)u_x] - \eta(t)v_x
\end{aligned}$$

where

$$P(u,v) = 2au^2v^2 + b(uv^2 + vu^2) + 2cuv + d(u+v) + 2e, \quad \eta(t) = \theta(t) - (r_1 + r_2 - r_3)/2$$

and

$$a = \frac{1}{8}(r_2 - r_1 + 2s), \quad b = \frac{1}{2}(p+q), \quad c = \frac{1}{4}(r_2 + r_1 - 2r_3), \quad d = \frac{1}{2}(q-p), \quad e = \frac{1}{8}(r_2 - r_1 - 2s).$$

The function $\eta(t)$ can be set to zero by the Galilean transformation $x \to x + \int \eta(t)\,dt$. The system obtained corresponds to the system (m) in the list in [25]. In the simplest case of vanishing constants $s = p = q = r_i = 0$ this system is equivalent (up to invertible point transformations) to the (complex) Heisenberg model and the corresponding algebra is $\mathcal{A}^0$.

5 Summary and Discussion

In this paper we have studied automorphic Lie algebras $\mathfrak{A}_\lambda^{\mathcal{G}}(\mathbf{\Gamma})$ where the Lie algebra $\mathfrak{A} = sl_2(\mathbb{C})$, the parameter λ belongs to the Riemann sphere, the reduction group $\mathcal{G}$ is finite and the set $\mathbf{\Gamma}$ is a finite union of simple orbits $\mathbf{\Gamma} = \bigcup \Gamma_s$ of the group $\mathcal{G}$ on the Riemann sphere. The algebra $\mathfrak{A}_\lambda^{\mathcal{G}}(\mathbf{\Gamma})$ is covered by the algebras $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_s)$ corresponding to simple orbits (21). Automorphic Lie algebras are almost graded and can be viewed as finitely generated $\mathcal{R}(\mathbf{\Gamma})^{\mathcal{G}}$ modules.

In this paper we have introduced a definition of *graded isomorphism* (Definition 1), which is stronger than isomorphism and algorithmically verifiable for almost graded algebras. We have shown that up to graded isomorphism all algebras $\mathfrak{A}_\lambda^{\mathcal{G}}(\Gamma_s)$, where $\mathfrak{A} = sl_2(\mathbb{C})$, are isomorphic to one of the five algebras listed in Section 3.2:

- If the reduction group is trivial, then any point μ on the Riemann sphere can be taken as its simple "orbit" $\Gamma_\mu = \{\mu\}$ and $\mathfrak{A}_\lambda(\Gamma_\mu) = \mathcal{R}_\lambda(\mu) \bigotimes_{\mathbb{C}} sl_2(\mathbb{C}) \sim \mathfrak{A}_\lambda(\infty) = \mathcal{A}^0$.
- For reduction groups $\mathbb{Z}_N,\ N \geq 2$ there are two degenerate orbits $\Gamma_0 = \{0\}$ and $\Gamma_\infty = \{\infty\}$, and a generic orbit $\Gamma_\mu = \{\mu\omega^k \mid k = 1, \ldots, N,\ \omega = \exp(2\pi i/N)\}$. These result in two algebras which are not graded isomorphic. One, $\mathcal{A}^1$, corresponds to degenerate orbits and the other, $\mathcal{A}^1_1$, to a generic orbit.
- The reduction group $\mathbb{D}_2 \sim \mathbb{Z}_2 \times \mathbb{Z}_2$ has three degenerate orbits $\Gamma_0 = \{0, \infty\},\ \Gamma_1 = \{\pm 1\}$ and $\Gamma_i = \{\pm i\}$, and a generic orbit $\Gamma_\mu = \{\pm\mu, \pm\mu^{-1}\}$. The degenerate orbits result in the automorphic Lie algebra graded isomorphic to $\mathcal{A}^2$, while a generic orbit results in the algebra $\mathcal{A}^2_\mu$. The algebras $\mathcal{A}^2_\mu$ and $\mathcal{A}^2_\nu$ are isomorphic if and only if
$$\nu \in \Gamma_\mu \cup \Gamma_{i\mu} \cup \Gamma_{\frac{\mu-1}{\mu+1}} \cup \Gamma_{i\frac{\mu-1}{\mu+1}} \cup \Gamma_{\frac{i\mu-1}{i\mu+1}} \cup \Gamma_{i\frac{i\mu-1}{i\mu+1}}\,.$$
All other possible finite reduction groups are isomorphic to the groups $\mathbb{D}_N, \mathbb{T}, \mathbb{O}$ and $\mathbb{I}$ and result in automorphic Lie algebras which are graded isomorphic to $\mathcal{A}^2$ for degenerate orbits and to $\mathcal{A}^2_\mu$ for a generic orbit for an appropriate choice of the parameter μ.

There are many directions of research, which are due to be developed and where only partial results are known:

Restrictions of Lax operators to automorphic Lie algebras can be seen as reductions of bigger systems using the reduction group approach [4, 5]. The reduced systems may get lacunae in the hierarchy of local conservation laws [5, 28], more complicated Hamiltonian structures and increased orders of the recursion operators [9, 29].

In this paper we assume that algebras are over the complex field $\mathbb{C}$ and the variables in the associated integrable systems are complex valued. In applications we often need to study equations with real valued functions or with a certain complex structure. For example in (58) we could assume that the variables u_1, u_2 are real or alternatively, after the complex change of the dependent variables $t \mapsto it,\ x \mapsto ix$, we could assume that $u_1 = u_2^*$, which would lead to an equation for one complex variable $u = u_1$
$$iu_t + u_{xx} - i(|u|^2 u)_x + 4iu_x^* = 0.$$
The above choices correspond to different real forms of the algebra $\mathcal{A}^2$. In this paper we did not study real forms of automorphic Lie algebras, although it is an interesting problem important for applications.

In Section 4 we presented examples of Lax pairs for automorphic Lie algebras with a single simple orbit. The integrable systems obtained are of evolutionary type. The well known integrable massive Thirring model is of hyperbolic type [30]. The operators of the Lax pair in this case are based on two sub-algebras of the type $\mathcal{A}^1$ corresponding to two degenerate orbits, namely $\mathfrak{A}_\lambda^{\mathbb{Z}_2}(\{0\})$ and $\mathfrak{A}_\lambda^{\mathbb{Z}_2}(\{\infty\})$.

One of the operators of the massive Thirring model coincides with the L operator of the derivative nonlinear Schrödinger equation (61), which is a symmetry of the Thirring model. An interesting and important problem is to describe and classify all possible Lax pairs associated with automorphic Lie algebras and their real forms, as well as corresponding integrable systems and their interrelations.

The spectral transform method can be applied to find solutions of integrable equations corresponding to automorphic Lax operators. It can be reduced to a matrix Riemann–Hilbert problem for piece-wise analytic fundamental solutions of the Lax equation [31]. The domains of analyticity of the analytic fundamental solutions and spectral data very much depend on the choice of the reduction group (see for example [4, 5]). Lax operators corresponding to isomorphic Lie algebras result in the same systems of integrable equations, but the analytic properties of fundamental solutions are different. It would be interesting to compare the spectral transforms for different presentations of graded isomorphic automorphic Lax operators.

In this paper we have only studied finite reduction groups acting on a Riemann sphere. There are generalisations of this construction. A finite reduction group acting on a torus was introduced to integrate the Landau-Lifshitz model of anisotropic ferromagnets [6]. The possibility of considering finite reduction groups on Riemann surfaces was discussed in [7]. The idea to consider infinite Fuchsian groups and the modular group for the construction of reduction groups and automorphic Lie algebras is not new, but up to now it has not yielded any non-trivial examples. In the recent work [32] the authors have constructed an automorphic Lie algebra associated with the modular group and $\mathcal{G} = sl_2$, which is graded isomorphic to the algebra $\mathcal{A}^1$ in this paper. It would be interesing to find automorphic Lie algebras corresponding to congruence subgroups of the modular group.

Acknowledgments

The authors are grateful to the EPSRC grant EP/P012655/1 for partial support.

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