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Joint Eigenfunctions for the Relativistic Calogero–Moser Hamiltonians of Hyperbolic Type. III. Factorized Asymptotics

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In the two preceding parts of this series of papers, we introduced and studied a recursion scheme for constructing joint eigenfunctions $J_N(a_+, a_-, b; x, y)$ of the Hamiltonians arising in the integrable N-particle systems of hyperbolic relativistic Calogero–Moser type. We focused on the 1st steps of the scheme in Part I and on the cases $N = 2$ and $N = 3$ in Part II. In this paper, we determine the dominant asymptotics of a similarity-transformed function $\mathrm{E}_N(b; x, y)$ for $y_j - y_{j+1} \to \infty$, $j = 1, \ldots, N-1$ and thereby confirm the long-standing conjecture that the particles in the hyperbolic relativistic Calogero–Moser system exhibit soliton scattering. This result generalizes a main result in Part II to all particle numbers $N > 3$.

1 Introduction

This paper is the 3rd part in a series of papers dedicated to the explicit diagonalization and Hilbert space transform theory for the integrable N-particle systems of hyperbolic relativistic Calogero–Moser type. The classical version of these systems was introduced in [13], whereas a quantization prescription preserving integrability was obtained in

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[10]. It is given by the commuting analytic difference operators (henceforth AΔOs)

$$S_k(x) = \sum_{\substack{I\subset\{1,\ldots,N\}\\ |I|=k}} \prod_{\substack{m\in I\\ n\notin I}} f_-(x_m - x_n) \prod_{l\in I} \exp(-i\hbar\beta\partial_{x_l}) \prod_{\substack{m\in I\\ n\notin I}} f_+(x_m - x_n), \quad k = 1, \ldots, N, \tag{1}$$

where

$$f_\pm(z) = \big(\sinh(\mu(z \pm i\beta g)/2)/\sinh(\mu z/2)\big)^{1/2} \tag{2}$$

and $\beta = 1/mc$, with m the particle rest mass and c the speed of light. In the nonrelativistic limit $c \to \infty$, these operators give rise to quantum integrals of the ordinary nonrelativistic hyperbolic Calogero–Moser systems, see for example the survey [11].

We reparametrize the two length scales in the Hamiltonians (1) as

$$a_+ = 2\pi/\mu \quad \text{(imaginary period / interaction length)} \tag{3}$$

$$a_- = \hbar/mc \quad \text{(shift step size / Compton wave length)} \tag{4}$$

and replace the coupling parameter g with the parameter

$$b = \beta g. \tag{5}$$

Interchanging a_+ and a_-, we obtain new Hamiltonians that commute with the given ones, since the shift operators in the former alter the arguments of the coefficients in the latter by a period and vice versa. The resulting $2N$ commuting Hamiltonians are given by

$$H_{k,\delta}(b;x) = \sum_{\substack{I\subset\{1,\ldots,N\}\\ |I|=k}} \prod_{\substack{m\in I\\ n\notin I}} f_{\delta,-}(x_m - x_n) \prod_{l\in I} \exp(-ia_{-\delta}\partial_{x_l}) \prod_{\substack{m\in I\\ n\notin I}} f_{\delta,+}(x_m - x_n), \tag{6}$$

where $k = 1, \ldots, N$, $\delta = +, -$ and

$$f_{\delta,\pm}(z) = \left(\frac{s_\delta(z \pm ib)}{s_\delta(z)}\right)^{1/2}. \tag{7}$$

Here we have used the functions

$$s_\delta(z) = \sinh(\pi z/a_\delta), \quad c_\delta(z) = \cosh(\pi z/a_\delta), \quad e_\delta(z) = \exp(\pi z/a_\delta), \quad \delta = +, -, \tag{8}$$

which will appear frequently throughout the paper.

From now on, we take $a_+, a_- \in (0, \infty)$, use further parameters

$$\alpha \equiv 2\pi/a_+a_-, \quad a \equiv (a_+ + a_-)/2, \tag{9}$$

$$a_s \equiv \min(a_+, a_-), \quad a_l \equiv \max(a_+, a_-), \tag{10}$$

and work with b-values in the strip

$$S_a \equiv \{b \in \mathbb{C} \mid \mathrm{Re}\, b \in (0, 2a)\}. \tag{11}$$

In addition, we make extensive use of the generalized Harish–Chandra c-function

$$c(b; z) \equiv \frac{G(z + ia - ib)}{G(z + ia)} = c(b; -z - 2ia + ib) \tag{12}$$

and its multivariate version

$$C_N(b; x) \equiv \prod_{1 \le j < k \le N} c(b; x_j - x_k), \quad N \ge 2. \tag{13}$$

Here $G(z) \equiv G(a_+, a_-; z)$ denotes the hyperbolic gamma function, whose salient features are reviewed in the 1st two parts of this series of papers. In particular, in (12) and frequently below, we use the reflection equation $G(-z) = 1/G(z)$. (To unburden notation, we usually suppress the dependence on the parameters a_+, a_-; also, the dependence on N and b is often omitted when ambiguities are unlikely to arise.)

In many instances, it is convenient to use one of two further incarnations of the Hamiltonians $H_{k,\delta}$, obtained by similarity transformation with either a weight function or a scattering function. More specifically, letting

$$w(z) = 1/c(z)c(-z), \quad W(x) = 1/C(x)C(-x), \tag{14}$$

$$u(z) = -c(z)/c(-z), \quad U(x) = (-)^{N(N-1)/2} C(x)/C(-x), \tag{15}$$

they read

$$A_{k,\delta}(x) \equiv W(x)^{-1/2} H_{k,\delta}(x) W(x)^{1/2}, \tag{16}$$

$$\mathcal{A}_{k,\delta}(x) \equiv U(x)^{-1/2} H_{k,\delta}(x) U(x)^{1/2} = C(x)^{-1} A_{k,\delta}(x) C(x), \tag{17}$$

where $k = 1, \ldots, N$, and $\delta = +, -$. Using the difference equations $G(z + ia_\delta/2)/G(z - ia_\delta/2) = 2c_{-\delta}(z)$ satisfied by the hyperbolic gamma function, it is a straightforward exercise to deduce their explicit expressions

$$A_{k,\delta}(x) = \sum_{\substack{I\subset\{1,\ldots,N\}\\|I|=k}} \prod_{\substack{m\in I\\n\notin I}} \frac{s_\delta(x_m - x_n - ib)}{s_\delta(x_m - x_n)} \prod_{l\in I} \exp(-ia_{-\delta}\partial_{x_l}) \tag{18}$$

and

$$\mathcal{A}_{k,\delta}(x) = \sum_{\substack{I\subset\{1,\ldots,N\}\\|I|=k}} \prod_{\substack{m\in I,n\notin I\\m>n}} \frac{s_\delta(x_m - x_n - ib)}{s_\delta(x_m - x_n)} \frac{s_\delta(x_m - x_n + ib - ia_{-\delta})}{s_\delta(x_m - x_n - ia_{-\delta})} \prod_{l\in I} \exp(-ia_{-\delta}\partial_{x_l}). \tag{19}$$

In particular, it follows that these similarity-transformed AΔOs preserve the space of meromorphic functions. Moreover, if $x \in \mathbb{R}^N$ and $(a_+, a_-, b) \in (0,\infty)^3$ with $b < 2a$, the weight function $W(x)$ is positive and the "S-matrix" $U(x)$ has modulus one. Consequently, the AΔOs $A_{k,\delta}$ and $\mathcal{A}_{k,\delta}$ are then formally self-adjoint, when viewed as operators on the Hilbert spaces $L^2(\mathbb{R}^N, W(x)dx)$ and $L^2(\mathbb{R}^N, dx)$, resp.

In Part I [4] of this series of papers, we took the 1st steps in developing a recursion scheme for constructing joint eigenfunctions $J_N(a_+, a_-, b; x, y)$ of the commuting AΔOs $A_{k,\delta}$. More specifically, we presented the formal features of the scheme, explicitly demonstrated its arbitrary-N viability for the "free" cases and established holomorphy domains and uniform decay bounds that were sufficient to render the scheme rigorous. Motivated by results on the "free" cases as well as the $N = 2$ case, which can be gleaned from [12], we also detailed several conjectured features of the joint eigenfunctions J_N.

In Part II [5], we proved a number of these conjectures in the cases $N = 2$ and $N = 3$. Indeed, we established global meromorphy, a number of invariance properties and a duality relation, and undertook a detailed study of asymptotic behavior. The purpose of this 3rd part is to generalize the results on asymptotics to all particle numbers $N > 3$. We shall make use of previous results in this series of papers without further ado, referring back to sections and equations in [4] and [5] by using the prefix I and II, respectively.

To a large extent, we can follow our approach in the $N = 3$ case, but the technical difficulties we encounter are considerably more involved. Important auxiliary results have been isolated in Lemma 2.3 and Theorem A.1. The latter theorem allows us to avoid the use of the bound II (2.73) on E_2 that we used for the $N = 3$ case, cf. the proof of II Theorem 3.7. This amounts to one of several simplifications of our $N = 3$ results in II Section 3. We could not obtain a counterpart of the bound II (2.73) for E_N with $N > 2$, but fortunately Theorem A.1 obviates this snag as well.

In order to describe the results and organization of this paper in more detail, we recall that the construction of J_N from J_{N-1} in I Section 6 produced the representation

$$J_N(b;x,y) = \frac{\exp(i\alpha y_N(x_1+\cdots+x_N))}{(N-1)!}\int_{\mathbb{R}^{N-1}} \mathrm{d}z\, I_N(b;x,y,z), \quad b\in S_a, \quad x,y\in\mathbb{R}^N, \tag{20}$$

where the integrand is given by

$$I_N(b;x,y,z) \equiv W_{N-1}(b;z)\mathcal{S}_N^\sharp(b;x,z)J_{N-1}(b;z,(y_1-y_N,\ldots,y_{N-1}-y_N)), \tag{21}$$

with kernel function (cf. I (A.6))

$$\begin{aligned}\mathcal{S}_N^\sharp(b;x,z) &\equiv \prod_{j=1}^{N}\prod_{k=1}^{N-1}\frac{G(z_k-x_j-ib/2)}{G(z_k-x_j+ib/2)}\\ &= \prod_{j=1}^{N}\prod_{k=1}^{N-1} c(b;z_k-x_j-ia+ib/2).\end{aligned} \tag{22}$$

When taking the 1st steps in developing the recursive scheme that led to the representation (20), we were inspired by earlier work on related integrable quantum many-body systems. To the best of our knowledge, the 1st indication that such a scheme could be possible can be found in work by Gutzwiller [2], who used it to connect eigenfunctions for the periodic and nonperiodic nonrelativistic Toda systems. Among more recent works, we drew particular inspiration from a number of papers by authors from the group of Gerasimov, Kharchev, Lebedev, Oblezin, and Semenov–Tian–Shansky, which can be traced from what we believe is their most recent paper [1] on the subject. References to further related works can be found in the introductions to I and II.

Defining

$$X_N \equiv \frac{1}{N}\sum_{j=1}^N x_j, \quad Y_N \equiv \frac{1}{N}\sum_{j=1}^N y_j, \quad x_j^{(N)} \equiv x_j - X_N, \quad y_j^{(N)} \equiv y_j - Y_N, \quad j=1,\ldots,N, \tag{23}$$

a straightforward induction argument revealed another important representation that we have occasion to invoke below, namely,

$$J_N(x,y) = \exp(Ni\alpha X_N Y_N)J_N^r(x,y), \tag{24}$$

$$J_N^r(x,y) \equiv \frac{1}{(N-1)!}\int_{\mathbb{R}^{N-1}} \mathrm{d}z\, W_{N-1}(z)\mathcal{S}_N^\sharp(x^{(N)},z)J_{N-1}(z,(y_1-y_N,\ldots,y_{N-1}-y_N)), \tag{25}$$

cf. I (6.27)–(6.28). Note that the function $J_N^r(x,y)$ depends only on the differences $x_j - x_{j+1}$ and $y_j - y_{j+1}$, $j = 1, \ldots, N-1$.

By performing simultaneous contour shifts in the former representation (20), we showed in I Theorem 6.1 that for fixed $y \in \mathbb{R}^N$ the function $J_N(b;x,y)$ is holomorphic in

$$D_N \equiv \left\{(b,x) \in S_a \times \mathbb{C}^N \mid \max_{1\le j<k\le N} |\mathrm{Im}\,(x_j - x_k)| < 2a - \mathrm{Re}\,b\right\}. \tag{26}$$

Moreover, after restricting attention to a subdomain of D_N for the dependence on (b,x), we could allow $y \in \mathbb{C}^N$ such that $|\mathrm{Im}\,(y_j - y_k)| < \mathrm{Re}\,b$, $1 \le j < k \le N$. Specifically, introducing the restricted domain

$$D_N^r \equiv \left\{(b,x) \in S_a \times \mathbb{C}^N \mid |\mathrm{Im}\,x_j^{(N)}| < a - \mathrm{Re}\,b/2,\ \ j = 1,\ldots,N\right\} \subset D_N, \tag{27}$$

we used the latter representation (24) to prove that $J_N(b;x,y)$ is holomorphic in (b,x,y) on the domain

$$\mathcal{D}_N \equiv \left\{(b,x,y) \in D_N^r \times \mathbb{C}^N \mid \max_{1\le j<k\le N} |\mathrm{Im}\,(y_j - y_k)| < \mathrm{Re}\,b\right\}, \tag{28}$$

cf. I Theorem 6.4.

In Section 2, we study the asymptotic behavior of the function

$$\mathrm{E}_N(b;x,y) \equiv \left(\frac{\phi(b)G(ib-ia)}{\sqrt{a_+a_-}}\right)^{N(N-1)/2} \frac{J_N(b;x,y)}{C_N(b;x)C_N(2a-b;y)}, \tag{29}$$

where

$$\phi(b) \equiv \exp(i\alpha b(b-2a)/4) = \phi(2a-b). \tag{30}$$

Note that E_N is a joint eigenfunction of the AΔOs $\mathcal{A}_{k,\delta}$, cf. (17) and I Theorem 6.2. Since the c-function is not even, E_N lacks some of the invariance properties of J_N. However, the multipliers in (29) are meromorphic functions whose features are known in great detail. Hence, the analyticity properties of E_N follow from those of J_N. Moreover, E_N is particularly well suited for Hilbert space purposes.

As the principal result of Section 2 and of this paper, we prove in Theorem 2.4 that E_N has the "unitary asymptotics"

$$\mathrm{E}_N(b;x,y) \sim \mathrm{E}_N^{\mathrm{as}}(b;x,y) \equiv \sum_{\sigma\in S_N} \prod_{\substack{j<k \\ \sigma^{-1}(j)>\sigma^{-1}(k)}} (-u(b;x_k - x_j)) \cdot \exp\left(i\alpha \sum_{j=1}^N x_{\sigma(j)}y_j\right), \tag{31}$$

for $y_j - y_{j+1} \to \infty$, $j = 1, \ldots, N-1$. Here the scattering function u is given by

$$u(b;z) \equiv -\frac{c(b;z)}{c(b;-z)} = -\prod_{\delta=+,-} \frac{G(z+\delta i(a-b))}{G(z+\delta ia)}. \tag{32}$$

It clearly satisfies

$$u(b;z)u(b;-z) = 1, \tag{33}$$

and we also have

$$|u(b;z)| = 1, \quad b, z \in \mathbb{R}, \tag{34}$$

due to the reflection equation I (A.6) and the conjugation relation I (A.9). Moreover, we obtain a uniform bound on $\mathrm{E}_N(x,y)$ for suitably restricted $(x,y) \in \mathbb{C}^N \times \mathbb{R}^N$, which plays a crucial role in the inductive step $N-1 \to N$.

The asymptotic behavior (31) confirms a long-standing conjecture. In physical parlance, it says that the particles in the relativistic Calogero–Moser systems of hyperbolic type exhibit soliton scattering (conservation of momenta and factorization of the S-matrix), cf. I Section 7. For a survey of the A_{N-1} type Calogero–Moser systems and their relation to soliton PDEs, we refer to [11]. In particular, the sine-Gordon soliton scattering corresponds to choosing b equal to $a_+/2$ or $a_-/2$ in (32). See also the recent paper [6] for more information on this "sine-Gordon" perspective.

To be precise, we establish the factorized asymptotics of $\mathrm{E}_N(x,y)$ in the "spectral" variables y, whereas the eigenvalue equations that follow from I Theorem 6.2 are with respect to the "geometric" variables x. In the $N=2$ and $N=3$ cases, we proved in II Lemma 2.5 and II Lemma 3.5, respectively, the duality property

$$\mathrm{E}_N(b;x,y) = \mathrm{E}_N(2a-b;y,x), \tag{35}$$

which immediately implies that $\mathrm{E}_N(x,y)$ has the same factorized asymptotics in the "geometric" variables x. (Note that the scattering function $u(b;z)$ is invariant under $b \to 2a-b$, cf. (32).) We certainly expect this duality property to hold true also for $N > 3$, but it remains a challenging open problem to supply a proof.

Within the context of harmonic analysis, factorized asymptotics was first established by Harish–Chandra for the spherical functions associated with certain symmetric spaces. Viewed from the A_{N-1} perspective of this paper, the Harish–Chandra work pertains to the nonrelativistic Calogero–Moser systems for a few special coupling constants (see [3] for a comprehensive account of the general Harish–Chandra results, as well as related ones, and [9] for their relevance to Calogero–Moser systems). Factorized asymptotics for the hyperbolic case with arbitrary positive coupling was first proved by Opdam [8], working within the arbitrary root system context developed by him and Heckman, a summary of which can be found in [7]. A crucial aspect of the asymptotic analysis in these references is the existence and exploitation of series expansions. By contrast, no such expansions are known for the eigenfunctions at issue in this paper. As in our previous work, a key point is rather to use their recursive structure.

2 Asymptotic Behavior

Using II Theorems 3.7–3.8 as the starting point for an induction argument, we proceed to determine the asymptotics of the function $\mathrm{E}_N(b;x,y)$ (29) for $m_N(y)\to\infty$, where

$$m_N(y)\equiv\min_{1\le j<k\le N}(y_j-y_k),\quad y\in\mathbb{R}^N. \tag{36}$$

More specifically, Theorems 2.4– 2.5 below are a consequence of the former for $N=3$, and our induction assumption is that they hold true if we replace N by $N-1$. In the present general-N setting, however, we restrict attention to $\mathrm{Re}\, b$ varying over a subinterval of $(0,2a)$, namely $(0,a_l]$. Thus, we introduce the strip

$$S_l\equiv\{b\in\mathbb{C}\mid \mathrm{Re}\, b\in(0,a_l]\}. \tag{37}$$

We start with some auxiliary results about $J_N(b;x,y)$.

Proposition 2.1. For fixed $y\in\mathbb{R}^N$, the function $J_N(b;x,y)$ is holomorphic in

$$D_N^l\equiv\Big\{(b,x)\in S_l\times\mathbb{C}^N\mid \max_{1\le j<k\le N}|\mathrm{Im}\,(x_j-x_k)|<a_s\Big\}. \tag{38}$$

Furthermore, for all $(b,x,y)\in\mathcal{D}_N$ (28) and $\eta\in\mathbb{C}$, we have symmetry properties

$$J_N(x,y)=J_N(-x,-y), \tag{39}$$

$$\begin{aligned}J_N(x,y)&=\exp(-i\alpha\eta(y_1+\cdots+y_N))J_N((x_1+\eta,\ldots,x_N+\eta),y)\\&=\exp(-i\alpha\eta(x_1+\cdots+x_N))J_N(x,(y_1+\eta,\ldots,y_N+\eta)),\end{aligned} \tag{40}$$

$$J_N(\sigma x,y)=J_N(x,y),\quad \sigma\in S_N. \tag{41}$$

Proof. The 1st assertion is an easy consequence of the readily verified inclusion

$$D_N^l \subset D_N, \tag{42}$$

cf. (26).

Letting $x, y \in \mathbb{R}^N$ to begin with, the permutation invariance (41) is immediate from the defining representation (20). To establish the invariance properties (39)–(40), we assume inductively that they hold true for $N \geq 3$. (In the case $N = 3$, this is the content of II Proposition 3.1.) From (14), (21)–(22), and (39) with $N \to N-1$ and the reflection equation I (A.6) for $G(z)$, we infer

$$I_N(-x, -y, -z) = I_N(x, y, z). \tag{43}$$

Changing variable $z \to -z$ in the representation (20), the invariance property (39) is a direct consequence of (43). Requiring in addition $\eta \in \mathbb{R}$, we deduce (40) from the alternative representation given by (24)–(25). Since (39)–(41) are preserved under analytic continuation, the proof is complete. ∎

This proposition has the following corollary.

Corollary 2.2. Letting $y \in \mathbb{R}^N$, the function $\mathrm{E}_N(b; x, y)$ is meromorphic in D_N^l and holomorphic in

$$D_{N,\beta}^l \equiv \left\{(b, x) \in D_N^l \mid \mathrm{Im}\,(x_j - x_k) < \beta, \ \ 1 \leq j < k \leq N\right\}, \tag{44}$$

where

$$\beta \equiv \min(\mathrm{Re}\, b, a_s). \tag{45}$$

Moreover, for all $(b, x, y) \in \mathcal{D}_N$ (28) and $\eta \in \mathbb{C}$, it satisfies

$$\mathrm{E}_N(-x, -y) = \mathrm{E}_N(x, y) \prod_{1 \leq j < k \leq N} u(x_j - x_k) u(y_j - y_k), \tag{46}$$

$$\begin{aligned} \mathrm{E}_N(x, y) &= \exp(-i\alpha\eta(y_1 + \cdots + y_N))\mathrm{E}_N((x_1 + \eta, \ldots, x_N + \eta), y) \\ &= \exp(-i\alpha\eta(x_1 + \cdots + x_N))\mathrm{E}_N(x, (y_1 + \eta, \ldots, y_N + \eta)), \end{aligned} \tag{47}$$

$$\mathrm{E}_N(\sigma x, y) = \mathrm{E}_N(x, y) \prod_{\substack{j<k \\ \sigma^{-1}(j) > \sigma^{-1}(k)}} (-u(x_j - x_k)), \quad \sigma \in S_N, \tag{48}$$

where $(\sigma x)_j \equiv x_{\sigma(j)}$.

Proof. The zeros of $C_N(b;x)$ are located at

$$x_j - x_k = -2ia - ima_+ - ina_-,\ ib + ima_+ + ina_-,\ 1 \le j < k \le N,\quad m,n \in \mathbb{N}, \tag{49}$$

so the poles of $1/C_N(b;x)$ do not belong to $D^l_{N,\beta}$. Hence, the 1st assertion is clear from the relation (29) between J_N and E_N.

Keeping in mind (13) and (32), the symmetry features are readily inferred from (29) and Proposition 2.1. ■

Recalling from I (2.11) the kernel function

$$\mathcal{K}^\sharp_N(b;x,z) \equiv [C_N(b;x)C_{N-1}(b;-z)]^{-1}\mathcal{S}^\sharp_N(b;x,z), \tag{50}$$

it is easily seen that (21)–(20) and (29) yield the representation

$$\begin{aligned}\mathrm{E}_N(b;x,y) &= \frac{1}{(N-1)!}\left(\frac{\phi(b)G(ib-ia)}{\sqrt{a_+a_-}}\right)^{N-1}\\ &\quad\times \frac{\exp(i\alpha y_N(x_1+\cdots+x_N))}{\prod_{n=1}^{N-1}c(2a-b;y_n-y_N)}\int_{\mathbb{R}^{N-1}} dz\,\mathrm{I}_N(b;x,y,z),\quad b\in S_a,\quad x,y\in\mathbb{R}^N,\end{aligned} \tag{51}$$

with integrand

$$\mathrm{I}_N(b;x,y,z) \equiv \mathcal{K}^\sharp_N(b;x,z)\mathrm{E}_{N-1}(b;z,(y_1-y_N,\ldots,y_{N-1}-y_N)). \tag{52}$$

Following our treatment of the $N=2$ and $N=3$ cases in II, we determine the dominant asymptotics of E_N by shifting the z_k-contours $\mathbb{R}$ in (51) up past the poles of I_N located at

$$z_k = x_j + ia - ib/2,\quad k=1,\ldots,N-1,\ \ j=1,\ldots,N. \tag{53}$$

Using (29) and (12)–(13), we find that the G-zero $G(ia)=0$ (cf. I (A.12)) ensures that E_N vanishes whenever $x_j = x_k$, $1 \le j < k \le N$. Hence, no generality is lost by assuming

$$x_j \ne x_k,\quad 1 \le j < k \le N, \tag{54}$$

so that the poles (53) are simple.

In order to keep track of the residues that appear, it will be important to shift the $N-1$ contours one at a time. Doing so, we must ensure that we retain sufficient decay of I_N on the contour tails and that we do not meet any of its x-independent poles.

To control the tail decay, we first use the c-definition (12) and the G-asymptotics specified in I (A.14)–(A.16) to infer

$$|\phi(b)^{\mp 1}\exp(\pm\alpha bz/2)c(b;z)-1| \le C_1(\rho,b,\operatorname{Im} z)\exp(-\alpha\rho|\operatorname{Re} z|), \quad \operatorname{Re} z\to\pm\infty, \tag{55}$$

where the decay rate ρ can be chosen in $[a_s/2, a_s)$ and where C_1 is continuous on $[a_s/2, a_s)\times S_a\times\mathbb{R}$.

Next, by the induction assumption, we may invoke Theorem 2.5 with $N\to N-1$. Requiring at first $\operatorname{Im}(z_j-z_k)\in(-a_s,0]$, $1\le j<k\le N-1$, we can use the resulting bound on E_{N-1}, together with (22) and (55), to deduce that the integrand I_N decays exponentially for $|\operatorname{Re} z_k|\to\infty$. Indeed, we have $N-1$ factors of the form $c(z_k\cdots)$ in the numerator and $N-2$ factors of the form $c(z_k\cdots)$ or $c(-z_k\cdots)$ in the denominator, cf. (22) and (13) with $N\to N-1$.

Now from (55) and the u-definition (32) we readily obtain

$$|u(b;z)\phi(b)^{\mp 2}+1| \le C_2(\rho,b,\operatorname{Im} z)\exp(-\alpha\rho|\operatorname{Re} z|), \quad \operatorname{Re} z\to\pm\infty, \tag{56}$$

with C_2 continuous on $[a_s/2, a_s)\times S_a\times\mathbb{R}$. Furthermore, using (52), (50), and (48), we find

$$\mathrm{I}_N(x,y,\tau z)=\mathrm{I}_N(x,y,z), \quad \tau\in S_{N-1}. \tag{57}$$

Combining this with (56), we conclude that I_N has the same decay for $\operatorname{Im}(z_j-z_k)\in[0,a_s)$, $1\le j<k\le N-1$.

The upshot of this analysis is that the shift of a single contour causes no problems at the tail ends, as long as the contours are separated by a distance less than a_s. Moreover, since we require $b\in S_l$, the x-independent poles of I_N are not met for $|\operatorname{Im}(z_j-z_k)|<\beta$, $1\le j<k\le N-1$, cf. Corollary 2.2.

Finally, for a given vector $t\equiv(t_1,\ldots,t_M)\in\mathbb{C}^M$, $M>1$, we use the notation

$$t(\nu_1,\ldots,\nu_L), \quad 1\le\nu_j\ne\nu_k\le M, \ \ 1\le j<k\le L, \tag{58}$$

to denote the vector in $\mathbb{C}^{M-L}$ obtained by omitting the entries $t_{\nu_1}, \ldots, t_{\nu_L}$ in t. Introducing the additional notation

$$z_{>L} \equiv z(1, \ldots, L) = (z_{L+1}, \ldots, z_{N-1}), \quad L = 1, \ldots, N-2, \tag{59}$$

and the functions

$$M_N(b; y) \equiv \frac{\phi(b)^{N-1}}{\prod_{n=1}^{N-1} c(2a - b; y_n - y_N)} \rho_N(b; y), \tag{60}$$

$$\rho_N(b; y) \equiv \exp\left(-\alpha(a - b/2) \sum_{n=1}^{N-1} (y_n - y_N)\right), \tag{61}$$

we are now ready to implement the contour shift procedure.

Lemma 2.3. Letting $(r, b) \in (0, a_s) \times S_l$ and $x, y \in \mathbb{R}^N$ with the x-restriction (54) in effect, we have

$$\begin{aligned}
&\frac{\mathrm{E}_N(x, y)}{M_N(y)} \exp(-i\alpha y_N(x_1 + \cdots + x_N)) \\
&\quad = \frac{1}{\rho_N(y)} \Bigg[\frac{1}{(N-1)!} \left(\frac{G(ib - ia)}{\sqrt{a_+ a_-}} \right)^{N-1} \int_{(C_b + ir)^{N-1}} \mathrm{d}z \, \mathrm{I}_N(x, y, z) \\
&\quad + \sum_{L=1}^{N-2} \frac{1}{(N-1-L)!} \left(\frac{G(ib - ia)}{\sqrt{a_+ a_-}} \right)^{N-1-L} \sum_{1 \le \nu_1 < \cdots < \nu_L \le N} \mathcal{U}_{\nu_1, \ldots, \nu_L}(x) \\
&\quad \times \int_{(C_b + ir)^{N-1-L}} \mathrm{d}z_{>L} \, \hat{\mathrm{I}}_{N; \nu_1, \ldots, \nu_L}(x, y, z_{>L}) \Bigg] \\
&\quad + \sum_{\nu=1}^{N} \frac{C_N(x(\nu), x_\nu)}{C_N(x)} \mathrm{E}_{N-1}(x(\nu), (y_1 - y_N, \ldots, y_{N-1} - y_N)).
\end{aligned} \tag{62}$$

Here, $\mathrm{I}_N(x, y, z)$ is given by (52), we have set

$$\begin{aligned}
\hat{\mathrm{I}}_{N; \nu_1, \ldots, \nu_L}(b; x, y, z_{>L}) &\equiv \mathcal{K}^\sharp_{N-L}(b; x(\nu_1, \ldots, \nu_L), z_{>L}) \\
&\times \mathrm{E}_{N-1}(b; (x_{\nu_1} + ia - ib/2, \ldots, x_{\nu_L} + ia - ib/2, z_{>L}), (y_1 - y_N, \ldots, y_{N-1} - y_N)),
\end{aligned} \tag{63}$$

$$\mathcal{U}_{\nu_1, \ldots, \nu_L}(b; x) \equiv \prod_{\ell=1}^{L} \prod_{\substack{j < \nu_\ell \\ j \neq \nu_1, \ldots, \nu_{\ell-1}}} (-u(b; x_{\nu_\ell} - x_j)), \tag{64}$$

and C_b denotes the contour

$$C_b \equiv \mathbb{R} + i(a - \mathrm{Re}\, b/2). \tag{65}$$

Proof. To start with, we write the left-hand side of (62) as

$$\frac{1}{(N-1)!}\frac{1}{\rho_N(y)}\mathcal{G}^{N-1}\int_{\mathbb{R}^{N-1}} \mathrm{d}z\, \mathcal{K}_N^\sharp(x,z)\mathrm{E}_{N-1}(z,\hat{y}), \tag{66}$$

cf. (51)–(52) and (60). Here we have introduced

$$\hat{y} \equiv (y_1 - y_N, \ldots, y_{N-1} - y_N), \quad \mathcal{G} \equiv \frac{G(ib - ia)}{\sqrt{a_+ a_-}}. \tag{67}$$

We find it convenient to work at first with $J_{N-1}(z,\hat{y})$, since it is S_{N-1}-invariant in z. Therefore, we use (29) with $N \to N-1$ to get (cf. (50) and (14))

$$\frac{1}{(N-1)!}\frac{1}{\rho_N(b;y)}\mathcal{G}^{N-1}\big(\phi(b)\mathcal{G}\big)^{(N-1)(N-2)/2}\frac{1}{C_N(b;x)}\frac{\mathcal{L}_N(x,y)}{C_{N-1}(2a-b;\hat{y})}, \tag{68}$$

with

$$\mathcal{L}_N(b;x,y) \equiv \int_{\mathbb{R}^{N-1}} \mathrm{d}z\, W_{N-1}(b;z)\mathcal{S}_N^\sharp(b;x,z)J_{N-1}(b;z,\hat{y}). \tag{69}$$

Letting

$$0 < \epsilon < \beta/2, \tag{70}$$

(with β defined by (45)), we move the $N-1$ contours $\mathbb{R}$ simultaneously to $C_b - i\epsilon$ without meeting poles. Shifting the z_1-contour to $C_b + i\epsilon$, we pick up residues at the poles (53) with $k = 1$. These poles arise from the factor

$$c(z_1 - x_j - ia + ib/2) = G(z_1 - x_j - ib/2)G(x_j - z_1 - ib/2) \tag{71}$$

in $\mathcal{S}_N^\sharp(x,z)$ (22), and the assumption (54) ensures that they are simple. Recalling the G-residue I (A.13), we have

$$\lim_{z_1 \to x_j + ia - ib/2}(z_1 - x_j - ia + ib/2)G(x_j - z_1 - ib/2) = \lim_{z \to -ia}(-z - ia)G(z) = \frac{\sqrt{a_+ a_-}}{2\pi i}, \tag{72}$$

so that

$$2\pi i \,\mathrm{Res}\, c(z_1 - x_j - ia + ib/2)|_{z_1 = x_j + ia - ib/2} = \frac{\sqrt{a_+ a_-}}{G(ib - ia)} = \mathcal{G}^{-1}. \tag{73}$$

Thus, we infer that $\mathcal{L}_N$ is given by

$$\mathcal{L}_N(x,y) = \int_{C_b+i\epsilon} \mathrm{d}z_1 \int_{(C_b-i\epsilon)^{N-2}} \mathrm{d}z_{>1}\, W_{N-1}(z)\mathcal{S}_N^\sharp(x,z)J_{N-1}(z,\hat{y})$$
$$+\mathcal{G}^{-1}\int_{(C_b-i\epsilon)^{N-2}} \mathrm{d}z_{>1} \sum_{\nu_1=1}^{N} \mathcal{R}_{\nu_1}(x,z_{>1})J_{N-1}((x_{\nu_1}+ia-ib/2,z_{>1}),\hat{y}), \quad (74)$$

with remainder residue

$$\begin{aligned}
\mathcal{R}_{\nu_1}(x,z_{>1}) &= \prod_{\substack{m,n=2\\ m\neq n}}^{N-1} \frac{1}{c(z_m-z_n)} \cdot \prod_{n=2}^{N-1} \frac{1}{c(x_{\nu_1}-z_n+ia-ib/2)c(z_n-x_{\nu_1}-ia+ib/2)} \\
&\quad \times \prod_{j=1}^{N}\prod_{k=2}^{N-1} c(z_k-x_j-ia+ib/2)\cdot \prod_{\substack{j=1\\ j\neq\nu_1}}^{N} c(x_{\nu_1}-x_j) \\
&= W_{N-2}(z_{>1}) \prod_{\substack{j=1\\ j\neq\nu_1}}^{N}\prod_{k=2}^{N-1} c(z_k-x_j-ia+ib/2) \\
&\quad \times \frac{\prod_{j=1, j\neq\nu_1}^{N} c(x_{\nu_1}-x_j)}{\prod_{k=2}^{N-1} c(x_{\nu_1}-z_k+ia-ib/2)} \\
&= W_{N-2}(z_{>1})\mathcal{S}_{N-1}^\sharp(x(\nu_1),z_{>1}) \frac{\prod_{j=1, j\neq\nu_1}^{N} c(x_{\nu_1}-x_j)}{\prod_{k=2}^{N-1} c(x_{\nu_1}-z_k+ia-ib/2)}. \quad (75)
\end{aligned}$$

We note that the ϵ-choice (70) guarantees that the factors $1/c(x_{\nu_1}-z_k+ia-ib/2)$ are analytic in z_k for $|\mathrm{Im}\, z_k-(a-\mathrm{Re}\, b/2)|\leq\epsilon$. Hence, moving the z_2-contours in (74) up by 2ϵ, we only encounter the poles (53) with $k=2$. In the residues spawned by the 1st integral we replace z_1 by z_2 and use the S_{N-1}-invariance of $J_{N-1}(z,\hat{y})$ in z to obtain

$$\int_{(C_b+i\epsilon)^2} \mathrm{d}z_1\, \mathrm{d}z_2 \int_{(C_b-i\epsilon)^{N-3}} \mathrm{d}z_{>2}\, W_{N-1}(z)\mathcal{S}_N^\sharp(x,z)J_{N-1}(z,\hat{y})$$
$$+\mathcal{G}^{-1}\int_{C_b+i\epsilon} \mathrm{d}z_2 \int_{(C_b-i\epsilon)^{N-3}} \mathrm{d}z_{>2} \sum_{\nu_1=1}^{N} \mathcal{R}_{\nu_1}(x,z_{>1})J_{N-1}((x_{\nu_1}+ia-ib/2,z_{>1}),\hat{y}). \quad (76)$$

From the 2nd integral in (74), we get a copy of the second integral in (76) plus a residue term

$$\mathcal{G}^{-2}\int_{(C_b-i\epsilon)^{N-3}} \mathrm{d}z_{>2} \sum_{\substack{\nu_1,\nu_2=1\\ \nu_1\neq\nu_2}}^{N} \mathcal{R}_{\nu_1,\nu_2}(x,z_{>2})J_{N-1}((x_{\nu_1}+ia-ib/2, x_{\nu_2}+ia-ib/2, z_{>2}),\hat{y}), \quad (77)$$

which is readily determined by adapting the computations in (75):

$$\mathcal{R}_{\nu_1,\nu_2}(x,z_{>2}) = W_{N-3}(z_{>2})\mathcal{S}^{\sharp}_{N-2}(x(\nu_1,\nu_2),z_{>2})\prod_{\ell=1}^{2}\frac{\prod_{\substack{j=1\\ j\neq\nu_1,\nu_2}}^{N} c(x_{\nu_\ell}-x_j)}{\prod_{k=3}^{N-1} c(x_{\nu_\ell}-z_k+ia-ib/2)}. \quad (78)$$

The upshot is that $\mathcal{L}_N(x,y)$ can be written

$$\begin{aligned}\mathcal{L}_N(x,y) &= \int_{(C_b+i\epsilon)^2} \mathrm{d}z_1\,\mathrm{d}z_2 \int_{(C_b-i\epsilon)^{N-3}} \mathrm{d}z_{>2}\, W_{N-1}(z)\mathcal{S}^{\sharp}_N(x,z)J_{N-1}(z,\hat{y})\\ &\quad + 2\mathcal{G}^{-1}\int_{C_b+i\epsilon}\mathrm{d}z_2\int_{(C_b-i\epsilon)^{N-3}}\mathrm{d}z_{>2}\sum_{\nu_1=1}^{N}\mathcal{R}_{\nu_1}(x,z_{>1})J_{N-1}((x_{\nu_1}+ia-ib/2,z_{>1}),\hat{y})\\ &\quad + \mathcal{G}^{-2}\int_{(C_b-i\epsilon)^{N-3}}\mathrm{d}z_{>2}\sum_{\substack{\nu_1,\nu_2=1\\ \nu_1\neq\nu_2}}^{N}\mathcal{R}_{\nu_1,\nu_2}(x,z_{>2})J_{N-1}((x_{\nu_1}+ia-ib/2,x_{\nu_2}+ia-ib/2,z_{>2}),\hat{y}),\end{aligned} \quad (79)$$

with $\mathcal{R}_{\nu_1}$ and $\mathcal{R}_{\nu_1,\nu_2}$ given by (75) and (78), respectively.

More generally, introducing the integration domains

$$V_L^M \equiv (C_b+i\epsilon)^{M-L}\times(C_b-i\epsilon)^{N-1-M}, \quad 1\leq M\leq N-1, \quad 0\leq L\leq M, \quad (80)$$

we claim that $\mathcal{L}_N(x,y)$ can be written

$$\begin{aligned}\mathcal{L}_N(x,y) &= \int_{V_0^M}\mathrm{d}z\, W_{N-1}(z)\mathcal{S}^{\sharp}_N(x,z)J_{N-1}(z,\hat{y})\\ &\quad + \sum_{L=1}^{M}\mathcal{G}^{-L}\binom{M}{L}\int_{V_L^M}\mathrm{d}z_{>L}\sum_{\substack{\nu_1,\ldots,\nu_L=1\\ \nu_j\neq\nu_k}}^{N}\mathcal{R}_{\nu_1,\ldots,\nu_L}(x,z_{>L})\\ &\quad \times J_{N-1}((x_{\nu_1}+ia-ib/2,\ldots,x_{\nu_L}+ia-ib/2,z_{>L}),\hat{y}),\end{aligned} \quad (81)$$

for any $M = 1, \dots, N-1$. Here we have introduced

$$\mathcal{R}_{\nu_1,\dots,\nu_L}(x, z_{>L}) \equiv W_{N-1-L}(z_{>L}) \mathcal{S}^{\sharp}_{N-L}(x(\nu_1, \dots, \nu_L), z_{>L}) \times \prod_{\ell=1}^{L} \frac{\prod_{j=1, j\neq \nu_1,\dots,\nu_L}^{N} c(x_{\nu_\ell} - x_j)}{\prod_{k=L+1}^{N-1} c(x_{\nu_\ell} - z_k + ia - ib/2)}, \quad L = 1, \dots, N-2, \quad L \leq M, \tag{82}$$

whereas for $L = M = N-1$ the integral should be omitted and we have

$$\mathcal{R}_{\nu_1,\dots,\nu_{N-1}}(x) \equiv \prod_{\ell=1}^{N-1} c(x_{\nu_\ell} - x_{\nu_N}), \quad \{\nu_1, \dots, \nu_N\} = \{1, \dots, N\}. \tag{83}$$

By (74)–(75) and (78)–(79), we know already that the claim holds true for $M = 1, 2$. Assuming (81) for $1 \leq M \leq N-2$, we now prove its validity for $M \to M+1$.

To this end, we move the z_{M+1}-contours up by 2ϵ, meeting the simple poles

$$z_{M+1} = x_{\nu_1} + ia - ib/2, \quad \nu_1 = 1, \dots, N, \tag{84}$$

in the 1st integral, and the simple poles

$$z_{M+1} = x_{\nu_{L+1}} + ia - ib/2, \quad \nu_{L+1} = 1, \dots, N, \quad \nu_{L+1} \neq \nu_1, \dots, \nu_L, \tag{85}$$

in the remaining integrals. Using S_{N-1}-invariance of $J_{N-1}(z, \hat{y})$ in z, it is readily seen that the 1st integral yields, upon taking $z(M+1) \to z_{>1}$ in the residue integral,

$$\int_{V_0^{M+1}} \mathrm{d}z\, W_{N-1}(z) \mathcal{S}^{\sharp}_N(x, z) J_{N-1}(z, \hat{y}) + \mathcal{G}^{-1} \int_{V_1^{M+1}} \mathrm{d}z_{>1} \sum_{\nu_1=1}^{N} \mathcal{R}_{\nu_1}(x, z_{>1}) J_{N-1}((x_{\nu_1} + ia - ib/2, z_{>1}), \hat{y}). \tag{86}$$

Similarly, the L-summand with $L = 1, \ldots, M$ yields, after taking $z_{>L}(M+1) \to z_{>L+1}$ in the residue integral,

$$\mathcal{G}^{-L}\binom{M}{L}\int_{V_L^{M+1}} \mathrm{d}z_{>L} \sum_{\substack{\nu_1,\ldots,\nu_L=1\\ \nu_j\neq\nu_k}}^{N} \mathcal{R}_{\nu_1,\ldots,\nu_L}(x, z_{>L})$$
$$\times J_{N-1}((x_{\nu_1}+ia-ib/2,\ldots,x_{\nu_L}+ia-ib/2, z_{>L}),\hat{y})$$
$$+\mathcal{G}^{-L-1}\binom{M}{L}\int_{V_{L+1}^{M+1}} \mathrm{d}z_{>L+1} \sum_{\substack{\nu_1,\ldots,\nu_{L+1}=1\\ \nu_j\neq\nu_k}}^{N} \mathcal{R}_{\nu_1,\ldots,\nu_{L+1}}(x, z_{>L+1})$$
$$\times J_{N-1}((x_{\nu_1}+ia-ib/2,\ldots,x_{\nu_{L+1}}+ia-ib/2, z_{>L+1}),\hat{y}). \quad (87)$$

Summing the terms (87) over $L = 1, \ldots, M$ and adding the resulting expression to (86), we arrive at the right-hand side of (81) with $M \to M+1$ by invoking Pascal's rule

$$\binom{M}{L} + \binom{M}{L-1} = \binom{M+1}{L}. \quad (88)$$

Hence, our claim is proved.

Next, we specialize (81) to $M = N-1$ and shift all contours up to $C_b + ir$ without encountering further poles. Using symmetry under permutations of $x_{\nu_1}, \ldots, x_{\nu_L}$, we thus obtain

$$\mathcal{L}_N(x,y) = \int_{(C_b+ir)^{N-1}} \mathrm{d}z\, W_{N-1}(z)\mathcal{S}_N^{\sharp}(x,z)J_{N-1}(z,\hat{y})$$
$$+(N-1)!\sum_{L=1}^{N-2}\mathcal{G}^{-L}\frac{1}{(N-1-L)!}\int_{(C_b+ir)^{N-1-L}} \mathrm{d}z_{>L} \sum_{1\le\nu_1<\cdots<\nu_L\le N} \mathcal{R}_{\nu_1,\ldots,\nu_L}(x, z_{>L})$$
$$\times J_{N-1}((x_{\nu_1}+ia-ib/2,\ldots,x_{\nu_L}+ia-ib/2, z_{>L}),\hat{y})$$
$$+(N-1)!\,\mathcal{G}^{1-N}\sum_{1\le\nu_1<\cdots<\nu_{N-1}\le N} R_{\nu_1,\ldots,\nu_{N-1}}(x)$$
$$\times J_{N-1}((x_{\nu_1}+ia-ib/2,\ldots,x_{\nu_{N-1}}+ia-ib/2),\hat{y}). \quad (89)$$

In order to establish the representation (62), we now reformulate (89) in terms of E_{N-1}. From (29) and (13), we infer

$$\begin{aligned}
&J_{N-1}((x_{\nu_1}+ia-ib/2,\dots,x_{\nu_L}+ia-ib/2,z_{>L}),\hat{y})\\
&\quad=(\phi(b)\mathcal{G})^{-(N-1)(N-2)/2}\mathrm{E}_{N-1}((x_{\nu_1}+ia-ib/2,\dots,x_{\nu_L}+ia-ib/2,z_{>L}),\hat{y})\\
&\qquad\times C_{N-1}(2a-b;\hat{y})C_L(x_{\nu_1},\dots,x_{\nu_L})C_{N-1-L}(z_{>L})\\
&\qquad\times\prod_{\ell=1}^{L}\prod_{k=L+1}^{N-1}c(x_{\nu_\ell}-z_k+ia-ib/2).
\end{aligned}\tag{90}$$

Combining (82) with (14) and (50), we deduce

$$\begin{aligned}
&\mathcal{R}_{\nu_1,\dots,\nu_L}(x,z_{>L})\\
&\quad=\mathcal{K}^{\sharp}_{N-L}(x(\nu_1,\dots,\nu_L),z_{>L})\frac{C_{N-L}(x(\nu_1,\dots,\nu_L))}{C_{N-1-L}(z_{>L})}\prod_{\ell=1}^{L}\frac{\prod_{j=1,\ j\neq\nu_1,\dots,\nu_L}^{N}c(x_{\nu_\ell}-x_j)}{\prod_{k=L+1}^{N-1}c(x_{\nu_\ell}-z_k+ia-ib/2)}.
\end{aligned}\tag{91}$$

It follows that

$$\begin{aligned}
&\mathcal{R}_{\nu_1,\dots,\nu_L}(x,z_{>L})J_{N-1}((x_{\nu_1}+ia-ib/2,\dots,x_{\nu_L}+ia-ib/2,z_{>L}),\hat{y})/C_{N-1}(2a-b;\hat{y})\\
&\quad=(\phi(b)\mathcal{G})^{-(N-1)(N-2)/2}\mathrm{E}_{N-1}((x_{\nu_1}+ia-ib/2,\dots,x_{\nu_L}+ia-ib/2,z_{>L}),\hat{y})\\
&\qquad\times\mathcal{K}^{\sharp}_{N-L}(x(\nu_1,\dots,\nu_L),z_{>L})C_L(x_{\nu_1},\dots,x_{\nu_L})C_{N-L}(x(\nu_1,\dots,\nu_L))\\
&\qquad\times\prod_{\ell=1}^{L}\prod_{\substack{j=1\\ j\neq\nu_1,\dots,\nu_L}}^{N}c(x_{\nu_\ell}-x_j).
\end{aligned}\tag{92}$$

Since $\nu_1<\cdots<\nu_L$ in (89), we can write

$$\begin{aligned}
C_N(x)&=C_L(x_{\nu_1},\dots,x_{\nu_L})C_{N-L}(x(\nu_1,\dots,\nu_L))\\
&\quad\times\prod_{\ell=1}^{L}\left(\prod_{\substack{j<\nu_\ell\\ j\neq\nu_1,\dots,\nu_{\ell-1}}}c(x_j-x_{\nu_\ell})\prod_{\substack{j>\nu_\ell\\ j\neq\nu_{\ell+1},\dots,\nu_L}}c(x_{\nu_\ell}-x_j)\right).
\end{aligned}\tag{93}$$

Multiplying (89) by the prefactors in (68) and using (92)–(93), (32), and (47), we arrive at the right-hand side of (62). ■

We proceed to analyze the asymptotic behavior of $\mathrm{E}_N(x, y)$ for $m_N(y) \to \infty$ using the representation (62). To this end, we need several bounds on the c- and u-functions, which we derive from the asymptotic estimates (55) and (56).

First, combining (55) with holomorphy of $c(b; z)$ for $(b, \mathrm{Im}\, z) \in S_a \times (0, a_s)$, we obtain a majorization

$$|c(b; p + ir)| \leq c(r, b) \exp(-\gamma |p|), \quad (r, b, p) \in (0, a_s) \times S_a \times \mathbb{R}, \tag{94}$$

where we have set

$$\gamma \equiv \alpha \mathrm{Re}\, b/2 = \frac{\pi \mathrm{Re}\, b}{a_+ a_-} \tag{95}$$

and where $c(r, b)$ is continuous on $(0, a_s) \times S_a$. Likewise, recalling $G(ia) = 0$, we get

$$|1/c(b; z)| \leq C(b) |\sinh(\gamma z)|, \quad (b, z) \in S_a \times \mathbb{R}, \tag{96}$$

with $C(b)$ continuous on S_a. Finally, letting $b \in S_a$, we note that $1/c(b; z)$ is holomorphic for $\mathrm{Im}\, z \in (-2a, \mathrm{Re}\, b)$. Combining this with (55), we conclude

$$|1/c(b; z)| \leq c(b) \exp(\gamma |\mathrm{Re}\, z|), \quad (b, \mathrm{Im}\, z) \in S_a \times [-a_s, 0], \tag{97}$$

with $c(b)$ continuous on S_a.

Turning to the u-function (32), we let $b \in S_a$. Then $u(b; z)$ is holomorphic in the strip $\mathrm{Im}\, z \in (-\min(\mathrm{Re}\, b, 2a - \mathrm{Re}\, b), a_s)$. Combining this with (56), we readily infer

$$|u(b; -z)| \leq c(b, \mathrm{Im}\, z), \quad (b, \mathrm{Im}\, z) \in S_a \times (-a_s, 0], \tag{98}$$

where $c(b, \mathrm{Im}\, z)$ is continuous on $S_a \times (-a_s, 0]$.

With these preliminaries out of the way, we return to the function $\mathrm{E}_N(x, y)$. Recalling the symmetry relation $\phi(2a - b) = \phi(b)$ (cf. (30)) and combining this with (55) and (96), we find

$$|M_N(b; y) - 1| \leq c(b, \rho) \exp(-\alpha \rho m_N(y)), \quad (b, y, \rho) \in S_a \times \mathbb{R}^N \times [a_s/2, a_s), \quad m_N(y) \geq 0, \tag{99}$$

where $c(b, \rho)$ is continuous on $S_a \times [a_s/2, a_s)$. Moreover, by the induction assumption, we may invoke Theorem 2.5 after substituting $N \to N - 1$. Combining the resulting bound on E_{N-1} with the c-function estimates just assembled, it is readily verified that both $\rho_N(y)^{-1} \mathrm{I}_N(x, y, z)$ and $\rho_N(y)^{-1} \hat{\mathrm{I}}_{N; \nu_1, \ldots, \nu_L}(x, y, z_{>L})$, $L = 1, \ldots, N - 2$, decay exponentially as

$m_N(y) \to \infty$. This suggests that the dominant asymptotics of $\mathrm{E}_N(x,y)$ arises from the last sum in (62).

To show that this is indeed the case, we first observe that the function $\mathrm{E}^{\mathrm{as}}_{N-1}(z,w)$ (31) can be rewritten

$$\mathrm{E}^{\mathrm{as}}_{N-1}(z,w) = \sum_{\tau\in S_{N-1}} \frac{C_{N-1}(z_\tau)}{C_{N-1}(z)} \exp(i\alpha z_\tau \cdot w). \tag{100}$$

Next, taking $N \to N-1$ in Theorem 2.4, we deduce from the induction assumption and (100) that we have

$$\exp(i\alpha y_N(x_1+\cdots+x_N))\mathrm{E}_{N-1}(x(\nu),(y_1-y_N,\ldots,y_{N-1}-y_N))$$
$$= \sum_{\substack{\sigma\in S_N\\ \sigma(N)=\nu}} \frac{C_{N-1}(x_{\sigma(1)},\ldots,x_{\sigma(N-1)})}{C_{N-1}(x(\nu))} \exp(i\alpha x_\sigma \cdot y) + R_\nu(x,y), \tag{101}$$

where the remainder satisfies a bound

$$|R_\nu(b;x,y)| \le C(r,b)P_{N-1}(\gamma|x(\nu)_1|,\ldots,\gamma|x(\nu)_{N-1}|)\exp(-\alpha r d_{N-1}(y_1,\ldots,y_{N-1})), \tag{102}$$

which holds for all $(b,x,y) \in S_l \times \mathbb{R}^N \times \mathbb{R}^N$ with $d_{N-1}(y_1,\ldots,y_{N-1}) \ge 0$. Here $C(r,b)$ is continuous on $[a_s/2, a_s) \times S_l$ and P_{N-1} is a polynomial of degree $\le (N-1)(N-2)/2$ with positive and constant coefficients. Now, for any $\sigma \in S_N$ such that $\sigma(N) = \nu$, we have an identity

$$\begin{aligned}\frac{C_N(x(\nu),x_\nu)C_{N-1}(x_{\sigma(1)},\ldots,x_{\sigma(N-1)})}{C_{N-1}(x(\nu))} &= \prod_{\substack{j=1\\ j\ne\nu}}^{N} c(x_j - x_{\sigma(N)}) \cdot \prod_{1\le j<k\le N-1} c(x_{\sigma(j)} - x_{\sigma(k)})\\ &= C_N(x_\sigma).\end{aligned} \tag{103}$$

Thus, we obtain, using (100) with $N-1 \to N$,

$$\exp(i\alpha y_N(x_1+\cdots+x_N))\sum_{\nu=1}^{N} \frac{C_N(x(\nu),x_\nu)}{C_N(x)}\mathrm{E}_{N-1}(x(\nu),(y_1-y_N,\ldots,y_{N-1}-y_N))$$
$$= \sum_{\sigma\in S_N} \frac{C_N(x_\sigma)}{C_N(x)} \exp(i\alpha x_\sigma \cdot y) + R(x,y) = E^{\mathrm{as}}_N(x,y) + R(x,y), \tag{104}$$

with remainder

$$R(x,y) \equiv \sum_{\nu=1}^{N} \frac{C_N(x(\nu),x_\nu)}{C_N(x)} R_\nu(x,y). \tag{105}$$

We note that an exponential decay bound for R is readily inferred from the bound (102) for R_ν. Indeed, after multiplying $|R_\nu|$ by $|C_N(x(\nu),x_\nu)/C_N(x)|$ and summing over $\nu = 1,\ldots,N$, we need only invoke the u-bound (98).

In the following theorem, our starting point is (62), rewritten as

$$\begin{aligned}
(\mathrm{E}_N - \mathrm{E}_N^{\mathrm{as}})(x,y) &= (M_N(y)-1)\mathrm{E}_N^{\mathrm{as}}(x,y) + M_N(y)R(x,y) \\
&+ \exp(i\alpha y_N(x_1+\cdots+x_N))\frac{M_N(y)}{\rho_N(y)}\Bigg[\frac{1}{(N-1)!}\left(\frac{G(ib-ia)}{\sqrt{a_+a_-}}\right)^{N-1}\int_{(C_b+ir)^{N-1}} \mathrm{d}z\,\mathrm{I}_N(x,y,z) \\
&+ \sum_{L=1}^{N-2}\frac{1}{(N-1-L)!}\left(\frac{G(ib-ia)}{\sqrt{a_+a_-}}\right)^{N-1-L}\sum_{1\le\nu_1<\cdots<\nu_L\le N}\mathcal{U}_{\nu_1,\ldots,\nu_L}(x) \\
&\times\int_{(C_b+ir)^{N-1-L}}\mathrm{d}z_{>L}\,\hat{\mathrm{I}}_{N;\nu_1,\ldots,\nu_L}(x,y,z_{>L})\Bigg],
\end{aligned} \tag{106}$$

where we have used (104). In view of our considerations above, we need only majorize the expression in square brackets on the right-hand side to infer exponential decay of the left-hand side with rate αr as $m_N(y)\to\infty$. As an immediate corollary, we obtain the "unitary asymptotics" (31) of E_N.

Theorem 2.4. Letting $(r,b)\in[a_s/2,a_s)\times S_l$, we have

$$|(\mathrm{E}_N-\mathrm{E}_N^{\mathrm{as}})(b;x,y)| < C(r,b)P_N(\gamma|x_1|,\ldots,\gamma|x_N|)\exp(-\alpha r m_N(y)), \tag{107}$$

for all $x,y\in\mathbb{R}^N$ with $m_N(y)>0$, where C is continuous on $[a_s/2,a_s)\times S_l$ and P_N is a polynomial of degree $\le N(N-1)/2$ with positive and constant coefficients.

Proof. In view of (98) (with $\mathrm{Im}\, z = 0$), it suffices to establish the bounds

$$\left| \int_{(C_b+ir)^{N-1}} dz\, \mathrm{I}_N(x, y, z) \right| \leq C_0(r, b) |\rho_N(y)| P_{N,0}(\gamma |x_1|, \ldots, \gamma |x_N|) \exp(-\alpha r m_N(y)), \tag{108}$$

$$\left| \int_{(C_b+ir)^{N-1-L}} dz_{>L}\, \hat{\mathrm{I}}_{N;\nu_1,\ldots,\nu_L}(x, y, z_{>L}) \right| \\ \leq C_L(r, b) |\rho_N(y)| P_{N,L}(\gamma |x_1|, \ldots, \gamma |x_N|) \exp(-\alpha r m_N(y)), \quad L = 1, \ldots, N-2, \tag{109}$$

for all $x, y \in \mathbb{R}^N$ with $m_N(y) > 0$. Here the functions C_0, C_L are continuous on $[a_s/2, a_s) \times S_l$ and $P_{N,0}$, $P_{N,L}$ are polynomials of degree $\leq N(N-1)/2 - L$ with positive and constant coefficients.

Taking $z_k \to z_k + i(a - b/2 + r)$, we infer from the identity (47) with $N \to N-1$ that

$$\int_{(C_b+ir)^{N-1}} dz\, \mathrm{I}_N(x, y, z) = \rho_N(y) \exp\left(-\alpha r \sum_{m=1}^{N-1} (y_m - y_N) \right) C_N(x)^{-1} \\ \times \int_{\mathbb{R}^{N-1}} dz\, \frac{\mathrm{E}_{N-1}(z, (y_1 - y_N, \ldots, y_{N-1} - y_N))}{C_{N-1}(-z)} \prod_{j=1}^{N} \prod_{k=1}^{N-1} c(z_k + ir - x_j). \tag{110}$$

Now by the induction assumption, Theorem 2.5 holds true when N is replaced by $N-1$. Combining the resulting bound on E_{N-1} with (94) and (96), we deduce

$$\left| \int_{(C_b+ir)^{N-1}} dz\, \mathrm{I}_N(x, y, z) \right| \leq C_0(r, b) |\rho_N(y)| \exp\left(-\alpha r \sum_{m=1}^{N-1} (y_m - y_N) \right) \\ \times \int_{\mathbb{R}^{N-1}} dz\, P_{N-1}(\gamma |z_1|, \ldots, \gamma |z_{N-1}|) \exp(F_{N-1}(\gamma x, \gamma z)), \tag{111}$$

where F_{N-1} is given by (A2) and P_{N-1} is a polynomial of degree $\leq (N-1)(N-2)/2$ with positive and constant coefficients. The bound (108) is now a direct consequence of Theorem A.1.

We proceed to prove (109). Taking $z_k \to z_k + i(a - b/2 + r)$, $L < k \leq N-1$, and using once more (47), we obtain

$$\int_{(C_b+ir)^{N-1-L}} dz_{>L}\, \hat{\mathrm{I}}_{N;\nu_1,\ldots,\nu_L}(x, y, z_{>L}) = \rho_N(y) C_{N-L}(x(\nu_1, \ldots, \nu_L))^{-1}$$
$$\times \int_{\mathbb{R}^{N-1-L}} dz_{>L}\, \mathrm{E}_{N-1}((x_{\nu_1}, \ldots, x_{\nu_L}, z_{L+1} + ir, \ldots, z_{N-1} + ir), (y_1 - y_N, \ldots, y_{N-1} - y_N))$$
$$\times \frac{1}{C_{N-1-L}(-z_{>L})} \prod_{\substack{j=1 \\ j \neq \nu_1,\ldots,\nu_L}}^{N} \prod_{k=L+1}^{N-1} c(z_k + ir - x_j). \quad (112)$$

By Theorem 2.5 with $N \to N-1$ and (94)–(96), it follows that

$$\left| \int_{(C_b+ir)^{N-1-L}} dz_{>L}\, \hat{\mathrm{I}}_{N;\nu_1,\ldots,\nu_L}(x, y, z_{>L}) \right| \leq C_L(r, b) |\rho_N(y)| \exp\left(-\alpha r \sum_{m=L+1}^{N-1} (y_m - y_N) \right)$$
$$\times \int_{\mathbb{R}^{N-1-L}} dz_{>L}\, P_{N-1}(\gamma |x_{\nu_1}|, \ldots, \gamma |x_{\nu_L}|, \gamma |z_{L+1}|, \ldots, \gamma |z_{N-1}|)$$
$$\times \exp\left(F_{N-1-L}(\gamma x(\nu_1, \ldots, \nu_L), \gamma z_{>L}) \right). \quad (113)$$

Since P_{N-1} is a polynomial of degree $\leq (N-1)(N-2)/2$ with positive, constant coefficients, we have

$$P_{N-1}(\gamma |x_{\nu_1}|, \ldots, \gamma |x_{\nu_L}|, \gamma |z_{L+1}|, \ldots, \gamma |z_{N-1}|)$$
$$= \sum_{\substack{k \in \mathbb{N}^L \\ |k| \leq (N-1)(N-2)/2}} \gamma^{|k|} |x_{\nu_1}|^{k_1} \cdots |x_{\nu_L}|^{k_L} P^k_{N-1,L}(\gamma |z_{L+1}|, \ldots, \gamma |z_{N-1}|), \quad (114)$$

for some polynomials $P^k_{N-1,L}$ of degree $\leq (N-1)(N-2)/2 - |k|$ with positive, constant coefficients, where $|k| \equiv k_1 + \cdots + k_L$. Substituting this expansion in (113), we can use Theorem A.1 to bound each term separately. Indeed, from (A1)–(A3), we get

$$\int_{\mathbb{R}^{N-1-L}} dz_{>L}\, P^k_{N-1,L}(\gamma |z_{L+1}|, \ldots, \gamma |z_{N-1}|) \exp\left(F_{N-1-L}(\gamma x(\nu_1, \ldots, \nu_L), \gamma z_{>L}) \right)$$
$$< P^k_{N,L}((\gamma |x_j|)_{j \neq \nu_1,\ldots,\nu_L}), \quad (115)$$

for some polynomials $P^k_{N,L}$ of degree

$$\deg P^k_{N,L} \le (N-1)(N-2)/2 - |k| + N - 1 - L = N(N-1)/2 - |k| - L, \tag{116}$$

with positive, constant coefficients. The bounds (113) and (115) clearly imply the desired majorization (109). ■

We proceed to obtain a bound on $\mathrm{E}_N(x,y)$ for $x,y \in \mathbb{C}^N \times \mathbb{R}^N$ satisfying

$$v_j - v_k \in (-a_s, 0], \quad 1 \le j < k \le N, \quad m_N(y) > 0, \quad v = \operatorname{Im} x. \tag{117}$$

Like in the $N = 2$ and $N = 3$ cases treated in II, we take as a starting point the representation for E_N given by (62).

We first derive the desired bound for the last sum in (62). To begin with, from (99) we easily get

$$|M_N(b;y)\exp(i\alpha y_N(x_1+\cdots+x_N))| < c(b)\exp\left(-\alpha\sum_{j=1}^N y_j v_j\right)\exp\left(\alpha\sum_{k=1}^{N-1}(y_k-y_N)v_k\right), \tag{118}$$

for all $(b,x,y) \in S_a \times \mathbb{C}^N \times \mathbb{R}^N$, with $c(b)$ continuous on S_a. Using next Theorem 2.5 with $N \to N-1$, we get an estimate

$$\begin{aligned} |\mathrm{E}_{N-1}(x(\nu),(y_1-y_N,\ldots,y_{N-1}-y_N))| \quad &< \quad C(\delta,b)P_{N-1}(\gamma|\operatorname{Re} x(\nu)_1|,\ldots,\gamma|\operatorname{Re} x(\nu)_{N-1}|) \\ &\quad \times \exp\left(-\alpha\sum_{k=1}^{N-1}(y_k-y_N)\operatorname{Im} x(\nu)_k\right), \end{aligned} \tag{119}$$

where P_{N-1} is a polynomial of degree $\le (N-1)(N-2)/2$ with positive and constant coefficients. Now when we take the product Π_ν of the functions on the left-hand sides of (118) and (119), we can use the majorization

$$\begin{aligned} &\exp\left(\alpha\sum_{k=1}^{N-1}(y_k-y_N)v_k\right)\exp\left(-\alpha\sum_{k=1}^{N-1}(y_k-y_N)\operatorname{Im} x(\nu)_k\right) \\ &= \exp\left(\alpha\sum_{k=\nu}^{N-1}(y_k-y_N)(v_k-v_{k+1})\right) \le 1, \quad m_N(y) > 0,\ v_k - v_{k+1} \le 0,\ k = 1,\ldots,N-1, \end{aligned} \tag{120}$$

to conclude that the product of Π_ν and the pertinent u-function product satisfies a bound of the type occurring in (121), cf. (62) and (98). (Indeed, from (32) and the G-pole locations I (A.11), we infer regularity of $u(b; x_k - x_j)$ for $-a_s < v_j - v_k < \min(\mathrm{Re}\, b, 2a - \mathrm{Re}\, b)$.)

Theorem 2.5. Letting $(\delta, b) \in (0, a_s] \times S_l$, we have

$$|\mathrm{E}_N(b; x, y)| < C(\delta, b) P_N(\gamma |\mathrm{Re}\, x_1|, \ldots, \gamma |\mathrm{Re}\, x_N|) \exp\left(-\alpha \sum_{j=1}^{N} y_j v_j\right), \tag{121}$$

for all $(x, y) \in \mathbb{C}^N \times \mathbb{R}^N$ satisfying

$$v_j - v_k \in [-a_s + \delta, 0], \quad 1 \le j < k \le N, \quad m_N(y) > 0, \quad v = \mathrm{Im}\, x, \tag{122}$$

where $C(\delta, b)$ is a continuous function on $(0, a_s] \times S_l$ and P_N is a polynomial of degree $\le N(N-1)/2$ with positive and constant coefficients.

Proof. Since we have already shown that the last sum in (62) satisfies a bound of this type, the assertion will follow once we prove that the integrals on the right-hand side of (62) are bounded by

$$C(\delta, b) |\rho_N(b; y)| P_N(\gamma |\mathrm{Re}\, x_1|, \ldots, \gamma |\mathrm{Re}\, x_N|) \exp\left(-\alpha \sum_{k=1}^{N-1} (y_k - y_N) v_k\right), \tag{123}$$

for all $(x, y) \in \mathbb{C}^N \times \mathbb{R}^N$ satisfying (122). Indeed, by the induction assumption, (121) holds true with N replaced by $N-1$, and when combined with the c-bound (97), it becomes clear that we can find a polynomial P_N of the required form such that the remaining sum is majorized by (123) without the factor $|\rho_N(b; y)|$.

Due to the identity (47), we may and shall restrict attention to

$$0 \le v_1 \le \cdots \le v_N \le a_s - \delta. \tag{124}$$

Requiring at first $x \in \mathbb{R}^N$, we repeat the steps leading to the $(N-1)$-fold integral (110). Allowing next $v_j \neq 0$, we require

$$\delta' \le r - v_j \le a_s - \delta', \quad \delta' \in (0, a_s/2], \quad j = 1, \ldots, N, \tag{125}$$

so that we stay clear of the poles of the c-functions for $z_k + ir - x_j = 0, a_s$. Choosing

$$r = a_s - \delta/2, \quad \delta' = \delta/2, \tag{126}$$

we can allow any $x \in \mathbb{C}^N$ satisfying (124). Invoking (121) with $N \to N-1$ and the bounds (94)–(96), we thus infer

$$\begin{aligned} \left| \int_{(C_b+ir)^{N-1}} dz\, \mathrm{I}_N(x,y,z) \right| &\leq c_2(\delta,b)|\rho_N(y)| \exp\left(-\alpha r \sum_{k=1}^{N-1} (y_k - y_N) \right) \\ &\times \int_{\mathbb{R}^{N-1}} dz\, P_{N-1}(\gamma|z_1|,\ldots,\gamma|z_{N-1}|) \exp\left(F_{N-1}((\gamma \mathrm{Re}\, x_1,\ldots,\gamma \mathrm{Re}\, x_N),\gamma z)\right), \end{aligned} \tag{127}$$

where c_2 is continuous on $(0,a_s]\times S_l$. Using Theorem A.1 to bound the remaining integral, we arrive at the desired majorization.

We turn now to the $(N-1-L)$-fold integral (112). Assuming (125)–(126), we can again allow any $x \in \mathbb{C}^N$ satisfying (124). Indeed, we stay clear of the pertinent poles of the c-functions and can use (121) with $N \to N-1$ and $\delta \to \delta/2$ to bound the E_{N-1}-factor. Using also the bounds (94) and (97), we obtain

$$\begin{aligned} \left| \int_{(C_b+ir)^{N-1-L}} dz_{>L}\, \hat{\mathrm{I}}_{N;\nu_1,\ldots,\nu_L}(x,y,z_{>L}) \right| &< c_3(\delta,b)|\rho_N(y)| \\ &\times \exp\left(-\alpha \sum_{j=1}^{L} (y_j - y_N) v_{\nu_j} - \alpha r \sum_{k=L+1}^{N-1} (y_k - y_N) \right) \\ &\times \int_{\mathbb{R}^{N-1-L}} dz_{>L}\, P_{N-1}(\gamma|\mathrm{Re}\, x_{\nu_1}|,\ldots,\gamma|\mathrm{Re}\, x_{\nu_L}|,\gamma|z_{L+1}|,\ldots,\gamma|z_{N-1}|) \\ &\times \exp\left(F_{N-1-L}(\gamma \mathrm{Re}\, x(\nu_1,\ldots,\nu_L),\gamma z_{>L})\right), \end{aligned} \tag{128}$$

with c_3 continuous on $(0,a_s] \times S_l$. Now we have

$$v_{\nu_j} \geq v_j,\ j=1,\ldots,L, \quad r > v_j,\ j=1,\ldots,N, \quad m_N(y) > 0, \tag{129}$$

whence we infer

$$\exp\left(-\alpha \sum_{j=1}^{L} (y_j - y_N) v_{\nu_j} - \alpha r \sum_{k=L+1}^{N-1} (y_k - y_N) \right) < \exp\left(-\alpha \sum_{k=1}^{N-1} (y_k - y_N) v_k \right). \tag{130}$$

Also, substituting the expansion (114) with $x_{\nu_j} \to \operatorname{Re} x_{\nu_j}$ in (128), each term is readily bounded using Theorem A.1. Hence, the majorization (123) results. ■

A Polynomial Bounds

In Section 2, we use the following theorem to bound remainder terms when studying the asymptotic behavior of the functions E_N, cf. Theorems 2.4– 2.5.

Theorem A.1. Let $z_1, \ldots, z_L, u_1, \ldots, u_{L+1} \in \mathbb{R}$, and let $\mathcal{P}_{L,M}(|z_1|, \ldots, |z_L|)$ be a polynomial of degree M with positive coefficients. Setting

$$I_{\mathcal{P},L}(u_1, \ldots, u_{L+1}) \equiv \int_{\mathbb{R}^L} \mathrm{d}z\, \mathcal{P}_{L,M}(|z_1|, \ldots, |z_L|) \exp(F_L(u,z)), \tag{A.1}$$

where

$$F_L(u,z) \equiv \sum_{1 \le m < n \le L+1} |u_m - u_n| + \sum_{1 \le m < n \le L} |z_m - z_n| - \sum_{j=1}^{L+1} \sum_{k=1}^{L} |u_j - z_k|, \tag{A.2}$$

we have a bound

$$I_{\mathcal{P},L}(u_1, \ldots, u_{L+1}) < Q_{L,M}(|u_1|, \ldots, |u_{L+1}|), \tag{A.3}$$

where $Q_{L,M}$ is a polynomial of degree $\le M + L$ with positive coefficients.

Proof. We prove this by induction on L. For $L = 1$, we have

$$I_{\mathcal{P},1}(u_1, u_2) = \int_{\mathbb{R}} \mathrm{d}z\, \mathcal{P}_{1,M}(|z|) \exp(|u_1 - u_2| - |u_1 - z| - |u_2 - z|). \tag{A.4}$$

We have symmetry under swapping u_1 and u_2, so we may take $u_2 \le u_1$. We write the integral as the sum of three integrals over $(-\infty, u_2)$, $[u_2, u_1]$, and (u_1, ∞), denoted by I^-, I^μ and I^+, resp. Then we have

$$I^+ = \int_{u_1}^{\infty} \mathrm{d}z\, \mathcal{P}_{1,M}(|z|) \exp(u_1 - u_2 - (z - u_1) - (z - u_2)) = \int_0^{\infty} \mathrm{d}z\, \mathcal{P}_{1,M}(|z + u_1|) e^{-2z}. \tag{A.5}$$

Now we need only use $|z + u_1| \le z + |u_1|$ to see that I^+ is bounded by a polynomial of degree M in $|u_1|$ with positive coefficients.

Likewise, since

$$I^- = \int_{-\infty}^{u_2} \mathrm{d}z\, \mathcal{P}_{1,M}(|z|) \exp(u_1 - u_2 - (u_1 - z) - (u_2 - z)) = \int_{-\infty}^{0} \mathrm{d}z\, \mathcal{P}_{1,M}(|z + u_2|) e^{2z}, \tag{A.6}$$

we infer that I^- is bounded by a polynomial of degree M in $|u_2|$ with positive coefficients.

Finally, we have for the middle integral

$$I^\mu = \int_{u_2}^{u_1} \mathrm{d}z\, \mathcal{P}_{1,M}(|z|) \exp(u_1 - u_2 - (u_1 - z) - (z - u_2)) = \int_{u_2}^{u_1} \mathrm{d}z\, \mathcal{P}_{1,M}(|z|), \tag{A.7}$$

and since we have

$$\int_{u_2}^{u_1} dz\,|z|^k \le \frac{1}{k+1}\Big(|u_1|^{k+1} + |u_2|^{k+1}\Big), \quad k \in \mathbb{N}, \tag{A.8}$$

we see that I^μ is bounded by a polynomial of degree $M+1$ in $|u_1|, |u_2|$, with positive coefficients. Thus, the assertion holds true for $L = 1$.

Next, we inductively assume the assertion has been proved up to $L-1$, $L > 1$. First, we claim that the function $F_L(u,z)$ (A.2) satisfies

$$F_L(u,z) \le 0, \quad \forall (u,z) \in \mathbb{R}^{L+1} \times \mathbb{R}^L. \tag{A.9}$$

Clearly, F has permutation symmetry in $u_1, \ldots, u_{L+1}$ and in $z_1, \ldots, z_L$. Therefore, we need only prove (A.9) under the assumptions $z_L \le z_{L-1} \le \cdots \le z_1$ and

$$u_{L+1} \le u_L \le \cdots \le u_1. \tag{A10}$$

Then we have

$$F_L(u,z) \le \sum_{1\le m<n\le L+1} (u_m - u_n) + \sum_{1\le m<n\le L} (z_m - z_n) - \sum_{j=1}^{L+1} \left(\sum_{j\le k} (u_j - z_k) + \sum_{j>k} (z_k - u_j) \right) = 0, \tag{A.11}$$

and so (A.9) follows.

We are now prepared to prove the bound (A.3). By permutation invariance of $I_{\mathcal{P},L}(u)$, we need only show its validity under the assumption (A.10). We write each z_k-integral as the sum of three integrals over $(-\infty, u_{L+1})$, $[u_{L+1}, u_1]$, and (u_1, ∞), denoted by I^-, I^μ and I^+, resp. We denote by $\hat{z}^k$ the vector in $\mathbb{R}^{L-1}$ arising by omitting the coordinate z_k from $z \in \mathbb{R}^L$. Then we have

$$\begin{aligned} I_{\mathcal{P},L}(u) &= \left(\prod_{k=1}^{N} \left(I^- + I^\mu + I^+\right) dz_k \right) \mathcal{P} \exp(F_L) \\ &< \sum_{k=1}^{L} \left(I^- \, dz_k \int_{\mathbb{R}^{L-1}} d\hat{z}^k + I^+ \, dz_k \int_{\mathbb{R}^{L-1}} d\hat{z}^k \right) \mathcal{P} \exp(F_L) + \prod_{k=1}^{L} I^\mu \, dz_k \, \mathcal{P} \exp(F_L). \end{aligned} \tag{A.12}$$

Next, using the bound (A.9), we note that the integral over $[u_{L+1}, u_1]^L$ is bounded by a sum of terms of the form

$$c \prod_{k=1}^{L} I^\mu \, dz_k \, |z_k|^{n_k}, \quad c > 0, \quad \sum_{k=1}^{L} n_k \le M. \tag{A.13}$$

In turn, such a term is bounded by

$$c_n \prod_{k=1}^{L} \left(|u_1|^{n_k+1} + |u_{L+1}|^{n_k+1} \right), \quad c_n > 0. \tag{A.14}$$

Hence, the integral over $[u_{L+1}, u_1]^L$ is majorized by a polynomial in $|u_1|, |u_{L+1}|$ of degree $\leq M + L$ with positive coefficients.

We proceed to study the z_k-integral I^+. We have $u_1 < z_k$, so we may write F_L as

$$\sum_{j=2}^{L+1}(u_1 - u_j) + \sum_{l\neq k}|z_k - z_l| - \sum_{j=1}^{L+1}(z_k - u_j) - \sum_{l\neq k}|u_1 - z_l| + F^+_{L-1}((u_2, \ldots, u_{L+1}), \hat{z}^k). \quad \text{(A.15)}$$

Taking $z_k \to z_k + u_1$ in the integral, we then get

$$e^{F^+_{L-1}} \int_0^\infty \mathrm{d}z_k\, \mathcal{P}(|z_1|, \ldots, |z_k + u_1|, \ldots, |z_L|) \exp\left(-(L+1)z_k + \sum_{l\neq k}(|z_k + u_1 - z_l| - |u_1 - z_l|)\right). \quad \text{(A.16)}$$

Majorizing the exponential by $\exp(-2z_k)$, we can bound each monomial term as a polynomial in $|u_1|$ of degree $\leq M$. The induction assumption now applies to the remaining $\hat{z}^k$-integrals over $\mathbb{R}^{L-1}$, yielding polynomials of the announced form.

The L integrals $I^- \,\mathrm{d}z_k$ can be estimated in a similar way, first writing F_L as

$$\sum_{j=1}^{L}(u_j - u_{L+1}) + \sum_{l\neq k}|z_k - z_l| - \sum_{j=1}^{L+1}(u_j - z_k) - \sum_{l\neq k}|u_{L+1} - z_l| + F^-_{L-1}((u_1, \ldots, u_L), \hat{z}^k), \quad \text{(A17)}$$

and then taking $z_k \to z_k + u_{L+1}$. ■

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