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Introduction to Special Issue: Foundations of Mathematical Structuralism

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Structuralism, the view that mathematics is the science of structures, can be characterized as a philosophical response to a general structural turn in modern mathematics. Structuralists aim to understand the ontological, epistemological, and semantical implications of this structural approach in mathematics. Theories of structuralism began to develop following the publication of Paul Benacerraf's paper "What numbers could not be" in 1965. These theories include non-eliminative approaches, formulated in a background ontology of *sui generis* structures, such as Stewart Shapiro's *ante rem* structuralism and Michael Resnik's pattern structuralism. In contrast, there are also eliminativist accounts of structuralism, such as Geoffrey Hellman's modal structuralism, which avoids *sui generis* structures. These research projects have guided more systematic focus on philosophical topics related to mathematical structuralism, including the identity criteria for objects in structures, dependence relations between objects and structures, and also, more recently, structural abstraction principles. Parallel to these developments are approaches that describe mathematical structure in category-theoretic terms (e.g. in work by Steve Awodey, Elaine Landry, and Colin McLarty). Category-theoretic approaches have been further developed using tools from homotopy type theory. Here we find a strong relationship between mathematical structuralism and the univalent foundations project, an approach to the foundations of mathematics based on higher category theory.

This brief overview of structuralist positions indicates that different theories of mathematical structure are based on strikingly different (and often incompatible) assumptions concerning the nature of mathematical knowledge and its objects. For instance, several version of structuralism have conflicting views on mathematical ontology and the nature of abstract structures. Do such structures, and the objects in them, exist as mathematical entities? If so, then are structures essentially sets, or are they entities of an altogether different nature, such as Platonic universals or unlabeled graphs?

Other background assumptions reflected in different approaches to structuralism concern the semantics of mathematical discourse. Do mathematical statements about structures and structural properties have objective truth values? Should we specify the truth-conditions of such statements in terms of classic model-theoretic semantics? Or perhaps a non-classical approach, e.g., in terms of a supervaluationist semantics, is more appropriate. Do the singular terms of a theory have unique reference, or do they behave semantically like variables or arbitrary names?

Finally, several theories of structuralism are based on specific assumptions concerning the methodology of mathematics. The focus here is on different structural methods, such as those used to introduce abstract structures in a given mathematical field. What is the best approach to speak about, say, the structure of real number fields or of hyperbolic space? Should we think about such structures as introduced through implicit definitions by axiom systems or rather through (structural) abstraction principles? Alternatively, is the structure of a space best characterized in terms of invariants under groups of transformations, as first suggested in Klein's Erlangen program?

This special issue addresses some of these topics in the philosophical and logical foundations of mathematical structuralism. It is based on the conference *Foundations of Mathematical Structuralism*, which was held at Ludwig-Maximilians-Universität Munich on October 12-14, 2016. The objective of the conference was to reassess the different perspectives on mathematical structuralism and its role in the foundations of mathematics and in mathematical practice. Specifically, the conference focused on the following research questions:

1. Does structuralism offer a philosophically viable foundation for modern mathematics?
2. What role do key notions such as structural abstraction, invariance, or structural identity play in different theories of structuralism?
3. To what degree does structuralism as a philosophical position describe actual mathematical practice and its history?
4. Does set theory, category theory, or homotopy type theory provide the most adequate structuralist foundation for mathematics?

The program for the three-day conference comprised talks by the following scholars: Steve Awodey, Francesca Biagioli, Francesca Boccuni, Jessica Carter, José Ferreirós, Gerhard Heinzmann, Geoffrey Hellman, James Ladyman, Hannes Leitgeb, Mary Leng, Øystein Linnebo, Josef Menŝík, Erich Reck, Daniel Waxman, and Jack Woods. The event was funded by the Deutsche Forschungsgemeinschaft (DFG) and organized by the Munich Center for Mathematical Philosophy (LMU Munich).

This special issue includes a selection of four research articles presented at the conference. Francesca Boccuni's & Jack Woods' article "Structuralist Neologicism" connects two dominant debates in modern philosophy of mathematics: structuralism and neologicism (or neo-Fregeanism). The focus of the article is on the semantics of mathematical discourse and addresses the question of how mathematical terms refer to objects. The starting point of their discussion are three desidarata that the authors argue any philosophical theory of mathematics should satisfy. These are (a) that mathematical objects are referred to by singular terms, (b) that truths about these objects are conceptual truths, and (c) that these objects have only mathematical properties. The authors show that both the neo-logicist program as well as theories of structuralism fail to meet all three intuitions. In particular, based on a critical discussion of the Caesar problem and related problems, it is argued that neo-logicism falls short of meeting conditions (a) and (b). Similarly, existing versions of structuralism (including *ante rem* and *in re* structuralism) fail to satisfy conditions (a) and (c).

Based on this critical discussion, Boccuni and Woods present their own account, "structuralist neo-logicism", as a theory yielding the best of both worlds in the sense of satisfying the three intuitions about mathematical objects. The approach gives up the assumption that mathematical terms have canonical reference, which is the case if the referent of a term can be perfectly individuated. Instead, their account is built on the notion of arbitrary reference, according to which mathematical terms refer "genuinely but arbitrarily" to abstract objects. Given the (neo-logicist) discussion of abstraction principles, Boccuni and Woods argue that number terms, as implicitly defined by such a principle, function semantically as arbitrary names. These are constant expressions that refer only arbitrarily to objects. The article presents two ways in which the arbitrariness of reference can be specified, either epistemologically or in terms of a supervaluationist semantics. Based on this presentation of structuralist neo-logicism, Boccuni and Woods finally suggest a new interpretation of Hume's principle. According to their view, the abstraction principle specifies the arbitrary reference of number terms and, at the same time, meets the desiderata (a)-(c).

Hannes Leitgeb's article "On Non-Eliminative Structuralism. Unlabeled Graphs as a Case Study, Part A" is the first of two connected articles presenting a novel account of structuralism based on graph theory. Part B will be published in another forthcoming issue of *Philosophia Mathematica*. Part A first charactizes non-eliminative structuralism, the view that mathematical structures are *sui generis* objects, as the most adequate structuralist account of modern mathematics. Leitgeb's own proposal for a theory of sui generis structures is based on the theory of unlabeled graphs. He argues that such graphs can be viewed as (pure) structures similarly to the ways non-eliminativist structuralists speak about *the* natural number structure (as specified by a categorical second-order axiomatization). As the central contribution in Part A, Leitgeb develops a new axiomatic theory of unlabeled and undirected graphs: UGT. This theory is formulated in a second-order logic and presents an alternative to the usual set-theoretic presentation of graphs. As Leitgeb points out, UGT can be viewed as a general theory of mathematical structures. In particular, unlabeled graphs introduced in this way are *sui generis* objects because it is not necessary to reduce or represent them as set-theoretical systems. Thus, the structure of an unlabeled graph need not be expressed set-theoretically. Moreover, they are structural in character because, unlike labelled graphs, the nodes in them can be understood as pure positions. Finally, UGT contains as an axiom a genuinely structuralist identity criterion for such objects, according to which any two unlabeled graphs are identical if and only if there exists a graph isomorphism between them.

Given the axiomatic theory of unlabeled graphs as sui generis structures presented in Part A, Part B then turns to a more philosophical discussion of Leitgeb's structuralism. In particular, he shows that some of the main philosophical objections against non-eliminative structuralism are unproblematic in his account. Leitgeb identifies three types of objections in this respect, namely "Problems of Identity, Objects, and Reference". The problems of identity concern two issues: first, the well-known debate on whether structurally indiscernible positions in a structure should be identified or not; second, the cross-structural identification of positions (i.e. the question whether the natural number 2 is identical to the real number 2). As Leitgeb shows, both problems can be convincingly solved if one takes the underlying structures to be unlabeled graphs. In turn, the problems of objects and reference for non-eliminative structuralism are based on what is called Benacerraf's permutation argument (and also on the problem of non-rigid structures). In particular, it is usually held that the permutability of structures leads to a kind of indeterminacy of reference of mathematical terms. Here again, Leitgeb shows that such semantic worries are unproblematic for the objects of his theory UGT.

The third article of the special issue, Mary Leng's "An '*i*' for an *i*, a Truth for a Truth" is thematically connected to both Boccuni's & Woods' article as well as to Leitgeb's Part B article. Leng also focuses on what Leitgeb describes as a second "problem of reference" often associated with non-eliminative structuralism. This is the question of how singular mathematical terms can uniquely refer to positions in non-rigid structures, that is, in structures that possess non-trivial automorphisms. Leng first introduces a general distinction between two views of mathematics, namely the algebraic and the assertory view. Versions of non-eliminative structuralism such as Shapiro's ante rem structuralism can be viewed as (at least partly) assertory approaches to mathematics. According to them, a theory such as Peano arithmetic describes a fixed subject matter, namely the natural number structure. Moreover, the singular terms of the language of arithmetic have fixed referents, namely positions in this structure (as well as relations between them). In contrast, forms of eliminative structuralism and of mathematical fictionalism are usually characterized in terms of an algebraic understanding of mathematics. As such, they lack the kind of realist semantics for singular terms often associated with Shapiro's view.

Leng's paper criticizes this standard assessment of the theoretical landscape. She shows that Shapiro's ante rem structuralism, viewed as a partially assertory approach, has in fact no advantage over fictionalism and other globally algebraic theories with respect to providing a face value semantics for mathematical terms. The main objection against ante rem structuralism analyzed in this respect is based on the well-known problem of non-rigid structures. As Leng points out, a semantic version of this problem shows that it is not possible to fix a unique reference of singular terms relative to such non-rigid structures. In particular, terms such as *i* in complex number theory function semantically not as constants but rather as (disguised) parameters. Given that Shapiro's approach does not account for the canonical reference to positions in such non-rigid structures, it follows that fictionalism and ante rem structuralism are theoretically on par, Leng argues, at least on the issue of providing a uniform semantics for mathematical discourse.

The forth article in the volume is Francesca Biagioli's "Structuralism and Mathematical Practice in Felix Klein's Work on Non-Euclidean Geometry". Biagioli investigates one of the important strands in the history of structural mathematics, namely Klein's Erlangen program and the conception of geometry as the study of invariants under transformation groups. The first part of the article focuses on the development of Klein's views on geometry, retracing the transition from his early writings on projective and non-Euclidean Geometry between 1871-74 to his group-theoretic classification of geometries in 1872 until the epistemological considerations on geometrical knowledge in his late publications from the 1890s. Biagioli describes Klein's approach to geometry as a kind of "methodological structuralism", where the central structural method is the use of transfer principles for the comparison of different geometries in terms of their corresponding transformation groups. The second thematic part of the article (starting with section 3) then turns to a closer philosophical analysis of Klein's structuralism implicit in his geometrical work. A focus in Biagioli's analysis is put here on Klein's understanding of the concept of mathematical abstraction. Based on a comparison with Dedekind's protoaxiomatic approach in the latter's work on the foundations of number theory and analysis, Biagioli discusses Klein's scattered methodological remarks on structural axiomatics and on the nature of axioms as definitions of abstract concepts. By connecting Klein's geometrical structuralism with the modern philosophical debates, she holds that Klein defended a version of *in rebus* (or *in re*) structuralism that becomes visible in his discussion of abstraction from physical to mathematical concepts.

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