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A Unified Approach to The Orbital Tracking Problem

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Abstract— **Consider an object in orbit about the earth for which a sequence of angles-only measurements is made. This paper looks in detail at a one-step update for the filtering problem. Although the problem appears very nonlinear at first sight, it can be almost reduced to the standard linear Kalman filter by a careful formulation. The key features of this formulation are (1) the use of a local or adapted basis rather than a fixed basis for three-dimensional Euclidean space and the use of structural rather than ambient coordinates to represent the state, (2) the development of a novel "normal:conditional-normal" distribution to described the propagated position of the state, and (3) the development of a novel "Observation-Centered" Kalman filter to update the state distribution.**

A major advantage of this unified approach is that it gives a closed form filter which is highly accurate under a wide range of conditions, including high initial uncertainty, high eccentricity and long propagation times.

I. INTRODUCTION

Orbital uncertainty propagation and orbital object tracking are a key themes in Space Situational Awareness (SSA) and a number of papers have been published in recent years to deal with the nonlinearity of the system equation when expressed in Cartesian coordinates. There are two basic strategies to deal with nonlinearity: (i) transform the coordinate system to remove the nonlinearity, or (ii) develop sophisticated methods or use a higher order polynomial to accommodate it. The current paper uses the first approach.

On the other hand, many other papers have taken the second approach. For example, Park and Scheeres [2] used a mixture (hybrid approach) of a simplified dynamic system (SDS) model and the state transition tensor (STT) model to propagate and model the uncertainty with higher order Taylor series terms [2], [3], [4]. Vittaldev, Russell and Linares [5] proposed a mixture of polynomial chaos expansion (PCE) and Gaussian Mixture Models (GMMs) based on Hermite polynomials. Several other papers [6], [7] also used the polynomial chaos model (PCM) and PCE for representing orbital uncertainty. Horwood and Poore [8] proposed a Gauss von Mises (GVM) filter using second order trigonometric terms.

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This paper gives a unified treatment for a one-step update in the orbital tracking or filtering problem. That is, an "initial" distribution is specified at an initial time $t = t_0 = 0$ for the state of an orbiting object, a noisy observation is made at a later time $t = t_1$, and an updated state distribution is required. It is *unified* because it combines several ideas that have appeared separately in earlier papers (e.g. [14], [15]).

The motivation for this paper is the large amounts of debris orbiting the earth. Many pieces of debris can be detected using ground-based angles-only measurements. However, the smaller debris may be observed only intermittently, leading to long (e.g. weeks, rather than hours or days) propagation times in successive observations. Hence there is a pressing need for quick, accurate and automatic filtering algorithms.

To emphasize the key ideas in our approach, some simplifying assumptions are made. If a filtering method breaks down under these idealized conditions, there is little hope of it doing well under more realistic conditions.

(a) Keplerian dynamics for the evolution of the state of the object. Thus an orbiting object follows an exactly elliptical orbit and its initial state determines its future state for all time. Perturbation effects such as atmospheric drag and gravitational distortions are not considered here.
(b) Notional observer located at the center of the earth. Thus we ignore issues of perspective when the observer is on the surface of the earth (or indeed in space).
(c) No issues of identifiability. It is assumed known that the object at time t_0 is the same as at time t_1.

If the initial uncertainty is small and the propagation time and the orbital eccentricity are is not too large, then the problem can be solved by the unscented [11], [13] or extended Kalman filter [12], [16] (UKF or EKF) in Cartesian coordinates. However, for longer propagation times, the propagated state distribution exhibits a pronounced banana shape in position [15]. The location of the object becomes spread out along an appreciable angular arc of its elliptical orbital path, and a point cloud for position becomes roughly banana-shaped.

One way to deal with longer propagation times and high initial uncertainties is to use equinoctial coordinates [10]. Using a UKF or EKF in these coordinates can be very successful in a wide range of circumstances. But there are still scenarios where severe problems arise, e.g. (a) retrograde orbits, (b) high eccentricity, and (c) specialized situations such as break-up events [10], [15].

There are three key contributions in this paper that fit together to give a "complete" and "universal" solution to the filtering problem.

(1) *Choice of coordinate system for the orbital state*. Throughout the paper an earth centered inertial (ECI) representation of space is used. The standard ECI basis consists of three orthonormal vectors pointing towards specified directions relative to the stars. The basis is fixed for all orbital objects. However, there are advantages in using a local basis adapted to the orbital object being studied. We propose the latter approach and give the name CRTN (central radial-tangential-normal) to the adapted basis (Section II).
Also, it is useful to make a distinction between ambient coordinates (describing what you see) and structural coordinates (describing more abstract features). For example, Cartesian coordinates are the prime example of ambient coordinates. Keplerian and equinoctial elements are examples of structural coordinates. We propose using AST (adapted structural) coordinates, a variant of equinoctial coordinates defined with respect to the CRTN basis. An attractive property of AST coordinates is that the system equation is exactly linear (Section III).
(2) *Closed form representation for the distribution of the propagated position of the orbital state*. A key component of the filter is the distribution of the propagated position of the orbiting object. This distribution can be described in closed form in terms of a new *normal:conditional-normal (NCN)* distribution. This representation is fundamental to the filtering problem, especially for break-up events, because it enables the uncertainty for the propagated position to be reformulated using the Gaussian (i.e. normal) distribution (Section IV).
(3) *Updating calculation for the filter*. The essence of any filter involves combining the propagated distribution of the state with the observation distribution to get an updated distribution for the state. It turns out that the commonly used UKF or EKF can be very misleading in this orbital setting under high eccentricity. Two modifications are proposed here: (a) a new closed form *observation-centered Kalman filter (OCKF)* for the position of the object along its orbital path, and (b) a conditioning argument using the NCN distribution for the remaining state variables (Section V).

The underlying principle behind our approach is to recast the problem so that as far as possible the underlying uncertainties are Gaussian and the system and observation equations are linear; hence something close to a standard Kalman filter can be applied and hence is optimal.

Here is an summary of how this methodology can be used in practice for a one-step update in the filtering problem. Except for step (1), the phrase "Gaussian distribution" is shorthand for "a distribution closely approximated by a Gaussian distribution."

(1) Start with uncertainty in the initial state at time t_0 specified in terms of a 6-dimensional Gaussian distribution in Cartesian-ECI coordinates.
(2) After transformation to AST coordinates at time t_0, the uncertainty can still be described by a 6-dimensional Gaussian distribution.
(3) Propagate the uncertainty in (2) by the linear system equation to a 6-dimensional Gaussian distribution in AST coordinates at the later time t_1.
(4) From the 6-dimensional distribution in (3), extract the two-dimensional propagated angles-only position distribution in terms of the NCN distribution.
(5) Combine the uncertainty in (4) with a noisy angles-only observation at time t_1 using the new OCKF and the conditioning argument to give the updated state distribution as a 6-dimensional Gaussian distribution in AST coordinates.
(6) Finally, if desired, the updated uncertainty can be mapped back as a a 6-dimensional Gaussian distribution in Cartesian-ECI coordinates.

The next sections give more details about each of the four main contributions. However, before that we briefly describe true, mean and eccentric anomalies.

A small object orbiting the earth follows an exact elliptical orbit under Keplerian dynamics, with the center of the earth at one of the focal points of the ellipse. There are three angles of mathematical interest in this setting to describe the angular position of the object along its orbit: the *eccentric anomaly* (E), the *mean anomaly* (M) and the *true anomaly* (T), where all three angles are measured from perigee. The true anomaly describes the actual angular position of the object, as measured from the center of the earth. The mean anomaly simplifies the mathematical development because it changes at a constant rate in time, and the eccentric anomaly is an intermediate angle of no direct interest. The relation between the angles is given as follows [9], where e is the ellipticity, $0 \le e < 1$:

$$\tan\frac{1}{2}T = \sqrt{\frac{1+e}{1-e}}\tan\frac{1}{2}E, \quad M = E - e\sin E.$$

These mappings are bijective, so any one angle determines the other two. The calculations are all straightforward, except that a numerical iteration is needed to solve for E from M. The notation $T = F_{\text{M-to-E}}(M,e)$ is used to describe the transformation between M and T and similar notation for the transformations between other pairs of angles.

II. Choice of Basis

The earth centered inertial (ECI) point of view provides a way to represent points in space. The center of the earth lies at the origin and three orthonormal directions specify the ECI basis. The second direction points towards the vernal direction and the third points to celestial north. The first is at right angles to both these. This basis is fixed in two senses: it does not change in time and it is the same for all orbiting objects being studied.

Next consider the *state* of an object orbiting the earth. The state at time t can be described in Cartesian-ECI coordinates (Cartesian coordinates with respect to the ECI basis) by three-dimensional position and three-dimensional velocity vectors $\boldsymbol{x}^{\text{ECI}}(t)$, $\dot{\boldsymbol{x}}^{\text{ECI}}(t)$. Under Keplerian dynamics the state at the initial time $t=0$ determines the state at all other times and the object follows an elliptical orbit. The initial state can also be used to define a *radial-tangential-normal (RTN)* orthonormal basis,

$$\boldsymbol{u} = \boldsymbol{u}^{\mathrm{RTN}} \propto \boldsymbol{x}^{\mathrm{ECI}}(0), \tag{II.1}$$

$$\boldsymbol{v} = \boldsymbol{v}^{\mathrm{RTN}} \propto \dot{\boldsymbol{x}}^{\mathrm{ECI}}(0) - \{\dot{\boldsymbol{x}}^{\mathrm{ECI}}(0)^T \boldsymbol{u}\}\boldsymbol{u}, \tag{II.2}$$

$$\boldsymbol{w} = \boldsymbol{w}^{\mathrm{RTN}} = \boldsymbol{u} \times \boldsymbol{v} \propto \boldsymbol{x}^{\mathrm{ECI}}(0) \times \dot{\boldsymbol{x}}^{\mathrm{ECI}}(0), \tag{II.3}$$

This RTN basis depends on the object being studied at time $t=0$. As defined here, it remains fixed for all later times.

To study uncertainty in a state at time $t=0$, it is convenient to think of a point cloud of states. Choose one particular state near the middle of point cloud and call it the *central state*. The central state is not regarded as random; it serves as a reference state. The exact choice of central state does not matter; changing it has a negligible effect on the later analysis. Then random states in the point cloud (called *deviated states*) can be described in terms of their differences from the central state. Uncertainty can be represented by specifying a distribution (typically a Gaussian distribution) for the deviated states.

Let $\boldsymbol{u}^{\mathrm{CRTN}}, \boldsymbol{v}^{\mathrm{CRTN}}, \boldsymbol{w}^{\mathrm{CRTN}}$ denote the RTN basis for the central state. The CRTN basis forms the reference basis for the construction of AST coordinates in the next section.

III. The AST coordinate system

The simplest way to represent the state of an orbiting object is in Cartesian coordinates, either with respect to the ECI basis as in (II.2) or with respect to the CRTN basis. These representations can be called *ambient* coordinates because they describe directly where the object is.

Given an orthonormal basis $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$, call $\boldsymbol{u}$ the *reference direction*, the $\boldsymbol{u}-\boldsymbol{v}$ plane the *reference plane* and $\boldsymbol{w}$ the *reference normal direction*.

A second way to represent the state is using *structural* coordinates which represent deeper features in the state. For example, the six Keplerian elements are the RAAN angle Ω (the angle in the reference plane from the reference direction to the RAAN direction), the argument of perigee ω (more specifically, the angle in the orbital plane from the RAAN direction to perigee), the true anomaly $T = T(t)$ (the angle in the orbital plane from the angle of perigee to the orbiting object), the inclination angle i (between the reference plane and the orbital plane), the eccentricity e, and the mean motion n (sometimes the semi-major axis is used for the final element). Note that the angles Ω, ω, i depend on the choice of reference basis. Conventionally, Keplerian elements are defined with respect to the ECI basis, so a more complete name is Keplerian-ECI elements. As the object evolves in time only the true anomaly $T(t)$ changes; the other elements remain fixed.

Keplerian-ECI elements can be transformed into equinoctial-ECI coordinates, which are generally much more suitable for Gaussian modelling. However, equinoctial-ECI coordinates are not completely adequate. In particular, one of the key coordinates is the *break angle* $\Omega+\omega+T$, which combines angles in two different planes. This construction causes complications when the ECI reference basis is used because the planes can be oriented in very different directions (a large value of i), e.g. for polar (i near 90^o) or retrograde (i near 180^o) orbits. Another complication is that the representation of the normal direction to the orbital plane becomes singular for an exactly retrograde orbit.

The use of the CRTN reference basis does not have these problems. For the central state, the inclination angle $i^{(c)}$ is exactly 0^o and for any deviated state the inclination angle $i^{(d)}$ will be small; hence the orbital and reference planes are always close together.

The proposal in this paper is to use "Adapted STructural (AST)" coordinates to describe the state of the orbiting object. These are essentially equinoctial-CRTN coordinates, with one important modification. Let $\theta(t)$ denote the break angle. For the central state, it vanishes at the initial time, $\theta^{(c)}(0) = 0^o$. For the deviated states $\theta^{(d)}(0)$ will be close to 0. Then $\theta^{(d)}(t)$ describes the angular position of the deviated object on the *true anomaly scale*. Similarly, let $\phi^{(d)}(t)$ denote the angular position of the object on the *mean anomaly scale*. AST coordinates use $\phi^{(d)}(t)$ instead of the break angle $\theta^{(d)}(t)$ to describe the angular position. A major advantage of this choice is that the system equation becomes exactly linear,

$$\theta(t) = \theta_p + F_{\text{M-to-T}}(\phi(t) - \phi_p, e), \tag{III.4}$$

$$\begin{aligned}\phi(t) &= \phi_p + F_{\text{T-to-M}}(\theta(0) - \theta_p, e) + nt \\ &= \phi(0) + nt.\end{aligned} \tag{III.5}$$

That is the derivative of the mean anomaly with respect to time is constant. On the other hand the derivative of the true anomaly is larger at perigee than at apogee.

It has been shown in earlier work ([15]) that the mapping between Cartesian coordinates and AST coordinates at time $t=0$ is approximately linear to a high level of approximation. Hence a Gaussian distribution in Cartesian coordinates corresponds closely to a Gaussian distribution in AST coordinates. Since the system equation is exactly linear, this means the propagated AST coordinates will also be close to a Gaussian distribution.

Here is a complete list of AST coordinates for a deviated state (dropping the $^{(d)}$ for simplicity),

$$A_1 = 2\tan(i/2)\cos\Omega,\ A_2 = 2\tan(i/2)\sin\Omega,\ A_3(t) = \phi(t),$$
$$A_4 = e\cos\theta_p,\ A_5 = e\sin\theta_p,\ A_6 = n,$$

where the Keplerian elements for the deviated state are defined with respect to the CRTN basis and θ_p denotes the direction of perigee on the true anomaly scale. Only $A_3(t)$, the break angle on the mean anomaly scale, varies with t.

When the initial uncertainties are small, A_1, A_2 and $A_3(0)$ will be small.

Let

$$R^{(c)} = [\boldsymbol{u}^{\mathrm{CRTN}} \boldsymbol{v}^{\mathrm{CRTN}} \boldsymbol{w}^{\mathrm{CRTN}}] \tag{III.6}$$

denote the 3×3 rotation matrix constructed from the central RTN basis using (II.1-II.3). Then *standardize* all the deviated and the central states by defining

$$\boldsymbol{x}(t) = R^{(c)T}\boldsymbol{x}^{\mathrm{ECI}(c)}(t), \quad \boldsymbol{x}^{(c)}(t) = R^{(c)T}\boldsymbol{x}^{\mathrm{ECI}(c)}(t), \tag{III.7}$$

$$\dot{\boldsymbol{x}}(t) = R^{(c)T}\dot{\boldsymbol{x}}^{\mathrm{ECI}}(t), \quad \dot{\boldsymbol{x}}^{(c)}(t) = R^{(c)T}\dot{\boldsymbol{x}}^{\mathrm{ECI}(c)}(t). \tag{III.8}$$

After standardization, a deviated state can then be written in the form

$$\boldsymbol{x}(0) = \begin{bmatrix} A+\varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix}, \quad \dot{\boldsymbol{x}}(0) = \begin{bmatrix} B+\delta_1 \\ C+\delta_2 \\ \delta_3 \end{bmatrix}, \tag{III.9}$$

where $A > 0$, $B \in \mathbb{R}$ and $C > 0$ are positive constants for the central state, and $\boldsymbol{\varepsilon} = [\varepsilon_1, \varepsilon_2, \varepsilon_3]^T$ and $\boldsymbol{\delta} = [\delta_1, \delta_2, \delta_3]^T$ represent small deviations from the central state which are modelled by the initial Gaussian distribution.

The difference in AST coordinates between the deviated and the central state can be approximated by linear expressions of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\delta}$ (using the first order Taylor series expansion),

$$\boldsymbol{A} - \boldsymbol{A}^{(c)} = J\begin{bmatrix}\varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \delta_1 & \delta_2 & \delta_3\end{bmatrix}^T = J[\boldsymbol{E} - \boldsymbol{E}^{(c)}]^T, \tag{III.10}$$

where, J is the 6×6 Jacobian matrix from Cartesian-CRTN ($\boldsymbol{E}$) to AST coordinates ($\boldsymbol{A}$) at $t = 0$. An explicit formula for J can be constructed [15].

IV. The NCN distribution

As time increases, it is helpful to represent the state in spherical-CRTN coordinates as

$$\boldsymbol{x}(t) = \begin{bmatrix} r(t)\cos\eta(t)\cos\psi(t) \\ r(t)\sin\eta(t)\cos\psi(t) \\ r(t)\sin\psi(t)], \end{bmatrix} \tag{IV.11}$$

where $\eta(t)$ is the "longitude", and $\psi(t)$ is the "latitude" with respect to the CRTN basis.

The longitude in (IV.11) and the break angle on the true anomaly scale are very similar, $\eta(t) \approx \theta(t)$, up to a first-order approximation. Hence either can be replaced by the other when convenient.

After a bit of computation the latitude can be written as

$$\psi(t) \approx A_1 \sin\eta(t) - A_2\cos\eta(t). \tag{IV.12}$$

Hence, conditional on $\eta(t)$ (or equivalently on $\theta(t)$ or on $A_3(t) = \phi(t)$), $\psi(t)$ is a linear combination of A_1 and A_2; hence it follows a conditional normal distribution and its conditional mean and variance can be computed explicitly.

A similar, but more complicated, expansion can be make for the radial component. Again it follows a conditional Gaussian distribution. It turns out that $1/r(t)$ is more Gaussian than $r(t)$ when the errors are not sufficiently small.

The joint distribution $(\phi(t), \psi(t), 1/r(t))$ can be called a *normal:conditional-normal (NCN)* distribution, since the marginal distribution of $\phi(t)$ is normal and the conditional distribution of $(\psi(t), r(t))$ given $\phi(t)$ is normal. Define *standardized* versions of the latitude and radial component by subtracting their conditional means dividing by their conditional standard deviations. Then the joint distribution of $\phi(t)$ and the standardized versions of $\psi(t)$ and $1/r(t)$ is trivariate normal.

Example 1. Consider a central orbit with eccentricity $e^{(c)} = 0.7$ and an orbital period of 12 hours. These parameters correspond to a highly eccentric orbit (HEO) with A = 9078 km, B = 2.6 km/sec and C = 8.1 km/sec. Set the initial error standard deviations to be 0 km for the position (corresponding to a break-up event) and 0.05 km/sec for the velocity coordinates. A point cloud of $N = 4000$ data points has been propagated for 1 central orbital period. The results are displayed in Fig. 1 as a pairs plot. The first three variables are $\phi(t)$ ("phi"), $\psi(t)$ ("psi") and $1/r(t)$ ("ri"). The final two variables are the standardized versions of $\psi(t)$ ("psi1") and $1/r(t)$ ("ri1"). Several conclusions can be noted.

(a) The distribution of $\phi(t)$ (panel (1,1)) is approximately normal.
(b) The distribution of $\psi(t)$ is clearly not normal (too much mass near 0; see panel (2,2)). Further, the joint distribution of the latitude and the longitude (panel (1,2)) shows a severe "pinching" pattern.
(c) The distribution of $1/r(t)$ is severely skewed and hence not normal. The joint distribution of $\phi(t)$ and $1/r(t)$ shows a very strong, but nonlinear, dependence (panel (1,3)).
(d) On the other hand, the standardized versions of the latitude and inverse radial component look normal (panels (4,4) and (5,5)) and the trivariate distribution for these variables and $\phi(t)$ looks trivariate normal (panels (1,4), (1,5) and (4,5)).

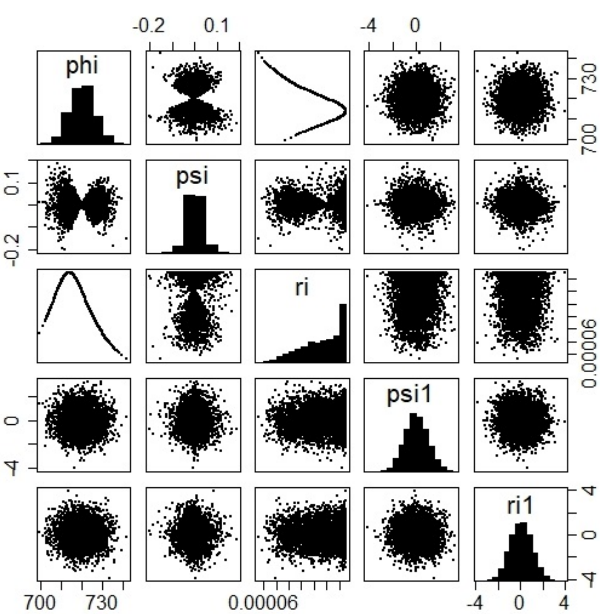

Fig. 1. **Example 1, propagated point cloud in spherical-CRTN coordinates, illustrating the NCN distribution.** The first, second and third elements are the longitude (on the mean anomaly scale), the latitude and the inverse radial component, respectively. The fourth and fifth elements are the standardized latitude and standardized radial component, respectively.

V. THE FILTERING PROBLEM

Each step of a filter has four main ingredients: (i) a state vector with an associated "initial distribution", (ii) a system equation leading to a "propagated distribution", (iii) an observation with an associated "observation distribution", and (iv) an observation equation linking the propagated state to the observation. Then an application of Bayes' Theorem leads to an updated "posterior" distribution for the state.

Using AST coordinates, the system equation is linear and it is usually reasonable to assume the initial state distribution is Gaussian; hence the propagated state distribution is also Gaussian. An angles-only observation is usually assumed to have a Fisher distribution on the sphere with a high concentration, or equivalently, a two-dimensional isotropic Gaussian distribution in a tangent plane to the sphere with a small variance.

The observation can be split into a longitude and a latitude with respect the CRTN basis. However, the observation longitude is on the true anomaly scale whereas the propagated angular position $A_3(t)$ is on the mean anomaly scale. When the eccentricity is high, the map between these two scales can be highly nonlinear.

Further the measurement error is always small, whereas the the propagated variance of $A_3(t)$ can be large when the propagation time is large. In this setting the commonly used unscented and extended Kalman filters (UKF and EKF) can lead to very inaccurate updates when the eccentricity is large. A new nonlinear *observation-centered* Kalman filter, with unscented and extended versions (OCUKF and OCEKF), has been developed to deal with the problem [14]. The two versions behave very similarly to one another so the distinction between them is not emphasized here, and the acronym OCKF is used to refer to both of them. In addition the OCKFs are very similar in performance to the well-established iterated unscented and iterated extended Kalman filters. However the OCKFs have the advantage of having a closed form and not needing iteration.

The OCKF in the current context is essentially a one-dimensional filter for which the state variable $A_3(t)$ is the angular position or longitude on the mean anomaly scale, and the observation is the observed longitude on the true anomaly scale. To deal with all the state variables a two-stage procedure is proposed for the update step. The first stage updates the distribution of $A_3(t)$ using the OCKF. For the second stage, $A_3(t)$ is treated as known and equal to its posterior mean. As noted in Section IV on the NCN distribution, the conditional distribution the latitude $\psi(t)$ given $A_3(t)$ is Gaussian. Hence a linear Kalman filter can be used for the remaining state variables.

The complications of the update step are needed to deal with two different issues, either separately or together. The OCKF stage is needed to deal with highly elliptical orbits for which the mapping between true and mean anomaly is very non-Gaussian. The conditioning stage is designed to deal with break-up events and similar situations, for which the propagated NCN distribution of $(A_3(t), \psi(t))$ is highly non-Gaussian.

Here are further details about the two stages of the update step.

A. Stage 1: OCKF

Replace the observed longitude $\theta_{\text{obs}}(t)$ on the true anomaly scale by its value $\phi_{\text{obs}}(t)$ on the mean anomaly scale,

$$\begin{aligned} \phi_{\text{obs}} &= \phi_p^{(c)}(t) + F_{\text{T-to-M}}(\theta_{\text{obs}} - \theta_p^{(c)}, e^{(c)}), \\ \theta_p^{(c)} &= F_{\text{M-to-T}}(\phi_p^{(c)}, e^{(c)}), \end{aligned} \tag{V.13}$$

where the prior central values are used for the unknown structural parameters.

In addition, the measurement variance for $\phi_{\text{obs}}(t)$ can be obtained from the the measurement variance for the $\theta_{\text{obs}}(t)$ by a first order Taylor series expansion,

$$\text{Var}(\phi_{\text{obs}}) \approx \left[\frac{\left(1 - e^{(c)2}\right)^{3/2}}{\left(1 + e^{(c)} \cos T^{(c)}\right)^2} \right]^2 \text{Var}(\theta_{\text{obs}}). \tag{V.14}$$

Then using the "pseudo-observation", $\phi_{\text{obs}}(t)$, a linear Kalman update can be carried out for $A_3(t)$.

B. Stage 2: conditioning on $A_3(t)$

From Equation (IV.12), the observed latitude can be written as a linear function of A_1 and A_2,

$$\psi_{\text{obs}}(t) \approx A_1 \sin \theta_{\text{true}}(t) - A_2 \cos \theta_{\text{true}}(t), \tag{V.15}$$

assuming $\theta_{\text{true}}(t)$ is known. After Stage 1, to a good level of approximation, $\theta_{\text{true}}(t)$ can be replaced by

$$\theta_{\text{post}}(t) = \phi_p^{(c)}(t) + F_{\text{M-to-T}}(\phi_{\text{post}}(t) - \theta_p^{(c)}, e^{(c)}),$$

where $\phi_{\text{post}}(t)$ is the posterior mean for $A_3(t)$ after Stage 1, and $\theta_{\text{post}}(t)$ is its transformation to the true anomaly scale.

Hence a linear Kalman update can be carried out for the conditional distribution of the remaining state variables, given $A_3(t)$.

Example 2 Assume the same object considered in Examples 1 with central eccentricity $e^{(c)} = 0.7$. This time consider initial standard deviations 30 km for the position and 0.05 km/sec for the velocity coordinates. For simplicity assume the central angle of perigee vanishes, $\theta_p^{(c)} = 0^o$. Recall from (III.5) that the propagated variance of $A_3(t)$ increases linearly with t. Choose the propagation time t_1 large enough that the standard deviation of $A_3(t_1)$ equals $\sigma^* = 25^o$. Also suppose that the propagated mean of $A_3(t_1)$ is $\mu^* = 260^o$.

Next, consider an angles-only observation with longitude θ_{obs} = 225.5^o and latitude ψ_{obs} = 0^o in the CRTN frame of reference, with measurement standard deviation 5.5e-04^o (2 arc-seconds) for both. After transformation to the mean anomaly scale, the observed longitude takes the value

$$\phi_{obs} = F_{\text{T-to-M}}(225.5^o, 0.7) = 310^o, \tag{V.16}$$

which is located at the 2.5% upper tail of the propagated distribution for $A_3(t_1)$ since $\phi_{obs} = \mu^* + 2\sigma^* = 260^o + 2 \times$

$25^o = 310^o$. Such cases are mildly unusual but not unlikely. The results in Table I show that the UKF and EKF yield posteriors that are far from the exact posterior distributions. The iterated filters (IUKF and IEKF) and the observation-centered filters (OCUKF and OCEKF) are virtually the same as one another and the exact result. The exact result was computed using a particle filter.

TABLE I

Table 1. Posterior means and standard deviations for $A_3(t_1)$ in Example 2, computed using various filters.

Moment	UKF	EKF	"Exact"	
mean (A_3)	327.1^o	329.8^o	310^o	
s.d (A_3)	4.1e-04^o	5e-04^o	3.2e-02^o	
Moment	IUKF	IEKF	OCUKF	OCEKF
mean (A_3)	310^o	310^o	310^o	310^o
s.d (A_3)	3.2e-02^o	3.1e-02^o	3.2e-02^o	3.3e-02^o

VI. Conclusion

To summarize, this paper has investigated various issues related to the orbital uncertainty analysis and tracking. The AST coordinate system has been developed to overcome the limitations of the equinoctial coordinate system.

This paper has also highlighted various issues related to the propagated uncertainty associated with the angular position and the radial component. In particular, standardization is sometimes needed to ensure the Gaussianity of the distributions of the latitude and the inverse radial distance.

These results have been combined into a procedure to compute a one-step update for the tracking problem using the OCKF. After suitable transformations and conditioning, the computations are nearly the same as those for the classic linear Kalman filter. The evidence from Example 2 and other investigations shows that the OCKF performs very well under a wide range of circumstances.

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