Research Article

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Simultaneous identification and reconstruction of the space-dependent reaction coefficient and source term

Abstract: The inverse problem of simultaneously determining, i.e., identifying and reconstructing, the space-dependent reaction coefficient and source term component from time-integral temperature measurements is investigated. This corresponds to thermal applications in which the heat is generated from a source depending linearly on the temperature, but with unknown space-dependent coefficients. For the resulting nonlinear inverse problem, we first prove the existence of solution based on the Schauder fixed point theorem. Then, under certain additional conditions, the solution is also proved to be unique. For the numerical reconstruction of solution, the problem is reformulated as a least-squares minimisation whose Fréchet gradients with respect to the two unknowns are derived in terms of the solution of an adjoint problem. The conjugate gradient method (CGM) to calculate the numerical solution is developed, and its convergence is proved from the Lipschitz continuity of these gradients. Three numerical examples for one- and two-dimensional inverse problems are illustrated to reveal the accuracy and stability of the solutions applying the CGM regularised by the discrepancy principle when noisy data are inverted.

Keywords: Inverse problem, parabolic equation, conjugate gradient method

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1 Introduction

In the mathematical modelling and simulation of complex phenomena, it is often the case that there are more than one physical property of the system that are unknown and they have to be determined (or estimated) from extra measurements of other available physical quantities. Nevertheless, such multiple coefficient inverse problems are more difficult to solve than the single ones due to the extra system-coupling and nonlinearity that arise. In addition, the degree of ill-posedness is expected to increase. When multiple coefficients appearing in the governing parabolic heat transfer equation are unknown, but they depend on the time variable only, their simultaneous identification in terms of local existence and uniqueness was initiated by Professor M. Ivanchov and his co-workers, e.g., [18], followed by numerical implementations realised in [16]. The key starting point in these works is the employment of Green's functions for the leading dominant part $\partial_t - a(t)\nabla^2$ (where $a(t) > 0$ is the diffusivity) of the governing parabolic heat equation. However, this approach is not extendable to situations when the unknown coefficients depend on the space variables, in

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which case the non-uniqueness of recovering multiple spacewise dependent properties becomes a main difficulty to overcome, [2, 9, 14]. In [19, 37], the initial temperature and the space-dependent reaction coefficient or the heat source, respectively, were identified and reconstructed from various temperature measurements. Also, very recently, the authors have investigated the simultaneous numerical reconstruction of the spacewise dependent reaction coefficient and heat source, (as well as the initial temperature), from temperature measurements at three distinct time instants [5]. Encouraged by this preliminary numerical study, in this paper, we investigate in more detail the existence and uniqueness of solution, as well as consider the more general (and practical) case of time-integral average temperature measurements.

Thus, we consider the heat transfer process in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, with sufficiently smooth boundary $\partial\Omega$, over the time interval from the initial time 0 to $T > 0$, governed by

$$\begin{cases} u_t(x, t) = \nabla^2 u(x, t) - q(x)u(x, t) + f(x)h(x, t) + g(x, t), & (x, t) \in \Omega \times (0, T) =: Q, \\ u(x, t) = \mu(x, t), & (x, t) \in \partial\Omega \times (0, T) =: S, \\ u(x, 0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \qquad (1.1)$$

where u represents the temperature, q is the spatially-dependent reaction coefficient, f, h and g are heat source components, μ is the Dirichlet boundary data and u_0 is the initial temperature. For simplicity, in (1.1), the heat capacity and thermal conductivity were assumed to be constant and taken to be unity.

When all the thermal properties and the initial and boundary conditions are specified, the well-posedness of the solution u to the initial-boundary value direct problem (1.1) is classical, see, e.g., [13]. However, in many engineering situations, the thermal coefficients as well as initial and/or boundary data cannot be measured directly. Thus, a wide range of inverse problems have been concerned to determine the physical properties and heat transfer coefficients, boundary and/or initial conditions, [27]. For instance, in the heat transfer in biological tissue, problem (1.1) becomes the well-known Pennes' bio-heat model [29], which is obtained by a balance between the accumulation of energy due to the blood flowing through capillary network and the heat generation due to cell metabolism. The reaction coefficient q, which is also known as the perfusion coefficient, plays an important role in the heat generation in carcinogenic skin and tumours due to the increased nutrition and oxygen demand, [28], and helps to understand the heat transfer through such biological tissues.

The nonlinear inverse problem of determining the reaction coefficient $q(x)$ has been extensively studied, when all the source components f, g and h, the Dirichlet boundary data μ and the initial temperature u_0 are known. Its existence and uniqueness have been established from certain measurements, including the temperature at the terminal time $t = T$ and the time-average integral of temperature, [17, 23, 26, 30, 32, 33]. Besides, there are several numerical techniques that have been developed to numerically reconstruct $q(x)$ from the above additional measurements, such as the Tikhonov regularization method [8], the Armijo algorithm combined with the finite element method (FEM) [6], the NAG routine E04FCF together with the finite-difference method (FDM) [36], and the CGM [3].

If the source component $f(x)$ is unknown, and the reaction coefficient q and the input data h, g, μ and u_0 are given, the linear inverse problem of identifying it from final time or time-integral temperature measurements has been extensively considered, e.g., [11, 20, 21, 31]. Such inverse source problems arise in various physical and engineering applications, e.g., the identification of sources of water and air pollution in the environment, or the heat sources in thermal processes.

In this work, we assume that the heat source components $h(x, t)$ and $g(x, t)$, the boundary temperature $\mu(x, t)$ and the initial temperature $u_0(x)$ are known, but the reaction coefficient $q(x)$ and the source component $f(x)$ are both unknown and have to be determined from some additional information. Previously, such additional information was taken to be the measurement of u and its time derivative u_t at the fixed instant $T_0 \in (0, T]$, namely,

$$u(x, T_0) = \zeta_1(x), \quad u_t(x, T_0) = \zeta_2(x), \quad x \in \Omega,$$

see [24], or the measurement of u at two different fixed times $t = T_1 \in (0, T)$ and $t = T$, namely,

$$u(x, T_1) = \xi_1(x), \quad u(x, T) = \xi_2(x), \quad x \in \Omega, \qquad (1.2)$$

see [25]. In this paper, we generalise the fixed time (instant) observations (1.2), which in practice may be difficult to be realised accurately, to time-average recordings, which smooth out the otherwise possibly large measurement errors in u. More precisely, we investigate the inverse problem of determining the triplet $(u(x, t), q(x), f(x))$ satisfying problem (1.1) together with the time-integral temperature measurements,

$$\int_0^T \omega_1(t) u(x, t)\, dt = \phi_1(x), \quad x \in \Omega, \tag{1.3}$$

$$\int_0^T \omega_2(t) u(x, t)\, dt = \phi_2(x), \quad x \in \Omega, \tag{1.4}$$

where $\omega_1(t)$ and $\omega_2(t)$ are two given weight functions, and $\phi_1(x)$ and $\phi_2(x)$ are given time-averaged measured data. If one weight function is a close approximation to the Dirac delta function $\delta(t - T)$, the time-integral temperature becomes the final time temperature observation. In this sense, the integral observations (1.3) and (1.4) can be viewed as a generalisation of the final-time observations (1.2). On the other hand, note that the choice of the weight functions in (1.3) and (1.4) is important to extract useful information for the recovery of the two unknown quantities, see [4] for more details.

Efficient schemes of reconstructing two quantities simultaneously have already been established, e.g., the space-dependent reaction coefficient and the initial temperature were simultaneously identified from the final time observation of temperature in [37] and from the time-integral temperature measurement in [4]. Such methods can also be applied to simultaneously determine the reaction coefficient $q(x)$ and the source component $f(x)$ in the inverse problem (1.1), (1.3) and (1.4).

The paper is organised as follows. The approach of dealing with the existence and uniqueness of the non-linear inverse problem (1.1), (1.3) and (1.4) is to transform it into a nonlinear non-classical direct problem, as described in Section 2. For the numerical solution, the least-squares objective functional is minimised to obtain the quasi-solution of the unknown reaction coefficient $q(x)$ and source component $f(x)$, as described in Section 3. In Section 4, the Fréchet differentiability of this objective functional is established, and the Fréchet gradients are derived and proved to be Lipschitz continuous. The CGM is established based on these gradients and the adjoint problem to reconstruct the unknown coefficients simultaneously. The convergence of the CGM with the Fletcher-Reeves formula [12] is considered based on the arguments in [38]. Since the inverse problem considered in this paper is unstable, the CGM regularised by the discrepancy principle [1] is utilised to obtain a stable numerical solution to the inverse problem. Three numerical examples of one- and two-dimensional inverse problems are presented and discussed in Section 5. Finally, Section 6 highlights the conclusions of the paper.

2 Unique solvability of the inverse problem

In this section, we shall prove that there exists a unique solution $(q(x), f(x), u(x, t))$ to the inverse problem (1.1), (1.3) and (1.4), when the source components $h(x, t)$ and $g(x, t)$, the Dirichlet boundary data $\mu(x, t)$ and the initial temperature $u_0(x)$ are specified. In the following mathematical analysis, we will use the standard Sobolev spaces $H^1(\Omega)$, $H_0^1(\Omega)$, $H^2(\Omega)$ and $H^{2,1}(Q)$, which have been defined in many textbooks, e.g., [35]. In order to obtain the existence and uniqueness of the solution to the inverse problem (1.1), (1.3) and (1.4), we introduce the following assumptions:

(a) the input data $h, g \in L_\infty(Q)$, $\mu \in L_\infty(S)$ and $u_0 \in H^1(\Omega) \cap L_\infty(\Omega)$,

(b) there exists a function $\mathcal{M}(x, t) \in H^{2,1}(Q) \cap L_\infty(0, T; H^1(\Omega) \cap L_\infty(\Omega)) =: \mathcal{V}$ satisfying $\mathcal{M}(x, t)|_S = \mu(x, t)$,

(c) $\phi_1, \phi_2 \in H^2(\Omega) \cap L_\infty(\Omega)$ and $\omega_1, \omega_2 \in C^1[0, T]$ satisfy the compatibility conditions

$$\int_0^T \omega_1(t)\mu(x, t)\, dt = \phi_1(x) \quad \text{and} \quad \int_0^T \omega_2(t)\mu(x, t)\, dt = \phi_2(x)$$

for $x \in \partial\Omega$,

(d) for all $x \in \overline{\Omega}$,

$$\phi_1(x) \int_0^T \omega_2(t)h(x, t)\, dt - \phi_2(x) \int_0^T \omega_1(t)h(x, t)\, dt \neq 0,$$

(e) $A_2 a_1 - A_1 a_2 \geq M_1$ and $\Phi_2 a_1 - \Phi_1 a_2 \leq M_2$ a.e. in $\overline{\Omega}$, for some positive constants M_1 and M_2, where

$$
\begin{cases}
a_i(x) = \displaystyle\int_0^T \omega_i(t)g(x, t)\, dt + \nabla^2 \phi_i(x) + \omega_i(0)u_0(x), \\[2mm]
A_i(x) = \dfrac{\int_0^T \omega_i(t)h(x, t)\, dt}{\phi_1(x) \int_0^T \omega_2(t)h(x, t)\, dt - \phi_2(x) \int_0^T \omega_1(t)h(x, t)\, dt}, \quad i = 1, 2. \\[4mm]
\Phi_i(x) = \dfrac{\phi_i(x)}{\phi_1(x) \int_0^T \omega_2(t)h(x, t)\, dt - \phi_2(x) \int_0^T \omega_1(t)h(x, t)\, dt},
\end{cases}
\tag{2.1}
$$

We also define the Banach space

$$\mathcal{V}_0 := H^{2,1}(Q) \cap L_\infty(0, T; H_0^1(\Omega) \cap L_\infty(\Omega)) \subset \mathcal{V}$$

equipped with the norm

$$\|u\|_{\mathcal{V}_0} = \|u\|_{H^{2,1}(Q)} + \|u\|_{L_\infty(0,T;H_0^1(\Omega) \cap L_\infty(\Omega))}. \tag{2.2}$$

We now state the well-posedness for the direct initial-boundary value problem (1.1) when all the coefficients and input data are specified, [26], and the Schauder fixed point theorem, which is employed to establish the existence of solution to the inverse problem.

Lemma 2.1. *Let assumptions* (a) *and* (b) *hold, and suppose that* $0 < q_- \leq q \leq q_+ < \infty$ *in* $\overline{\Omega}$ *for some prescribed constants* q_- *and* q_+, $q \in L_\infty(\Omega)$ *and* $f \in L_\infty(\Omega)$. *Then the initial-boundary value direct problem* (1.1) *satisfying the compatibility condition* $u_0(x) = \mu(x, 0)$ *for all* $x \in \partial\Omega$, *has a unique solution* $u \in \mathcal{V}$, *which satisfies the following estimate:*

$$|u(x, t)| \leq \max\left\{\frac{1}{q_-}(\|f\|_{L_\infty(\Omega)}\|h\|_{L_\infty(Q)} + \|g\|_{L_\infty(Q)}), \|\mu\|_{L_\infty(S)}, \|u_0\|_{L_\infty(\Omega)}\right\} \quad a.e. \text{ in } \overline{Q}. \tag{2.3}$$

Theorem 2.1 (Schauder fixed point theorem). *Let X be a closed convex subset of a Banach space V and let $\mathcal{F} : X \mapsto X$ be a continuous and compact mapping. Then $\mathcal{F}$ has a fixed point.*

In order to obtain the existence and uniqueness to the solution of the inverse problem (1.1), (1.3) and (1.4), we reformulate it as a nonlinear non-classical parabolic problem. Thus, multiplying the first equation in (1.1) by $\omega_1(t)$ and $\omega_2(t)$, respectively, integrating the resulting relations with respect to t from 0 to T, and using (1.3) and (1.4), we have

$$
\omega_i(T)u(x, T) - \omega_i(0)u_0(x) - \int_0^T \omega_i'(t)u(x, t)\, dt
$$
$$
= \nabla^2 \phi_i(x) - q(x)\phi_i(x) + f(x) \int_0^T \omega_i(t)h(x, t)\, dt + \int_0^T \omega_i(t)g(x, t)\, dt, \quad i = 1, 2.
\tag{2.4}
$$

By the notations in (2.1) and using (2.4), we have

$$q(x) = A_2(x)(a_1(x) + \overline{u}_1(x)) - A_1(x)(a_2(x) + \overline{u}_2(x)), \tag{2.5}$$
$$f(x) = \Phi_2(x)(a_1(x) + \overline{u}_1(x)) - \Phi_1(x)(a_2(x) + \overline{u}_2(x)), \tag{2.6}$$

where

$$\overline{u}_i(x) = -\omega_i(T)u(x, T) + \int_0^T \omega_i'(t)u(x, t)\, dt, \quad i = 1, 2. \tag{2.7}$$

Substituting (2.5) and (2.6) into (1.1), we obtain the nonlinear non-classical parabolic problem given by

$$
\begin{cases}
u_t = \nabla^2 u - \{A_2(x)(a_1(x) + \overline{u}_1(x)) - A_1(x)(a_2(x) + \overline{u}_2(x))\}u \\
\quad + \{\Phi_2(x)(a_1(x) + \overline{u}_1(x)) - \Phi_1(x)(a_2(x) + \overline{u}_2(x))\}h(x, t) + g(x, t), & (x, t) \in Q, \\
u(x, t) = \mu(x, t), & (x, t) \in S, \\
u(x, 0) = u_0(x), & x \in \overline{\Omega}.
\end{cases}
\tag{2.8}
$$

Thus, the inverse problem (1.1), (1.3) and (1.4) of determining the unknown quantities $q(x)$ and $f(x)$ simultaneously is equivalent to obtaining the solution $u(x, t)$ to the nonlinear parabolic problem (2.8).

We utilise the methodology applied in [26], see also [24, 25], and consider the following two auxiliary parabolic problems:

$$
\begin{cases}
U_t = \nabla^2 U - (A_2 a_1 - A_1 a_2)U + g, & (x, t) \in Q, \\
U(x, t) = \mu(x, t), & (x, t) \in S, \\
U(x, 0) = u_0(x), & x \in \overline{\Omega},
\end{cases}
\tag{2.9}
$$

and

$$
\begin{cases}
W_t = \nabla^2 W - \{A_2 a_1 - A_1 a_2 + c(A_2(\overline{W}_1 + \overline{U}_1) - A_1(\overline{W}_2 + \overline{U}_2))\}W \\
\quad - c(A_2(\overline{W}_1 + \overline{U}_1) - A_1(\overline{W}_2 + \overline{U}_2))U \\
\quad + \{\Phi_2 a_1 - \Phi_1 a_2 + c(\Phi_2(\overline{W}_1 + \overline{U}_1) - \Phi_1(\overline{W}_2 + \overline{U}_2))\}h, & (x, t) \in Q, \\
W(x, t) = 0, & (x, t) \in S, \\
W(x, 0) = 0, & x \in \overline{\Omega},
\end{cases}
\tag{2.10}
$$

where

$$
\overline{W}_i(x) := -\omega_i(T)W(x, T) + \int_0^T \omega_i'(t)W(x, t)\, dt,
$$

$$
\overline{U}_i(x) := -\omega_i(T)U(x, T) + \int_0^T \omega_i'(t)U(x, t)\, dt, \quad i = 1, 2,
\tag{2.11}
$$

and $c(\cdot)$ is a Lipschitz continuous function on $\mathbb{R}$ defined by

$$
c(\xi) = \begin{cases}
\xi & \text{if } |\xi| \le M_0, \\
M_0 & \text{if } \xi > M_0, \\
-M_0 & \text{if } \xi < -M_0,
\end{cases}
\tag{2.12}
$$

and M_0 is some fixed number in the interval $(0, M_1)$. Remark that from (2.11) we obtain

$$
\overline{U}_i(x) + \overline{W}_i(x) = -\omega_i(T)(U(x, T) + W(x, T)) + \int_0^T \omega_i'(t)(U(x, t) + W(x, t))\, dt, \quad i = 1, 2,
\tag{2.13}
$$

whose form, in relation to (2.7), suggests to prove that $u = U + W$ solves problem (2.8).

Theorem 2.2. *Let assumptions* (a)–(e) *hold. Then there exists a solution* $W(x, t) \in \mathcal{V}_0$ *to problem* (2.10).

Proof. Taking the linear initial-boundary value problem (2.9) into consideration and using Lemma 2.1, there exists a unique solution $U(x, t) \in \mathcal{V}$, satisfying the following estimate:

$$
|U(x, t)| \le \max\left\{\frac{1}{M_1}\|g\|_{L_\infty(Q)}, \|\mu\|_{L_\infty(S)}, \|u_0\|_{L_\infty(\Omega)}\right\} =: N_1.
\tag{2.14}
$$

Consider the following linear initial-boundary value problem:

$$
\begin{cases}
W_t = \nabla^2 W - \{A_2 a_1 - A_1 a_2 + c(A_2(\overline{v}_1 + \overline{U}_1) - A_1(\overline{v}_2 + \overline{U}_2))\}W - c(A_2(\overline{v}_1 + \overline{U}_1) \\
\quad - A_1(\overline{v}_2 + \overline{U}_2))U + \{\Phi_2 a_1 - \Phi_1 a_2 + c(\Phi_2(\overline{v}_1 + \overline{U}_1) - \Phi_1(\overline{v}_2 + \overline{U}_2))\}h, & (x, t) \in Q, \\
W(x, t) = 0, & (x, t) \in S, \\
W(x, 0) = 0, & x \in \overline{\Omega},
\end{cases}
\tag{2.15}
$$

where the functions $\bar{v}_1$ and $\bar{v}_2$ are defined by the same way as for problem (2.10), namely,

$$\bar{v}_i(x) := -\omega_i(T)v(x, T) + \int_0^T \omega_i'(t)v(x, t)\, dt, \quad i = 1, 2, \tag{2.16}$$

and $v(x, t)$ is an arbitrary function belonging to the space $\mathcal{V}_0$.

Multiplying the first equation in (2.15) by W, integrating the result over the space-time domain Q and using the definition of function $c(\cdot)$ in (2.12), we have

$$\frac{1}{2}\|W(\cdot, T)\|_{L_2(\Omega)}^2 + \int_Q (|\nabla W|^2 + (M_1 - M_0)W^2)\, dx\, dt$$

$$\leq \frac{(M_0 + M_2)^2}{M_1 - M_0}\|h\|_{L_2(Q)}^2 + \frac{M_0^2 N_1^2}{M_1 - M_0}|Q| + \frac{M_1 - M_0}{2}\|W\|_{L_2(Q)}^2, \tag{2.17}$$

where $|Q| = T \times \operatorname{diam}(\Omega)$ is the size of the solution domain Q, and we have used the inequality $\alpha\beta \leq \frac{1}{4\epsilon}\alpha^2 + \epsilon\beta^2$ for $\epsilon = \frac{1}{4}(M_1 - M_0)$, twice to estimate the third and fourth-terms on the right-hand side of the first equation in (2.15). Thus, we obtain

$$\|W(\cdot, T)\|_{L_2(\Omega)}^2 + \int_Q (W^2 + |\nabla W|^2)\, dx\, dt \leq C_1(\|h\|_{L_2(Q)}^2 + 1),$$

where C_1 is a positive constant depending on M_0, M_1, M_2, N_1 and $|Q|$. On dividing inequality (2.17) by $\min\{M_1 - M_0, 2\} > 0$ and re-denoting the resulting constant still by C_1 then, we get the estimate

$$\int_Q (W^2 + |\nabla W|^2)\, dx\, dt \leq C_1(\|h\|_{L_2(Q)}^2 + 1).$$

From the homogeneous Dirichlet boundary condition and initial condition on W in (2.15) we have

$$\int_0^t \int_\Omega (W_\tau - \nabla^2 W)^2\, dx\, d\tau = \int_0^t \int_\Omega (W_\tau^2 + (\nabla^2 W)^2 - 2W_\tau \nabla^2 W)\, dx\, d\tau$$

$$= \int_0^t \int_\Omega (W_\tau^2 + (\nabla^2 W)^2 + 2\nabla W_\tau \cdot \nabla W)\, dx\, d\tau$$

$$= \int_0^t \int_\Omega (W_\tau^2 + (\nabla^2 W)^2)\, dx\, d\tau + \int_\Omega |\nabla W(x, t)|^2\, dx. \tag{2.18}$$

Using assumption (e), equation (2.12), and the homogeneous boundary and initial conditions in (2.15), we have

$$-\int_0^t \int_\Omega \{A_2 a_1 - A_1 a_2 + c(A_2(\bar{v}_1 + \overline{U}_1) - A_1(\bar{v}_2 + \overline{U}_2))\}W(W_\tau - \nabla^2 W)\, dx\, d\tau$$

$$\leq \frac{M_0 - M_1}{2}\|W(\cdot, t)\|_{L_2(\Omega)}^2 + (M_0 - M_1)\int_0^t \int_\Omega |\nabla W|^2\, dx\, d\tau \leq 0, \tag{2.19}$$

$$-\int_0^t \int_\Omega c(A_2(\bar{v}_1 + \overline{U}_1) - A_1(\bar{v}_2 + \overline{U}_2))U(W_\tau - \nabla^2 W)\, dx\, d\tau$$

$$\leq \int_0^t \int_\Omega \left[2c^2(A_2(\bar{v}_1 + \overline{U}_1) - A_1(\bar{v}_2 + \overline{U}_2))U^2 + \frac{1}{8}(W_\tau - \nabla^2 W)^2\right] dx\, d\tau$$

$$\leq \frac{1}{4}\int_0^t \int_\Omega (W_\tau^2 + (\nabla^2 W)^2)\, dx\, d\tau + 2M_0^2 N_1^2|Q|, \tag{2.20}$$

and

$$\int_0^t \int_\Omega \{\Phi_2 a_1 - \Phi_1 a_2 + c(\Phi_2(\overline{v}_1 + \overline{U}_1) - \Phi_1(\overline{v}_2 + \overline{U}_2))\} h(W_\tau - \nabla^2 W) \, dx \, d\tau$$

$$\le \frac{1}{4} \int_0^t \int_\Omega (W_\tau^2 + (\nabla^2 W)^2) \, dx \, d\tau + 2(M_0 + M_2)^2 \int_0^t \int_\Omega h^2 \, dx \, d\tau, \tag{2.21}$$

where we have used that for any two positive numbers α and β we have that $\alpha\beta \le 2\alpha^2 + \frac{\beta^2}{8}$. Then, multiplying the first equation in (2.15) by $(W_\tau - \nabla^2 W)$ and integrating over $\Omega \times (0, t)$, along with using (2.18)–(2.21), and that $M_0 < M_1$, we get

$$\int_0^t \int_\Omega (W_\tau^2 + (\nabla^2 W)^2) \, dx \, d\tau + \int_\Omega |\nabla W(x, t)|^2 \, dx \le C_2\left(\int_0^t \int_\Omega h^2 \, dx \, d\tau + 1\right) \le C_2(\|h\|_{L_2(Q)}^2 + 1), \tag{2.22}$$

where $C_2 > 0$ depends on M_0, M_2, N_1 and $|Q|$. This implies that

$$\max_{t \in [0,T]} \int_\Omega |\nabla W(x, t)|^2 \, dx \le C_2(\|h\|_{L_2(Q)}^2 + 1), \tag{2.23}$$

and (taking $t = T$ in (2.22))

$$\int_Q (W_t^2 + (\nabla^2 W)^2) \, dx \, dt \le C_2(\|h\|_{L_2(Q)}^2 + 1). \tag{2.24}$$

Using inequality (2.3) in the Lemma 2.1 applied to the solution W of problem (2.15), equations (2.14), (2.15) and the assumption (e), we obtain

$$|W(x, t)| \le \frac{1}{M_1 - M_0}(M_0 N_1 + (M_0 + M_2)\|h\|_{L_\infty(Q)}) =: N_2 \quad \text{a.e. } (x, t) \in \overline{Q}. \tag{2.25}$$

Using (2.2), estimates (2.18), (2.23)–(2.25) imply that the solution $W(x, t)$ of the initial-boundary value problem (2.15) belongs to the space $\mathcal{V}_0$, and satisfies the following estimate:

$$\|W\|_{\mathcal{V}_0} \le R, \tag{2.26}$$

where

$$R := C_0(\|h\|_{L_2(Q)} + 1) + N_2,$$

and C_0 is a positive constant depending on M_0, M_1, M_2, N_1 and $|Q|$. Thus, problem (2.15) generates an operator $\mathcal{F}$ from $\mathcal{V}_0$ to itself and $\mathcal{F}(v) = W$.

Let

$$X = \{v \in \mathcal{V}_0 : \|v\|_{\mathcal{V}_0} \le R\}.$$

Then X is a bounded and closed convex subset of $\mathcal{V}_0$. Since the solution W to problem (2.15) satisfies the estimate (2.26), the operator $\mathcal{F}$ takes the set X into itself.

It remains to show $\mathcal{F} : X \mapsto X$ is a continuous and compact operator. Let $\{v_n(x, t)\}$ be a sequence of functions from the set X. Then according to (2.16), $\{(\overline{v}_1)_n(x)\}$ and $\{(\overline{v}_2)_n(x)\}$ are two bounded sequences in $H^2(\Omega) \cap L_\infty(\Omega)$. Clearly, the sequence $\{W_n(x, t)\}$ given by $W_n := \mathcal{F}(v_n)$ is a bounded sequence in X by (2.25). By the boundedness of the sequences $\{v_n(x, t)\}$, $\{W_n(x, t)\}$, $\{(\overline{v}_1)_n(x)\}$ and $\{(\overline{v}_2)_n(x)\}$, there exist four subsequences, still denoted the same, such that

- $v_n(x, t) \to v(x, t)$ and $W_n(x, t) \to W(x, t)$ weakly in $H^{2,1}(Q) \cap L_\infty(Q)$ a.e. $(x, t) \in \overline{Q}$,
- $(\overline{v}_1)_n(x) \to \overline{v}_1(x)$ and $(\overline{v}_2)_n(x) \to \overline{v}_2(x)$ in $L_\infty(\Omega)$ a.e. $x \in \overline{\Omega}$.

Denoting $w_n(x, t) = W_n(x, t) - W(x, t)$, we have

$$\begin{cases} (w_n)_t = \nabla^2 w_n - s_n^{(1)} w_n - s_n^{(2)}(W + U) + s_n^{(3)} h, & (x, t) \in Q, \\ w_n(x, t) = 0, & (x, t) \in S, \\ w_n(x, 0) = 0, & x \in \overline{\Omega}, \end{cases} \tag{2.27}$$

where

$$s_n^{(1)} := A_2 a_1 - A_1 a_2 + c(A_2((\overline{v}_1)_n + \overline{U}_1) - A_1((\overline{v}_2)_n + \overline{U}_2)),$$

$$s_n^{(2)} := c(A_2((\overline{v}_1)_n + \overline{U}_1) - A_1((\overline{v}_2)_n + \overline{U}_2)) - c(A_2(\overline{v}_1 + \overline{U}_1) - A_1(\overline{v}_2 + \overline{U}_2)),$$

$$s_n^{(3)} := c(\Phi_2((\overline{v}_1)_n + \overline{U}_1) - \Phi_1((\overline{v}_2)_n + \overline{U}_2)) - c(\Phi_2(\overline{v}_1 + \overline{U}_1) - \Phi_1(\overline{v}_2 + \overline{U}_2)).$$

Applying the same technique used to derive (2.17) for the function W satisfying problem (2.15), to the function w_n satisfying problem (2.27), we obtain

$$\frac{1}{2}\|w_n(\cdot, T)\|_{L_2(\Omega)}^2 + \int_Q (|\nabla w_n|^2 + (M_1 - M_0)w_n^2)\, dx\, dt$$

$$\leq \frac{\|W\|_{L_2(Q)}^2 + \|U\|_{L_2(Q)}^2}{M_1 - M_0}\|s_n^{(2)}\|_{L_\infty(\Omega)}^2 + \frac{\|h\|_{L_2(Q)}^2}{M_1 - M_0}\|s_n^{(3)}\|_{L_\infty(\Omega)}^2 + \frac{M_1 - M_0}{2}\|w_n\|_{L_2(Q)}^2.$$

It then yields that

$$\int_Q (w_n^2 + |\nabla w_n|^2)\, dx\, dt \leq \frac{1}{\min\{1, \frac{M_1-M_0}{2}\}}\left\{\frac{\|W\|_{L_2(Q)}^2 + \|U\|_{L_2(Q)}^2}{M_1 - M_0}\|s_n^{(2)}\|_{L_\infty(\Omega)}^2 + \frac{\|h\|_{L_2(Q)}^2}{M_1 - M_0}\|s_n^{(3)}\|_{L_\infty(\Omega)}^2\right\}. \tag{2.28}$$

Invoking the approaches utilised in deriving (2.18)–(2.24), we have

$$\int_0^t \int_\Omega ((w_n)_\tau - \nabla^2 w_n)^2\, dx\, d\tau = \int_0^t \int_\Omega ((w_n)_\tau^2 + (\nabla^2 w_n)^2)\, dx\, d\tau + \int_\Omega |\nabla w_n(x, t)|^2\, dx, \tag{2.29}$$

$$-\int_0^t \int_\Omega s_n^{(1)} w_n((w_n)_\tau - \nabla^2 w_n)\, dx\, d\tau \leq \frac{M_0 - M_1}{2}\|w_n(\cdot, t)\|_{L_2(\Omega)}^2 + (M_0 - M_1)\int_0^t \int_\Omega |\nabla w_n|^2\, dx\, d\tau \leq 0, \tag{2.30}$$

$$-\int_0^t \int_\Omega s_n^{(2)}(W + U)((w_n)_\tau - \nabla^2 w_n)\, dx\, d\tau \leq \frac{1}{4}\int_0^t \int_\Omega ((w_n)_\tau^2 + (\nabla^2 w_n)^2)\, dx\, d\tau$$

$$+ 2\|s_n^{(2)}\|_{L_\infty(\Omega)}^2 \int_0^t \int_\Omega (W^2 + U^2)\, dx\, d\tau, \tag{2.31}$$

$$\int_0^t \int_\Omega s_n^{(3)} h((w_n)_\tau - \nabla^2 w_n)\, dx\, d\tau \leq \frac{1}{4}\int_0^t \int_\Omega ((w_n)_\tau^2 + (\nabla^2 w_n)^2)\, dx\, d\tau + 2\|s_n^{(3)}\|_{L_\infty(\Omega)}^2 \int_0^t \int_\Omega h^2\, dx\, d\tau. \tag{2.32}$$

Multiplying the first equation in (2.27) by $((w_n)_\tau - \nabla^2 w_n)$ and integrating over $\Omega \times (0, t)$, along with using (2.29)–(2.32), we obtain

$$\int_0^t \int_\Omega ((w_n)_\tau^2 + (\nabla^2 w_n)^2)\, dx\, d\tau + \int_\Omega |\nabla w_n(x, t)|^2\, dx$$

$$\leq 4\|s_n^{(2)}\|_{L_\infty(\Omega)}^2 \int_0^t \int_\Omega (W^2 + U^2)\, dx\, d\tau + 4\|s_n^{(3)}\|_{L_\infty(\Omega)}^2 \int_0^t \int_\Omega h^2\, dx\, d\tau,$$

which implies that

$$\left\{\int_Q ((w_n)_t^2 + (\nabla^2 w_n)^2)\, dx\, dt,\ \max_{t\in[0,T]} \int_\Omega |\nabla w_n(x, t)|^2\, dx\right\}$$

$$\leq 4\|s_n^{(2)}\|_{L_\infty(\Omega)}^2 (\|W\|_{L_2(Q)}^2 + \|U\|_{L_2(Q)}^2) + 4\|s_n^{(3)}\|_{L_\infty(\Omega)}^2 \|h\|_{L_2(Q)}^2. \tag{2.33}$$

Finally, using Lemma 2.1 and estimates (2.14) and (2.25) to problem (2.27), we have

$$|w_n(x, t)| \leq \frac{1}{M_1 - M_0}((N_1 + N_2)\|s_n^{(2)}\|_{L_\infty(\Omega)} + \|h\|_{L_\infty(Q)}\|s_n^{(3)}\|_{L_\infty(\Omega)}) \quad \text{a.e. } (x, t) \in \overline{Q}. \tag{2.34}$$

By the definition (2.12) of the function $c(\xi)$, we observe that

$$|s_n^{(2)}(x)| \leq |(\bar{v}_1)_n(x) - \bar{v}_1(x)| \max_{\xi \in \bar{\Omega}} |A_2(\xi)| + |(\bar{v}_2)_n(x) - \bar{v}_2(x)| \max_{\xi \in \bar{\Omega}} |A_1(\xi)|.$$

Then, as $n \to \infty$, it follows that $\|s_n^{(2)}\|_{L_\infty(\Omega)} \to 0$. Similarly, $\|s_n^{(3)}\|_{L_\infty(\Omega)} \to 0$, as $n \to \infty$. Thus, from (2.28), (2.33) and (2.34), we obtain that, as $n \to \infty$,

$$\int_Q (|\nabla w_n|^2 + w_n^2) \, dx \, dt \to 0, \quad \int_Q ((w_n)_t^2 + (\nabla^2 w_n)^2) \, dx \, dt \to 0,$$

and

$$\max_{t \in [0,T]} \int_\Omega |\nabla w_n(x,t)|^2 \, dx \to 0 \quad \text{and} \quad |w_n(x,t)| \to 0 \quad \text{a.e. } (x,t) \in \bar{Q}.$$

The above results together with (2.2) imply that $\|w_n\|_{\mathcal{V}_0} \to 0$, as $n \to \infty$, and $W(x,t) \in X$ due to the convexity and closedness of X. This shows that for any bounded sequence $\{v_n(x,t)\}$ in X, the sequence $\{W_n = \mathcal{F}(v_n)\}$ has a convergent subsequence in X. Thus, the operator $\mathcal{F}$ is a continuous and compact mapping of X into itself. Since X is a convex closed subset of $\mathcal{V}_0$, the Schauder fixed point Theorem 2.1 implies that the operator $\mathcal{F}$ has a fixed point W in $\mathcal{V}_0$, which is the solution $W(x,t)$ of problem (2.10). Moreover, W satisfies inequality (2.25) rewritten as

$$|W(x,t)| \leq N_2 \quad \text{a.e. } (x,t) \in \bar{Q}. \tag{2.35}$$

This conclude the proof of the Existence Theorem 2.2. □

Next, we will prove the existence and uniqueness of the solution to the inverse problem (1.1), (1.3) and (1.4). We set

$$M_3 := |\omega_1(T)| + |\omega_2(T)| + \int_0^T |\omega_1'(t)| \, dt + \int_0^T |\omega_2'(t)| \, dt,$$

$$M_4 := \max_{x \in \bar{\Omega}} \{|A_1(x)|, |A_2(x)|, |\Phi_1(x)|, |\Phi_2(x)|\}, \tag{2.36}$$

$$N_3 := \frac{2M_3 M_4 (N_1 + N_2 + \|h\|_{L_\infty(Q)})}{M_1 - 2M_3 M_4 (N_1 + N_2)}. \tag{2.37}$$

Theorem 2.3. *Suppose that assumptions* (a)–(e) *are satisfied. Assume that there exists a number* $M_0 \in (0, M_1)$ *satisfying*

$$2M_3 M_4 (N_1 + N_2) \leq M_0, \quad N_3 < 1. \tag{2.38}$$

Then there exists a unique solution $(u(x,t), q(x), f(x)) \in \mathcal{V} \times L_\infty(\Omega) \times L_\infty(\Omega)$ *and* $q(x) > 0$ *a.e.* $x \in \bar{\Omega}$ *to the inverse problem* (1.1), (1.3) *and* (1.4).

Proof. According to Lemma 2.1 and Theorem 2.2, we know that the parabolic problems (2.9) and (2.10) have solutions $U(x,t) \in \mathcal{V}$ and $W(x,t) \in \mathcal{V}_0$ satisfying (2.14) and (2.35), respectively.

By the definition of $\bar{W}_1$ in (2.11), we have

$$|\bar{W}_1| \leq \left(|\omega_1(T)| + \int_0^T |\omega_1'(t)| \, dt \right) \|W\|_{L_\infty(Q)} \leq M_3 N_2, \tag{2.39}$$

and by the same way, we have $|\bar{W}_2| \leq M_3 N_2$, $|\bar{U}_1| \leq M_3 N_1$ and $|\bar{U}_2| \leq M_3 N_1$, which imply that

$$|\bar{W}_i + \bar{U}_i| \leq M_3 (N_1 + N_2), \quad i = 1, 2.$$

Thus,

$$|A_2(\bar{W}_1 + \bar{U}_1) - A_1(\bar{W}_2 + \bar{U}_2)| \leq M_3 (N_1 + N_2)(|A_1| + |A_2|) \leq 2M_3 M_4 (N_1 + N_2),$$
$$|\Phi_2(\bar{W}_1 + \bar{U}_1) - \Phi_1(\bar{W}_2 + \bar{U}_2)| \leq M_3 (N_1 + N_2)(|\Phi_1| + |\Phi_2|) \leq 2M_3 M_4 (N_1 + N_2).$$

Then these inequalities and the first inequality in (2.38) along with the definition (2.12) of the function $c(\cdot)$

imply that
$$c(A_2(\overline{W}_1 + \overline{U}_1) - A_1(\overline{W}_2 + \overline{U}_2)) = A_2(\overline{W}_1 + \overline{U}_1) - A_1(\overline{W}_2 + \overline{U}_2),$$
$$c(\Phi_2(\overline{W}_1 + \overline{U}_1) - \Phi_1(\overline{W}_2 + \overline{U}_2)) = \Phi_2(\overline{W}_1 + \overline{U}_1) - \Phi_1(\overline{W}_2 + \overline{U}_2).$$

Then problem (2.10) becomes
$$
\begin{cases}
W_t = \nabla^2 W - \{A_2 a_1 - A_1 a_2 + A_2(\overline{W}_1 + \overline{U}_1) - A_1(\overline{W}_2 + \overline{U}_2)\}W \\
\qquad - (A_2(\overline{W}_1 + \overline{U}_1) - A_1(\overline{W}_2 + \overline{U}_2))U \\
\qquad + \{\Phi_2 a_1 - \Phi_1 a_2 + \Phi_2(\overline{W}_1 + \overline{U}_1) - \Phi_1(\overline{W}_2 + \overline{U}_2)\}h, & (x, t) \in Q, \\
W(x, t) = 0, & (x, t) \in S, \\
W(x, 0) = 0, & x \in \overline{\Omega},
\end{cases}
\tag{2.40}
$$

which has a solution $W(x, t) \in \mathcal{V}_0$ such that $|W(x, t)| \leq N_2$ a.e. $(x, t) \in Q$.

From (2.9) and (2.40), taking $u(x, t) = W(x, t) + U(x, t)$, it is easy to obtain that $u(x, t)$ belongs to the space $\mathcal{V}$ and satisfies problem (2.8). Moreover, $u(x, t)$ satisfies the estimate
$$|u(x, t)| \leq N_1 + N_2 \quad \text{a.e. } (x, t) \in \overline{Q}. \tag{2.41}$$

Suppose now that $w(x, t)$ and $v(x, t) \in \mathcal{V}$ are two solutions to problem (2.8), and set
$$z(x, t) = v(x, t) - w(x, t).$$

Then $z(x, t)$ satisfies the following problem:
$$
\begin{cases}
z_t = \nabla^2 z - (A_2 a_1 - A_1 a_2 + A_2 \overline{w}_1 - A_1 \overline{w}_2)z - (A_2 \overline{z}_1 - A_1 \overline{z}_2)v + (\Phi_2 \overline{z}_1 - \Phi_1 \overline{z}_2)h, & (x, t) \in Q, \\
z(x, t) = 0, & (x, t) \in S, \\
z(x, 0) = 0, & x \in \overline{\Omega},
\end{cases}
\tag{2.42}
$$
where $\overline{z}_i = \overline{v}_i - \overline{w}_i$, $\overline{v}_i$ is defined by (2.16) and
$$\overline{w}_i = -\omega_i(T)w(x, T) + \int_0^T \omega_i'(t)w(x, t)\,dt, \quad i = 1, 2.$$

By the first inequality in (2.38) and $M_0 \in (0, M_1)$, as in (2.39), we have
$$|A_2 a_1 - A_1 a_2 + A_2 \overline{w}_1 - A_1 \overline{w}_2| \geq M_1 - \max_{x \in \overline{\Omega}} |A_2 \overline{w}_1 - A_1 \overline{w}_2|$$
$$\geq M_1 - 2M_3 M_4(N_1 + N_2) > M_0 - 2M_3 M_4(N_1 + N_2) \geq 0, \tag{2.43}$$

and using that (as in (2.39)), $|\overline{z}_i| \leq M_3 \|z\|_{L_\infty(Q)}$ for $i = 1, 2$,
$$\max\{|A_2 \overline{z}_1 - A_1 \overline{z}_2|, |\Phi_2 \overline{z}_1 - \Phi_1 \overline{z}_2|\} \leq 2M_3 M_4 \|z\|_{L_\infty(Q)}.$$

Thus from Lemma 2.1 applied to the solution z of problem (2.42), and inequality (2.41) applied to the function v, we obtain that
$$\|z\|_{L_\infty(Q)} \leq \frac{2M_3 M_4 \|z\|_{L_\infty(Q)}(\|v\|_{L_\infty(Q)} + \|h\|_{L_\infty(Q)})}{M_1 - 2M_3 M_4(N_1 + N_2)}$$
$$\leq \frac{2M_3 M_4(N_1 + N_2 + \|h\|_{L_\infty(Q)})}{M_1 - 2M_3 M_4(N_1 + N_2)} \|z\|_{L_\infty(Q)} = N_3 \|z\|_{L_\infty(Q)}.$$

Since $N_3 < 1$, we obtain that $\|z\|_{L_\infty(Q)} = 0$. Thus, there exists at most one solution to problem (2.8) in the space $\mathcal{V}$.

This result means that $q(x)$ given by (2.5) and $f(x)$ given by (2.6) uniquely satisfy the inverse problem (1.1), (1.3) and (1.4), and $q(x), f(x) \in L_\infty(\Omega)$, since the determination of the solution (u, q, f) to the inverse problem (1.1), (1.3) and (1.4) is equivalent to that of the solution to the nonlinear parabolic problem (2.8). Furthermore, $q(x) > 0$, a.e. $x \in \overline{\Omega}$, due to (2.5) and the first inequality in (2.38) similar as deriving (2.43). The proof is complete. $\qquad\square$

2.1 Example

Let us now give an example for which the sufficient conditions (a)–(e) and (2.38) of the existence and unique-ness of solution in Theorem 2.3 are satisfied. Consider the one-dimensional ($N = 1$) case such that $\Omega = (0, l)$ with $l > 0$ and $Q = \Omega \times (0, T)$ with $T > 0$. Take the Dirichlet data

$$\mu_1(t) = u(0, t) = \frac{\pi l}{2T}, \quad \mu_2(t) = u(l, t) = \frac{2l}{T} \sin\left(\frac{2\pi t}{T}\right), \quad t \in (0, T), \tag{2.44}$$

and the initial temperature

$$u_0(x) = u(x, 0) = \frac{\pi(l - x)}{2T}, \quad x \in [0, l]. \tag{2.45}$$

Also, take the weight functions

$$\omega_1(t) = \sin\left(\frac{2\pi t}{T}\right), \quad \omega_2(t) = \frac{2}{\pi} + \frac{\pi}{2} - \frac{\pi t}{T}, \quad t \in [0, T], \tag{2.46}$$

and the measurements (1.3) and (1.4) given by

$$\phi_1(x) = \int_0^T \omega_1(t)u(x, t) \, dt = x, \quad \phi_2(x) = \int_0^T \omega_2(t)u(x, t) \, dt = l, \quad x \in [0, l].$$

Finally, take the source component terms

$$h(x, t) = \sin\left(\frac{\pi t}{2T}\right), \quad g(x, t) = \sin\left(\frac{2\pi t}{T}\right), \quad (x, t) \in Q. \tag{2.47}$$

Let us first check assumptions (a)–(e) by hand (or, in some cases, using symbolic computations in MAPLE). The regularity assumptions (a) and (b) are obviously satisfied. Condition (c) is also satisfied since

$$0 = \phi_1(0) = \int_0^T \omega_1(t)\mu_1(t) \, dt, \quad l = \phi_1(l) = \int_0^T \omega_1(t)\mu_2(t) \, dt,$$

and

$$l = \phi_2(0) = \int_0^T \omega_2(t)\mu_1(t) \, dt, \quad l = \phi_2(l) = \int_0^T \omega_2(t)\mu_2(t) \, dt.$$

Condition (d) is satisfied since

$$\Delta(x) := \phi_1(x) \int_0^T \omega_2(t)h(x, t) \, dt - \phi_2(x) \int_0^T \omega_1(t)h(x, t) \, dt = \frac{xT(\pi - 2)^2}{\pi^2} + \frac{8lT}{15\pi} \neq 0, \quad x \in [0, l].$$

To check that assumption (e) is satisfied, we first calculate the quantities given by expression (2.1), namely,

$$a_1(x) = \int_0^T \omega_1(t)g(x, t) \, dt + \nabla^2\phi_1(x) + \omega_1(0)u_0(x) = \frac{T}{2},$$

$$a_2(x) = \int_0^T \omega_2(t)g(x, t) \, dt + \nabla^2\phi_2(x) + \omega_2(0)u_0(x) = \frac{T}{2} + \frac{(l - x)(\pi^2 + 4)}{4T},$$

$$A_1(x) = \frac{\int_0^T \omega_1(t)h(x, t) \, dt}{\Delta(x)} = -\frac{8T}{15\pi\Delta(x)} = -\frac{8\pi}{8l\pi + 15x(\pi - 2)^2},$$

$$A_2(x) = \frac{\int_0^T \omega_2(t)h(x, t) \, dt}{\Delta(x)} = \frac{T(\pi - 2)^2}{\pi^2\Delta(x)} = \frac{15(\pi - 2)^2}{8l\pi + 15x(\pi - 2)^2},$$

$$\Phi_1(x) = \frac{x}{\Delta(x)} = \frac{15\pi^2 x}{T(8l\pi + 15x(\pi - 2)^2)},$$

$$\Phi_2(x) = \frac{l}{\Delta(x)} = \frac{15\pi^2 l}{T(8l\pi + 15x(\pi - 2)^2)}.$$

The function $A_2(x)a_1(x) - A_1(x)a_2(x)$ is a decreasing function since its derivative is negative. Hence, its minimum value is achieved at $x = l$, which yields

$$A_2(x)a_1(x) - A_1(x)a_2(x) \geq A_2(l)a_1(l) - A_1(l)a_2(l) = \frac{T}{2l} =: M_1.$$

The function $\Phi_2(x)a_1(x) - \Phi_1(x)a_2(x)$ is decreasing if $\frac{T^2}{l} > 7$. Hence, its maximum is achieved at $x = 0$, which yields

$$\Phi_2(x)a_1(x) - \Phi_1(x)a_2(x) \leq \Phi_2(0)a_1(0) - \Phi_1(0)a_2(0) = \frac{15\pi}{16} =: M_2.$$

From (2.44), (2.45) and (2.47), we have that

$$\|g\|_{L_\infty(Q)} = 1, \quad \|u_0\|_{L_\infty(\Omega)} = \frac{\pi l}{2T}, \quad \|\mu_1\|_{L_\infty(0,T)} = \frac{\pi l}{2T}, \quad \|\mu_2\|_{L_\infty(0,T)} = \frac{2l}{T},$$

and from (2.14) we get

$$N_1 = \frac{2l}{T}.$$

Next, letting $M_0 = \frac{T}{4l} < M_1 = \frac{T}{2l}$ in (2.12), from (2.25) we obtain that (note that $\|h\|_{L_\infty(Q)} = 1$)

$$N_2 = \frac{4l}{T}\left(\frac{1}{2} + \frac{T}{4l} + \frac{15\pi}{16}\right) = 1 + \left(2 + \frac{15\pi}{4}\right)\frac{l}{T}.$$

Finally, from (2.36) and (2.37) we obtain that

$$M_3 = 0 + \left|\frac{2}{\pi} - \frac{\pi}{2}\right| + \frac{2\pi}{T}\int_0^T \left|\cos\left(\frac{2\pi t}{T}\right)\right| dt + \int_0^T \frac{\pi}{T} dt = \frac{3\pi}{2} - \frac{2}{\pi} + 4 \approx 8.076,$$

$$M_4 = \max\left\{\frac{1}{l}, \frac{15(\pi-2)^2}{8l\pi}, \frac{15\pi^2}{T(15\pi^2 - 52\pi + 60)}, \frac{15\pi}{8T}\right\} = \max\left\{\frac{1}{l}, \frac{15\pi}{8T}\right\},$$

and

$$2M_3M_4(N_1 + N_2) = 2\left(\frac{3\pi}{2} - \frac{2}{\pi} + 4\right)\max\left\{\frac{1}{l}, \frac{15\pi}{8T}\right\}\left(1 + \left(4 + \frac{15\pi}{4}\right)\frac{l}{T}\right),$$

$$N_3 = \frac{2\left(\frac{3\pi}{2} - \frac{2}{\pi} + 4\right)\max\left\{\frac{1}{l}, \frac{15\pi}{8T}\right\}\left(2 + \left(4 + \frac{15\pi}{4}\right)\frac{l}{T}\right)}{\frac{T}{2l} - 2\left(\frac{3\pi}{2} - \frac{2}{\pi} + 4\right)\max\left\{\frac{1}{l}, \frac{15\pi}{8T}\right\}\left(1 + \left(4 + \frac{15\pi}{4}\right)\frac{l}{T}\right)}.$$

Then conditions in (2.38) require that

$$2\left(\frac{3\pi}{2} - \frac{2}{\pi} + 4\right)\max\left\{\frac{1}{l}, \frac{15\pi}{8T}\right\}\left(1 + \left(4 + \frac{15\pi}{4}\right)\frac{l}{T}\right) \leq \frac{T}{4l},$$

$$2\left(\frac{3\pi}{2} - \frac{2}{\pi} + 4\right)\max\left\{\frac{1}{l}, \frac{15\pi}{8T}\right\}\left(3 + 2\left(4 + \frac{15\pi}{4}\right)\frac{l}{T}\right) < \frac{T}{2l}.$$

From this, we get that

$$4\left(\frac{3\pi}{2} - \frac{2}{\pi} + 4\right)\max\left\{1, \frac{15\pi l}{8T}\right\}\left(3T + \left(8 + \frac{15\pi}{2}\right)l\right) < T^2. \tag{2.48}$$

For fixed l and T sufficiently large, this condition is clearly satisfied. For example, if $T \geq 270$ and $0 < l \leq \frac{144}{\pi}$, then

$$\frac{15\pi l}{8T} \leq 1$$

and (2.48) is satisfied. We also satisfy that

$$l < \frac{T^2}{7}$$

which was previously needed to ensure that the function $\Phi_2 a_1 - \Phi_1 a_2 \leq M_2$ is decreasing.

In the next section, we shall numerically recover the unknown coefficients $q(x)$ and $f(x)$, simultaneously, from the time-average temperature measurements ϕ_1 given by (1.3) and ϕ_2 given by (1.4).

3 Optimisation method

In practical cases, the time-average temperature measurements ϕ_1 given by (1.3) and ϕ_2 given by (1.4) may not be smooth functions, but contain some noise. This leads to the main difficulty in the numerical reconstruction of the solution since the inverse problem is ill-posed such that small errors into the input data (1.3) and (1.4) lead to large errors in the output coefficients $q(x)$ and $f(x)$. From the view of numerical implementation, we in fact search for the solution approximated from the noisy measurements $(\phi_1^\epsilon, \phi_2^\epsilon) \in L_\infty(\Omega) \times L_\infty(\Omega)$ of the exact data (ϕ_1, ϕ_2) satisfying

$$\|\phi_1^\epsilon - \phi_1\|_{L_2(\Omega)} \leq \epsilon, \quad \|\phi_2^\epsilon - \phi_2\|_{L_2(\Omega)} \leq \epsilon, \tag{3.1}$$

where $\epsilon \geq 0$ represents the noise level. In this noisy case, instead of (1.3) and (1.4) we impose

$$\int_0^T \omega_1(t) u(x, t) \, dt = \phi_1^\epsilon(x), \quad x \in \Omega, \tag{3.2}$$

$$\int_0^T \omega_2(t) u(x, t) \, dt = \phi_2^\epsilon(x), \quad x \in \Omega. \tag{3.3}$$

Therefore, the inverse problem is to recover the reaction coefficient $q(x)$, the source term $f(x)$ and the temperature $u(x, t)$ from (1.1), (1.3) and (1.4), given ϕ_i^ϵ, for $i = 1, 2$, are such that (3.1) holds. Conditions (a)–(e) and (2.38) are assumed whenever necessary.

Let $u(x, t; q, f)$ or $u(q, f)$, if there is no confusion, represent the unique solution in $\mathcal{V} \subset H^{2,1}(Q)$ of the direct problem (1.1) with respect to a particular pair $(q, f) \in \mathcal{A}_q \times \mathcal{A}_f$, as guaranteed by Lemma 2.1 where the admissible sets $\mathcal{A}_q$ and $\mathcal{A}_f$ are defined by

$$\mathcal{A}_q := \{q \in L_\infty(\Omega) : 0 < q_- \leq q(x) \leq q_+ \text{ a.e. } x \in \Omega\},$$
$$\mathcal{A}_f := \{f \in L_\infty(\Omega) : |f(x)| \leq \kappa \text{ a.e. } x \in \Omega\}$$

and q_-, q_+ and κ are given positive constants. Then the quasi-solution of the inverse problem (1.1), (3.2) and (3.3) can be obtained by minimising the least-squares objective functional $J(q, f) : \mathcal{A}_q \times \mathcal{A}_f \mapsto \mathbb{R}_+$ defined by

$$J(q, f) = \frac{1}{2} \left\| \int_0^T \omega_1(t) u(\cdot, t; q, f) \, dt - \phi_1^\epsilon \right\|_{L_2(\Omega)}^2 + \frac{1}{2} \left\| \int_0^T \omega_2(t) u(\cdot, t; q, f) \, dt - \phi_2^\epsilon \right\|_{L_2(\Omega)}^2. \tag{3.4}$$

In the present study we do not debate on the issue of discretise then regularise or, regularise then discretise, see, e.g., [10]; for simplicity, we adopt the latter approach.

Next, we intend to derive the Fréchet derivatives $J_q'(q, f)$ and $J_f'(q, f)$ of the objective functional $J(q, f)$, and express these gradients via the solution $\lambda(x, t)$ of a certain appropriate adjoint problem to problem (1.1) given by

$$\begin{cases} \lambda_t = -\nabla^2 \lambda + q\lambda - \sum_{i=1}^{2} \omega_i(t) \left(\int_0^T \omega_i(\tau) u(x, \tau) \, d\tau - \phi_i^\epsilon(x) \right), & (x, t) \in Q, \\ \lambda(x, t) = 0, & (x, t) \in S, \\ \lambda(x, T) = 0, & x \in \overline{\Omega}. \end{cases} \tag{3.5}$$

Lemma 3.1. *Suppose that $q \in \mathcal{A}_q$, $\phi_1^\epsilon, \phi_2^\epsilon \in L_\infty(\Omega)$ and $\omega_1(t), \omega_2(t) \in L_\infty[0, T]$. Then the adjoint problem (3.5) has a unique solution $\lambda \in \mathcal{V}_0 \subset \mathcal{V}$, which satisfies*

$$\|\lambda\|_{L_2(Q)} \leq C \sum_{i=1}^{2} \|\omega_i\|_{L_\infty[0,T]} (\|\phi_i^\epsilon\|_{L_\infty(\Omega)} + \|\omega_i\|_{L_\infty[0,T]} \|u\|_{L_\infty(Q)}), \tag{3.6}$$

and C is a positive constant depending on T, $|Q|$ and q_-.

Proof. The existence and uniqueness of the solution $\lambda \in \mathcal{V}_0 \subset \mathcal{V}$ follows from Lemma 2.1. Multiplying the adjoint problem (3.5) by λ, and integrating the resulting relation over Q, we have

$$q_-\|\lambda\|_{L_2(Q)}^2 \le \frac{1}{2}\|\lambda(\cdot,0)\|_{L_2(\Omega)}^2 + \int_Q (|\nabla\lambda|^2 + q\lambda^2)\,dx\,dt$$

$$= \sum_{i=1}^{2} \int_Q \lambda(x,t)\omega_i(t)\left(\int_0^T \omega_i(\tau)u(x,\tau)\,d\tau - \phi_i^\epsilon(x)\right)dx\,dt.$$

Since

$$\left|\int_Q \lambda(x,t)\omega_i(t)\left(\int_0^T \omega_i(\tau)u(x,\tau)\,d\tau - \phi_i^\epsilon(x)\right)dx\,dt\right| \le C\left(\|\omega_i\|_{L_\infty[0,T]}\|\phi_i^\epsilon\|_{L_\infty(\Omega)} + \|\omega_i\|_{L_\infty[0,T]}^2\|u\|_{L_\infty(Q)}\right)\|\lambda\|_{L_2(Q)}$$

for some constant $C \ge 0$, this yields (3.6). The lemma is proved   □

Theorem 3.1. *The objective functional $J(q,f)$ is Fréchet differentiable, and the derivatives $J_q'(q,f)$ and $J_f'(q,f)$ at $(q,f) \in \mathcal{A}_q \times \mathcal{A}_f$ are given by*

$$J_q'(q,f) = -\int_0^T u(x,t)\lambda(x,t)\,dt, \tag{3.7}$$

$$J_f'(q,f) = \int_0^T h(x,t)\lambda(x,t)\,dt. \tag{3.8}$$

Proof. Take $\Delta q \in L_\infty(\Omega)$ such that $q + \Delta q \in \mathcal{A}_q$, and denote by $\Delta u_q := u(q+\Delta q, f) - u(q,f)$ the increment of $u(q,f)$ with respect to q. Based on (1.1), Δu_q satisfies the following problem:

$$\begin{cases} (\Delta u_q)_t = \nabla^2(\Delta u_q) - q\Delta u_q - \Delta q\,u(q+\Delta q,f), & (x,t) \in Q, \\ \Delta u_q(x,t) = 0, & (x,t) \in S, \\ \Delta u_q(x,0) = 0, & x \in \overline{\Omega}. \end{cases} \tag{3.9}$$

Multiplying the first equation in (3.9) by Δu_q, and integrating it on Q, we have

$$\frac{1}{2}\|\Delta u_q(\cdot,T)\|_{L_2(\Omega)}^2 + \int_Q |\nabla(\Delta u_q)|^2\,dx\,dt + \int_Q q|\Delta u_q|^2\,dx\,dt = -\int_Q \Delta q\Delta u_q u(q+\Delta q,f)\,dx\,dt.$$

Then

$$\min\{1, q_-\}\|\Delta u_q\|_{H^{1,0}(Q)}^2 \le \|\Delta q\|_{L_\infty(\Omega)}\|\Delta u_q\|_{L_2(Q)}\|u(q+\Delta q,f)\|_{L_2(Q)},$$

which implies that, using also (2.41),

$$\|\Delta u_q\|_{L_2(Q)} \le c\|\Delta q\|_{L_\infty(\Omega)}, \tag{3.10}$$

where c is a positive constant independent of q and f.

Denoting $\Delta J_q := J(q+\Delta q, f) - J(q,f)$, by the adjoint problem (3.5), we have

$$\Delta J_q = \int_Q \Delta u_q(x,t)\left\{\omega_1(t)\left(\int_0^T \omega_1(\tau)u(x,\tau)\,d\tau - \phi_1^\epsilon(x)\right) + \omega_2(t)\left(\int_0^T \omega_2(\tau)u(x,\tau)\,d\tau - \phi_2^\epsilon(x)\right)\right\}dx\,dt$$

$$+ \frac{1}{2}\sum_{i=1}^{2}\left\|\int_0^T \omega_i(t)\Delta u_q(\cdot,t)\,dt\right\|_{L_2(\Omega)}^2$$

$$= \int_Q \Delta u_q(-\lambda_t - \nabla^2\lambda + q\lambda)\,dx\,dt + \frac{1}{2}\sum_{i=1}^{2}\left\|\int_0^T \omega_i(t)\Delta u_q(\cdot,t)\,dt\right\|_{L_2(\Omega)}^2.$$

Via problems (3.5) and (3.9), the first term of the right-hand side of the above equation becomes

$$\int_Q \Delta u_q (-\lambda_t - \nabla^2 \lambda + q\lambda) \, dx \, dt = -\int_\Omega \Delta u_q \lambda|_{t=0}^T \, dx + \int_Q \lambda (\Delta u_q)_t \, dx \, dt - \int_0^T \Delta u_q \nabla \lambda|_S \, dt + \int_0^T \lambda \Delta u_q|_S \, dt$$

$$- \int_Q \lambda \nabla^2 (\Delta u_q) \, dx \, dt + \int_Q \lambda q \Delta u_q \, dx \, dt$$

$$= \int_Q \lambda [(\Delta u_q)_t - \nabla^2 (\Delta u_q) + q \Delta u_q] \, dx \, dt$$

$$= -\int_Q \lambda \Delta q u(q + \Delta q, f) \, dx \, dt$$

$$= -\int_Q \lambda \Delta q \Delta u_q \, dx \, dt - \int_Q \lambda \Delta q u(q, f) \, dx \, dt.$$

Thus,

$$\Delta J_q = -\int_Q \lambda \Delta q \Delta u_q \, dx \, dt - \int_Q \lambda \Delta q u(q, f) \, dx \, dt + \frac{1}{2} \sum_{i=1}^2 \left\| \int_0^T \omega_i(t) \Delta u_q(\cdot, t) \, dt \right\|_{L_2(\Omega)}^2.$$

By using (3.10), we have

$$\left\| \int_0^T \omega_i(t) \Delta u_q(\cdot, t) \, dt \right\|_{L_2(\Omega)} \leq c \|\omega_i\|_{L_\infty[0,T]} \|\Delta u_q\|_{L_2(Q)} \leq c \|\omega_i\|_{L_\infty[0,T]} \|\Delta q\|_{L_\infty(\Omega)}, \quad i = 1, 2,$$

and

$$\left| \int_Q \lambda \Delta q \Delta u_q \, dx \, dt \right| \leq \|\Delta q\|_{L_\infty(\Omega)} \|\lambda\|_{L_2(Q)} \|\Delta u_q\|_{L_2(Q)} \leq c \|\Delta q\|_{L_\infty(\Omega)}^2 \|\lambda\|_{L_2(Q)}.$$

Consequently, we obtain

$$\Delta J_q = \int_\Omega \Delta q \left(-\int_0^T \lambda u \, dt \right) dx + o(\|\Delta q\|_{L_\infty(\Omega)}),$$

which means that the Fréchet derivative $J_q'(q, f)$ is given by (3.7).

Similarly, take $\Delta f \in L_\infty(\Omega)$ such that $f + \Delta f \in \mathcal{A}_f$, and denote by $\Delta u_f := u(q, f + \Delta f) - u(q, f)$ the increment of u with respect to f. Then Δu_f satisfies the problem

$$\begin{cases} (\Delta u_f)_t = \nabla^2 (\Delta u_f) - q \Delta u_f + \Delta f h, & (x, t) \in Q, \\ \Delta u_f(x, t) = 0, & (x, t) \in S, \\ \Delta u_f(x, 0) = 0, & x \in \overline{\Omega}, \end{cases} \tag{3.11}$$

and

$$\|\Delta u_f\|_{L_2(Q)} \leq c \|\Delta f\|_{L_\infty(\Omega)} \|h\|_{L_2(Q)}.$$

Denoting $\Delta J_f := J(q, f + \Delta f) - J(q, f)$, and by the same approach, we have

$$\Delta J_f = \int_Q \Delta u_f (-\lambda_t - \nabla^2 \lambda + q\lambda) \, dx \, dt + \frac{1}{2} \left\| \int_0^T \omega_1(t) \Delta u_f(\cdot, t) \, dt \right\|_{L_2(\Omega)}^2 + \frac{1}{2} \left\| \int_0^T \omega_2(t) \Delta u_f(\cdot, t) \, dt \right\|_{L_2(\Omega)}^2$$

and

$$\int_Q \Delta u_f (-\lambda_t - \nabla^2 \lambda + q\lambda) \, dx \, dt = \int_Q \lambda [(\Delta u_f)_t - \nabla^2 (\Delta u_f) + q \Delta u_f] \, dx \, dt = \int_Q \Delta f \lambda h \, dx \, dt.$$

Finally, we obtain that

$$\Delta J_f = \int_\Omega \Delta f \left(\int_0^T \lambda h \, dt \right) dx + o(\|\Delta f\|_{L_\infty(\Omega)}),$$

thus the Fréchet gradient $J_f'(q, f)$ is given by (3.8). The theorem is proved. □

Theorem 3.2. *The Fréchet gradients $J_q'(q, f)$ and $J_f'(q, f)$ are Lipschitz continuous, i.e., there exist two positive constants L_q and L_f such that*

$$\|J_q'(q_1, f_1) - J_q'(q_2, f_2)\|_{L_2(\Omega)} \le L_q(\|q_1 - q_2\|_{L_2(\Omega)} + \|f_1 - f_2\|_{L_2(\Omega)}),$$
$$\|J_f'(q_1, f_1) - J_f'(q_2, f_2)\|_{L_2(\Omega)} \le L_f(\|q_1 - q_2\|_{L_2(\Omega)} + \|f_1 - f_2\|_{L_2(\Omega)})$$

for any $q_1, q_2 \in \mathcal{A}_q$ and $f_1, f_2 \in \mathcal{A}_f$.

Proof. For any $q_1, q_2 \in \mathcal{A}_q$, and $f_1, f_2 \in \mathcal{A}_f$, denoting $\Delta J_q' := J_q'(q_1, f_1) - J_q'(q_2, f_2)$, from (3.7), we have

$$\|\Delta J_q'\|_{L_2(\Omega)} \le \left\| \int_0^T \lambda(q_1, f_1)[u(q_1, f_1) - u(q_2, f_2)] \, dt \right\|_{L_2(\Omega)} + \left\| \int_0^T u(q_2, f_2)[\lambda(q_1, f_1) - \lambda(q_2, f_2)] \, dt \right\|_{L_2(\Omega)}. \quad (3.12)$$

From (1.1) and (3.5), $\overline{u} := u(q_1, f_1) - u(q_2, f_2)$ satisfies the following problem:

$$\begin{cases} \overline{u}_t = \nabla^2 \overline{u} - q_1 \overline{u} - (q_1 - q_2)u(q_2, f_2) + (f_1 - f_2)h, & (x, t) \in Q, \\ \overline{u}(x, t) = 0, & (x, t) \in S, \\ \overline{u}(x, 0) = 0, & x \in \overline{\Omega}, \end{cases} \quad (3.13)$$

and $\overline{\lambda} := \lambda(q_1, f_1) - \lambda(q_2, f_2)$ satisfies the following problem:

$$\begin{cases} \overline{\lambda}_t = -\nabla^2 \overline{\lambda} + q_1 \overline{\lambda} + (q_1 - q_2)\lambda(q_2, f_2) - \sum_{i=1}^2 \omega_i(t) \int_0^T \omega_i(\tau)\overline{u}(x, \tau) \, d\tau, & (x, t) \in Q, \\ \overline{\lambda}(x, t) = 0, & (x, t) \in S, \\ \overline{\lambda}(x, T) = 0, & x \in \overline{\Omega}. \end{cases}$$

From (2.3) in Lemma 2.1, for any $(q, f) \in \mathcal{A}_q \times \mathcal{A}_f$, we have

$$\|u(q, f)\|_{L_\infty(Q)} \le \max\left\{ \frac{\kappa}{q_-}\|h\|_{L_\infty(Q)} + \frac{1}{q_-}\|g\|_{L_\infty(Q)}, \|\mu\|_{L_\infty(S)}, \|u_0\|_{L_\infty(\Omega)} \right\} =: K_1.$$

Also, since $\omega_1, \omega_2 \in L_\infty[0, T]$ and $\phi_1^\epsilon, \phi_2^\epsilon \in L_\infty(\Omega)$, from Lemma 2.1 applied to the function λ satisfying (3.5), we have

$$\|\lambda(q, f)\|_{L_\infty(Q)} \le \frac{1}{q_-}\left(TK_1 \sum_{i=1}^2 \|\omega_i\|_{L_\infty[0, T]}^2 + \sum_{i=1}^2 \|\omega_i\|_{L_\infty[0, T]}\|\phi_i^\epsilon\|_{L_\infty(\Omega)} \right) =: K_2.$$

Multiplying by $\overline{u}$ the first equation in (3.13), integrating over Q, and using integration by parts, we obtain

$$\frac{1}{2}\|\overline{u}(\cdot, T)\|_{L_2(\Omega)}^2 + \int_Q (|\nabla\overline{u}|^2 + q_1|\overline{u}|^2) \, dx \, dt = \int_Q ((q_1 - q_2)u\overline{u} + (f_1 - f_2)h\overline{u}) \, dx \, dt.$$

Since $q_1 \in \mathcal{A}_q$, the above relation implies that

$$q_-\|\overline{u}\|_{L_2(Q)}^2 \le \sqrt{T}\|\overline{u}\|_{L_2(Q)}(\|u\|_{L_\infty(Q)}\|q_1 - q_2\|_{L_2(\Omega)} + \|h\|_{L_\infty(Q)}\|f_1 - f_2\|_{L_2(\Omega)}).$$

Thus, we obtain

$$\|\overline{u}\|_{L_2(Q)} \le \frac{\sqrt{T}}{q_-}(\|u\|_{L_\infty(Q)}\|q_1 - q_2\|_{L_2(\Omega)} + \|h\|_{L_\infty(Q)}\|f_1 - f_2\|_{L_2(\Omega)}) \le K_3(\|q_1 - q_2\|_{L_2(\Omega)} + \|f_1 - f_2\|_{L_2(\Omega)}),$$

and similarly,

$$\|\overline{\lambda}\|_{L_2(Q)} \le \frac{\sqrt{T}}{q_-}\left(\|\lambda(q_2, f_2)\|_{L_\infty(Q)}\|q_1 - q_2\|_{L_2(\Omega)} + \|\overline{u}\|_{L_2(Q)} \sum_{i=1}^2 \|\omega_i\|_{L_\infty[0, T]}^2 \right)$$

$$\le K_4(\|q_1 - q_2\|_{L_2(\Omega)} + \|f_1 - f_2\|_{L_2(\Omega)}),$$

where

$$K_3 := \max\{K_1, \|h\|_{L_\infty(Q)}\}\frac{\sqrt{T}}{q_-} > 0, \quad K_4 := \left(K_2 + K_3 \sum_{i=1}^2 \|\omega_i\|_{L_\infty[0, T]}^2 \right)\frac{\sqrt{T}}{q_-} > 0.$$

From (3.12), for $\Delta J'_q$, we then have

$$\|\Delta J'_q\|_{L_2(\Omega)} \le \left\|\int_0^T \lambda(q_1, f_1)\overline{u}\, dt\right\|_{L_2(\Omega)} + \left\|\int_0^T u(q_2, f_2)\overline{\lambda}\, dt\right\|_{L_2(\Omega)}$$

$$\le \sqrt{T}\left(\|\lambda(q_1, f_1)\|_{L_\infty(Q)}\|\overline{u}\|_{L_2(Q)} + \|u(q_2, f_2)\|_{L_\infty(Q)}\|\overline{\lambda}\|_{L_2(Q)}\right)$$

$$\le L_q\left(\|q_1 - q_2\|_{L_2(\Omega)} + \|f_1 - f_2\|_{L_2(\Omega)}\right),$$

where $L_q := \sqrt{T}(K_2 K_3 + K_1 K_4) > 0$ is independent of q_1, q_2, f_1 and f_2.

Similarly, taking $\Delta J'_f = J'_f(q_1, f_1) - J'_f(q_2, f_2)$, we have

$$\|\Delta J'_f\|_{L_2(\Omega)} = \left\|\int_0^T h\overline{\lambda}\, dt\right\|_{L_2(\Omega)} \le \sqrt{T}\|h\|_{L_\infty(Q)}\|\overline{\lambda}\|_{L_2(Q)} \le L_f\left(\|q_1 - q_2\|_{L_2(\Omega)} + \|f_1 - f_2\|_{L_2(\Omega)}\right),$$

where $L_f := \sqrt{T}K_4\|h\|_{L_\infty(Q)} > 0$ is independent of q_1, q_2, f_1 and f_2. The theorem is proved. □

4 Conjugate gradient method

The following iteration process based on the CGM is utilised for the simultaneous reconstruction of the space-dependent reaction coefficient $q(x)$ and the source term $f(x)$ by minimising the objective functional $J(q, f)$ given by (3.4):

$$q^{k+1} = q^k + \beta_q^k s_q^k, \quad f^{k+1} = f^k + \beta_f^k s_f^k, \quad k = 0, 1, 2, \dots, \tag{4.1}$$

where k denotes the number of iterations, $q^0(x)$ and $f^0(x)$ are the initial guesses for $q(x)$ and $f(x)$, respectively, β_q^k and β_f^k are the step sizes, and s_q^k and s_f^k are the search directions for q and f given by

$$s_q^k = \begin{cases} -J'^0_q, & k = 0, \\ -J'^k_q + \gamma_q^k s_q^{k-1}, & k \ge 1, \end{cases} \quad s_f^k = \begin{cases} -J'^0_f, & k = 0, \\ -J'^k_f + \gamma_f^k s_f^{k-1}, & k \ge 1, \end{cases} \tag{4.2}$$

where $J'^k_q := J'_q(q^k, f^k)$, $J'^k_f := J'_f(q^k, f^k)$, γ_q^k and γ_f^k are the conjugate coefficients, and the Fletcher–Reeves formula [12] is applied, i.e.,

$$\gamma_q^k = \frac{\|J'^k_q\|_{L_2(\Omega)}}{\|J'^{k-1}_q\|_{L_2(\Omega)}}, \quad \gamma_f^k = \frac{\|J'^k_f\|_{L_2(\Omega)}}{\|J'^{k-1}_f\|_{L_2(\Omega)}}, \quad k \ge 1. \tag{4.3}$$

The step sizes β_q^k and β_f^k can be determined by the line search method, i.e.,

$$J(q^{k+1}, f^{k+1}) = \min_{\beta_q, \beta_f \ge 0} J(q^k + \beta_q s_q^k, f^k + \beta_f s_f^k). \tag{4.4}$$

Then, by using (4.1), we have

$$\frac{\partial J(q^{k+1}, f^{k+1})}{\partial \beta_q^k} = \lim_{\beta_q^k \to 0} \frac{J(q^{k+1}, f^{k+1}) - J(q^k, f^{k+1})}{\beta_q^k}$$

$$= \lim_{\beta_q^k \to 0} \frac{1}{\beta_q^k}\left(\int_\Omega \beta_q^k s_q^k\left(-\int_0^T \lambda(q^k, f^{k+1})u(q^k, f^{k+1})\, dt\right)dx + o(\|\beta_q^k s_q^k\|_{L_\infty(\Omega)})\right)$$

$$= \int_\Omega J'^{k+1}_q s_q^k\, dx,$$

and similarly

$$\frac{\partial J(q^{k+1}, f^{k+1})}{\partial \beta_f^k} = \int_\Omega J'^{k+1}_f s_f^k\, dx.$$

This means that the minimisation (4.4) implies that the step sizes β_q^k and β_f^k satisfy

$$\int_\Omega J'^{k+1}_q s_q^k\, dx = 0, \quad \int_\Omega J'^{k+1}_f s_f^k\, dx = 0, \quad k \ge 0. \tag{4.5}$$

4.1 Convergence

In this subsection, inspired by the arguments in [7], we discuss the convergence of the above iterative process (4.1)–(4.3).

Theorem 4.1. *Let $\{q^k, f^k\}_{k\geq 0} \in \mathcal{A}_q \times \mathcal{A}_f$ be the iterations defined by the (4.1), and let β_q^k and β_f^k satisfy (4.4). Then $\{J(q^k, f^k)\}_{k\geq 0}$ defined by (3.4) is a monotone decreasing convergence sequence.*

Proof. Since the search step sizes β_q^k and β_f^k satisfy the condition (4.4), they also satisfy (4.5). For $k = 0$, by (4.2) we have

$$\int_\Omega J_q'^0 s_q^0 \, dx = -\|J_q'^0\|_{L_2(\Omega)}^2.$$

For $k \geq 1$, we can obtain

$$\int_\Omega J_q'^k s_q^k \, dx = -\|J_q'^k\|_{L_2(\Omega)}^2 + \gamma_q^k \int_\Omega J_q'^k s_q^{k-1} \, dx = -\|J_q'^k\|_{L_2(\Omega)}^2.$$

Therefore, we have that

$$\int_\Omega J_q'^k s_q^k \, dx = -\|J_q'^k\|_{L_2(\Omega)}^2, \quad k \geq 0.$$

Similarly, we have

$$\int_\Omega J_f'^k s_f^k \, dx = -\|J_f'^k\|_{L_2(\Omega)}^2, \quad k \geq 0.$$

By (4.4), we have

$$J(q^{k+1}, f^{k+1}) \leq J(q^k + \beta_q s_q^k, f^k + \beta_f s_f^k) \quad \text{for all } \beta_q, \beta_f \geq 0.$$

Then, for the case $\beta_q \geq 0$ and $\beta_f = 0$, using the mean value theorem and the Lipschitz continuity of the gradient J_q' given by Theorem 3.2, we obtain

$$J(q^k, f^k) - J(q^k + \beta_q s_q^k, f^k) = -\beta_q \int_0^1 \int_\Omega J_q'(q^k + \theta\beta_q s_q^k, f^k) s_q^k \, dx \, d\theta$$

$$= -\beta_q \int_\Omega J_q'^k s_q^k \, dx - \beta_q \int_0^1 \int_\Omega (J_q'(q^k + \theta\beta_q s_q^k, f^k) - J_q'(q^k, f^k)) s_q^k \, dx \, d\theta$$

$$\geq \beta_q \|J_q'^k\|_{L_2(\Omega)}^2 - \beta_q \|s_q^k\|_{L_2(\Omega)} \int_0^1 \|J_q'(q^k + \theta\beta_q s_q^k, f^k) - J_q'(q^k, f^k)\|_{L_2(\Omega)} \, d\theta$$

$$\geq \beta_q \|J_q'^k\|_{L_2(\Omega)}^2 - \frac{1}{2}\beta_q^2 L_q \|s_q^k\|_{L_2(\Omega)}^2.$$

It is obvious that the function of β_q on the right-hand side of the above formula attains its maximum at

$$\beta_q = \frac{\|J_q'^k\|_{L_2(\Omega)}^2}{L_q \|s_q^k\|_{L_2(\Omega)}^2},$$

i.e.,

$$J(q^k, f^k) - J(q^k + \beta_q s_q^k, f^k) \geq \frac{\|J_q'^k\|_{L_2(\Omega)}^4}{2L_q \|s_q^k\|_{L_2(\Omega)}^2}.$$

Similarly, taking

$$\beta_q = 0 \quad \text{and} \quad \beta_f = \frac{\|J_f'^k\|_{L_2(\Omega)}^2}{L_f \|s_f^k\|_{L_2(\Omega)}^2},$$

we have

$$J(q^k, f^k) - J(q^k, f^k + \beta_f s_f^k) \geq \frac{\|J_f'^k\|_{L_2(\Omega)}^4}{2L_f \|s_f^k\|_{L_2(\Omega)}^2}.$$

Thus,

$$J(q^k, f^k) - J(q^{k+1}, f^{k+1}) \geq \max\left\{\frac{\|J_q'^k\|_{L_2(\Omega)}^4}{2L_q \|s_q^k\|_{L_2(\Omega)}^2}, \frac{\|J_f'^k\|_{L_2(\Omega)}^4}{2L_f \|s_f^k\|_{L_2(\Omega)}^2}\right\},$$

and hence, $J(q^{k+1}, f^{k+1}) \leq J(q^k, f^k)$ for all $k \geq 0$. Furthermore, since $\{J(q^k, f^k)\}_{k\geq0}$ defined by (3.4) is non-negative, it follows that it is a monotonic decreasing convergent sequence. □

According to the arguments in [4, 7], and the results in Theorems 3.2 and 4.1, we obtain the following result:

Theorem 4.2. *Under the assumptions of Theorem 4.1, we have that the CGM* (4.1)–(4.3) *either terminates at a stationary point or converges in the following senses:*

$$\liminf_{k\to\infty} \|J_q'^k\|_{L_2(\Omega)} = 0, \quad \liminf_{k\to\infty} \|J_f'^k\|_{L_2(\Omega)} = 0.$$

4.2 Iterative algorithm

According to the above discussion, all the parameters of the iterative process (4.1) are expressed explicitly except for the search step sizes β_q^k and β_f^k. By using the line search (4.4), these two parameters can be obtained by minimising

$$J(q^{k+1}, f^{k+1}) = \frac{1}{2} \int_\Omega \left(\int_0^T \omega_1 u(q^k + \beta_q^k s_q^k, f^k + \beta_f^k s_f^k) - \phi_1^\epsilon\right)^2 dx$$

$$+ \frac{1}{2} \int_\Omega \left(\int_0^T \omega_2 u(q^k + \beta_q^k s_q^k, f^k + \beta_f^k s_f^k) - \phi_2^\epsilon\right)^2 dx.$$

Since in this expression the search step sizes β_q^k and β_f^k are in implicit form, we linearise it to approximate $J(q^{k+1}, f^{k+1})$ such that the search step sizes β_q^k and β_f^k become explicit in the new formulation. Thus

$$u(x, t; q^k + \beta_q^k s_q^k, f^k + \beta_f^k s_f^k) \approx u(x, t; q^k, f^k) + \beta_q^k \Delta u_q(x, t; q^k, f^k) + \beta_f^k \Delta u_f(x, t; q^k, f^k),$$

where $\Delta u_q(x, t; q^k, f^k)$ and $\Delta u_f(x, t; q^k, f^k)$ are obtained by solving the sensitivity problems (3.9) and (3.11) with $\Delta q^k = s_q^k$ and $\Delta f^k = s_f^k$, respectively. We now define

$$u_i^k := \int_0^T \omega_i u(q^k, f^k) \, dt,$$

$$\Delta u_{q,i}^k := \int_0^T \omega_i \Delta u_q(q^k, f^k) \, dt,$$

$$\Delta u_{f,i}^k := \int_0^T \omega_i \Delta u_f(q^k, f^k) \, dt, \quad i = 1, 2.$$

Then

$$J(q^{k+1}, f^{k+1}) = \frac{1}{2} \int_\Omega (u_1^k + \beta_q^k \Delta u_{q,1}^k + \beta_f^k \Delta u_{f,1}^k - \phi_1^\epsilon)^2 \, dx + \frac{1}{2} \int_\Omega (u_2^k + \beta_q^k \Delta u_{q,2}^k + \beta_f^k \Delta u_{f,2}^k - \phi_2^\epsilon)^2 \, dx.$$

The derivatives of $J(q^{k+1}, f^{k+1})$ with respect to β_q^k and β_f^k are given by

$$\frac{\partial J(q^{k+1}, f^{k+1})}{\partial \beta_q^k} = A_1 \beta_q^k + A_2 \beta_f^k + A_3, \quad \frac{\partial J(q^{k+1}, f^{k+1})}{\partial \beta_f^k} = A_2 \beta_q^k + A_4 \beta_f^k + A_5,$$

where

$$A_1 = \int_\Omega \{(\Delta u_{q,1}^k)^2 + (\Delta u_{q,2}^k)^2\}\, dx, \qquad\qquad A_2 = \int_\Omega (\Delta u_{q,1}^k \Delta u_{f,1}^k + \Delta u_{q,2}^k \Delta u_{f,2}^k)\, dx,$$

$$A_3 = \int_\Omega \{(u_1^k - \phi_1^\epsilon)\Delta u_{q,1}^k + (u_2^k - \phi_2^\epsilon)\Delta u_{q,2}^k\}\, dx, \quad A_4 = \int_\Omega \{(\Delta u_{f,1}^k)^2 + (\Delta u_{f,2}^k)^2\}\, dx,$$

$$A_5 = \int_\Omega \{(u_1^k - \phi_1^\epsilon)\Delta u_{f,1}^k + (u_2^k - \phi_2^\epsilon)\Delta u_{f,2}^k\}\, dx.$$

By setting

$$\frac{\partial J(q^{k+1}, f^{k+1})}{\partial \beta_q^k} = \frac{\partial J(q^{k+1}, f^{k+1})}{\partial \beta_f^k} = 0,$$

the search step sizes β_q^k and β_f^k are given as follows:

$$\beta_q^k = \frac{A_3 A_4 - A_2 A_5}{A_2^2 - A_1 A_4}, \quad \beta_f^k = \frac{A_1 A_5 - A_2 A_3}{A_2^2 - A_1 A_4}. \tag{4.6}$$

The iteration process given by (4.1) does not provide the CGM with the regularisation necessary for the minimisation of the objective functional (3.4) to be classified as well-posed because of the errors inherent in the time-average temperature measurements (1.3) and (1.4). However, the method may become well-posed if the discrepancy principle is applied to stop the iteration procedure. According to the discrepancy principle, the iterative procedure is stopped when the following criterion is satisfied:

$$J(q^k, f^k) \le \overline{\epsilon}, \tag{4.7}$$

where $\overline{\epsilon} = \frac{1}{2} \sum_{i=1,2} \|\phi_i^\epsilon - \phi_i\|_{L_2(\Omega)}^2 \le \epsilon^2$.

Then the iteration algorithm based on CGM for the numerical recovery of the reaction coefficient $q(x)$ and the source component $f(x)$ is as follows:

Step 1. Set $k = 0$ and choose initial guesses $q^0(x)$ and $f^0(x)$ for the unknown space-dependent coefficients $q(x)$ and $f(x)$, respectively.

Step 2. Solve the direct problem (1.1) numerically by utilising the finite-difference scheme to compute the temperature $u(x, t; q^k, f^k)$, and the objective functional $J(q^k, f^k)$ given by (3.4).

Step 3. Solve the adjoint problem (3.5) numerically to obtain $\lambda(x, t; q^k, f^k)$, and the Fréchet gradients $J_q'(q^k, f^k)$ by (3.7) and $J_f'(q^k, f^k)$ by (3.8). Calculate the conjugate coefficients γ_q^k and γ_f^k in (4.3), and the search directions s_q^k and s_f^k in (4.2).

Step 4. Solve the problems $\Delta u_q(x, t; q^k, f^k)$ given by (3.9) and $\Delta u_f(x, t; q^k, f^k)$ given by (3.11) by using $\Delta q^k = s_q^k$ and $\Delta f^k = s_f^k$, and compute the search step sizes β_q^k and β_f^k given by (4.6).

Step 5. Update q^{k+1} and f^{k+1} by (4.1).

Step 6. If the stopping condition (4.7) is satisfied, then go to **Step 7**. Else set $k = k + 1$, and go to **Step 2**.

Step 7. End.

5 Numerical results and discussions

In this section, we perform one- and two-dimensional numerical computations to simultaneously reconstruct the space-dependent reaction coefficient $q(x)$ and source component $f(x)$ based on the CGM described in the previous section. The finite-difference method (FDM) based on the Crank–Nicolson scheme is employed in one-dimension ($N = 1$), whilst the alternating direction implicit (ADI) method is employed in two-dimension ($N = 2$), see [5], for solving the direct, sensitivity and adjoint problems involved in the CGM. In the two-dimensional case we decided to switch to an ADI (of Peaceman–Rachford type) method because, as pointed

out in [15], it is more suitable, in the sense of convergence and stability, than the Crank–Nicolson scheme for two-dimensional parabolic problems with self-adjoint elliptic spatial operator, see [34]. The Simpson's rule is utilised for computing the integrals involved in the objective functional $J(q, f)$, its Fréchet derivatives J'_q and J'_f, the adjoint problem for λ, and the conjugate coefficients γ_q and γ_f. We have also tested the accuracy using the trapezoidal rule obtaining no significant differences in the numerical results than when using the Simpson's rule.

The accuracy errors $E_q(k)$ and $E_f(k)$, as functions of the iteration number k, for the reaction coefficient $q(x)$ and the source component $f(x)$, respectively, are defined as

$$E_q(k) = \|q^k - q\|_{L_2(\Omega)}, \tag{5.1}$$
$$E_f(k) = \|f^k - f\|_{L_2(\Omega)}, \tag{5.2}$$

where q^k and f^k are the CGM iterates, and q and f are the analytical expressions of the reaction coefficient and source component, if available.

The noisy integral temperature observations ϕ_1^ϵ and ϕ_2^ϵ in (3.2) and (3.3) are simulated by adding to their analytical values, if available, Gaussian additive noise as

$$\phi_1^\epsilon = \phi_1 + \sigma \times \text{random}(1), \quad \phi_2^\epsilon = \phi_2 + \sigma \times \text{random}(1), \tag{5.3}$$

where σ is the standard deviation given by $\sigma = \frac{p}{100} \times \max_{x \in \overline{\Omega}}\{|\phi_1(x)|, |\phi_2(x)|\}$, $p\%$ represents the percentage of noise, and the term random(1) generates random values from a Gaussian distribution with zero mean and standard deviation equal to unity.

In the one-dimensional ($N = 1$) examples we consider the final time $T = 1$, the domain $\Omega = (0, 1)$, and take the FDM mesh size $\Delta x = \Delta t = 0.01$ for solving the direct, sensitivity and adjoint problems in the CGM iterative procedure. In the two-dimensional ($N = 2$) example, we consider $T = 1$, $\Omega = (0, 1) \times (0, 1)$, and the ADI mesh size $\Delta x_1 = \Delta x_2 = \Delta t = 0.01$. As for the weight functions $\omega_1(t)$ and $\omega_2(t)$ in (1.3) and (1.4), we choose the functions

$$\omega_i(t) = \frac{1}{a\sqrt{\pi}} e^{-\frac{(t-t_i)^2}{a^2}}, \quad t \in [0, T], \ i = 1, 2, \tag{5.4}$$

where a is a small positive constant, and $t_1 \neq t_2 \in [0, T]$. It is obvious that $\omega_i(t) \approx \delta(t - t_i)$ for small values of a, where $\delta(\cdot)$ is the Dirac delta function. Then, according to the properties of the Dirac delta function, equations (1.3) and (1.4) would become

$$\phi_i(x) = \int_0^T \omega_i(t)u(x, t)\, dt \approx \int_0^T \delta(t - t_i)u(x, t)\, dt = u(x, t_i), \quad i = 1, 2, \tag{5.5}$$

which implies that the time-average temperature measurements ϕ_1 and ϕ_2 can be regarded as the measured temperature at the distinct time instants t_1 and t_2, see [5]. For all the three numerical examples presented in the following subsections, we take the weight functions as

$$\omega_1(t) = \frac{1}{10^{-3}\sqrt{\pi}} e^{-\frac{(t-0.2)^2}{10^{-6}}}, \quad \omega_2(t) = \frac{1}{10^{-3}\sqrt{\pi}} e^{-\frac{(t-1)^2}{10^{-6}}}, \tag{5.6}$$

i.e., $a = 10^{-3}$, $t_1 = 0.2$ and $t_2 = T = 1$ in (5.4).

5.1 Example 1

In the one-dimensional ($N = 1$) case, we take the input data as

$$h(x, t) = (1 + x)t^3, \quad \mu(0, t) = e^{-t}, \quad \mu(1, t) = 2e^{-t}, \quad u_0(x) = 1 + x^2,$$
$$g(x, t) = (1 + \sin(\pi x) + \sin(2\pi x))(1 + x^2)e^{-t} - 2e^{-t} - \left(\frac{1}{2} - 4x + 4x^2\right)(1 + x)t^3,$$

and since the weight functions are chosen as in (5.6), the measured temperatures are taken as in (5.5),

namely,

$$\phi_1(x) = \int_0^1 \omega_1(t)u(x, t)\, dt \approx e^{-0.2}(1 + x^2), \quad \phi_2(x) = \int_0^1 \omega_2(t)u(x, t)\, dt \approx e^{-1}(1 + x^2), \quad x \in (0, 1).$$

These approximations already introduce some (numerical) noise into the input data (1.3) and (1.4). An analytical solution of the inverse problem (1.1), (1.3) and (1.4) with the above data is given by

$$q(x) = 2 + \sin(\pi x) + \sin(2\pi x), \quad f(x) = \frac{1}{2} - 4x(1 - x), \quad u(x, t) = (1 + x^2)e^{-t}. \tag{5.7}$$

Although the conditions in (2.38) of Theorem 2.3 are not satisfied to guarantee the uniqueness of the solution, the numerically obtained results indicate that (5.7) is the closest solution to the chosen initial guess.

From (3.5), one can observe that the gradients (3.7) and (3.8) vanish for x on the boundary $\partial\Omega$, hence we need to specify the initial guesses for q^0 and f^0 on $\partial\Omega$ be equal to their true values, otherwise, from (4.1) and (4.2) there will be no progress with iterations at the boundary points. Therefore, we take $q^0(x) = 2$ and $f^0(x) = 0.5$ for $x \in \overline{\Omega}$, which ensures that $q^0(x) = q^{\text{true}}(x)$ and $f^0(x) = f^{\text{true}}(x)$ for $x \in \partial\Omega$, and also that these initial guesses are rather far from their true values (5.7) for $x \in \Omega$.

We also note that the weight functions $\omega_1(t)$ and $\omega_2(t)$ in (1.3) and (1.4) can be chosen as other functions rather than our choices in (5.6). For instance, the errors E_q and E_f given by (5.1) and (5.2) after 50 CGM iterations were obtained to be $E_q(50) \in \{0.0343, 0.0484, 0.0441\}$ and $E_f(50) \in \{0.0635, 0.0977, 0.0807\}$ for the choices $(\omega_1(t), \omega_2(t)) \in \{(1, t^2), (1, t^3), (t^2, e^t)\}$, respectively. These small values of errors indicate that accurate numerical results can be obtained for rather arbitrary functionally independent weight functions ω_1 and ω_2 in (1.3) and (1.4), respectively.

Figure 1 illustrates the behaviour of the objective functional $J(q^k, f^k)$ defined in (3.4), as a function of the iteration number k, for the simultaneous recovery of the two space-dependent unknowns $q(x)$ and $f(x)$, in case of no noise, i.e., $p = 0$, and with $p = \{1, 2\}$ noise. It is obvious that the objective functional is a monotonic decreasing function of k, and converges rapidly to a small positive value, as the result in Theorem 4.1 predicts. The stopping iteration numbers $k^* \in \{22, 3, 3\}$ for $p \in \{0, 1, 2\}$, respectively, are obtained based on the discrepancy principle (4.7). We have chosen the threshold $\overline{\epsilon} = 10^{-8}$, which is a small value close to 0 for $p = 0$, whilst the values $\overline{\epsilon} \in \{0.0047, 0.0187\}$ were computed utilising $\overline{\epsilon} = \frac{1}{2} \sum_{i=1,2} \|\phi_i^\epsilon - \phi_i\|_{L_2(\Omega)}^2$ and (5.3) for $p = \{1, 2\}$ noise, respectively. The errors (5.1) and (5.2) for the unknowns $q(x)$ and $f(x)$ are found as $E_q \in \{0.0028, 0.1891, 0.2715\}$ and $E_f \in \{0.0064, 0.0811, 0.1539\}$ for $p \in \{0, 1, 2\}$ noise, respectively. These small values indicate that the numerical solutions are reasonably accurate for both $q(x)$ and $f(x)$. In addition, the norms of the Fréchet gradients were obtained as $\|J_q'(q^{k^*}, f^{k^*})\|_{L_2(\Omega)} \in \{1.8 \times 10^{-5}, 0.0011, 0.0019\}$
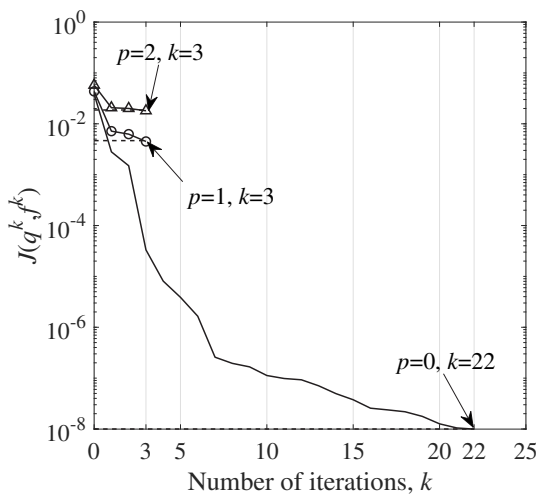

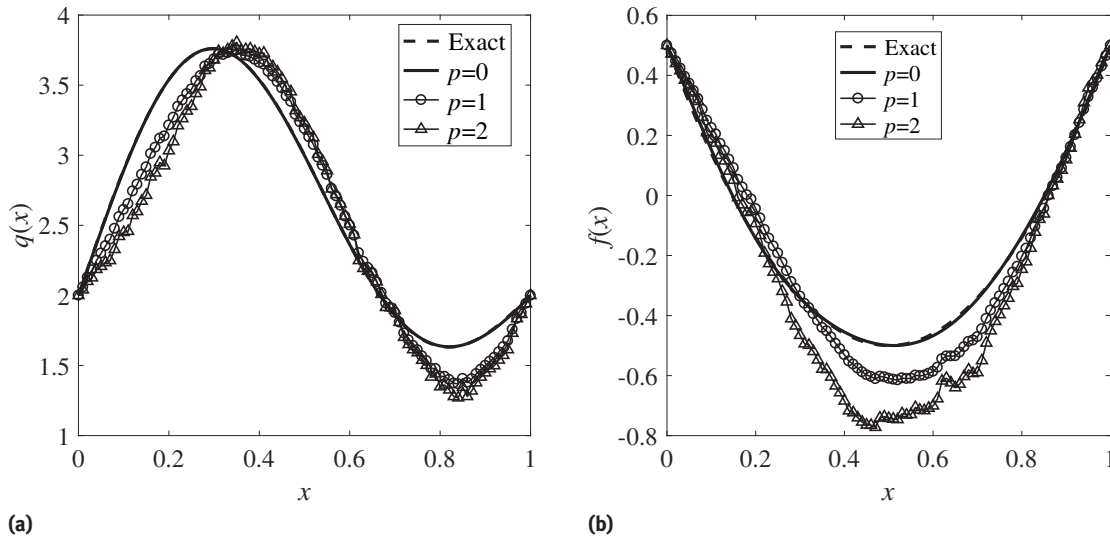**Figure 1:** The objective functional $J(q^k, f^k)$, for $p \in \{0, 1, 2\}$ noise, for Example 1.

Figure 2: The analytical and numerical results for: (a) the reaction coefficient $q(x)$ and (b) the source component $f(x)$, for $p \in \{0, 1, 2\}$ noise, for Example 1.

and $\|J'_f(q^{k^*}, f^{k^*})\|_{L_2(\Omega)} \in \{7.1 \times 10^{-6}, 6 \times 10^{-4}, 0.0018\}$ for $p \in \{0, 1, 2\}$ noise, respectively. These small values indicate that the CGM, with iterations stopped by the discrepancy principle (4.7), is a semi-convergent regularisation method.

In Figures 2 (a) and 2 (b), the numerical solutions for the space-dependent reaction coefficient $q(x)$ and the source term $f(x)$ are plotted at the stopping iteration numbers k^* inferred from Figure 1, for the levels of noise $p \in \{0, 1, 2\}$. Higher levels of noise produced less accurate numerical results and therefore they are not presented. For $p = 0$ noiseless data, the analytical and numerical solutions overlap and they are graphically undistinguishable. Overall, from Figure 2 it can be seen that the stable and accurate solutions are obtained for both coefficients.

5.2 Example 2

In the previous example, we have considered a test with a smooth analytical solution given by (5.7). In this example, we consider a more severe test involving reconstructing a discontinuous reaction coefficient, as follows. We take

$$h(x, t) = (1 + \sin(\pi x))e^{-t}, \quad \mu(0, t) = 1, \quad \mu(1, t) = 1 + t, \quad u_0(x) = 1,$$

$$g(x, t) = x^2 - 2t - (4x - 1)\sin(\pi x)(1 + \sin(\pi x)e^{-t} + \begin{cases} 2(1 + x^2 t), & x \in [\frac{1}{4}, \frac{3}{4}], \\ 1 + x^2 t, & \text{otherwise}, \end{cases}$$

and the measurements (5.5) given by

$$\phi_1(x) = \int_0^1 \omega_1(t)u(x, t)\, dt \approx 1 + 0.2x^2, \quad \phi_2(x) = \int_0^1 \omega_2(t)u(x, t)\, dt \approx 1 + x^2, \quad x \in (0, 1).$$

Then an analytical solution of the inverse problem (1.1), (1.3) and (1.4) with this input data is given by

$$q(x) = \begin{cases} 2, & x \in [\frac{1}{4}, \frac{3}{4}], \\ 1, & \text{otherwise}, \end{cases}, \quad f(x) = (4x - 1)\sin(\pi x), \quad u(x, t) = 1 + x^2 t.$$

The initial guesses are taken as $q^0(x) = 1$ and $f^0(x) = 0$. As in the previous example, the iterative CGM is stopped at the iterations $k^* \in \{60, 5, 4\}$ for $p \in \{0, 1, 2\}$, by using the discrepancy principle (4.7) with
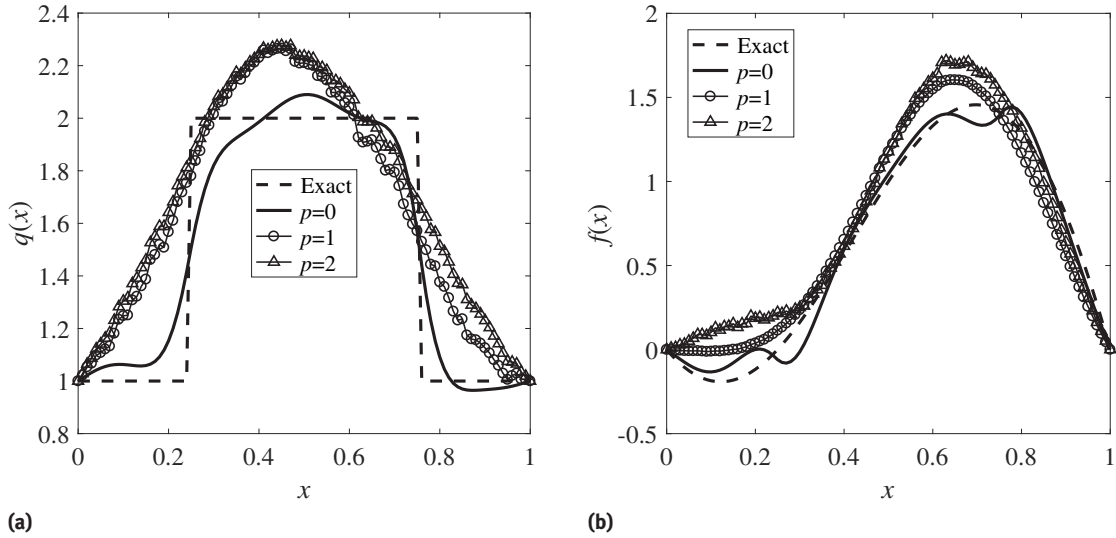
Figure 3: The analytical and numerical results for: (a) the reaction coefficient $q(x)$ and (b) the source component $f(x)$, for $p \in \{0, 1, 2\}$ noise, for Example 2.

$\bar{\epsilon} \in \{10^{-7}, 0.0025, 0.01\}$, respectively. The errors (5.1) and (5.2) for $q(x)$ and $f(x)$ at these stopping iteration numbers are $E_q \in \{0.1454, 0.2726, 0.3121\}$ and $E_f \in \{0.0774, 0.1533, 0.1944\}$ for $p \in \{0, 1, 2\}$ noise, respectively. Figures 3 (a) and 3 (b) present the numerical solutions for the reaction coefficient $q(x)$ and the source term $f(x)$ at these stopping iteration numbers, respectively. From Figure 3 we can observe that the numerical results are stable and reasonably accurate bearing in mind the severe discontinuous reaction coefficient $q(x)$ that had to be retrieved simultaneously with the source term $f(x)$.

5.3 Example 3

We now consider a two-dimensional ($N = 2$) example and take the input data as

$$h(x_1, x_2, t) = (1 + x_1 + x_2)t^2, \quad u_0(x_1, x_2) = 1 + x_1^2 + x_2^2,$$

$$g(x_1, x_2, t) = (x_1^2 + x_2^2 - 4)e^{-t} + (1 + \sin(\pi x_1)\sin(\pi x_2))(1 + (x_1^2 + x_2^2)e^{-t})$$

$$+ \begin{cases} -(1 + x_1 + x_2)t^2, & (x_1, x_2) \in [\frac{1}{4}, \frac{3}{4}] \times [\frac{1}{4}, \frac{3}{4}], \\ (1 + x_1 + x_2)t^2, & \text{otherwise}, \end{cases}$$

$$\mu(0, x_2, t) = 1 + x_2^2 e^{-t}, \quad \mu(1, x_2, t) = 1 + (1 + x_2^2)e^{-t},$$

$$\mu(x_1, 0, t) = 1 + x_1^2 e^{-t}, \quad \mu(x_1, 1, t) = 1 + (1 + x_1^2)e^{-t},$$

and the measurements (5.5) given by

$$\phi_1(x_1, x_2) = \int_0^1 \omega_1(t)u(x_1, x_2, t)\, dt \approx 1 + e^{-0.2}(x_1^2 + x_2^2),$$

$$\phi_2(x_1, x_2) = \int_0^1 \omega_2(t)u(x_1, x_2, t)\, dt \approx 1 + e^{-1}(x_1^2 + x_2^2), \quad (x_1, x_2) \in (0, 1) \times (0, 1).$$

Then an analytical solution of the inverse problem (1.1), (1.3) and (1.4) with this input data is given by

$$q(x_1, x_2) = 1 + \sin(\pi x_1)\sin(\pi x_2), \quad f(x_1, x_2) = \begin{cases} 1, & (x_1, x_2) \in [\frac{1}{4}, \frac{3}{4}] \times [\frac{1}{4}, \frac{3}{4}], \\ -1, & \text{otherwise}, \end{cases}$$

$$u(x_1, x_2, t) = 1 + (x_1^2 + x_2^2)e^{-t}.$$
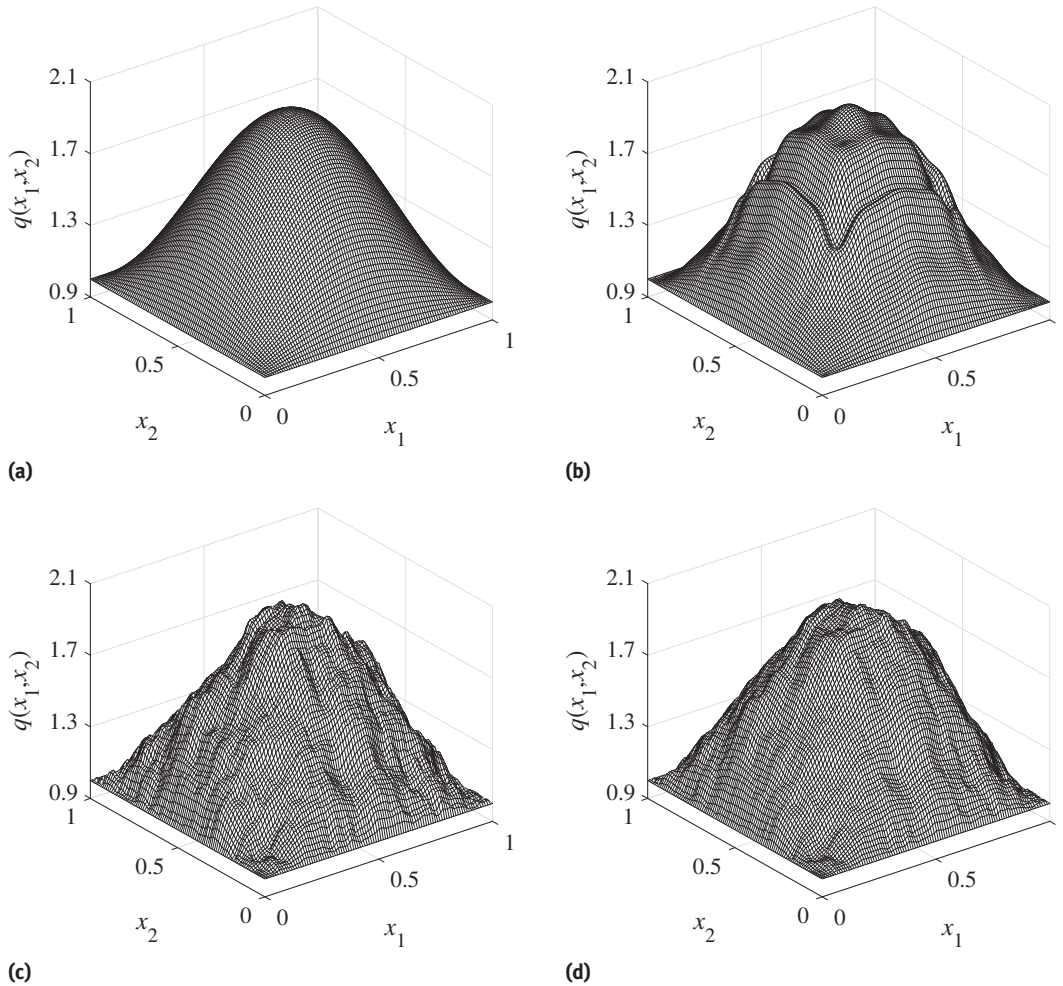
(a)

(b)

(c)

(d)

Figure 4: (a) The analytical and the numerical reaction coefficient $q(x_1, x_2)$ for: (b) $p = 0$, (c) $p = 1$ and (d) $p = 2$ noise, for Example 3.

The initial guesses are taken as $q^0(x_1, x_2) = 1$ and $f^0(x_1, x_2) = -1$. The CGM is stopped at $k^* \in \{70, 6, 4\}$ for $p \in \{0, 1, 2\}$, using the discrepancy principle (4.7) with $\overline{\epsilon} \in \{10^{-8}, 0.0096, 0.0386\}$, respectively. The corresponding accuracy errors for the reaction coefficient q and source term f are $E_q \in \{0.0263, 0.0501, 0.0568\}$ and $E_f \in \{0.1946, 0.4147, 0.4461\}$ for $p \in \{0, 1, 2\}$, respectively. The errors in f are relatively large since in this example the source component is a discontinuous function. The corresponding numerical solutions for q and f are shown in Figures 4 and 5. It is clear that the retrieved solutions are reasonably accurate and stable for both the reaction coefficient q and the source term f.

6 Conclusions

In this paper, the simultaneous identification and reconstruction of the space-dependent reaction coefficient and source term from time-integral temperature measurements has been investigated. The existence and uniqueness of the solution have been established. Then the two unknown quantities have been reconstructed simultaneously by minimising the least-squares objective functional, and the CGM has been developed using the newly derived Fréchet gradients, along with the adjoint and sensitivity problems. The convergence of the nonlinear CGM has been considered using the approaches employed in [7]. Three numerical examples in both one- and two-dimensions have been illustrated, and the results indicate the accuracy and stability of
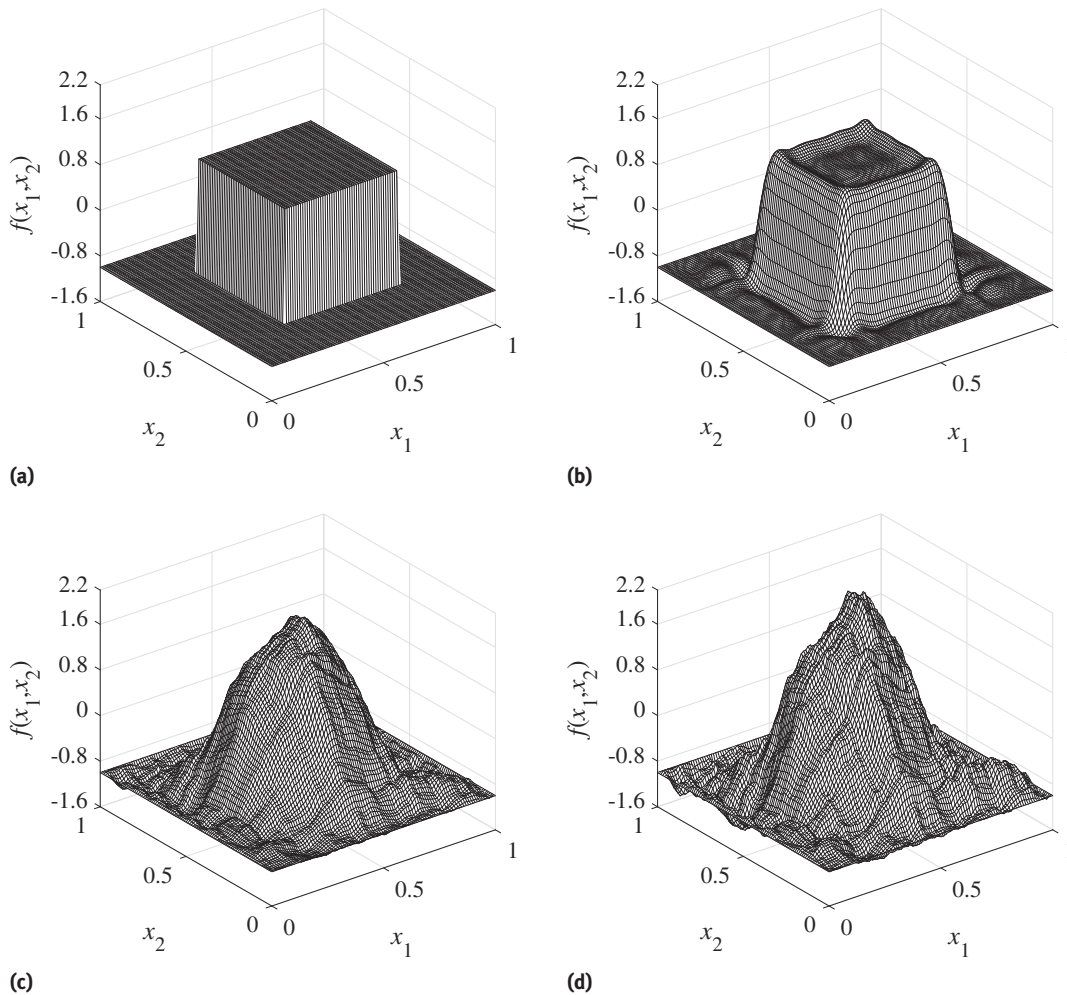
(a)

(b)

(c)

(d)

Figure 5: (a) The analytical and the numerical source component $f(x_1, x_2)$ for: (b) $p = 0$, (c) $p = 1$ and (d) $p = 2$ noise, for Example 3.

the numerical recovery of the two unknown spatially-dependent quantities of interest. Further work will consider the simultaneous numerical reconstruction of the space-dependent convection and reaction coefficients from time-integral temperature measurements, [22].

References

[1] O. M. Alifanov, *Inverse Heat Transfer Problems*, Springer, Berlin, 2012.

[2] S. R. Arridge and W. R. B. Lionheart, Nonuniqueness in diffusion-based optical tomography, *Optics Lett.* **23** (1998), no. 11, 882–884.

[3] K. Cao and D. Lesnic, Reconstruction of the space-dependent perfusion coefficient from final time or time-average temperature measurements, *J. Comput. Appl. Math.* **337** (2018), 150–165.

[4] K. Cao and D. Lesnic, Simultaneous reconstruction of the perfusion coefficient and initial temperature from time-average integral temperature measurements, *Appl. Math. Model.* **68** (2019), 523–539.

[5]   K. Cao and D. Lesnic, Simultaneous reconstruction of the spatially-distributed reaction coefficient, initial temperature and heat source from temperature measurements at different times, *Comput. Math. Appl.* **78** (2019), no. 10, 3237–3249.

[6]   Q. Chen and J. Liu, Solving an inverse parabolic problem by optimization from final measurement data, *J. Comput. Appl. Math.* **193** (2006), no. 1, 183–203.

[7]   Y. H. Dai and Y. Yuan, Convergence properties of the Fletcher–Reeves method, *IMA J. Numer. Anal.* **16** (1996), no. 2, 155–164.

[8]   Z.-C. Deng, L. Yang and J.-N. Yu, Identifying the radiative coefficient of heat conduction equations from discrete measurement data, *Appl. Math. Lett.* **22** (2009), no. 4, 495–500.

[9]   H. Egger and M. Schlottbom, Numerical methods for parameter identification in stationary radiative transfer, *Comput. Optim. Appl.* **62** (2015), no. 1, 67–83.

[10] H. W. Engl and C. W. Groetsch, Projection-regularization methods for linear operator equations of the first kind, in: *Special Program on Inverse Problems*, Proc. Centre Math. Anal. Austral. Nat. Univ. 17, Australian National University, Canberra (1988), 17–31.

[11] A. Erdem, D. Lesnic and A. Hasanov, Identification of a spacewise dependent heat source, *Appl. Math. Model.* **37** (2013), no. 24, 10231–10244.

[12] R. Fletcher and C. M. Reeves, Function minimization by conjugate gradients, *Comput. J.* **7** (1964), 149–154.

[13] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, 1964.

[14] B. Harrach, On uniqueness in diffuse optical tomography, *Inverse Problems* **25** (2009), no. 5, Article ID 055010.

[15] A. Hasanov and J. L. Mueller, A numerical method for backward parabolic problems with non-selfadjoint elliptic operators, *Appl. Numer. Math.* **37** (2001), no. 1–2, 55–78.

[16] M. S. Hussein, D. Lesnic, M. I. Ivanchov and H. A. Snitko, Multiple time-dependent coefficient identification thermal problems with a free boundary, *Appl. Numer. Math.* **99** (2016), 24–50.

[17] V. Isakov, Inverse parabolic problems with the final overdetermination, *Comm. Pure Appl. Math.* **44** (1991), no. 2, 185–209.

[18] M. I. Ivanchov and N. V. Pabirivs'ka, Simultaneous determination of two coefficients in a parabolic equation in the case of nonlocal and integral conditions, *Ukr. Math. J.* **53** (2001), no. 5, 674–684.

[19] B. T. Johansson and D. Lesnic, A procedure for determining a spacewise dependent heat source and the initial temperature, *Appl. Anal.* **87** (2008), no. 3, 265–276.

[20] T. Johansson and D. Lesnic, Determination of a spacewise dependent heat source, *J. Comput. Appl. Math.* **209** (2007), no. 1, 66–80.

[21] V. L. Kamynin, On an inverse problem of determining the right-hand side of a parabolic equation with the integral overdetermination condition, *Math. Notes* **77** (2005), no. 4, 482–493.

[22] V. L. Kamynin, The inverse problem of simultaneous determination of the two lower space-dependent coefficients in a parabolic equation, *Math. Notes* **106** (2019), no. 2, 235–247.

[23] V. L. Kamynin and A. B. Kostin, Two inverse problems of the determination of a coefficient in a parabolic equation, *Differ. Equ.* **46** (2010), no. 3, 375–386.

[24] A. I. Kozhanov, On solvability of an inverse problem with an unknown coefficient and right-hand side for a parabolic equation, *J. Inverse Ill-Posed Probl.* **10** (2002), no. 6, 611–627.

[25] A. I. Kozhanov, An inverse problem with an unknown coefficient and right-hand side for a parabolic equation. II, *J. Inverse Ill-Posed Probl.* **11** (2003), no. 5, 505–522.

[26] A. I. Kozhanov, A nonlinear loaded parabolic equation and a related inverse problem, *Math. Notes* **76** (2004), no. 5, 784–795.

[27] K. Kurpisz and A. J. Nowak, *Inverse Thermal Problems*, Int. Ser. Comput. Eng., Computational Mechanics Publications, Southampton, 1995.

[28] L. Marin, L. Elliott, P. J. Heggs, D. B. Ingham, D. Lesnic and X. Wen, Analysis of polygonal fins using the boundary element method, *Appl. Thermal Eng.* **24** (2004), no. 8, 1321–1339.

[29] H. H. Pennes, Analysis of tissue and arterial blood temperatures in the resting human forearm, *J. Appl. Physiol.* **1** (1948), no. 2, 93–122.

[30] A. I. Prilepko and A. B. Kostin, On certain inverse problems for parabolic equations with final and integral observation, *Russ. Acad. Sci. Sib. Math.* **75** (1992), no. 4, 473–490.

[31] A. I. Prilepko, D. G. Orlovsky and I. A. Vasin, *Methods for Solving Inverse Problems in Mathematical Physics*, Monogr. Textb. Pure Appl. Math. 231, CRC Press, New York, 2000.

[32] A. I. Prilepko and V. V. Solov'ev, Solvability of the inverse boundary-value problem of finding a coefficient of a lower-order derivative in a parabolic equation, *Differ. Equ.* **23** (1987), no. 1, 101–107.

[33] W. Rundell, The determination of a parabolic equation from initial and final data, *Proc. Amer. Math. Soc.* **99** (1987), no. 4, 637–642.

[34] A. A. Samarskii, *The Theory of Difference Schemes*, Monogr. Textb. Pure Appl. Math. 240, Marcel Dekker, New York, 2001.

[35] F. Tröltzsch, *Optimal Control of Partial Differential Equations: Theory, Methods, and Applications*, Grad. Stud. Math. 112, American Mathematical Society, Providence, 2010.

[36] D. Trucu, D. B. Ingham and D. Lesnic, Space-dependent perfusion coefficient identification in the transient bio-heat equation, *J. Engrg. Math.* **67** (2010), no. 4, 307–315.

[37] M. Yamamoto and J. Zou, Simultaneous reconstruction of the initial temperature and heat radiative coefficient, *Inverse Problems* **17** (2001), no. 4, 1181–1202,

[38] G. Zoutendijk, Nonlinear programming, computational methods, *Integer Nonlinear Programm.* **143** (1970), no. 1, 37–86.