SEMI-CLASSICAL MASS ASYMPTOTICS ON STATIONARY SPACETIMES

ALEXANDER STROHMAIER AND STEVE ZELDITCH

ABSTRACT. We study the spectrum of a timelike Killing vector field Z acting as a differential operator on the solution space $\mathcal{H}_m \coloneqq \{u \mid (\Box_g + m^2)u = 0\}$ of the Klein-Gordon equation on a globally hyperbolic stationary spacetime (M, g) with compact Cauchy hypersurface Σ. We endow $\mathcal{H}_m$ with a natural inner product, so that $D_Z = \frac{1}{i}\nabla_Z$ is a self-adjoint operator on $\mathcal{H}_m$ with discrete spectrum $\{\lambda_j(m)\}$. In earlier work, we proved a Weyl law for the number of eigenvalues $\lambda_j(m)$ in an interval for fixed mass m. In this sequel, we prove a Weyl law along 'ladders' $\{(m, \lambda_j(m) : m \in \mathbb{R}_+\}$ such that $\frac{\lambda_j(m)}{m} \simeq \nu$ as $m \to \infty$. More precisely, we given an asymptotic formula as $m \to \infty$ for the counting function $N_{\nu,C}(m) \coloneqq \#\{j \mid \frac{\lambda_j(m)}{m} \in [\nu - \frac{C}{m}, \nu + \frac{C}{m}]\}$ for $C > 0$. The asymptotics are determined from the dynamics of the Killing flow e^{tZ} on the hypersurface $\mathcal{N}_{1,\nu}$ in the space $\mathcal{N}_1$ of mass 1 geodesics γ where $\langle \dot{\gamma}, Z \rangle = \nu$. The method is to treat m as a semi-classical parameter h^{-1} and employ techniques of homogeneous quantization.

In a recent article [SZ18], the authors introduced a generalization of the Gutzwiller-Duistermaat-Guillemin trace formula and Weyl law on a globally hyperbolic, stationary spacetime (M, g) of dimension n with compact Cauchy hypersurface Σ. The trace formula was for the trace $\operatorname{Tr} e^{tZ}|_{\ker \Box_g}$ of the translation operator e^{tZ} by the flow of the timelike Killing vector field Z on the space $\ker \Box_g$ of null-solutions of the Klein-Gordon operator $\Box_g$. To define the trace, a Hilbert space structure is introduced on $\ker \Box_g$, and the trace is expressed as an integral over Σ of a spacelike $(n-1)$-form constructed from the solution operator E of the Cauchy problem. One of the principal ingredients was the description of E as a Fourier integral operator in [DH72, Theorem 6.5.3 and Theorem 6.6.1]. A second principal ingredient was the calculation by Duistermaat-Guillemin [DG75] of the trace $\operatorname{Tr} \exp it\sqrt{-\Delta}$ of the wave group of a compact Riemannian manifold (Σ, h) using the symbol calculus of Fourier integral operators. The authors of [DH72] write that one motivation for their article was to clarify the nature of 'propagators' in relativistic quantum mechanics, and their article has since become foundational for quantum field theory on a curved spacetime. The relativistic trace formula of [SZ18] grew naturally from the combination of results of [DH72, DG75], and suggests that much of spectral asymptotics in non-relativistic quantum mechanics has a generalizations to globally hyperbolic, stationary spacetimes.

The present article is a continuation of [SZ18], still in the setting of a globally hyperbolic, stationary spacetime (M, g) (time-oriented and oriented) of dimension $n \geq 2$ with compact Cauchy hypersurface Σ. In [SZ18], the spectral problem was that of $D_Z = \frac{1}{i}\nabla_Z$ in $\ker \Box_g$, or

---

more generally the kernel of $\Box_g + V$ where V is a D_Z-invariant potential V. In this article, we fix $V = m^2$ (the mass squared), and study the spectrum of D_Z in the kernel

$$\mathcal{H}^{(m)} := \ker(\Box_g + m^2), \tag{1}$$

of the Klein-Gordon operator

$$\Box_g = -\frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ik} \partial_k \right).$$

In Section 3, we equip $\mathcal{H}^{(m)}$ with an energy inner product (Definition 3.1) so that (1) becomes a Hilbert space when $m \neq 0$.

The main idea of this article is to treat m as a semi-classical parameter, so that in effect we study the semi-classical Klein-Gordon operator $m^{-2} \Box_g + 1$ in the semi-classical limit $m \to \infty$. Although it is rather obvious that m^{-1} is formally analogous to the Planck constant $\hbar$ in a non-relativistic Schrödinger operator, we have not found prior studies of semi-classical mass asymptotics for the Klein - Gordon operator. Equivalently, since $[\Box_g, D_Z] = 0$, we are studying the joint spectrum $\{(m^2, \lambda_j(m))\}$ of $(\Box_g, D_Z)$, i.e.

$$\begin{cases} (\Box_g + m^2)u = 0, \\ \\ D_Z u = \lambda u \end{cases} \tag{2}$$

in the spirit of equivariant spectral asymptotics. Since D_Z is first order, we reparametrize the joint spectrum as $\{(m, \lambda_j(m)), m \in \mathbb{R}\}$, and study the parametrized eigenvalues $m \to \lambda_j(m)$ as $m \to \infty, \lambda_j(m) \to \infty$ in such a way that $\frac{\lambda_j(m)}{m}$ tends to a limit value $\nu \in \mathbb{R}$. Following [GS82], we refer to such pairs as a 'ray' or 'ladder' L_ν in the joint spectrum. As is pointed out in [GU89], ladder asymptotics can be used to obtain semi-classical asymptotics, and that is our purpose in this article.

The basic goal is to determine the asymptotics as $m \to \infty$ of the mass-semi-classical Weyl eigenvalue counting function,

$$N_{\nu,C}(m) := \#\{j \mid \frac{\lambda_j(m)}{m} \in [\nu - \frac{C}{m}, \nu + \frac{C}{m}]\}, \tag{3}$$

where $C > 0$ is a given constant. Theorem 1.3 gives asymptotics as $m \to \infty$ of these Weyl spectral functions. To study this sharp counting function, we first smooth out the indicator function $\mathbf{1}_{[-C,C]}$ to a Schwartz function $\psi \in \mathcal{S}(\mathbb{R})$ with $\hat{\psi} \in C_0^\infty(\mathbb{R})$ and form,

$$N_{\nu,\psi}(m) := \sum_{j \in \mathbb{Z}} \psi(\lambda_j(m) - \nu m). \tag{4}$$

The asymptotics of this smoothed Weyl function are determined in Theorem 1.2, and the asymptotics of (3) are determined in Theorem 1.3.

We view the eigenvalues of D_Z on $\mathcal{H}_m$ as 'energy levels' of a mass m free particle. In special relativity the energy of particle is the time component of its 4-momentum. Here we view the classical energy E of a mass m geodesic γ as the constant value $\langle Z, \dot{\gamma} \rangle$, which plays the role of the time component of its 4-momentum. Mass semi-classical asymptotics determines how Klein-Gordon particles scale at large energies when the mass m and energy E are scaled in the

same way. Having a fixed ratio $\frac{E}{m}$ intuitively means roughly that the velocity of the particle is fixed while letting $m, E \to \infty$. This is explained in detail in Section 0.6.

The study of the joint spectrum of $(\Box_g, D_Z)$ may seem reminiscent of the study of the joint spectrum of the Laplacian Δ_h and a Killing vector field Z on a compact manifold (Σ, h), e.g. the Laplacian on a surface of revolution. Among the many articles studying equivariant spectral asymptotics and its applications to semi-classical asymptotics, the article [GU89] is most relevant to our results and we refer there for references to the literature. The equations (2) seem formally analogous to $(\Delta_h + \lambda^2)u = 0, D_Z = \lambda u$ in $L^2(\Sigma, dV_h)$, but there is a fundamental difference in the way that we treat the joint spectral problem: as in [SZ18], we do not study the joint spectrum on $L^2(M, dV)$, but rather endow (1) with its own Hilbertian inner product (Definition 3.1 in Section 3). Hence a 'ladder' in the joint spectrum involves the spectrum of D_Z in the one-parameter family (1) of Hilbert spaces. Since this type of joint spectral problem is non-standard even in the case of product ('ultra-static') spacetimes, we include a short Section 12 to explain what it boils down to in the (rather trivial) product case.

0.1. **Homogenizing and discretizing the m parameter.** To deal with the one-parameter family (1) of Hilbert spaces as $m \to \infty$, we construct the globally hyperbolic stationary space-time
$$(\widetilde{M}, \widetilde{g}) = (M, g) \times S_T^1,$$
where $S_T^1 = \mathbb{R}/T\mathbb{Z}$, and where we take the semi-Riemannian product of the Lorentzian manifold (M, g) with S_T^1. The circle factor S^1 is spacelike, so that the metric $\widetilde{g} = g \oplus (\mathrm{d}\theta^2)$ is Lorentzian of signature $(-1, +1, \ldots, +1)$, and has the compact Cauchy hypersurface $\Sigma \times S_T^1$. The group $\mathbb{R} \times S_T^1$ acts by isometries, where $\mathbb{R}$ acts by the Killing flow e^{tZ} and S_T^1 acts on the S_T^1 factor. The eigenvalues of D_θ are $m_k = \frac{2\pi}{T}k$ with $k \in \mathbb{Z}$. Since T is arbitrary we essentially obtain all mass parameter values, and since we are interested in ladders, little is lost by the discretization.

Notational Conventions To avoid cluttering up the notation, we usually assume that $T = 2\pi$ and write S^1 for $S^1 = S_{2\pi}^1$. We often use the notation C_n for any purely dimensional constant that may vary in each occurrence.

When it could be confusing whether a point ζ lies in T^*M or $T^*\widetilde{M}$, we use the notation $\zeta = (x, \xi)$ for the former and $\zeta = (\tilde{x}, \tilde{\xi})$ for the latter. When there is no possibility of confusion, we drop the tilde's. In homogeneous quantization, one often deletes the zero section of a cotangent bundle and works on $\dot{T}^*M = T^*M \backslash 0$.

Our signature convention is $(-, +, \cdots, +)$ but $\Box_g$ is minus the Lorentzian Laplacian. The Minkowski metric is thus $-dt^2 + \sum_j dx_j^2$ and its Klein-Gordon operator is $\frac{\partial^2}{\partial t^2} - \Delta + m^2$. Note that $-\Delta$ is a positive operator, and that the mass term m^2 is positive, so that the semi-classical symbol $-\tau^2 + |\xi|^2 + m^2$ has zeros on the two-sheeted mass hyperboloid (bundle) $\tau^2 = |\xi|^2 + m^2$.

Finally, we use the term *Liouville measure* for general surface measures μ_L induced by the symplectic volume measure Ω on co-isotropic submanifolds defined by some Poisson-commuting functions. In the case of a hypersurface $\{H = E\}$ in a symplectic manifold, the Liouville measure is given by $\mu_L = \frac{\Omega}{\mathrm{d}H}|_{H=E}$. It satisfies, $\mu_L(\{H = E\} = \frac{\mathrm{d}}{\mathrm{d}E}\mathrm{Vol}_\Omega(\{H \leq E\})$. For a co-dimension two surface $\{f_1 = E_1, f_2 = E_2\}$, the Liouville measure is given by

$\mu_{\{f_1=E_1, f_2=E_2\}} = \frac{\Omega}{df_1 \wedge df_2}|_{f_1=E_1, f_2=E_2}$. To avoid excessive subscripts, we often denote the Liouville (or, Leray) measure on a manifold Y by μ_Y.

0.2. Quantum ladders.

We define the null-space of the wave operator $\widetilde{\Box} = \widetilde{\Box}_{\widetilde{g}}$ of $(\widetilde{M}, \widetilde{g})$, $\widetilde{\Box} = \Box_g - \frac{\partial^2}{\partial \theta^2}$,

$$\mathcal{H}_{KG}^\infty := \ker \widetilde{\Box} \cap C^\infty = \ker(\Box_g - \frac{\partial^2}{\partial \theta^2}) \cap C^\infty, \tag{5}$$

i.e. the space of smooth solutions. Following [SZ18], we endow in Section 3 the space of smooth solutions $\ker \widetilde{\Box} \cap C^\infty$ with a positive semi-definite quadratic form with finite dimensional null-space and define $\mathcal{H}_{KG}$ as the completion in the associated topology. This therefore defines a notion of a trace $\operatorname{Tr}_{\mathcal{H}_{KG}}$. We are interested in the joint spectrum of $\mathbb{R} \times S_T^1$ on the null-space (5), where $\mathbb{R}$ acts by the Killing flow e^{tZ} and S_T^1 acts on the S_T^1 factor. The results of [SZ18] apply directly to the spectrum of D_Z or D_θ in (5), and in particular imply that the spectrum of the $\mathbb{R} \times S_T^1$ action on $\ker \widetilde{\Box}$ is discrete and consists of pairs $\{m_k, \lambda_j(m_k)\}$ with $m_k = \frac{2\pi}{T}k$ (see Theorem 3.4).

Under the S^1 action, (5) splits up into Fourier modes of S^1 (i.e. the eigenspaces of D_θ in (5)),

$$\mathcal{H}_{KG} = \bigoplus_{m \in \mathbb{Z}} \mathcal{H}_{KG}^{(m)}, \quad \mathcal{H}_{KG}^{(m)} := \mathcal{H}_{KG} \cap \{D_\theta = m\}, \tag{6}$$

A 'ladder subspace' of (6) is, roughly speaking, defined as the kernel,

$$\mathcal{H}_{\mathcal{L}_\nu} \text{ ``:=''} \{u \in \mathcal{H}_{KG} \mid P_\nu u = 0\}. \tag{7}$$

of the operator,

$$P_\nu := D_Z - \nu D_\theta. \tag{8}$$

Since the eigenvalues generally do not intersect lines through the origin this heuristic definition is not literally correct and only provides intuition. The precise definition is that the joint eigenvalues $\{(m, \lambda_j(m)\}$ lie in a strip (or thickening) of the ray $\{(m, \lambda) \mid \lambda = \nu m\}$. To make this precise, we introduce real valued test functions ψ with $\hat{\psi} \in C_0^\infty$ and define the the 'fuzzy' ladder subspace to be the range of the operator

$$\Pi_{\nu, \psi} := \psi(D_Z - \nu D_\theta) : \mathcal{H}_{KG} \to \mathcal{H}_{KG}. \tag{9}$$

The spectral problem is then to find the distribution of eigenvalues of D_Z in a ladder. The notion of a fuzzy ladder is due to Guillemin-Uribe [GU89] in the compact elliptic setting. We will see that our results have many overlaps, and also significant differences, with [GU89].

Remark 0.1. In [SZ18], the authors studied the distribution of eigenvalues of D_Z in a fixed Klein-Gordon space (1). In terms of ladders, the mass m is fixed, and if we think of m as the vertical axis, the corresponding ladder is horizontal. Horizontal ladders do not fit well with ladder notation. If we replace (8) by $\mu D_Z - D_\theta$, then a horizontal ladder cutoff would correspond to $\mu = 0$ and $\psi(D_\theta)$ can be thought of as localizing around $m = 0$ in the spectrum of D_θ. Since we assume that $\hat{\psi} \in C_0^\infty(\mathbb{R})$, it requires Tauberian theorems to make such localizations.

0.3. **Generating functions and their singularities.** To find the asymptotics as $m \to \infty$ of (4) and then (3), we form the generating functions,

$$
\begin{cases}
\Upsilon_{\nu,\psi}^{(1)}(s) := \sum_{m,j \in \mathbb{Z}} \psi(\lambda_j(m) - m\nu)e^{ims}, \\
\\
\Upsilon_{\nu,\psi}^{(2)}(s) := \sum_{m,j \in \mathbb{Z}} \psi(\lambda_j(m) - m\nu)e^{i\lambda_j(m)s}.
\end{cases}
\tag{10}
$$

The Weyl functions (4) are essentially the Fourier coefficients of these generating functions in the decomposition (6). More precisely, $\Upsilon_{\nu,\psi}^{(2)}$ is the Fourier transform of the periodic ladder spectral function,

$$
dN_{\nu,\psi}^{(1)}(x) = \sum_{j \in \mathbb{Z}, m \in \mathbb{Z}} \psi(\lambda_j(m) - \nu m)\delta(x - m),
\tag{11}
$$

and $\Upsilon_{\nu,\psi}^{(2)}$ is the Fourier transform of the ladder spectral function,

$$
dN_{\nu,\psi}(x) = \sum_{j,m \in \mathbb{Z}} \psi(\lambda_j(m) - \nu m)\delta(x - \lambda_j(m)).
\tag{12}
$$

To determine the behavior of (4) as $m \to \infty$, it suffices to determine the singularities at $s = 0$ of these generating functions.

The Weyl asymptotics of (4) follows from a singularity analysis of (10) by a Fourier Tauberian theorem. The results are reminiscent of the ladder asymptotics of Guillemin-Uribe [GU89] in the compact elliptic setting, and as they explain, the singularities of $\Upsilon_{\nu,\psi}^1$ resp. $\Upsilon_{\nu,\psi}^2$ at $s = 0$ are essentially the same. In fact, it is simpler to use $\Upsilon_{\nu,\psi}^{(1)}$ and only sum over $m \geq 0$. The sum is then a Hardy distribution, and as in [GU89, Section 7] one can easily determine the large m asymptotics from the singularities by matching periodic distributions on S^1.

0.4. **Classical ladders.** It is well-known in spectral asymptotics that the singularities of (10) are controlled by an underlying classical Hamiltonian dynamical system. We need to define it before stating our main results. More details will be given in Section 2.3.

The characteristic variety of $\widetilde{\square}$ is the null-cone bundle,

$$
\mathrm{Char}(\widetilde{\square}) = T_0^* \widetilde{M} = \{(\tilde{x}, \tilde{\xi}) \in \dot{T}^* \widetilde{M} \mid \tilde{g}(\tilde{\xi}, \tilde{\xi}) = 0\},
\tag{13}
$$

where $\tilde{g}(\tilde{\xi}, \tilde{\xi})$ is the Lorentzian inner product of $\widetilde{M}$. The quotient of $\mathrm{Char}(\widetilde{\square})$ by the (null) geodesic flow $\widetilde{G}^t : T^* \widetilde{M} \to T^* \widetilde{M}$ is the symplectic cone,

$$
\widetilde{\mathcal{N}} = \mathrm{Char}(\widetilde{\square})/ \sim = T_0^* \widetilde{M},
\tag{14}
$$

which is called the space of scaled null-geodesics of $(\widetilde{M}, \tilde{g})$. The space $\widetilde{\mathcal{N}}$ is a disjoint union $\widetilde{\mathcal{N}} = \widetilde{\mathcal{N}}_+ \cup \widetilde{\mathcal{N}}_-$ of future and past directed scaled null-geodesics. In [SZ18, Section 2.1] it is shown to be a homogeneous (i.e. conic) symplectic manifold, and it is explained that $\widetilde{\mathcal{N}}$ is the symplectic cone of dimension $2n$ corresponding to the Hilbert space $\mathcal{H}_{KG}$ in the sense of geometric quantization.

The $\mathbb{R} \times S^1$ action lifts to $T^* \widetilde{M}$ as a Hamiltonian action with classical moment map

$$
\mathcal{P} : T^* \widetilde{M} \to \mathbb{R}^2, \quad \mathcal{P}(\tilde{x}, \tilde{\xi}) = (\langle \tilde{\xi}, Z \rangle, \langle \tilde{\xi}, \frac{\partial}{\partial \theta} \rangle).
\tag{15}
$$

For notational simplicity, we set

$$p_Z(\tilde{x}, \tilde{\xi}) := \langle \tilde{\xi}, Z \rangle, \quad p_\theta(\tilde{x}, \tilde{\xi}) := \langle \tilde{\xi}, \frac{\partial}{\partial \theta} \rangle.$$

We also denote the Hamilton vector field of a function f (on any cotangent bundle or symplectic manifold) by H_f and we denote its Hamiltonian flow by $\exp t H_f$.

Given $\nu \in \mathbb{R}_+$, the principal symbol of (8) is the Hamiltonian

$$p_\nu(\tilde{x}, \tilde{\xi}) = p_Z(\tilde{x}, \tilde{\xi}) - \nu p_\theta(\tilde{x}, \tilde{\xi}). \tag{16}$$

Definition 0.2. The classical ladder L_ν is the double characteristic variety $D\mathrm{Char}_\nu = \mathrm{Char}(\widetilde{\square}, P_\nu)$, i.e. the set of $(\tilde{x}, \tilde{\xi}) \in \dot{T}^* \widetilde{M}$ such that

$$\begin{cases} \frac{1}{2} \widetilde{g}(\tilde{\xi}, \tilde{\xi}) = 0, \\[2mm] p_\nu(\tilde{x}, \tilde{\xi}) = 0. \end{cases}$$

We assume throughout that (M, g) is a standard stationary spacetime, so that we may choose a Cauchy hypersurface Σ of M and represent $M = \mathbb{R} \times \Sigma$ with $Z = \frac{\partial}{\partial t}$ (see Section 2). We let coordinates (t, x, θ) be corresponding coordinates on $M \times S^1 \simeq \mathbb{R} \times \Sigma \times S^1$, and let (τ, ξ, σ) be the dual symplectic coordinates. As explained in more detail in Sections 2.2-(2.3), $p_Z(t, x, \theta, \tau, \xi, \sigma) = \tau$, $p_\theta = \sigma$. The equations of the double characteristic variety are then,

$$\begin{cases} (i) \quad g((\tau, \xi), (\tau, \xi)) + \sigma^2 = 0, \\[2mm] (ii) \quad \tau - \nu \sigma = 0. \end{cases} \tag{17}$$

Thus, σ plays the role of the mass parameter.

Definition 0.3. We say that ν is admissible if 0 is in the range of p_ν on $\widetilde{\mathcal{N}}$ and is a regular value.

Criteria for admissibility in terms of Killing horizons are given in Section 2.4.

In Theorems 1.1 and 1.2 we assume that ν is admissible in the sense of Definition 0.3. To understand this condition, we use the equations to eliminate σ and find that $D\mathrm{Char}_\nu$ corresponds to points of $T^* \widetilde{M}$ satisfying

$$g((\tau, \xi), (\tau, \xi)) + \nu^{-2} \tau^2 = 0. \tag{18}$$

As will be proved in Proposition 2.4, the projection of $D\mathrm{Char}_\nu$ to $\widetilde{M}$ equals set (33) of $\widetilde{M}$ where $Z - \nu \frac{\partial}{\partial \theta}$ is spacelike, and thus is bounded by the Killing horizon. It follows that $D\mathrm{Char}_\nu$ is empty if ν is such that $Z - \nu \frac{\partial}{\partial \theta}$ is nowhere spacelike. For instance, if $M = \mathbb{R} \times \Sigma$ is a product spacetime with metric $-dt^2 + h$, then the equation states that $|\xi|_h^2 = (1 - \nu^{-2}) \tau^2$, which has no solutions unless $\nu \geq 1$.

To obtain a Hamiltonian flow on a symplectic manifold, we reduce the classical ladder. Since p_Z, p_θ are constant on orbits of the geodesic flow, the moment map is well defined as a map on (14), and defines a reduced moment map,

$$\mathcal{P}_{\widetilde{\mathcal{N}}} : \widetilde{\mathcal{N}} \to \mathbb{R}^2. \tag{19}$$

Definition 0.4. A *reduced classical ladder* $\mathcal{L}_\nu$ is the co-isotropic submanifold,

$$\mathcal{L}_\nu := \{(\tilde{x}, \tilde{\xi}) \in \widetilde{\mathcal{N}} \mid p_\nu(\tilde{x}, \tilde{\xi}) = 0\} = \mathcal{P}_{\widetilde{\mathcal{N}}}^{-1} \{(\nu\sigma, \sigma)) \mid \sigma \in \mathbb{R}_+\}. \tag{20}$$

$\mathcal{L}_\nu$ is a conic hypersurface of $\widetilde{\mathcal{N}}$ whose null-foliation is given by orbits of the Hamiltonian flow of p_ν (16) on the symplectic manifold $\widetilde{\mathcal{N}}$.

0.5. **The governing dynamical system.** As mentioned above, the singularities of (10) correspond to periods s of a Hamiltonian flow on a symplectic manifold. We say that this Hamiltonian system 'controls' the singularities of (10). The flow for massive ladders is obtained by reducing the joint Hamiltonian flow of p_Z, p_θ on $\widetilde{\mathcal{N}}$ with respect to $\sigma = 1$. For the massless ladder it is obtained by reducing with respect to $\sigma = 0$. Setting $\sigma = 1$ is arbitrary, reflecting the fact that taking the product $\widetilde{M} = M \times S^1$ is an artifice whose purpose is simply to homogenize the semi-classical theory. Reducing with respect to $\sigma = 1$ is one way to de-homogenize the theory. Since we will set $p_\nu = 0$ (16), setting $\sigma = 1$ is equivalent to setting $p_Z = \nu$ in $\dot{T}^*M$.

First let us consider the reduced symplectic manifold obtained from (17) by setting $p_\theta = \sigma = 1$ in $\widetilde{\mathcal{N}}$ and quotienting it by the Hamiltonian S^1 action. This of course un-does the product with S^1 that converts M to $\widetilde{M}$; the quotient depends on the level $\sigma = 1$. Evidently, the reduction of the equations (17) in T^*M gives,

$$\begin{cases} (i) & g((\tau, \xi), (\tau, \xi)) = -1, \\[2mm] (ii) & p_Z = \tau = \nu. \end{cases} \tag{21}$$

The first equation defines the unit 'hyperboloid bundle' of T^*M (see Section 2.2). The second equation gives a 'time-slice of each cotangent space hyperboloid, selecting out a 'sphere' in each cotangent space. We note that p_Z is constant on G^s-orbits in the hyperboloid bundle, so it descends to the quotient,

$$\mathcal{N}_1 := \{(x, \xi) \in T^*M \mid g(\xi, \xi) = -1\}/\sim, \tag{22}$$

where $\sim$ is the equivalent relation of lying on the same orbit of the geodesic flow.

Definition 0.5. Let ν be admissible in the sense of Definition 0.3. The governing dynamical system for (10) is the Killing flow of e^{tZ} on the level set

$$\mathcal{N}_1(\nu) := \{p_Z = \nu\} \subset \mathcal{N}_1. \tag{23}$$

Remark 0.6. In the massless case of [SZ18], the governing dynamical system is the Hamiltonian flow e^{tZ} of p_Z on the conic symplectic manifold $\mathcal{N}$ of null geodesics. Since p_Z is homogeneous, its Hamiltonian flow has the same periods on all non-zero level sets. In this case the classical dynamical system can be shown to be equivalent to a magnetic flow (see for example [SZ20] for a discussion of this).

We introduce the following notation for volume forms.

Definition 0.7. As in the notational introduction, we denote by Ω the symplectic volume form of a symplectic manifold. Given a Hamiltonian H we denote by $\mu_L = \frac{\Omega}{dH}$ the Liouville volume form on a level set of H; we also denote it by $\mu_{\{H=1\}}$ when we need to specify the

relevant manifold. Thus, $\Omega_{\mathcal{N}_1}$ denotes the symplectic volume form of $\mathcal{N}_1$, and $\mu_{\mathcal{N}_1(\nu)}$ denote Liouville measure on the set $\mathcal{N}_1(\nu)$.

The Liouville measure of $\mathcal{N}_1(\nu)$ can be computed explicitly for a standard spacetime. When ν is a regular value of p_Z, it is given by

$$\mu_{\mathcal{N}_1(\nu)}(\mathcal{N}_1(\nu)) = \text{Vol}(S_{n-1}) \int_\Sigma \frac{\nu N}{(N^2 - \beta^2)^{\frac{n}{2}}} \left(\nu^2 - (N^2 - |\beta|_h^2) \right)_+^{\frac{n-3}{2}} \, \mathrm{d}\text{Vol}_h, \qquad (24)$$

The derivation of this formula can be found in the proof of Prop. 2.5.

0.6. **Physical interpretation of the governing dynamical system.** Propagation of light and of particles at extremely high energy follows the rules of geometrical optics on spacetimes in the sense that singularities travel on lightlike geodesics. In this sense the natural dynamical system associated with the high energy limit of [SZ18] was the dynamics on the space of lightlike geodesics. Here the ladder theory corresponds to increasing the energy at the same time as the mass and keeping the ratio between mass and energy at a fixed rate ν. This amounts to fixing the ratio between energy and momentum. One therefore ends up with a dynamical system on a space of timelike geodesics, thus corresponding to the dynamics of particles traveling at less than the speed of light in the gravitational field. The parameter ν plays as similar role as the famous Lorentz factor $(1 - v^2)^{-\frac{1}{2}}$ does in Minkowski spacetime (here units are chosen so that the speed of light c equals 1). To see this in a general stationary spacetime assume that we have a metric of the form $g = -N^2 \mathrm{d}t^2 + h_{ij}(\mathrm{d}x^i + \beta^i \mathrm{d}t)(\mathrm{d}x^j + \beta^j \mathrm{d}t)$ as in (28) (see Section 2 for details) with a Lapse function N and a shift vector field β. Now assume that $\gamma(t)$ is a timelike geodesic in M corresponding to the dynamics in $\mathcal{N}_1(\nu)$, i.e. with the property that $g(\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t), \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)) = -1$ and such that $g(Z, \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)) = \nu$. If we fix the mass m to be one then the covector $\xi(t)$ associated to $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)$ corresponds to the relativistic momentum of the particle: in local coordinates this momentum has the energy as its time component and the momentum as its spatial component. If we choose a local orthonormal frame that identifies the tangent space at a point $x \in M$ with Minkowski space so that the Killing vector Z is given at x by the vector $\sqrt{N^2 - |\beta|_h^2}\partial_{t_0}$, then such a geodesic will have relativistic momentum with t_0-component equal to $\nu(N^2 - |\beta|_h^2)^{-\frac{1}{2}}$ and with spatial component of length $\sqrt{\frac{\nu^2}{N^2 - |\beta|_h^2} - 1}$. Therefore in this frame the spacelike component of the speed is

$$v(t) = \frac{\sqrt{\nu^2(N^2 - |\beta|_h^2)^{-\frac{1}{2}} - 1}}{\nu(N^2 - |\beta|_h^2)^{-\frac{1}{2}}} = \sqrt{1 - \frac{(N^2 - |\beta|_h^2)}{\nu^2}},$$

and thus,

$$\nu = \frac{\sqrt{N^2 - |\beta|_h^2}}{\sqrt{1 - v^2}}.$$

Note that the speed is not constant since the particle moves in a gravitational field, but ν is a conserved quantity and has frame-independent meaning once Z is fixed. In the product case we have $N = 1$ and $\beta = 0$ and this reduces to the Lorentz factor.

1. STATEMENT OF RESULTS

Our goal is to determine the singularity at $s = 0$ of $\Upsilon^{(1)}_{\nu,\psi}(s)$, and to apply the result to prove a Weyl law for the joint spectrum along the ray. By elaborating the techniques, we could develop the theory as in [SZ18] to prove a Gutzwiller formula giving the singularities of the trace at non-zero s in terms of periods of periodic orbits of the governing Hamiltonian flow of Section 0.5. However, we do not give the details of the calculations of the singularities at $s \neq 0$ for the sake of expository brevity. Henceforth we redefine (10) as a Hardy distribution,

$$\Upsilon^{(1)}_{+,\nu,\psi}(s) := \sum_{m=0}^{\infty} \sum_{j \in \mathbb{Z}} \psi(\lambda_j(m) - m\nu) e^{ims} = \operatorname{Tr}_{\mathcal{H}_{KG}} \Pi_{\nu,\psi} \Pi_+ e^{isD_\theta}, \tag{25}$$

where Π_+ is the spectral projection of D_θ onto the non-negative part of the spectrum. Then up to a constant $c = \sum_{j \in \mathbb{Z}} \psi(\lambda_j(0))$ the real part of $2\Upsilon^{(1)}_+$ equals $\Upsilon^{(1)}$. The singularities of $\Upsilon^{(1)}_+$ thus determine the singularities of $\Upsilon^{(1)}$. We show that $\Upsilon^{(1)}_+$ is a polyhomogeneous Hardy distribution with a co-normal singularity at $s = 0$, i.e. it has a singularity expansion as a sum of model periodic distributions defined by the Fourier series (cf. [GU89, Lemma 7.1])

$$b_k(s; \omega) = \sum_{m=0}^{\infty} m^k \omega^{-m} e^{ims},$$

where $\omega \in S^1$. In general the ω arise as holonomies corresponding to singularities at $s \neq 0$. For the singularity at $s = 0$, $\omega = 1$ and the singularity is the same as for the model homogeneous distributions $\mu_k(s)$ on $\mathbb{R}$, defined by the oscillatory integrals (cf. [GS77, GU89])

$$\mu_k(s) = \int_0^\infty e^{is\tau} \tau^k d\tau.$$

Theorem 1.1. *Let (M, g) be a spatially compact stationary globally hyperbolic spacetime with $\dim M = n$, with timelike Killing vector field Z, let ν be admissible in the sense of Definition 0.3. Let $\psi \in \mathcal{S}(\mathbb{R})$ with $\hat{\psi} \in C_0^\infty(\mathbb{R})$. Then the singular support of $\Upsilon^{(1)}_{+,\nu,\psi}(s)$ is contained in the set $\{s \in \mathbb{R} \mid \exists s' \in \mathcal{P}_\nu \cap \operatorname{supp}(\hat{\psi}), s - \nu s' \in 2\pi\mathbb{Z}\}$, where $\mathcal{P}_\nu$ is the set of periods of periodic orbits of e^{sZ} on $\mathcal{N}_1(\nu)$. In case $\operatorname{supp}\psi \subset (-\pi/\nu, \pi/\nu)$ the distribution $\Upsilon^{(1)}_{+,\nu,\psi}(s) = \operatorname{Tr}_{\mathcal{H}_{KG}} \Pi_{\nu,\psi} \Pi_+ e^{isD_\theta}$ (25) is a Hardy distribution on S^1 with a conormal singularity at $s = 0$. That is,*

$$\Upsilon^{(1)}_{+,\nu,\psi}(s) = e_{0;\nu,\psi}(s) + \rho_{0;\nu,\psi}(s)$$

where $\rho_{0;\nu,\psi}(s)$ is a distribution that is smooth near 0, and $e_{0;\nu,\psi}(s)$ is a Lagrangian distribution with singularity at $s = 0$ of the form

$$e_{0;\nu,\psi}(s) \sim (2\pi)^{-n+1} \mu_{\mathcal{N}_1(\nu)}(\mathcal{N}_1(\nu)) \hat{\psi}(0) \mu_{n-2}(s) + c_1(\nu, \psi) \mu_{n-3}(s) + \dots,$$

where μ_L is the Liouville measure on $\mathcal{N}_1(\nu)$ (see Definition 0.5).

There is very similar statement for $\Upsilon^{(2)}_{\nu,\psi}(s)$.

Dually, we obtain an asymptotic expansion for smoothed ladder Weyl spectral functions.

Theorem 1.2. *Let (M, g) satisfy the assumptions of Theorem 1.1. Let $\{\lambda_j(m)\}_{j=1}^\infty$ be the eigenvalues of D_Z in $\mathcal{H}_m$ (1), and let ν be admissible. Let $\psi \in \mathcal{S}(\mathbb{R})$ with $\hat{\psi} \in C_0^\infty(\mathbb{R})$ of*

sufficiently small support around $s = 0$ so that no other periods of periodic orbits of e^{sZ} on $\widetilde{\mathcal{N}}$ lie in its support. Then, there exists a complete asymptotic expansion as $m \to \infty$,

$$N_{\nu,\psi}(m) \simeq m^{n-2} \sum_{k=0}^{\infty} a_k(\nu, \psi) m^{-k},$$

where $a_0(\nu, \psi) = (2\pi)^{-n+1} \hat{\psi}(0) \mu_{\mathcal{N}_1(\nu)}(\mathcal{N}_1(\nu))$.

Theorem 1.2 is deduced from Theorem 1.1 using a Hardy-Fourier Tauberian argument, [GU89, Section 7]. For the corresponding compact elliptic case, see [GU89, Lemma 7.1-Corollary 7.2].

The sharp Weyl law for the the counting function (3) of the ladder $\mathcal{H}_\nu$ states:

Theorem 1.3. *With the same notation and assumptions as in Theorem 1.2, assume in addition that the closed orbits of the Killing flow on (23) are non-degenerate. Then the sharp Weyl function (3) admits the asymptotic expansion,*

$$N_{\nu,C}(m) = 2C(2\pi)^{-n+1} \mu_{\mathcal{N}_1(\nu)}(\mathcal{N}_1(\nu)) \, m^{n-2} + o(m^{n-2}),$$

as $m \to \infty$.

In fact, as in [DG75, Lemma 3.3], the hypothesis of non-degeneracy may be weakened to the statement that the fixed point sets of the Killing flow e^{tZ} on $\mathcal{N}_1(\nu)$ for $t \neq 0$ are of Liouville measure zero. The need for such an hypothesis is illustrated in the product case in Section 12, where the spectral problem coincides with the one studied in [DG75]. When the Hamilton flow is periodic, the ladder eigenvalues $\{\lambda_j(m)\}_{m=1}^{\infty}$ cluster along an arithmetic progression (depending on m), and one does not have two-term Weyl asymptotics. This phenomenon is well-known in spectral asymptotics and is discussed for compact elliptic ladder asymptotics on [GU89, page 420]. The sharp Weyl result is deduced from Theorem 1.2 by a Tauberian theorem as in [BrU91, Theorem 3.2]. To prove it, we will need a slight extension of Theorem 1.2, where we remove the assumption on supp $\hat{\psi}$.

Lemma 1.4. *With the same notation as in Theorem 1.2, assume that the set of periodic orbits of e^{tZ} of period $t \neq 0$ on (26) have measure zero. Let $\psi \in \mathcal{S}(\mathbb{R})$ with $\hat{\psi} \in C_0^\infty(\mathbb{R})$. Then,*

$$N_{\nu,\psi}(m) = m^{n-2} a_0(\nu, \psi) + o(m^{n-2}).$$

Remark 1.5. The Weyl asymptotics are consistent with the elementary product case (74). They are referred to as 'differentiated Weyl laws' in spectral asymptotics, because they count eigenvalues in short intervals. For this reason, the order of the singularity in Lemma 1.1 and the order of growth of eigenvalues in Theorem 1.3 differ by 1 from the orders in the massless case of [SZ18]. As mentioned above, the trace asymptotics of [SZ18] correspond to horizontal ladders, while the techniques of this paper require that $0 < \nu < \infty$.

1.1. **Relation to compact elliptic ladder asymptotics.** As mentioned above, Theorem 1.1 is not simply the ladder asymptotics of [GU89] transported to the Lorentzian setting, because $\mathcal{H}_{KG}$ is not an L^2 subspace of $L^2(\widetilde{M}, dV_{\widetilde{g}})$, and there does not exist an orthogonal projection from the latter to the former. In Section 12 we specialize the results to the case of product spacetimes in order to clarify the nature of the results in this special case and to tie

together the traces (10) on the Hilbert spaces (1) with the compact elliptic ladder asymptotics as in [GU89], as well as with traditional semi-classical asymptotics.

Let us explain the analogies between objects of this article and objects in [GU89]. The Hilbert space (5) corresponds to the homogeneous symplectic manifold (14). Since (5) is the entire Hilbert space in our spectral problem, it plays the role of $L^2(X)$ in [GU89]. Hence $\widetilde{\mathcal{N}}$ plays the role of T^*X in that article. As mentioned above, (15) generates a Hamiltonian $\mathbb{R} \times S^1$ action on $\widetilde{\mathcal{N}}$. Formally, the ladder Hilbert space (7) corresponds to the quotient,

$$\widetilde{\mathcal{N}}_\nu : \{\zeta \in \widetilde{\mathcal{N}} : p_\nu(\zeta) = 0\}/ \sim, \tag{26}$$

where $\sim$ is the equivalence relation of belonging to the same orbit of the Hamiltonian flow $e^{tZ}e^{-t\nu\frac{\partial}{\partial\theta}}$ of p_ν. We reduce this Hamiltonian system by setting $\sigma = 1$ on $\widetilde{\mathcal{N}}_\nu$ and dividing by the S^1 action, reducing to the Hamiltonian flow e^{tZ} acting on the set $\{p_Z = \nu\} \subset \mathcal{N}_1$ (22). In Section 2.3, we show that there exists a symplectic quotient for this action due to global hyperbolicity of the spacetime.

Remark 1.6. Since there exists a fibration of $\{\zeta \in \widetilde{\mathcal{N}} : p_\nu(\zeta) = 0\} \to \widetilde{\mathcal{N}}_\nu$, there exists an algebra (denoted $\mathcal{R}_\Sigma$ in [GS79]) of Fourier integral operators associated to this null foliation in the sense of [GS79]. However, the result of [GS79, Proposition 4.1] on existence of projections $\mathcal{R}_\Sigma$ is false in our setting (where the fibers are non-compact).

We defined the governing dynamical system in Section 0.5. Let us compare this flow with the 'governing Hamiltonian flow' in [GU89], especially in its applications to semi-classical Schrödinger operators. In place of (15) of this article, the $\mathbb{R} \times S^1$ action in [GU89] is generated by a positive elliptic operator P of order 1 and the same D_θ as in the present article. In the semi-classical application, one has Riemannian manifold N and defines $M = N \times S^1$. Let $P = \sqrt{-\Delta_N + VD_\theta^2}$ where V is the potential. Denote by $a = \sigma$, the generator of the S^1 action. Let $\{f_t\}$ be the Hamiltonian flow of the principal symbol p of P. Then the governing flow of [GU89] is the reduction of f_t on the co-isotropic submanifold $p = E, a = 1$. More precisely, let $Z = a^{-1}(1)$ and $B = Z/S^1$, and let $\pi : Z \to B$ be the natural projection, and let $\widetilde{p} \in C^\infty(B)$ be the reduced symbol. Let $\{\varphi_t\}$ be the Hamilton flow of $\widetilde{p}$ on B. The closed trajectories of φ_t on $\widetilde{p} = E$ determine the singularities of Υ. Since $Z = T^*N \times \{\sigma = 1\}$, $Z/S^1 = T^*N$, and the governing flow is that of $H = |\xi|^2 + V$ on $\{H = E^2\}$.

As explained above, in this article, $\widetilde{\mathcal{N}}$ replaces $T^*(N \times S^1)$. We set $\sigma = 1$ in $\widetilde{\mathcal{N}}$ and divide to get $\mathcal{N}_1$. We then set $p_Z = \nu$ on $\mathcal{N}_1$ and study the Hamiltonian flow e^{tZ} on this level set, analogously to the Hamiltonian flow on $\{H = E\}$ in the compact elliptic setting.

1.2. Semi-classical heuristics.
It would be natural to work semi-classically throughout rather than taking the product with S^1 and working homogeneously, which is rather artificial. To do this, one would need a semi-classical parametrix for the Green's functions of $\Box + m^2$ (see Section 4.) Such a parametrix exists on Minkowski space, but to our knowledge it has not been generalized to general stationary spacetimes. Perhaps for this reason, the physical interpretation of high mass asymptotics has not been developed. In some sense, the semi-classical parametrix is defined implicitly in this article as a Fourier coefficient of the

homogeneous parametrix on $\widetilde{M}$. We hope to develop this theory further in a later article. The strategy of studying semi-classical asymptotics on M by using homogeneous asymptotics on $M \times S^1$ was first used in [GU89].

Although we homogenized the ladder asymptotics problem in Section 0.1, it is useful to keep in mind the corresponding objects in the semi-classical, non-homogeneous, picture with large parameter m. The Klein-Gordon operator $m^{-2} \square_g + 1$ on M has semi-classical symbol $-g(\xi, \xi) + 1$, and the semi-classical characteristic variety is the unit mass hyperboloid (bundle) $g(\xi, \xi) = 1$ (see Section 2.2.) We denote its quotient by the (massive) geodesic flow by $\mathcal{N}_1$.

The semi-classical symbol of $m^{-1} D_Z$ is $p_Z(x, \xi) = \langle \xi, Z \rangle$ as above. Weyl asymptotics concerns the distribution of the eigenvalues $\lambda_j(m)$ with $m^{-1}\lambda_j(m) \simeq \nu$, which corresponds to the level set $p_Z = \nu$ on the quotient $\mathcal{N}_1$. Note that the homogeneous approach gives the same governing Hamiltonian flow.

1.3. **Organization.** In Section 2 we review the geometry of globally hyperbolic stationary spacetimes from a more 'global geometric' view than in [SZ18]. This will prove useful when we analyze the symplectic geometry of the null geodesics of $\widetilde{M} = M \times S^1$ and the symplectic reduction in Sections 2.2-2.3. In Section 4, we review the existence of advanced/retarded Green's functions for the Klein-Gordon equation on $\widetilde{M} = M \times S^1$ and the Hilbert space inner product (Definition 3.1), on $\mathcal{H}_{KG}$, which does not differ in any essential way from the discussion in [SZ18]. In Section 6 we discuss the quantum ladder subspaces $\mathcal{H}_{\nu,\psi}$ in relation to symplectic reduction. Using this material, we prove Theorem 1.1 in Section 5.1. We then derive Theorem 1.2 in Section 10. In Section 11 we prove Theorem 1.3. In Section 12 we specialize to product spacetimes to compare the result with standard Weyl asymptotics.

2. Geometry of globally hyperbolic stationary spacetimes

In this section, we briefly review the geometry of globally hyperbolic stationary spacetimes. Lorentzian manifolds with a complete timelike Killing vector field are called *stationary*. If (M, g) is a stationary globally hyperbolic spacetime then (cf. [SZ18])

$$(M, g) \simeq (\mathbb{R} \times \Sigma, -(N^2 - |\eta|_h^2)\mathrm{d}t^2 + \mathrm{d}t \otimes \eta + \eta \otimes \mathrm{d}t + h), \qquad (27)$$

where (Σ, h) is a Riemannian manifold, $N : \Sigma \to \mathbb{R}_+$ is a positive smooth function, and η a a covector field on Σ. In this case ∂_t is a Killing vector field. Such stationary spacetimes are sometimes referred to as *standard stationary spacetimes*. If (M, g) is a spatially compact stationary globally hyperbolic spacetime then (M, g) isometric to a product $\mathbb{R} \times \Sigma$ with metric (27). It is sometimes convenient to write this metric in the form

$$g = -N^2\mathrm{d}t^2 + h_{ij}(\mathrm{d}x^i + \beta^i\mathrm{d}t)(\mathrm{d}x^j + \beta^j\mathrm{d}t), \qquad (28)$$

where β (the shift vector field) is the vector field obtained from η by identifying vectors with covectors using h the above metric, i.e. $\beta^i = h^{ij}\eta_j$ in local coordinates. The coefficients are independent of t so that the vector field $Z = \frac{\partial}{\partial t}$ is a timelike Killing vector field.

We define $\mathcal{K} = M/\mathbb{R}$ to be the space of Killing orbits, i.e. the quotient of M by the $\mathbb{R}$ action defined by e^{tZ}. We thus have an $\mathbb{R}$-principal bundle

$$\pi : M \to \mathcal{K}$$

whose fibers are Killing orbits $[x] := \{e^{tZ} \cdot x \mid t \in \mathbb{R}\}$. A model for $\mathcal{K}$ is obtained by choosing a cross section to the Killing flow, and by definition any Cauchy hypersurface Σ defines such a cross section. A globally hyperbolic spacetime will be called *spatially compact* if there exists a compact Cauchy surface. In this case all Cauchy surfaces will be compact. Restriction of the the projection $M \to \mathcal{K}$ to Σ gives a diffeomorphism $\Sigma \simeq \mathcal{K}$. Thus, we obtain an identification of a product decomposition $\mathbb{R} \times \Sigma$ with $\mathbb{R} \times \mathcal{K}$. In terms of any choice of Cauchy hypersurface; we may also view the product decomposition as $\mathbb{R} \times \mathcal{K}$, namely the $\mathbb{R}$ fiber bundle $\pi : M \to \mathcal{K}$ is a product bundle.

As in the presentation of Cortier-Minerbe [CM16], the orthogonal distributions to the fibers determine a connection 1-form Θ for which $\Theta(Z) = 1$, $\mathcal{L}_Z \Theta = 0$. The metric g on the horizontal spaces $\ker \Theta$ induces a metric $g_{\mathcal{K}}$ on $\mathcal{K}$. Define u by $u^2 = -g(Z, Z)$. It is constant along the fibers (Killing orbits), hence defines a function on $\mathcal{K}$. The spacetime metric is then:

$$g = -u^2 \Theta \otimes \Theta + \pi^* g_{\mathcal{K}}, \quad u^2 = -g(Z, Z). \tag{29}$$

Indeed, by definition, g and $-u^2 \Theta \otimes \Theta + \pi^* g_{\mathcal{K}}$ agree on Z. Moreover they agree on the orthogonal complement to Z. Hence they agree on all vectors.

For $\widetilde{M} = M \times S^1$ as above, $\widetilde{g} = g + \mathrm{d}\theta^2$, so the standard metric on $\widetilde{M}$ has the form,

$$\widetilde{g} = -N^2 \mathrm{d}t^2 + h_{ij}(\mathrm{d}x^i + \beta^i \mathrm{d}t)(\mathrm{d}x^j + \beta^j \mathrm{d}t) + \mathrm{d}\theta^2. \tag{30}$$

2.1. Null geodesics.
Since $(\widetilde{M}, \widetilde{g})$ is the product of a Lorentzian space M and a Riemannian space S^1, its geodesic flow is the product flow, $\widetilde{G}^s(x, \xi, \theta, \sigma) = (G^s(x, \xi), G^s_{S^1}(\theta, \sigma))$ where $G^s_{S^1}$ is the Hamiltonian flow on $T^* S^1$ generated by σ^2. That is, the Hamiltonian generating the product flow is $q(x, t, \xi', \tau, \theta, \sigma) := g_x((\tau, \xi'), (\tau, \xi')) - \sigma^2$ and the Hamiltonian flow on $T^* \widetilde{M}$ is the product flow $G^s \times G^s_{S^1}$ on $T^* M \times T^* S^1$.

If $(\widetilde{M}, \widetilde{g})$ is globally hyperbolic and $\widetilde{\Sigma}$ a Cauchy surface then each element in $\widetilde{\mathcal{N}}$ intersects $\widetilde{\Sigma}$ exactly once. The tangent (co-)vector of the geodesic is lightlike, and its pullback to $\widetilde{\Sigma}$ defines a covector in $\dot{T}^* \widetilde{\Sigma}$. This defines a diffeomorphisms

$$\iota_\pm : \widetilde{\mathcal{N}}_\pm \to \dot{T}^* \widetilde{\Sigma}, \tag{31}$$

since, for each element $\eta \in \dot{T}^* \widetilde{\Sigma}$, there is precisely one lightlike future/past directed covector $\xi \in T^* \widetilde{M} \smallsetminus 0$ whose pull-back is η. We quote the following result from [SZ18, Proposition 2.1]:

Proposition 2.1. *If $(\widetilde{M}, \widetilde{g})$ is a globally hyperbolic spacetime then the symplectic structure on $\widetilde{\mathcal{N}}$ does not depend on the Cauchy surface, and for any Cauchy surface $\widetilde{\Sigma}$ the map (31) is a homogeneous symplectic diffeomorphism.*

2.2. Moment maps.
We return to Section 0.4 and continue the discussion there. There are three commuting operators, $\frac{1}{2}\widetilde{\Box}, D_Z, D_s$ on $\widetilde{M}$, whose principal symbols are Poisson-commuting

functions on $T^*\widetilde{M}$, given by

$$
\begin{cases}
\frac{1}{2}\sigma_{\widetilde{\square}}(\tilde{x}, \tilde{\xi}) = \frac{1}{2}\tilde{\xi} \cdot \tilde{\xi} := \frac{1}{2}\sum_{i,j} g^{ij}(x)\tilde{\xi}_i\tilde{\xi}_j, \\[2mm]
p_Z(\tilde{x}, \tilde{\xi}) = \sigma_{D_Z}(\tilde{x}, \tilde{\xi}) = \langle \tilde{\xi}, Z \rangle, \\[2mm]
p_\theta(\tilde{x}, \tilde{\xi}) = \sigma_{D_\theta} = \langle \tilde{\xi}, \frac{\partial}{\partial\theta} \rangle.
\end{cases}
\tag{32}
$$

They jointly define the moment map,

$$
\widetilde{\mathcal{P}} : T^*\widetilde{M} \to \mathbb{R}^3, \ \widetilde{\mathcal{P}}(\tilde{x}, \tilde{\xi}) = (\frac{1}{2}\sigma_{\widetilde{\square}}(\tilde{x}, \tilde{\xi}), \sigma_{D_Z}(\tilde{x}, \tilde{\xi}), p_\theta(\tilde{x}, \tilde{\xi})).
$$

If we omit the first component we get the homogeneous moment map (15).

Both Z and $\frac{\partial}{\partial\theta}$ are Killing vector fields, but Z is timelike and $\frac{\partial}{\partial\theta}$ is spacelike. This moment map is not homogeneous and it is better to reduce the dimension and define the degree one homogeneous moment map (19) of the Hamiltonian action of $\mathbb{R} \times S^1$ on $\widetilde{\mathcal{N}}$. The Hamiltonian flow of $\frac{1}{2}(\tilde{\xi}, \tilde{\xi})$ on $\mathrm{Char}(\widetilde{\square})$ is the null geodesic flow $\widetilde{G}^s$. We define the symplectic conic manifold of null geodesics of $\widetilde{M}$ as the orbit space (14). We refer to [SZ18, Section 2.1] for proof that it is symplectic. It inherits the conic structure from $\mathrm{Char}(\widetilde{\square})$. On $\widetilde{\mathcal{N}}$ we have the reduced moment map (19).

2.3. Ladder varieties, their projections and the Killing horizon.
The characteristic variety of $\widetilde{\square}$ is

$$
(\bigcup_{\sigma \in \mathbb{R}} H_\sigma \times S^1 \times \{\sigma\}) \smallsetminus 0 \subset \dot{T}^*\widetilde{M},
$$

where $H_\sigma = \{(x, t, \xi', \tau') \in T^*M : g_x((\tau, \xi'), (\tau, \xi')) = -\sigma^2\}$ is the mass hyperboloid bundle in T^*M. The additional equation $\tau = \nu\sigma$ of L_ν pulls out one sphere

$$
S_{\sigma,\nu} := \{(x, \xi) \in H_\sigma \mid \tau = \nu\sigma\}
$$

in the σ- hyperboloid. Thus, the double characteristic variety, the zero set of the map $(\frac{1}{2}\sigma_{\widetilde{\square}}, p_\nu)$, is

$$
\mathrm{DChar}_\nu = (\bigcup_{\sigma \in \mathbb{R}} S_{\sigma,\nu} \times S^1 \times \{\sigma\}) \smallsetminus 0.
$$

At the regular points of $(\frac{1}{2}\sigma_{\widetilde{\square}}, p_\nu)$ we have the Liouville density μ_{DChar_ν} induced by this map. The equations define a codimension two homogeneous co-isotropic submanifold of $\dot{T}^*\widetilde{M}$. The purpose of this section is, first, to specify this submanifold more precisely for a standard spacetime. Secondly, we determine its projection to $\widetilde{M}$ and compare it to the Killing horizon of $Z - \nu\frac{\partial}{\partial\theta}$. The projection may be viewed as the "classically allowed region" for our problem and we determine its boundary. We assume the metrics are in standard form (30) on $\widetilde{M}$ (see Definition 0.2) in coordinates (t, x, θ) relative to a splitting $M = \mathbb{R} \times \Sigma \times S^1 = \mathbb{R} \times \tilde{\Sigma}$ as in the previous section. We let τ, ξ, σ be the dual symplectic coordinates, i.e. they pick out the coefficients of $dt, dx_j, d\sigma$ in a covector $df(t, x, \theta)$. We denote by $\mathrm{DChar}_\nu(\theta, \sigma) \subset T^*M$ the slice of DChar_ν where (θ, σ) is held fixed.

A symplectic quotient of DChar_ν is obtained by setting $\sigma = 1$ and dividing by the S^1 action, and then dividing by the geodesic flow to obtain $\mathcal{N}_1(\nu) = \{p_Z = m\nu\} \subset \mathcal{N}_1$. This hypersurface also has a symplectic quotient since each mass 1 geodesic intersects Σ exactly once.

The projection of $\mathrm{DChar}_\nu(s, \sigma)$ is closely related to the set of points where the Lorentzian norm of the Killing vector field $Z - \nu \frac{\partial}{\partial \theta}$ vanishes. We will refer to this set as the Killing horizon $\widetilde{\mathcal{K}}$.

Definition 2.2. The Killing horizon $\widetilde{\mathcal{K}}$ of $Z - \nu \frac{\partial}{\partial \theta}$ is the subset in $\mathbb{R} \times \widetilde{M}$ where $g(Z, Z) = -\nu^2$.

Lemma 2.3. *The Killing horizon of $Z - \nu \frac{\partial}{\partial \theta}$ is the product $\mathbb{R} \times \mathcal{K}_\Sigma(\nu) \times S^1$, where*

$$\mathcal{K}_\Sigma(\nu) := \{|\beta|_h^2 = N^2 - \nu^2\}.$$

Proof. Consider $(x, \theta) \in \widetilde{\Sigma}$ and $V \in T_{(x,\theta)}\widetilde{\Sigma}$. Then $V = (V_M, V')$ where $V_M \in T_x M, V' \in T_\theta S^1$ and $\widetilde{g}(V, V) = g(V_M, V_M) + |V'|^2$. $\qquad \square$

The set

$$\mathcal{A}_\nu := \{(x, \theta) \in \widetilde{\Sigma} \mid N^2 - |\beta|_h^2 > \nu^2\} \tag{33}$$

is the set where $Z - \nu \frac{\partial}{\partial \theta}$ is spacelike, and we give it the term 'allowed region' by analogy with allowed regions $\{V \leq E\}$ for non-relativistic Schrödinger operators $-h^2\Delta + V$. The region where $Z - \nu \frac{\partial}{\partial \theta}$ is timelike plays the role of the 'forbidden' region.

We now determine DChar_ν in phase space. For any manifold X, let $\pi : T^*X \to X$ be the natural projection.

Proposition 2.4. *In a standard spacetime, for fixed (θ, σ), $\mathrm{DChar}_\nu(\theta, \sigma)$ is the set of solutions $(t, x; \tau, \xi, \sigma)$ of the equation,*

$$(\tau - (\beta, \xi))^2 - N^2 h^{-1}(\xi, \xi) - N^2\sigma^2 = 0, \quad \tau = \nu\sigma,$$

or equivalently,

$$(\tau(\nu^2 - N^2) - (\beta, \xi)\nu^2)^2 = \nu^2 N^2 \left((\beta, \xi)^2 + h^{-1}(\xi, \xi)\left(\nu^2 - N^2\right)\right), \quad \tau = \nu\sigma,$$

and $\mathrm{DChar}_\nu = \bigcup_{(\theta,\sigma)\in T^*S^1} \mathrm{DChar}_\nu(\theta, \sigma) \times \{(\theta, \sigma)\}$. The projection of $\mathrm{DChar}_\nu(\theta, \sigma)$ to M is the set (33) of points where $Z - \nu \frac{\partial}{\partial \theta}$ is spacelike. The boundary of the projection is the Killing horizon.

Proof. Let $\widetilde{H}(t, x, \theta; \tau, \xi, \sigma)$ denote the norm-square function $\widetilde{\xi} \in T^*\widetilde{M} \to \widetilde{g}^*(\widetilde{\xi}, \widetilde{\xi})$ of the dual co-metric metric $\widetilde{g}^*$ on $T^*\widetilde{M}$. The dual co-metric has the form,

$$\widetilde{g}^{-1} = N^{-2} \begin{pmatrix} -1 & \beta & 0 \\ \beta^T & N^2 h^{-1} - \beta \otimes \beta & 0 \\ 0 & 0 & N^2 \end{pmatrix},$$

and a little bit of algebra gives,

$$\begin{aligned} \widetilde{H} &= -N^{-2}\tau^2 + 2N^{-2}\tau(\beta, \xi) + (h^{-1} - N^{-2}\beta \otimes \beta)(\xi, \xi) + \sigma^2 \\ &= -N^{-2}[(\tau - (\beta, \xi))^2 - (\beta, \xi)^2 - (N^2 h^{-1} - \beta \otimes \beta)(\xi, \xi) - N^2\sigma^2]. \end{aligned}$$

hence the double characteristic variety is defined by

$$\begin{cases} (\tau - (\beta, \xi))^2 - N^2 h^{-1}(\xi, \xi) - N^2 \sigma^2 = 0, \\ \\ \tau = \nu\sigma. \end{cases}$$

Eliminating σ by the second equation, as in (18), we get a constraint equation for $(\tau, \xi) \in T^*M$:

$$\left(\tau(\nu^2 - N^2) - (\beta, \xi)\nu^2\right)^2 = \nu^2 N^2 \left((\beta, \xi)^2 + h^{-1}(\xi, \xi)(\nu^2 - N^2)\right), \tag{34}$$

Obviously, a necessary condition that $(t, x, \theta; \tau, \xi, \sigma)$ solve the equation is that (x, ξ) lies in the following set,

$$\mathcal{D}_\Sigma(\nu) := \{(x, \xi) \in T^*\Sigma \mid \frac{(\beta, \xi)^2}{h^{-1}(\xi, \xi)} \ge (N^2 - \nu^2)\}. \tag{35}$$

Since $h(\beta, \beta) \ge \frac{(\beta, \xi)^2}{h^{-1}(\xi, \xi)}$ for all $\xi \ne 0$, the projection of $\mathcal{D}_\Sigma$ to Σ lies in (33). When $Z - \nu\frac{\partial}{\partial\theta}$ is timelike, the orthogonal complement of $Z - \nu\frac{\partial}{\partial\theta}$ has empty intersection with the null cone bundle. It follows that the projection of DChar_ν is contained in (33) . Conversely, if $(t, x, \theta) \in \mathcal{A}_\nu$ then $\xi = \beta \in \mathcal{D}_\Sigma$. Hence, the projection equals (33).

Since the equations are independent of t, $\pi\mathrm{DChar}_\nu$ is the set $\mathbb{R} \times \mathcal{D}_\Sigma(\nu) \times S^1$. For $(x, \xi) \in \mathcal{D}_\Sigma(\nu)$, there exist two solutions $(t, x, \tau, \xi) \in \mathrm{DChar}_\nu(s, \sigma)$ (with multiplicity) of the defining equation in Lemma 2.4, and DChar_ν is the union over $\mathcal{D}_\Sigma(\nu) \times S^1 \times \mathbb{R}$ of the pair of solutions times $\{(\theta, \nu^{-1}\tau)\}$.

$$\square$$

The reduction DChar_ν with respect to the Hamiltonian flow of $\sigma_{\widetilde{\square}}$ and further reduction of the level set $\sigma = m$ gives the level set $\tau = \nu m$ on $\mathcal{N}_m$ and therefore $\mathcal{N}_m(\nu)$.

Proposition 2.5. *The Liouville measure μ of the level set $\tau - \nu\sigma = 0$ in $\mathcal{N}_m$ is given by*

$$\mu = m^{n-2}\mu_{\mathcal{N}_1(\nu)}(\mathcal{N}_1(\nu)) = m^{n-2}\mathrm{Vol}(S_{n-2}) \int_\Sigma \frac{N\nu}{(N^2 - \beta^2)^{\frac{n}{2}}} \left(\nu^2 - (N^2 - |\beta|_h^2)\right)_+^{\frac{n-3}{2}} \mathrm{dVol}_h.$$

Proof. We first remark that reduction with respect to the Hamiltonian flow of $\frac{1}{2}\sigma_{\widetilde{\square}}$ can be achieved by choosing $T^*\widetilde{\Sigma}$ as a cross section. The level $\sigma = m$ and further reduction shows that $\mathcal{N}_m$ can be identified with $H_m|_\Sigma \times \{1\} \times \{m\} \subset T^*\widetilde{M}|_\Sigma$.

We compute the volume over a point in $\mathbb{R} \times \Sigma \times S^1$ with respect to Riemann normal coordinates y on Σ, so that h at this point is the identity matrix. We can choose these coordinates in such a way that $dy_1(\beta) = |\beta|_h$ and $dy_k(\beta) = 0$ for $k > 1$. We denote the coordinates of a covector in $T^*\Sigma$ in this coordinate system by $(\xi_1, \xi_\perp)$, where $\xi_\perp = (\xi_2, \ldots, \xi_{n-1})$. Inserting $\tau = \nu m + q$ in the equation for the unit hyperboloid bundle we obtain

$$(\nu m + q - |\beta|_h\xi_1)^2 = N^2\xi^2 + N^2\sigma^2 = N^2\xi_1^2 + N^2\xi_\perp^2 + N^2 m^2.$$

A little bit of algebra gives

$$\left(\xi_1 + \frac{|\beta|_h(\nu m + q)}{N^2 - |\beta|_h^2}\right)^2 (N^2 - |\beta|_h^2) + N^2\xi_\perp^2 = \frac{N^2 m^2}{N^2 - |\beta|_h^2} \left((\nu + q/m)^2 - (N^2 - |\beta|_h^2)\right),$$

which defines a distorted ellipsoid of volume

$$\mathrm{Vol}(B_{n-1})\frac{Nm^{n-1}}{(N^2-|\beta|_h^2)^{\frac{n}{2}}}\left((\nu+q/m)^2-(N^2-|\beta|_h^2)\right)^{\frac{n-1}{2}},$$

taking the derivative in q gives

$$\mathrm{Vol}(S_{n-1})\frac{Nm^{n-2}(\nu+q/m)}{(N^2-|\beta|_h^2)^{\frac{n}{2}}}\left((\nu+q/m)^2-(N^2-|\beta|_h^2)\right)^{\frac{n-3}{2}},$$

which yields the claimed formula for $q = 0$. Setting $m = 1$ gives the Liouville measure for $\mathcal{N}_1(\nu)$. $\qquad\square$

We close this section with the following observation:

Lemma 2.6. *If $\widetilde{\mathcal{K}}$ is regular and $(t,x,\theta) \in \widetilde{\mathcal{K}}$ then $(t,x,\theta,Z-\nu\frac{\partial}{\partial\theta}) \in \mathrm{DChar}_\nu$. Hence, $N^*(\widetilde{\mathcal{K}})$ is a Lagrangian submanifold of $T^*\widetilde{M}$ contained in DChar_ν.*

Proof. If $(t,x,\theta) \in \widetilde{\mathcal{K}}$, then by definition, identifying covectors and vectors using the metric, $(t,x,\theta,Z-\nu\frac{\partial}{\partial\theta}) \in \mathrm{Char}(\widetilde{\Box})$. Also, if $(t,x,\theta) \in \widetilde{\mathcal{K}}$ then $p_\nu(Z-\nu\frac{\partial}{\partial\theta}) = \langle(Z-\nu\frac{\partial}{\partial\theta}),(Z-\nu\frac{\partial}{\partial\theta})\rangle = 0$. Since $\widetilde{\mathcal{K}}$ has codimension one in $\widetilde{M}$, $\{(t,x,\theta,r(Z-\nu\frac{\partial}{\partial\theta})) : (t,x,\theta) \in \widetilde{\mathcal{K}}, r \in \mathbb{R}\} = N^*(\widetilde{\mathcal{K}})$ is a Lagrangian submanifold of $T^*\widetilde{M}$ contained in DChar_ν. $\qquad\square$

2.4. **Admissibility.** We now give equivalent criteria for admissibility in the sense of Definition 0.3.

Proposition 2.7. *ν is admissible unless $\nu = \sqrt{N^2-|\beta|^2}$ at a point $x \in \Sigma$ where $d(N^2-|\beta|^2) = 0$. In particular, if $\nu > \max_{x\in\Sigma}\sqrt{N^2-|\beta|^2}$, then ν is admissible and $Z-\nu\frac{\partial}{\partial\theta}$ is everywhere spacelike.*

Proof. Setting $\sigma = 1$ in DChar produces the equation for $\mathcal{N}_1 \subset T^*M$:

$$(\tau - (\beta,\xi))^2 = N^2\left(h^{-1}(\xi,\xi)+1\right) \iff \tau = (\beta,\xi) \pm N\sqrt{(h^{-1}(\xi,\xi)+1)}.$$

This equation is independent of t and we may regard it as an equation on $T^*\Sigma$. Hence we would like levels ν and find (x,ξ) such $\tau = \nu$ and that

$$d\tau = \mathrm{d}(\beta,\xi) \pm \mathrm{d}\left[N\sqrt{(h^{-1}(\xi,\xi)+1)}\right] = 0.$$

Take the derivative in ξ first, a necessary condition is,

$$\beta = \pm\frac{Nd_\xi h^{-1}(\xi,\xi)}{\sqrt{(h^{-1}(\xi,\xi)+1)}} = \pm\frac{Nh^{-1}(\xi)}{\sqrt{(h^{-1}(\xi,\xi)+1)}} \iff \frac{\beta}{N} = \pm\frac{h^{-1}(\xi)}{\sqrt{(h^{-1}(\xi,\xi)+1)}}.$$

For $\beta(x) \neq 0$, there exist two distinct solutions in ξ. We claim that the solution in the future sheet is,

$$\xi = \frac{h(\beta,\cdot)}{\sqrt{N^2-|\beta|^2}} = \frac{\beta}{N}.$$

One has the corresponding values $\sqrt{h^{-1}(\xi,\xi)+1} = \frac{N}{\sqrt{N^2-|\beta|^2}}$, and the value of τ in the future pointing sheet is,

$$\tau_+ \;=\; \sqrt{N^2 - |\beta|^2}.$$

It is the minimum value of τ on the forward sheet, and this fiber-wise critical point exists and is unique (on the forward sheet) for all $(t,x) \in M$. It follows that, if $\nu > \max_{x\in\Sigma}\sqrt{N^2-|\beta|^2}$, then ν is a regular value.

We have not yet considered derivatives in x, so that these values of τ_+ are only necessary conditions for a critical point. Fully critical points are those points $x \in \Sigma$ which additionally satisfy $d(N^2 - |\beta|^2) = 0$.

$\square$

By Lemma 2.3, the Killing horizon of $Z - \nu\frac{\partial}{\partial\theta}$ is the product $\mathbb{R}\times\mathcal{K}_\Sigma(\nu)\times S^1$, where $\mathcal{K}_\Sigma(\nu) :=$ $\{(x,\theta) \in \widetilde{M} : \nu^2 = N^2 - |\beta|_h^2\}$. Thus it consists of points (t,x,θ) where $(t,x) \in M$ and ν is the 'height' τ of the bottom of the unit mass hyperboloid. Since the bottom exists over all $(t,x) \in M$, admissibility is not simply the condition that ν is never the bottom height, unless (as in the case of product spacetimes), the bottom height is constant. Indeed, by Sard's theorem, the set of critical values has measure zero, while the bottom height usually fills out an interval.

3. Hilbert space topology on $\mathcal{H}_{KG}$ and quantum ladders

The space $\ker\widetilde{\square}\cap C^\infty$ of smooth solutions of $\widetilde{\square}u = 0$ is naturally a symplectic space, with symplectic form defined by

$$\sigma(u,v) = \int_{\widetilde\Sigma}\left((\nu_{\tilde x}u)(\tilde x)v(\tilde x) - v(\tilde x)(\nu_{\tilde x}u)(\tilde x)\right)d\mathrm{Vol}_{\widetilde\Sigma}, \tag{36}$$

where integration is over any Cauchy surface $\widetilde\Sigma$, and ν denotes the future directed unit normal vector field to $\widetilde\Sigma$. By Green's identity, the definition does not depend on the choice of Cauchy surface.

We now equip $\ker\widetilde\square\cap C^\infty$, as in (5) with the energy inner product. Let $\widetilde g = g + d\theta^2$ be the Lorentzian metric on $\widetilde M$. For $u \in C^\infty(\widetilde M)$, we define the stress-energy tensor $T(u)$ by

$$T(u) := d\overline{u}\otimes du - \frac{1}{2}|du|^2\widetilde g. \tag{37}$$

Definition 3.1. The energy (quadratic) form on the space $\ker\widetilde\square\cap C^\infty(\widetilde M)$ is defined by polarization of

$$Q(u,u) = \int_{\widetilde\Sigma}\langle T(u)(Z),\nu\rangle d\mathrm{Vol}_{\widetilde\Sigma}$$

where ν is the unit normal to $\widetilde\Sigma$, a spacelike hypersurface and Z is the timelike Killing vector field.

The following is a standard observation:

Lemma 3.2. *If* $\widetilde\square u = 0$ *then the covector field* $T(u)(Z)$ *is divergence free.*

We extend the definition of the quadratic form in a sesquilinear manner to complex valued functions. This quadratic form is independent of the chosen Cauchy surface. As verified in [SZ18, Section 3.2],

Lemma 3.3. *The energy quadratic form is invariant under the Killing flow, i.e.*

$$Q(e^{\mathrm{i}sD_Z}u, e^{\mathrm{i}sD_Z}u) = Q(u, u)$$

for all $u \in \ker \widetilde{\square}$ and related to the symplectic form by

$$Q(u, v) = \frac{\mathrm{i}}{2}\sigma(\bar{u}, D_Z v) = \frac{1}{2}\sigma(\bar{u}, \mathcal{L}_Z v).$$

The quadratic form is positive definite in $\mathrm{d}u$ and therefore only positive semi-definite on $\ker \widetilde{\square}$. The kernel of Q is one dimensional and spanned by the constant function. In $\ker \widetilde{\square}$ we have the two dimensional invariant subspace spanned by the functions 1 and t. On this subspace the operator D_Z is not diagonalisable but has a nontrivial Jordan-block. We will denote this space spanned by $\{1, t\}$ by V_0 as it is the generalized eigenspace of the operator D_Z with eigenvalue 0. If V is the symplectic complement of V_0 then V is an invariant for D_Z and we have

$$\ker \widetilde{\square} \cap C^\infty = V_0 \oplus V.$$

This direct sum is orthogonal with respect to Q and Q restricted to V is positive definite. We can therefore complete V to a Hilbert space $\mathcal{H}^c_{KG}$. Choosing any inner product on V_0 this defines the topology of a Hilbert space (a Hilbertisable locally convex topology) on $\ker \widetilde{\square} \cap C^\infty$. The completion with respect to the uniform structure induced by this locally convex topology then is the space

$$\mathcal{H}_{KG} = V_0 \oplus \mathcal{H}^c_{KG}.$$

The symplectic form extends to $\mathcal{H}_{KG}$ and it is easy to see using the explicit representation in standard form that $\mathcal{H}_{KG}$ coincides with the space of solutions with Cauchy data in $H^1(\widetilde{\Sigma}) \oplus L^2(\widetilde{\Sigma})$ and that the Cauchy data map is a continuous bijection (see [SZ18, Section 3] for details). Since the space V_0 is finite dimensional $\mathcal{H}_{KG}$ is essentially a Hilbert space.

Using Theorem 5.1 below, the following is proved in [SZ18, Theorem 5.2]. We simplify the statement because the potential V in that article is zero here.

Theorem 3.4. *Suppose that $(\widetilde{M}, \widetilde{g})$ is a spatially compact globally hyperbolic stationary space-time. Then D_Z is a self-adjoint operator on $\mathcal{H}^c_{KG}$ with discrete spectrum that consists of infinitely many real eigenvalues of finite multiplicity that accumulate at $-\infty$ and $+\infty$.*

Since complex conjugation anti-commutes with D_Z the spectrum is symmetric about the origin. This means apart from the non-trivial Jordan block in V_0 the operator D_Z can be completely diagonalised.

4. Fundamental solutions and parametrices on $M \times S^1$

In this section, we review the basic results on retarded/advanced Green's functions on globally hyperbolic spacetimes, and apply them to $\widetilde{M} = M \times S^1$.

We consider the Klein-Gordon operator $\widetilde{\Box} = \Box_g - \frac{\partial^2}{\partial\theta^2}$. A *fundamental solution* of $\widetilde{\Box}$ is a distribution kernel $F : C_0^\infty(\widetilde{M}) \to C^\infty(\widetilde{M})$ such that $\widetilde{\Box}F = F\widetilde{\Box} = \mathrm{id}_{C_0^\infty(\widetilde{M})}$. A *parametrix* is a map $F : C_0^\infty(\widetilde{M}) \to C^\infty(\widetilde{M})$ such that $\widetilde{\Box}F = F\widetilde{\Box} = \mathrm{id}_{C_0^\infty(\widetilde{M})}$ mod C^∞. A fundamental solution $E_{\mathrm{ret/adv}}(x,y)$ is called *retarded/advanced* if $\mathrm{supp}\,(E_{\mathrm{ret/adv}}f) \subset J_\pm(\mathrm{supp}\,f)$, where $J_\pm(K)$ refers to the causal future/past of a subset $K \subset M$.

Let $T_0^*\widetilde{M}$ be the set of null covectors (the bundle of light-cones in cotangent space). Then $T_0^*\widetilde{M} \smallsetminus 0$ is a closed conic subset in $T^*\widetilde{M}$ and the set

$$\widetilde{C} = \{(\tilde{x},\tilde{\xi},\tilde{x}',\tilde{\xi}') \in (T_0^*\widetilde{M} \smallsetminus 0)^2 \mid (\tilde{x},\tilde{\xi}) \in \mathrm{Char}(\widetilde{\Box}), (\tilde{x},\tilde{\xi}) = \widetilde{G}^s(\tilde{x}',\tilde{\xi}') \text{ for some } s \in \mathbb{R}\} \quad (38)$$

defines a homogeneous canonical relation from $T^*\widetilde{M} \smallsetminus 0$ to $T^*\widetilde{M} \smallsetminus 0$. Here $\widetilde{G}^s$ denotes the null geodesic flow of $\widetilde{M}$.

A basic result of Duistermaat-Hörmander [DH72, Theorem 6.5.3] (see also Theorem 3.4.7 in [BGP] and [SZ18, Theorem 7.1]) is the following,

Theorem 4.1. *If $(\widetilde{M}, g)$ is a globally hyperbolic spacetime, then there exist unique retarded fundamental solutions $\widetilde{E}_{\mathrm{ret/adv}}$ for $\widetilde{\Box}$. The difference $\widetilde{E} = \widetilde{E}_{\mathrm{ret}} - \widetilde{E}_{\mathrm{adv}}$ is a Fourier integral operator in $I^{-\frac{3}{2}}(\widetilde{M} \times \widetilde{M}, \widetilde{C}')$ with principal symbol $\mp\frac{i}{2}\sqrt{2\pi}|\mathrm{d}s|^{\frac{1}{2}} \otimes |\mu_{\mathrm{char}(\frac{1}{2}\widetilde{\Box})}|^{\frac{1}{2}}$ on $C_\pm'$.*

4.1. Cauchy extension and restriction operators.

We now define Cauchy extension and restriction operators for general globally hyperbolic spacetimes $\widetilde{M}$ with compact Cauchy hypersurfaces $\widetilde{\Sigma}$.

For $f, g \in C_0^\infty(\widetilde{\Sigma})$ we define the distributions $\delta_{\widetilde{\Sigma},g}$ and $\delta_{\widetilde{\Sigma},f}'$ by

$$\delta_{\widetilde{\Sigma},g}(\varphi) = \int_{\widetilde{\Sigma}} g(\tilde{x})\varphi(\tilde{x})\,\mathrm{dVol}_{\widetilde{\Sigma}}(\tilde{x}), \quad \delta_{\widetilde{\Sigma},f}'(\varphi) = -\int_{\widetilde{\Sigma}} f(\tilde{x})(\nu_{\widetilde{\Sigma}}\varphi)(\tilde{x})\,\mathrm{dVol}_{\widetilde{\Sigma}}(\tilde{x}). \quad (39)$$

where $\nu_{\widetilde{\Sigma}}$ is the future directed normal vector field to $\widetilde{\Sigma}$.

Definition 4.2. We define the Cauchy extension operators $\mathcal{E}_0, \mathcal{E}_1$ and $\mathcal{E}$ as the operators

$$\mathcal{E}_0 : H^1(\widetilde{\Sigma}) \to \ker\widetilde{\Box}, \quad f \mapsto \widetilde{E}(\delta_{\widetilde{\Sigma},f}'),$$

$$\mathcal{E}_1 : L^2(\widetilde{\Sigma}) \to \ker\widetilde{\Box}, \quad g \mapsto \widetilde{E}(\delta_{\widetilde{\Sigma},g}),$$

$$\mathcal{E} : H^1(\widetilde{\Sigma}) \oplus L^2(\widetilde{\Sigma}) \to \ker\widetilde{\Box}, \quad (f,g) \mapsto \widetilde{E}(\delta_{\widetilde{\Sigma},f}' + \delta_{\widetilde{\Sigma},g}) = \mathcal{E}_0 f + \mathcal{E}_1 g.$$

By the mapping properties of Fourier integral operators in $I^{-\frac{3}{2}}(\widetilde{M} \times \widetilde{M}, \widetilde{C}')$ the function $u = \mathcal{E}(f,g)$ is well defined for $f, g \in C^\infty(\widetilde{\Sigma})$, and by Green's identity applied to the causal future of $\widetilde{\Sigma}$, u is the unique solution of the Cauchy problem

$$\widetilde{\Box}u = 0, \quad (f,g) = (u|_{\widetilde{\Sigma}}, \nu_{\widetilde{\Sigma}}u|_{\widetilde{\Sigma}}).$$

Indeed, by the support properties of retarded and advanced fundamental solutions we have $u = u_+ - u_-$, where $u_\pm = \widetilde{E}_{\mathrm{ret/adv}}(\delta_{\widetilde{\Sigma},f}' + \delta_{\widetilde{\Sigma},g})$ is supported in $J^\pm(\widetilde{\Sigma}) =: \widetilde{M}_\pm$. If $\varphi \in C_0^\infty(\widetilde{M})$ is an

arbitrary test function, then,

$$(\delta'_{\widetilde{\Sigma},f} + \delta_{\widetilde{\Sigma},g}, \varphi) = (u_+, \widetilde{\square}\varphi) = \int_{\widetilde{M}_+} (u\widetilde{\square}\varphi)\mathrm{dVol}_g$$

$$= \int_{\widetilde{M}_+} (u\widetilde{\square}\varphi - \varphi\widetilde{\square}u)\mathrm{dVol}_g = \int_{\widetilde{\Sigma}} -u(\tilde{x})(\nu_{\widetilde{\Sigma}}\varphi)(\tilde{x}) + \varphi(\tilde{x})(\nu_{\widetilde{\Sigma}}u)(\tilde{x})\mathrm{dVol}_{\widetilde{\Sigma}}(\tilde{x}).$$

Since this is true for all test functions φ this shows u has Cauchy data (f, g).

The following corollary is well known and can also be proved more directly [D96, Theorem 5.1.2].

Corollary 4.3. *If $(\widetilde{M}, g)$ is a globally hyperbolic spacetime, and $\widetilde{\Sigma}$ is a Cauchy hypersurface, then the Cauchy extension operators $\mathcal{E}_0$ and $\mathcal{E}_1$ of Definition 4.2 are Fourier integral operators in $I^{-\frac{1}{4}}(\widetilde{M} \times \widetilde{\Sigma}, \widetilde{C_\Sigma}')$ and $I^{-\frac{5}{4}}(\widetilde{M} \times \widetilde{\Sigma}, \widetilde{C_\Sigma}')$, respectively, where*

$$\widetilde{C_\Sigma} = \{(\tilde{x}, \tilde{\xi}, \tilde{q}, \tilde{p}) \in T^*\widetilde{M} \times T^*\widetilde{\Sigma} : g(\tilde{\xi}, \tilde{\xi}) = 0; \exists \tilde{\eta} \in T^*_{\widetilde{\Sigma}}\widetilde{M} : \tilde{\eta}|_{T_q\widetilde{\Sigma}} = \tilde{p}, \exists s : \widetilde{G}^s(\tilde{q}, \tilde{\eta}) = (\tilde{x}, \tilde{\xi})\}.$$

Proof. We may view $\mathcal{E} = \widetilde{E} \circ \delta$ where $\delta : H^1(\widetilde{\Sigma}) \oplus L^2(\widetilde{\Sigma}) \to \mathcal{D}'(\widetilde{M})$ is defined by $\delta(f, g) = (\delta'_{\widetilde{\Sigma},f} + \delta_{\widetilde{\Sigma},g})$. We then obtain the wave front relation of $\mathcal{E}$ as the composition of the wave front relations of $\widetilde{E}$ given in Theorem 4.1 and the wave front relation of δ, or equivalently the wave front relations of $\delta'_{\widetilde{\Sigma},f}, \delta_{\widetilde{\Sigma},g})$. Both of these are extension operators, extending a distribution on $\widetilde{\Sigma}$ to a distribution on $\widetilde{M}$; they are easily seen to be adjoints of the restriction operators $\gamma_{\widetilde{\Sigma}}, \gamma^\nu_{\widetilde{\Sigma}}$, where $\gamma_{\widetilde{\Sigma}}(u) = u|_{\widetilde{\Sigma}}, \gamma^\nu_{\widetilde{\Sigma}}(u) = (\partial_\nu u)|_{\widetilde{\Sigma}}$. It is well-known that the wave front relation of either restriction operator is the graph of the restriction of a covector ν on $\widetilde{M}$ to $T\widetilde{\Sigma}$, and the wave front relation of the extension operator is the adjoint relation. The composition is transversal and hence is a Fourier integral operator. $\qquad\square$

We further define the Cauchy restriction operator as follows:

Definition 4.4. Let Σ be any Cauchy hypersurface in M and let $\widetilde{\Sigma} = \Sigma \times S^1$. Define the associated Cauchy restriction map by,

$$\mathcal{R}_{\widetilde{\square}} : \ker \widetilde{\square} \to H^1(\widetilde{\Sigma}) \oplus L^2(\widetilde{\Sigma}), \quad \mathcal{R}_{\widetilde{\square}}(u) = (u|_{\widetilde{\Sigma}}, \partial_{\nu_{\widetilde{\Sigma}}} u)|_{\widetilde{\Sigma}}.$$

Comparison with (39) shows that if $u = \mathcal{E}(f, g)$ then, $(u|_{\widetilde{\Sigma}}, \partial_{\nu_{\widetilde{\Sigma}}} u)|_{\widetilde{\Sigma}} = (f, g)$, i.e. $R_{\widetilde{\square}}$ is invertible as a bounded operator on (5) and

$$\mathcal{R}_{\widetilde{\square}}^{-1} = \mathcal{E}. \tag{40}$$

Although $\mathcal{R}_{\widetilde{\square}}, \mathcal{E}$ obviously depend on the choice of Cauchy hypersurface, we suppress the dependence in the notation.

We also need the 'global' restriction operator on all of $C^1(\widetilde{M})$, defined by the same formula as in Definition 4.4.

Lemma 4.5. *The global restriction operator $\mathcal{R}$ is in $I^{1/4}(\widetilde{\Sigma} \times \widetilde{M}, \Lambda)$, where*

$$\Lambda = \{(\tilde{q}, \tilde{\eta}; \tilde{q}, \tilde{\xi}) \mid \tilde{q} \in \widetilde{\Sigma}, \tilde{\xi}|_{T\widetilde{\Sigma}} = \tilde{\eta}\}.$$

We refer to [D96, Page 113] for the proof. Another proof is given in [ToZ13, Section 5.2], along with a discussion of a complication in the statement due to normal and tangential directions to $\widetilde{\Sigma}$. Namely, the canonical relation contains points $(\tilde{q}, 0; \tilde{q}, \tilde{\xi})$ when $\tilde{\xi} \in N^*\widetilde{\Sigma}$ and such points are not allowed in the Hörmander definition of homogeneous Fourier integral operators in $I^k(X, \Lambda)$. To deal with this problem, one needs to introduce a cutoff away from normal direction, as in [ToZ13, Section 5.1], and then the cutoff restriction operator does satisfy the statement of Lemma 4.5. As long as such conormal directions do not arise in the compositions defining the trace, they may be ignored. For the sake of expository brevity, we ignore the conormal directions in Lemma 4.5 and only verify during the trace calculations that they do not arise in the compositions.

5. The wave trace of D_Z in $\mathcal{H}_{KG}$ and ladder Hilbert spaces

In this section we review [SZ18, Theorem 4.1] and [SZ18, Theorem 5.2], since we need these results in the present article. Define $\widetilde{U}(s)$ to be translation by e^{sZ} on $\ker \widetilde{\Box}$. This clearly extends to the spaces $\mathcal{H}_{KG}$ and $\mathcal{H}_{KG}^c$.

Theorem 5.1. *Suppose that $\varphi \in C_0^\infty(\mathbb{R})$. Then the operator*

$$\widetilde{U}_\varphi = \int_{\mathbb{R}} \varphi(s)\widetilde{U}(s)\mathrm{d}s : \mathcal{H}_{KG} \to \mathcal{H}_{KG}$$

is trace-class, and its trace equals

$$\mathrm{Tr}\left(\widetilde{U}_\varphi\right) = \int_{\widetilde{\Sigma}} \int_{\mathbb{R}} \varphi(s)\left(\nu_{\tilde{x}}e^{i(D_Z)_{\tilde{x}}t}\widetilde{E}(\tilde{x}, \tilde{y}) - \nu_{\tilde{y}}e^{i(D_Z)_{\tilde{x}}s}\widetilde{E}(\tilde{x}, \tilde{y})\right)\mathrm{d}s|_{\tilde{y}=\tilde{x}} \, \mathrm{dV}_{\widetilde{\Sigma}(\tilde{x})}.$$

As explained in [SZ18, Section 4], a coordinate invariant way to state Theorem 5.1 is that

$$\mathrm{Tr}\left(\widetilde{U}(s)\right) = \int_{\widetilde{\Sigma}} * \left(\mathrm{d}_{\tilde{x}}\widetilde{E}_s(\tilde{x}, \tilde{y}) - \mathrm{d}_{\tilde{y}}\widetilde{E}_s(\tilde{x}, \tilde{y})\right)|_{\tilde{y}=\tilde{x}}, \tag{41}$$

where $*$ is the Hodge star operator on M and where

$$\widetilde{E}_s(\tilde{x}, \tilde{y}) = e^{i(D_Z)_{\tilde{x}}s}\widetilde{E}(\tilde{x}, \tilde{y}). \tag{42}$$

Since E is skew-symmetric and commutes with the flow,

$$\widetilde{E}_s(\tilde{x}, \tilde{y}) = -\widetilde{E}_{-s}(\tilde{y}, \tilde{x}).$$

Hence, we also have

Corollary 5.2. *The distributional trace defined above equals*

$$\mathrm{Tr}\left(\widetilde{U}(s)\right) = \int_{\widetilde{\Sigma}} * \left(\mathrm{d}_{\tilde{x}}(\widetilde{E}_s(\tilde{x}, \tilde{y}) + \widetilde{E}_{-s}(\tilde{x}, \tilde{y}))\right)|_{\tilde{y}=\tilde{x}}. \tag{43}$$

5.1. **The trace of $U(t, s)$.** We now consider the joint propagator

$$U(t, s) := e^{itD_Z}e^{isD_\theta} \tag{44}$$

We also define,

$$\widetilde{E}_{t,s} := e^{i(D_Z)t}e^{i(D_\theta)s}\widetilde{E}. \tag{45}$$

Since $\widetilde{E}$ is skew-symmetric and commutes with the flow,

$$\widetilde{E}_{t,s} = -\widetilde{E}_{-t,-s}.$$

Similar to [SZ18, Theorem 4.1], we have,

Theorem 5.3. *Suppose that* $\varphi \in C_0^\infty(\mathbb{R} \times S^1)$. *Then the operator*

$$U_\varphi = \int_{\mathbb{R}} \int_{S^1} \varphi(t,s) U(t,s) \mathrm{d}t \mathrm{d}s : \mathcal{H}_{KG} \to \mathcal{H}_{KG}$$

is trace-class, and its trace equals

$$\operatorname{Tr}_{\mathcal{H}_{KG}}(U_\varphi) = \int_{\Sigma \times S^1} \int_{\mathbb{R}} \varphi(t,s) \left(\nu_{\tilde{x}} e^{i(D_Z)_{\tilde{x}} t} e^{i(D_\theta)s} \widetilde{E}(\tilde{x},\tilde{y}) - \nu_y e^{i(D_Z)_{\tilde{x}} t} e^{i(D_\theta)s} \widetilde{E}(\tilde{x},\tilde{y}) \right) \mathrm{d}t|_{\tilde{y}=\tilde{x}} \mathrm{d}\mathrm{Vol}_{\tilde{\Sigma}}$$

The integral is independent of the choice of Cauchy surface Σ.

There is a more coordinate invariant way to write this expression, namely,

$$\operatorname{Tr}_{\mathcal{H}_{KG}}(U(t,s)) = \int_{\Sigma \times S^1} \star \left(\mathrm{d}_{\tilde{x}} \widetilde{E}_{t,s}(\tilde{x},\tilde{y}) - \mathrm{d}_{\tilde{y}} \widetilde{E}_{t,s}(\tilde{x},\tilde{y}) \right)|_{\tilde{y}=\tilde{x}}, \tag{46}$$

where $\star$ is the Hodge star operator on $\widetilde{M}$.

We consider the commuting joint eigenvalue problem on $\widetilde{M}$,

$$\begin{cases} \widetilde{\Box} u = (\Box_g + D_\theta^2) u = 0, \\ \\ D_Z u = \lambda u \\ \\ D_\theta u = m u. \end{cases}$$

The following result follows from Theorem 3.4

Proposition 5.4. *The joint spectrum of the eigenvalue problem*

$$D_Z u = \lambda u, \quad D_\theta u = m u.$$

in $\mathcal{H}_{KG}$ *on* $\widetilde{M} = M \times S^1$ *is discrete in* $\mathbb{R} \times \mathbb{R}$, *and the eigenfunctions* $(u_{m,\lambda_j(m)})_{\lambda_j \neq 0}$ *are* C^∞ *and define an orthonormal basis of* $\mathcal{H}_{KG}^c$.

6. Quantum Ladders and ladder traces

Next we define the quantum ladders corresponding to the classical ladders in Section 2.3. Let $D_\theta = \frac{1}{i} \frac{\partial}{\partial \theta}$ and $D_Z = \frac{1}{i} \nabla_Z$. Suppose $\hat{\psi} \in C_0^\infty(\mathbb{R})$. We define the Fourier integral operator,

$$\psi(D_Z - \nu D_\theta) := \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\psi}(s) e^{-i s \nu D_\theta} \circ e^{i s D_Z} \mathrm{d}s : L^2(\widetilde{M}) \to L^2(\widetilde{M}). \tag{47}$$

Note that the operator $\psi(D_Z - \nu D_\theta)$ is a properly supported Fourier integral operator.

6.1. **Quantum ladder traces.** To prove Theorem 1.1, we express the generating functions (10) as ladder traces.

Definition 6.1. Let $\hat{\psi} \in C_0^\infty(\mathbb{R})$. The 'fuzzy ladder' projector (9) of slope ν defined by ψ is the approximate projector,

$$\Pi_{\nu,\psi} = \psi(D_Z - \nu D_\theta) : \mathcal{H}_{KG} \to \mathcal{H}_{KG}. \tag{48}$$

We can then represent the generating functions (10) by traces:

Lemma 6.2.

$$\begin{cases} \Upsilon_{\nu,\psi}^{(1)}(s) := \operatorname{Tr}_{\mathcal{H}_{KG}} \Pi_{\nu,\psi} e^{\mathrm{i}s D_\theta}, \\[2mm] \Upsilon_{\nu,\psi}^{(2)}(s) := \operatorname{Tr}_{\mathcal{H}_{KG}} \Pi_{\nu,\psi} e^{\mathrm{i}s D_Z} \end{cases}. \tag{49}$$

Proof. This follows directly from the definitions (10) and the spectral theory of Proposition 5.4.

$\square$

We concentrate on the first trace, since it is sufficient for the proof of our main results. We now express the traces in (49) explicitly in terms of kernels. As mentioned in the introduction, $\mathcal{H}_{KG}$ is not a subspace of $L^2(\widetilde{M})$ so we cannot express the restriction to $\mathcal{H}_{KG}$ by an orthogonal projection. But, the map $\widetilde{E}$ maps onto the space of smooth solutions we have

$$\Pi_{\nu,\psi}\widetilde{E} = \psi(D_Z - \nu D_\theta)\widetilde{E}. \tag{50}$$

We now express traces in terms of the kernel of this operator.

Lemma 6.3. *For any* $g \in C_0^\infty(\mathbb{R})$ *the distribution*

$$G = \int_{\mathbb{R}} g(s) e^{\mathrm{i}s D_\theta} \psi(D_Z - \nu D_\theta)\widetilde{E} \mathrm{d}s \tag{51}$$

is smooth. Hence, the operator G *is trace-class.*

Proof. This is proved by showing that the wave front set $\mathrm{WF}(G)$ of G is empty. More detailed wavefront set computations are performed in the next section and we therefore only sketch the argument. The wavefront set of the distributional kernel of $\widetilde{E}$ only contains non-zero lightlike covectors of the form (ζ, ζ') such that ζ and $-\zeta'$ are on the same orbit of the geodesic flow on $T^*\widetilde{M}$ (see for example [SZ18, Theorem 3.2]). One then computes directly (see for example the proof of Lemma 8.2) that the wavefront set of G is contained in the set of non-zero lightlike covectors (ζ, ζ') with $p_\nu(\zeta) = 0$, $p_\theta(\zeta) = 0$ such that there exists $t, s \in \mathbb{R}$ with $G^t \exp(s\partial_\theta)\zeta = -\zeta'$. The equations $p_\nu(\zeta) = 0, p_\theta(\zeta) = 0$ imply $\zeta = 0$ for lightlike covectors. $\square$

Theorem 6.4. *As a distribution on* $\mathbb{R}$,

$$\Upsilon_{\nu,\psi}^{(1)}(s) = \int_{\mathbb{R}} \int_{\Sigma \times S^1} \hat{\psi}(s')(\mathrm{d}_{\tilde{x}}\widetilde{E}_{s',s-\nu s'}(\tilde{x}, \tilde{y})|_{\tilde{y}=\tilde{x}} - \mathrm{d}_{\tilde{y}}\widetilde{E}_{s',s-\nu s'}(\tilde{x}, \tilde{y})|_{\tilde{x}=\tilde{y}} \, \mathrm{dVol}_{\widetilde{\Sigma}}\mathrm{d}s'. \tag{52}$$

Equivalently,

$$\Upsilon_{\nu,\psi}^{(1)}(s) = \int_{\widetilde{\Sigma}} e^{\mathrm{i}s D_\theta}\{\nu_{\tilde{x}}\psi(D_Z - \nu D_\theta)\widetilde{E}(\tilde{x}, \tilde{y}) - \nu_{\tilde{y}}\psi(D_Z - \nu D_\theta)\widetilde{E}(\tilde{x}, \tilde{y})\}|_{\tilde{x}=\tilde{y}} \, \mathrm{dVol}_{\widetilde{\Sigma}} \tag{53}$$

More precisely, for any $g \in C_0^\infty(\mathbb{R})$,

$$\int g(s)\Upsilon_{\nu,\psi}^{(1)}(s)\mathrm{d}s = \int_{\widetilde{\Sigma}} (\nu_{\tilde{x}}G(\tilde{x},\tilde{y}) - \nu_{\tilde{y}}G(\tilde{x},\tilde{y}))|_{\tilde{y}=\tilde{x}} \,\mathrm{dVol}_{\widetilde{\Sigma}}.$$

The integrals are independent of the Cauchy surface chosen.

Here, $\mathrm{dVol}_{\widetilde{\Sigma}}$ is the Lorentzian surface measure. In case g is even and ψ is even this simplifies to

$$\int g(s)\Upsilon_{\nu,\psi}^{(1)}(s)\mathrm{d}s = 2\int_{\tilde{\Sigma}} (\nu_{\tilde{x}}G(\tilde{x},\tilde{y}))|_{\tilde{y}=\tilde{x}} \,\mathrm{dVol}_{\widetilde{\Sigma}}(\tilde{x}).$$

Proof. The trace formula follows from Theorem 5.3, in particular the simpler form (46). To obtain (52) from (46), it suffices to observe that the trace in (52) replaces (s, s') by $\alpha(s', s) = (s', s - \nu s')$ and then integrates in s' against $\hat{\psi}(s')$. $\qquad\square$

We now give a second proof that does not use [SZ18, Theorem 4.1] or Theorem 5.3 explicitly.

Proof. We introduce the Σ-dependent ladder projector:

$$\Pi_{\nu,\psi}^{\widetilde{\Sigma}} := \int_{\mathbb{R}} \hat{\psi}(s)\mathcal{R}e^{-\mathrm{i}sD_\theta} \circ e^{\mathrm{i}s\nu D_Z}\mathcal{E}\mathrm{d}s : H^1(\widetilde{\Sigma}) \oplus L^2(\widetilde{\Sigma}) \to H^1(\widetilde{\Sigma}) \oplus L^2(\widetilde{\Sigma}). \qquad (54)$$

as an operator on the Cauchy data space $H^1(\widetilde{\Sigma}) \oplus L^2(\widetilde{\Sigma})$.

Note that under the extension-restriction identification in Definition 4.2-Definition 4.4, the S^1 action acts by rotations on $H^1(\widetilde{\Sigma}) \oplus L^2(\widetilde{\Sigma})$. Although (54) depends on the choice of $\widetilde{\Sigma}$, the trace formula it induces does not.

By (40), and by (48)-(54), $\Upsilon_{\nu,\psi}^{(1)}(s) = \mathrm{Tr}\,_{\mathcal{H}_{KG}}\Pi_{\nu,\psi}e^{\mathrm{i}sD_\theta}$ is given by,

$$
\begin{aligned}
\Upsilon_{\nu,\psi}^{(1)}(s) &= \mathrm{Tr}\,_{\widetilde{\Sigma}}\Pi_{\nu,\psi}^{\widetilde{\Sigma}}e^{\mathrm{i}sD_\theta} \\[2mm]
&= \int_{\mathbb{R}}\int_{\tilde{\Sigma}} \hat{\psi}(s') * (\mathrm{d}_{\tilde{x}}\widetilde{E}(\tilde{x}, e^{s\frac{\partial}{\partial\theta}} \circ e^{s'(Z-\nu\frac{\partial}{\partial\theta})}\tilde{y}) - \mathrm{d}_{\tilde{y}}\widetilde{E}(\tilde{x}, e^{s\frac{\partial}{\partial\theta}} \circ e^{s'(Z-\nu\frac{\partial}{\partial\theta})}\tilde{y})|_{\tilde{x}=\tilde{y}}\mathrm{d}s'\,\mathrm{dVol}_{\widetilde{\Sigma}}(\tilde{x}) \\[2mm]
&= \int_{\mathbb{R}}\int_{\tilde{\Sigma}} \hat{\psi}(s') * (\mathrm{d}_{\tilde{x}}\widetilde{E}_{s',s-\nu s'}(\tilde{x},\tilde{y}) - \mathrm{d}_{\tilde{y}}\widetilde{E}_{s',s-\nu s'}(\tilde{x},\tilde{y}))|_{\tilde{x}=\tilde{y}}\mathrm{d}s'\,\mathrm{dVol}_{\widetilde{\Sigma}}(\tilde{x}).
\end{aligned}
$$
$$(55)$$

In the first line, $\mathrm{Tr}\,_{\widetilde{\Sigma}}$ denotes the trace on the finite energy Cauchy data space. The first line holds because $\mathcal{R}_{\widetilde{\square}}$ is an invertible operator (see Definition 4.4) conjugating $\Psi_{\nu,\psi}$ on $\mathcal{H}_{KG}$ and (54). $\qquad\square$

7. MICROLOCAL CALCULATION OF THE SINGULARITY OF $\Upsilon_{\nu,\psi}^{(1)}(s)$

Our goal now is to determine the singular support of $\Upsilon_{\nu,\psi}^{(1)}(s)$, and to calculate the coefficient of the leading singularity at $s = 0$. We use the symbol calculus of Fourier integral operators as in [DG75, GU89] in the calculation. By Theorem 6.4, the distribution trace may be expressed in several ways as a composition of simpler Fourier integral operators. We consider two ways to group the compositions, each of some independent interest in related problems.

The first is to use (53) and to consider each term of the trace as the composition

$$(\Pi_{\widetilde{\Sigma}})_* \Delta_{\widetilde{\Sigma}}^* R_{\widetilde{\Sigma} \times \widetilde{\Sigma}} \nu_{\tilde{x}} e^{\mathrm{i}s D_\theta} \psi(D_Z - \nu D_\theta) \widetilde{E}$$

of the following Fourier integral operators:

- $\nu_{\tilde{x}}$, the unit normal vector field extended to a neighborhood of the Cauchy surface.
- $\widetilde{E}$.
- $\psi(D_Z - \nu D_\theta)\widetilde{E} \in \mathcal{D}'(\widetilde{M} \times \widetilde{M})$.
- $e^{\mathrm{i}s D_\theta}$, the translation operator on S^1 by $e^{\mathrm{i}s}$.
- $R_{\widetilde{\Sigma} \times \widetilde{\Sigma}} : \widetilde{M} \times \widetilde{M} \to \widetilde{\Sigma} \times \widetilde{\Sigma}$, the restriction operator, i.e. pullback under the embedding $\widetilde{\Sigma} \times \widetilde{\Sigma} \to \widetilde{M} \times \widetilde{M}$.
- $\Delta_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \widetilde{\Sigma} \times \widetilde{\Sigma}, \Delta_{\widetilde{\Sigma}}(\tilde{x}) = (\tilde{x}, \tilde{x})$, pullback under the diagonal embedding.
- $\Pi_{\widetilde{\Sigma}} : \mathbb{R} \times \widetilde{\Sigma} \to \mathbb{R} \times \mathbb{R}, \Pi_{\widetilde{\Sigma}}(t, \tilde{x}) = t$, i.e. the pushforward is integration over $\widetilde{\Sigma}$.

A second grouping is to compose in the order:

$$\Upsilon_{\nu,\psi}^{(1)}(t) = \rho_* \hat{\psi} (\Pi_{\widetilde{\Sigma}})_* \alpha^* \Delta_{\widetilde{\Sigma}}^* R_{\widetilde{\Sigma} \times \widetilde{\Sigma}} \nu_{\tilde{x}} \widetilde{E}_{t,s}, \tag{56}$$

with a sequence of pullbacks and pushforwards under the maps defined by:

- $\widetilde{E}_{t,s}(\tilde{x}, \tilde{y}) = U(t, s)\widetilde{E}(\tilde{x}, \tilde{y})$
- $\alpha(s, t) = (s, t - \nu s)$.
- $\rho(t, s) = t$.
- $\Pi_{\widetilde{\Sigma}} : \mathbb{R} \times \mathbb{R} \times \widetilde{\Sigma} \to \mathbb{R} \times \mathbb{R}, \Pi_{\widetilde{\Sigma}}(t, s, \tilde{x}) = (t, s)$.

The first method emphasizes the fundamental operator $\psi(D_Z - \nu D_\theta)\widetilde{E}$, and so we follow this approach. The second is useful if one would to study the singularities of the two-variable trace $\mathrm{Tr}_{\mathcal{H}_{KG}} U(t, s)\widetilde{E}$ first.

7.1. Background on Fourier integral operators and their symbols.

The advantage of expressing $\Upsilon_{\nu,\psi}^{(1)}(s)$ in terms of pullback and pushforward is the symbol calculus of Lagrangian distributions is more elementary to describe for such compositions. We refer to [HoIV, GS77, D96] for background but quickly review the basic definitions. Assume that $f : X \to Y$ is a smooth map between manifolds. Let $\Lambda \subset T^*Y$ be a Lagrangian submanifold. Then its pullback is defined by,

$$f^* \Lambda = \{(x, \xi) \in T^*X \mid \exists (y, \eta) \in \Lambda, f(x) = y, f^*\eta = \xi\}. \tag{57}$$

On the other hand, let $\Lambda \subset T^*X$. Then its pushforward is defined by,

$$f_* \Lambda = \{(y, \eta) \in T^*Y \mid y = f(x), (x, f^*\eta) \in \Lambda\}. \tag{58}$$

The principal symbol of a Fourier integral operator associated to a canonical relation C is a half-density times a section of the Maslov line bundle on C. We refer to [HoIV, Section 25.2] and to [D96, Definition 4.1.1] for the definition; see also [DG75, GU89] for further expositions and for several calculations of principal symbols closely related to those of this article.

The *order* of a homogeneous Fourier integral operator $A : L^2(X) \to L^2(Y)$ in the non-degenerate case is given in terms of a local oscillatory integral formula

$$K_A(x, y) = \frac{1}{(2\pi)^{n/4 + N/2}} \int_{\mathbb{R}^N} e^{\mathrm{i}\varphi(x, y, \theta)} a(x, y, \theta) \mathrm{d}\theta$$

by

$$\operatorname{ord} A = m + \frac{N}{2} - \frac{n}{4}, \ \text{ where } n = \dim X + \dim Y, \ m = \operatorname{ord} a \tag{59}$$

where the order of the amplitude $a(x, y, \theta)$ is the degree of the top order term of the polyhomogeneous expansion of a in θ, and N is the number of phase variables θ in the local Fourier integral representation (see [HoIV, Proposition 25.1.5]); in the general clean case with excess e, the order goes up by $\frac{e}{2}$ (see [HoIV, Proposition 25.1.5']). The order is designed to be independent of the specific representation of K_A as an oscillatory integral. The space of Fourier integral operators of order μ associated to a canonical relation C is denoted by

$$K_A \in I^{\mu}(Y \times X, C').$$

If $A_1 \in I^{\mu_1}(X \times Y, C'_1), A_2 \in I^{\mu_2}(Y \times Z, C'_2)$, and if $C_1 \circ C_2$ is a 'clean' composition, then by [HoIV, Theorem 25.2.3],

$$A_1 \circ A_2 \in I^{\mu_1 + \mu_2 + e/2}(X \times Z, C'), \ \ C = C_1 \circ C_2,$$

where e is the 'excess' of the composition, i.e. if $\gamma \in C$, then $e = \dim C_\gamma$, the dimension of the fiber of $C_1 \times C_2 \cap T^*X \times \Delta_{T^*Y} \times T^*Z$ over γ.

Pullback and pushforward of half-densities on Lagrangian submanifolds are more difficult to describe. They depend on the map f being a *morphism* in the language of [GS77, page 349]. Namely, if $f : X \to Y$ is a smooth map, we say it is a morphism on half-densities if it is augmented by a section $r(x) \in \operatorname{Hom}(|\Lambda|^{\frac{1}{2}}(TY_{f(x)}, |\Lambda|^{\frac{1}{2}}T_xX)$, that is, a linear transformation mapping densities on $TY_{f(x)}$ to densities on T_xX. As pointed out in [GS77, page 349], such a map is equivalent to augmenting f with a special kind of half-density on the co-normal bundle $N^*(\operatorname{graph}(f))$ to the graph of f, which is constant along the fibers of the co-normal bundle. In our application, the maps are all restriction maps or pushforwards under canonical maps, and they are morphisms in quite obvious ways. Note that under pullback by an immersion, or under a restriction, the number n of independent variables is decreased by the codimension k and therefore the order goes up by $\frac{k}{4}$. Pullbacks under submersions increase the number n. Pushforward is adjoint to pullback and therefore also decreases the order by the same amount.

As examples relevant to this article, we record some basic calculations of canonical relations and orders in [DG75] in the setting of half-wave groups e^{itQ} generated by positive elliptic first order pseudo-differential operators Q on a Riemannian manifold X. Let $\Delta : X \to X \times X$ be the diagonal embedding. The corresponding pullback operator, the operator of restriction to the diagonal, Δ^* is a Fourier integral operator of order $\frac{n}{4}$ where $n = \dim X$, and has canonical relation,

$$WF'(\Delta^*) = \{((x, \xi - \eta), (x, \xi), (y, \eta)) \in (T^*X \times T^*X \times T^*X) \smallsetminus 0\}.$$

Moreover, the wave group $U(t) = e^{itQ}$ of any positive elliptic operator is a Fourier integral operator of order $-\frac{1}{4}$ associated to the space-time graph of the bi-characteristic flow Φ^t, i.e. Hamilton flow of the principal symbol σ_Q of Q. It follows that Δ^*U is a Fourier integral distribution on $\mathbb{R} \times X$ with canonical relation,

$$WF'(\Delta^*(U)) = \{(((t, \tau), (x, \xi - \eta)), \tau + |\xi|_g = 0, (x, \xi) = G^t(x, \eta)\}.$$

Moreover, if $\Pi : \mathbb{R} \times X \to \mathbb{R}$ denotes the natural projection, then the associated push-forward Π_* has order $\frac{1}{2} - \frac{n}{4}$ when $n = \dim X$, and

$$WF'(\Pi_*) = \{(((t,\tau),((t,\tau),(x,0))))\}.$$

The wave trace $\hat{\sigma}(t) = \operatorname{Tr} e^{\mathrm{i}tQ}$ is a Fourier integral distribution on $\mathbb{R}$ of order $-\frac{1}{4} + \frac{d}{2} = -\frac{1}{4} + \frac{2n-1}{2}$ where $d = \dim S^* X$, and

$$WF'(\hat{\sigma}(t)) = \{(t,\tau) \mid \tau < 0, \exists (x,\xi), (x,\xi) = \Phi^t(x,\xi), \tau + |\xi|_g = 0\}.$$

The principal symbol of $\operatorname{Tr} e^{\mathrm{i}tQ}$ at $t = 0$ on $T_0^* \mathbb{R}$ is a half-density on the negative sub-cone and is calculated in [DG75, Proposition 2.1] and equals $C_n \operatorname{Vol}(S^* M)|d\tau|^{\frac{1}{2}}$ where $S^* M = \{\sigma_Q = 1\}$. Homogeneous Fourier integral distributions on $\mathbb{R}$ of order $\frac{2n-1}{2} - \frac{1}{4}$ that are microlocally supported on $\{0\} \times \mathbb{R}_-$ have the form $\int_0^\infty s^{\frac{d-1}{2}} e^{\mathrm{i}ts} \mathrm{d}s$, where $d = 2n - 1$.

8. Proof of Theorem 1.1

We now prove Theorem 1.1. In keeping track of the orders (and universal dimensional constants) of the many compositions to follow, it is helpful to keep in mind that they all must agree with the product case of Section 12. Although we provide details on orders and symbols of the partial compositions en route to the final result, it is not really necessary to do so because we only need to know the order of the final composition, and we have computed it in the product case in Section 12. For this reason, we only give full details on the canonical relations, and do not always give full details on orders. To simplify statements when dealing with principal symbols given with respect to parametrizations we introduce the following notation. If $\iota : M \to N$ is a local diffeomorphism and ρ a compactly supported half-density on M we define the *push-forward* $\iota_* \rho$ by $\iota_* \rho(y) = \sum_\kappa ((\iota_\kappa)^{-1})^* \rho(x_\kappa)$, where $\{x_\kappa\}$ is the set of elements in $\iota^{-1}(y) \cap \operatorname{supp} \rho$ and $(\iota_\kappa)^{-1}$ are local inverses of ι defined near x_κ. This sum is always finite and defines a half-density on N.

We will need the following calculation from [SZ18, Section 8] in all the computations below. The geodesic vector field, i.e. the Hamiltonian vector field of $\frac{1}{2} g^{-1}$, defines a local flow on the null covectors $T_{0,\pm}^* \widetilde{M}$ and the space of orbits is the symplectic manifold $\widetilde{\mathcal{N}}$. The flow parameters s_1, s_2 and the symplectic volume form $\mathrm{dV}_{\widetilde{\mathcal{N}}} = \frac{1}{d!} \omega^d$ on $\widetilde{\mathcal{N}}$ then define a half-density,

$$|\mathrm{d}_C|^{\frac{1}{2}} = |\mathrm{d}s_1|^{\frac{1}{2}} \otimes |\mathrm{d}s_2|^{\frac{1}{2}} \otimes |\mathrm{dV}_{\widetilde{\mathcal{N}}}|^{\frac{1}{2}} = |\mathrm{d}s_1|^{\frac{1}{2}} \otimes |\mathrm{d}\mu_{T_0^* \widetilde{M}}|^{\frac{1}{2}}, \tag{60}$$

where $\mathrm{d}\mu_{T_0^* \widetilde{M}}$ is the Liouville measure on $T_0^* \widetilde{M}$. We record

$$\dim M = n, \dim T^* M = 2n, \dim T^* M_{0,\pm} = 2n - 1, \dim C = 2n, \dim \widetilde{\mathcal{N}} = 2n - 2, \text{ and } d = n - 1.$$

The operator with kernel $\mathrm{d}_x \widetilde{E}_{t,s}(x,y)$ has order -1 and principal symbol equal to

$$\pm \frac{1}{2}(2\pi)(\xi \wedge)|\mathrm{d}s|^{\frac{1}{2}} \otimes |\mathrm{d}t|^{\frac{1}{2}} \otimes |\mathrm{d}_C|^{\frac{1}{2}},$$

on each component C_+ and C_-, where $(\xi \wedge)$ denotes the operator of exterior multiplication by ξ ($\mathrm{d}_x \widetilde{E}_t(x,y)$ is a one form). The proof is essentially the same as for [SZ18, Lemma 8.2]. The principal symbol of $\mathrm{d}_{\tilde{x}}(\widetilde{E}_{t,s}(\tilde{x},\tilde{y}) + \widetilde{E}_{-t,-s}(\tilde{y},\tilde{x}))|_{\tilde{x}=\tilde{y}}$ is easily computed using (64).

Remark 8.1. When computing the $*d_{\tilde{x}}\widetilde{E}(\tilde{x}, \tilde{y})$ and restricting to $\widetilde{\Sigma}$, we often use below the general formula that $\iota_{\Sigma}^*(*d_x E) = \nu_x E dV_{\widetilde{\Sigma}}$, where $\iota : \Sigma \to M$ is the embedding.

The first step in proving Theorem 1.1 in the first approach of Section 7 is to prove,

Lemma 8.2. *Let ν be admissible. Then, $\psi(D_Z - \nu D_\theta)\widetilde{E}$ is a Fourier integral operator,*

$$\psi(D_Z - \nu D_\theta)\widetilde{E} \in I^{-2}(\widetilde{M} \times \widetilde{M}, \widetilde{\mathcal{L}}_\nu),$$

with canonical relation,

$$\widetilde{\mathcal{L}}_\nu := \{(\zeta', \zeta) \in \dot{T}^*\widetilde{M} \times \dot{T}^*\widetilde{M} : \exists t \in \operatorname{supp}\hat{\psi}, s \in \mathbb{R} : \widetilde{G}^s \exp tH_{p_\nu}(\zeta) = \zeta', p_\nu(\zeta) = \tilde{g}(\zeta, \zeta) = 0\}.$$

This canonical relation is parametrized by the local diffeomorphism

$$(t, s, \zeta) \in \mathbb{R} \times \mathbb{R} \times \operatorname{Char}(\widetilde{\square}, P_\nu) \to (\widetilde{G}^s \exp tH_{p_\nu}(\zeta), \zeta)$$

and its principal symbol is the push-forward of the half-density $(-\frac{i}{2})\hat{\psi}(t)|dt|^{\frac{1}{2}} \otimes |ds|^{\frac{1}{2}} \otimes |\mu_{D\operatorname{Char}_\nu}|^{\frac{1}{2}}$, where $\operatorname{Char}(\frac{1}{2}\widetilde{\square}, P_\nu) = D\operatorname{Char}_\nu$ is the double-characteristic variety of Definition 0.2, and $\mu_{D\operatorname{Char}_\nu}$ is the natural Liouville surface measure on this variety.

Proof. By [TU92, Proposition 2.1], $\psi(D_Z - \nu D_\theta) \in I^{-\frac{1}{2}}(\widetilde{M} \times \widetilde{M}, \mathcal{I})$, where

$$\mathcal{I} = \{(\zeta_1, \zeta_2) \in \dot{T}^*\widetilde{M} \times \dot{T}^*\widetilde{M} \mid p_\nu(\zeta_1) = 0, \exists t \in \operatorname{supp}\hat{\psi}, \exp tH_{p_\nu}(\zeta_2) = \zeta_1\} \tag{61}$$

By [TU92, Lemma 2.6], the principal symbol of $\psi(D_Z - \nu D_\theta)$ is $(2\pi)^{-\frac{1}{2}}\hat{\psi}(t)|dt|^{\frac{1}{2}} \otimes |d\mu_L|^{\frac{1}{2}}$. We then compose this canonical relation with that of $\widetilde{E}$, given in Theorem 4.1, to obtain $\widetilde{\mathcal{L}}_\nu$. Further, we compose the principal symbol of $\nu_x\widetilde{E}$ (also computed from Theorem 4.1) with the principal symbol of $\psi(D_Z - \nu D_\theta)$ to obtain the stated order and principal symbol. $\square$

Next we compose with e^{itD_θ} to obtain,

Lemma 8.3. *$\nu_{\tilde{x}}e^{itD_\theta}\psi(D_Z - \nu D_\theta)\widetilde{E}$ is a Fourier integral operator,*

$$\nu_{\tilde{x}}e^{itD_\theta}\psi(D_Z - \nu D_\theta)\widetilde{E} \in I^{-\frac{5}{4}}(\mathbb{R} \times \widetilde{M} \times \widetilde{M}, \widetilde{\Gamma}_\nu),$$

with canonical relation,

$$\widetilde{\Gamma}_\nu := \begin{array}{l} \{(t, p_\theta(\zeta), \zeta', \zeta) \in T^*\mathbb{R} \times \dot{T}^*\widetilde{M} \times \dot{T}^*\widetilde{M} \mid \exists t' \in \operatorname{supp}\hat{\psi}, \exists s, t \in \mathbb{R}, \\ e^{t\frac{\partial}{\partial\theta}}\widetilde{G}^s \exp t'H_{p_\nu}(\zeta) = \zeta', p_\nu(\zeta) = \tilde{g}(\zeta, \zeta) = 0\} \end{array}.$$

Its principal symbol is the push-forward of

$$\sigma_{\nu_{\tilde{x}}e^{itD_\theta}\psi(D_Z - \nu D_\theta)\widetilde{E}} = |\langle\xi, \nu_x\rangle|(-\frac{i}{2}(2\pi)^{\frac{1}{4}})\hat{\psi}(t')|dt|^{\frac{1}{2}} \otimes |dt'|^{\frac{1}{2}} \otimes |ds|^{\frac{1}{2}} \otimes |\mu_{D\operatorname{Char}_\nu}|^{\frac{1}{2}}.$$

Here, we use Remark 8.1, the composition formula, and that e^{itD_θ} introduces the new variable t and is of order $-\frac{1}{4}$.

We then compose with $\mathcal{R}_{\widetilde{\Sigma}\times\widetilde{\Sigma}}$, the pullback under the embedding $\iota_{\widetilde{\Sigma}}\times\iota_{\widetilde{\Sigma}} = \iota_{\widetilde{\Sigma}\times\widetilde{\Sigma}}: \widetilde{\Sigma}\times\widetilde{\Sigma} \to \widetilde{M}\times\widetilde{M}$, and use (57) to calculate that the canonical relation of $\mathcal{R}_{\widetilde{\Sigma}\times\widetilde{\Sigma}}\nu_{\tilde{x}}e^{itD_\theta}\psi(D_Z - \nu D_\theta)\widetilde{E}$ is

$$\iota^*_{\widetilde{\Sigma}\times\widetilde{\Sigma}}\Gamma_\nu\circ\widetilde{\mathcal{C}} = \begin{aligned}&\{(t, p_\theta(q,\xi), q, \xi, q', \xi') \mid (q, \xi, q', \xi') \in \dot{T}^*\widetilde{\Sigma}\times\dot{T}^*\widetilde{\Sigma}, \exists t' \in \operatorname{supp}\hat{\psi}, s \in \mathbb{R}, \exists(\zeta,\zeta')\\ &\in T_q^*\widetilde{M}\times T_{q'}^*\widetilde{M}, \zeta|_{T_q\widetilde{\Sigma}} = \xi, \zeta'|_{T_{q'}\widetilde{\Sigma}} = \xi', p_\nu(\zeta) = g(\zeta,\zeta) = 0, e^{t\frac{\partial}{\partial\theta}}\widetilde{G}^s\exp t'H_{p_\nu}(\zeta) = \zeta')\}.\end{aligned}$$

We note that $p_\theta(q,\xi) = p_\theta(\zeta)$, where $\zeta|_{T\widetilde{\Sigma}} = \xi$, since $\widetilde{\Sigma} = \Sigma\times S^1$ and the restriction does not touch the S^1 factor.

The next step is to compose with the diagonal embedding $\Delta_{\widetilde{\Sigma}}$. Using its canonical relation in 7.1, the canonical relation of

$$\Delta_{\widetilde{\Sigma}}^*\mathcal{R}_{\widetilde{\Sigma}\times\widetilde{\Sigma}}\nu_{\tilde{x}}e^{itD_\theta}\psi(D_Z - \nu D_\theta)\widetilde{E}(x,y)$$

is

$$\Delta_{\widetilde{\Sigma}}^*\iota^*_{\widetilde{\Sigma}\times\widetilde{\Sigma}}\widetilde{\Gamma}_\nu = \begin{aligned}&\{(t, p_\theta(q,\xi), q, \xi - \xi') \mid (q,\xi), (q,\xi') \in \dot{T}^*\widetilde{\Sigma}\times\dot{T}^*\widetilde{\Sigma} : \exists t' \in \operatorname{supp}\hat{\psi},\\ &\exists(\zeta,\zeta') \in \dot{T}_q^*\widetilde{M}\times\dot{T}_q^*\widetilde{M}, \zeta|_{\dot{T}_q\widetilde{\Sigma}} = \xi, \zeta'|_{T_q\widetilde{\Sigma}} = \xi', p_\nu(\zeta) = \tilde{g}(\zeta,\zeta) = 0,\\ &e^{t\frac{\partial}{\partial\theta}}\widetilde{G}^s\exp t'H_{p_\nu}(\zeta) = \zeta')\}.\end{aligned}$$

The final step is to compose with $(\Pi_{\widetilde{\Sigma}})_*$ (integration over $\widetilde{\Sigma}$) and use the wave front relation for this operator in Section 7.1 to get that

$$(\Pi_{\widetilde{\Sigma}})_*\Delta_{\widetilde{\Sigma}}^*\iota^*_{\widetilde{\Sigma}\times\widetilde{\Sigma}}\widetilde{\Gamma}_\nu = \begin{aligned}&\{(t, \tau) \mid \exists t' \in \operatorname{supp}\hat{\psi}, \exists(q,\xi) \in \dot{T}^*\widetilde{\Sigma} : \exists\zeta \in \dot{T}_q^*\widetilde{M}, \zeta|_{\dot{T}_q\widetilde{\Sigma}} = \xi, \tau = p_\theta(\zeta),\\ &p_\nu(\zeta) = \tilde{g}(\zeta,\zeta) = 0, e^{t\frac{\partial}{\partial\theta}}\widetilde{G}^s\exp t'H_{p_\nu}(\zeta) = \zeta)\}.\end{aligned}$$

We re-write the fixed point equation as $e^{(t-\nu t')\frac{\partial}{\partial\theta}}e^{t'Z}G^s(\zeta) = \zeta$. Since $e^{(t-\nu t')\frac{\partial}{\partial\theta}}, e^{t'Z}$ act on different components of ζ and $\widetilde{G}^s = \exp sH_{\sigma^2}\circ G^s$ is a product flow, the fixed point equation splits into two equations:

- $\exists\zeta \in \mathcal{N}_1 \cap \{p_Z = \nu\sigma\}$ so that $e^{t'Z}(\zeta) = \zeta$, i.e. t' lies in the set $\mathcal{P}_\nu$ of periods of $e^{t'Z}$ on $\mathcal{N}_1 \cap \{p_Z = \nu\sigma\}$.
- $t - \nu t' \in 2\pi\mathbb{Z}$ for some $t' \in \mathcal{P}_\nu\cap\operatorname{supp}\hat{\psi}$. In particular, this includes $t = t' = 0$.

This proves the statement

$$\operatorname{sing-supp}\Upsilon^{(1)}(t) \subset \{t \in [0, 2\pi] : \exists t' \in \mathcal{P}_\nu\cap\operatorname{supp}\hat{\psi}, \ t - \nu t' \in 2\pi\mathbb{Z}\}$$

in Theorem 1.1.

Remark 8.4. If we change the circle S^1 from $\mathbb{R}/2\pi\mathbb{Z}$ to $\mathbb{R}/T\mathbb{Z}$, then the second condition becomes $t - \nu t' \in \frac{2\pi}{T}\mathbb{Z}$ for some $t' \in \mathcal{P}_\nu$.

Next we compute the singularity at $t = 0$ under the assumption that $\operatorname{supp}\hat{\psi} \in (-\pi/\nu, \pi/\nu)$. Then the equation $e^{t\frac{\partial}{\partial\theta}}\widetilde{G}^s\exp t'H_{p_\nu}(\zeta) = \zeta'$ f implies that $t - \nu t' = 0$. Since $\tilde{\Sigma}$ is a Cauchy surface the fixed point equation determines s uniquely as a function of t' as long as t is sufficiently close to zero.

As explained in [SZ18, Section 1.1], the restriction of a null covector to $\widetilde{\Sigma}$ defines a covector in $T^*\widetilde{\Sigma} \smallsetminus 0$, defining a smooth maps $\widetilde{\mathcal{N}}_\pm$ to $T^*\widetilde{\Sigma} \smallsetminus 0$, since for each element $\eta \in \dot{T}^*\widetilde{\Sigma}$ there is precisely one lightlike future/past directed covector $\xi \in T^*\widetilde{M} \smallsetminus 0$ whose pull-back is η. The Hamiltonian p_ν (16) also restricts to $T^*\widetilde{\Sigma}$ and thus defines a codimension one conic submanifold we denote by

$$p_{\nu,\widetilde{\Sigma}}^{-1}(0) = \{(q,\xi) \in T^*\widetilde{\Sigma} \mid p_\nu(q,\xi) = 0\}.$$

Remark 8.5. Note that Z is transverse to $\widetilde{\Sigma}$; in the special case of static spacetimes, one may define Σ to a hypersurface normal to Z. In that case, the restriction $p_\nu|_{T^*\widetilde{\Sigma}}$ would be equal $\nu\sigma|_{T^*\widetilde{\Sigma}}$ and its zero set would be $T^*\Sigma$.

The order of the singularity is given by $\frac{e}{2}$, where is the excess e of the composition defining the trace. As in [DG75, Theorem 4.5], if the dimension of the fixed point in the level set $\mathcal{N}_1(\nu)$ is d, then the principal singularity is the homogeneous distribution $\tilde{\mu}_{\frac{d-1}{2}} = \int_{\mathbb{R}} s^{\frac{d-1}{2}} e^{-ist} ds$. For $t = 0$, the fixed point set all of $\mathcal{N}_1(\nu) := \{(x,\xi) \in \mathcal{N}_1 : \langle \xi, Z \rangle = \nu\}$, and has dimension $d = 2n - 3$. Hence the leading singularity is of type $\tilde{\mu}_{n-2}$. Note that, as mentioned above, $\mathcal{N}_1 \simeq T^*\Sigma$ (of dimension $2n - 2$).

Finally, we must compute the symbol. As computed in [SZ18, Lemma 8.4 - Lemma 8.5], the factor of $|\langle \xi, \nu_x \rangle|$ in Lemma 8.3 is cancelled when in the composition with $R_\Sigma \Delta^*$, and the principal symbol is,

$$\sigma_{\mathcal{R}_{\widetilde{\Sigma}}\Delta^* \nu_{\tilde{x}} e^{itD_\theta} \psi(D_Z - \nu D_\theta)\widetilde{E}} = \left(-\frac{i}{2}(2\pi)^{-\frac{1}{2}}\right)\hat{\psi}(t')|dt|^{\frac{1}{2}} \otimes |dt'|^{\frac{1}{2}} \otimes \iota_{\widetilde{\Sigma}}^* |\mu_{\{p_\nu=0\}\cap T^*_{0,\widetilde{\Sigma}}}|^{\frac{1}{2}}.$$

Here, $\mu_{\{p_\nu=0\}\cap T^*_{0,\widetilde{\Sigma}}}$ is the Liouville measure on the codimension two submanifold of $T^*\widetilde{\Sigma}$ of $T^*\widetilde{M}$ defined by the pullback of the defining functions $\iota_{\widetilde{\Sigma}^{-1}}(\frac{1}{2}g(\xi,\xi), p_\nu(\xi))$ of the double characteristic variety.

It follows that the principal symbol of the pushforward to $\mathbb{R}$ on $T^*_0\mathbb{R}$ is given by,

$$\sigma_{\Upsilon^{(1)}_{\nu,\psi}}|_{T^*_0\mathbb{R}} = (2\pi)^{3/4-(n-1)}\hat{\psi}(0)\,\mu_{\mathcal{N}_1(\nu)}(\mathcal{N}_1(\nu))\,|\tau|^{n-2}|d\tau|^{\frac{1}{2}}.$$

This gives a leading singularity of the form

$$\Upsilon^{(1)}_{\nu,\psi}(s) \sim (2\pi)^{-n+1}\hat{\psi}(0)\,\mu_{\mathcal{N}_1(\nu)}(\mathcal{N}_1(\nu))\tilde{\mu}_{n-2}(s) + c_1\tilde{\mu}_{n-3}(s) + \dots,$$

where

$$\tilde{\mu}_k(s) = \int_{-\infty}^{\infty} e^{is\tau}|\tau|^k d\tau.$$

Since $\Upsilon^{(1)}_{+,\nu,\psi}$ can be constructed from $\Upsilon^{(1)}_{\nu,\psi}$ by projecting to the non-negative Fourier coefficients we obtain that $\Upsilon^{(1)}_{+,\nu,\psi}$ has leading singularity of the form

$$(2\pi)^{-n+1}\hat{\psi}(0)\,\mu_{\mathcal{N}_1(\nu)}(\mathcal{N}_1(\nu))\mu_{n-2}(t) \tag{62}$$

as stated in Theorem 1.1. This completes the proof of Theorem 1.1.

9. Further Fourier integral operator calculations

Although we do not need them to prove Theorem 1.1, we provide some further calculations of canonical relations and principal symbols using the second approach to the composition in Section 7, since they are also of interest and are closer to the calculations in [SZ18].

9.1. $\operatorname{Tr}_{\mathcal{H}_{KG}} U(t,s)$ **as a Fourier integral distribution.** The next Proposition is analogous to the calculations in [SZ18, Section 8]. In the following, we compose $\widetilde{E}$ with $U(t,s)$ rather than with $\psi(D_Z - \nu D_\theta)$ and take the trace. We will assume throughout that s and t are sufficiently close to zero.

Proposition 9.1. *The distributional trace*

$$\operatorname{Tr}_{\mathcal{H}_{KG}}(U(t,s)\widetilde{E}) = \int_{\Sigma \times S^1} * \left(\mathrm{d}_{\tilde{x}} (E_{t,s}(\tilde{x}, \tilde{y}) + E_{-t,-s}(\tilde{x}, \tilde{y})) \right)|_{\tilde{y}=\tilde{x}}, \tag{63}$$

is a Fourier integral distribution on $\mathbb{R} \times S^1$ of order $n-1$, with wave front relation,

$$\mathrm{WF}'(\operatorname{Tr}_{\mathcal{H}_{KG}}(U(t,s))) \subseteq \{(t,\tau,s,\sigma) \in T^*(\mathbb{R} \times S^1) \mid \exists \ell \in \mathbb{R}, (\tilde{x}, \tilde{\xi}) \in T^*\widetilde{M} :$$

$$G^\ell \circ e^{tZ} e^{\mathrm{i}s D_\theta}(\tilde{x}, \tilde{\xi}) = (\tilde{x}, \tilde{\xi}), \ (\tilde{\xi}, \tilde{\xi}) = 0, \tau = p_Z(\tilde{x}, \tilde{\xi}), \sigma = p_\theta(\tilde{x}, \tilde{\xi})\}$$

$$= \{(t,\tau,\theta,\sigma) \in T^*(\mathbb{R} \times S^1) \mid \exists [\gamma] \in \widetilde{\mathcal{N}} :$$

$$e^{tZ} e^{\mathrm{i}s D_\theta}[\gamma] = [\gamma], \tau = p_Z[\gamma], \sigma = p_\theta[\gamma]\}$$

Remark 9.2. Recalling the analogy with the trace $\hat{\sigma}$ of the wave group in [DG75], with $\widetilde{\mathcal{N}}$ playing the role of T^*X, we see that the order is consistent with the order $-\frac{1}{4} + \frac{2n-1}{2}$ with $2n-1 = \dim S^*X$ of $\hat{\sigma}$ mentioned in Section 7.1. There is an extra time variable by comparison with $\hat{\sigma}$ so the order should be $-\frac{1}{2} + \frac{\dim \widetilde{\mathcal{N}}-1}{2}$.

Proof. The proof is very similar to that of [SZ18, Theorem 1.4] for the case of $\operatorname{Tr} U(t)$ on $\mathcal{H}_{KG}$. The only difference is that we now consider the two-parameter flow $U(s,t)$ (44).

We first describe $U(t,s)$ as a Fourier integral operator. Define the 'moment Lagrangian',

$$\Gamma = \{(t,\tau,\theta,\sigma,\zeta,-(f_s \circ g_t)(\zeta)), \zeta \in T^*\widetilde{M}\},$$

where $\tau = p_Z(\zeta), \sigma = p_\theta(\zeta)$.

Lemma 9.3. $U \in I^{-\frac{1}{2}}(\mathbb{R} \times \mathbb{R} \times \widetilde{M} \times \widetilde{M}, \Gamma)$.

The statement and proof are the same as in [GU89, Lemma 3.1], where $U(t,s) = e^{\mathrm{i}(tP+sQ)}$. Here, $P = D_Z, Q = D_\theta$.

Remark 9.4. In the setting of compact Riemannian manifolds, the multi-variable trace formula was analyzed in [CdV79], and the analysis is similar in the present Lorentzian setting.

We next describe $\widetilde{E}_{t,s} = U(t,s)\widetilde{E}$ as a Fourier integral operator. The description is almost immediate from the Duistermaat - Hörmander theorem reviewed in Theorem 4.1.

Lemma 9.5. $\widetilde{E}_{t,s} \in I^{-2}(\mathbb{R} \times S^1 \times \widetilde{M} \times \widetilde{M}, \mathcal{C})$, *where*

$$\mathcal{C} = \{(t, p_Z(\zeta_1), s, p_\theta(\zeta_1), (\zeta_1, \zeta_2)) \in T^*(\mathbb{R} \times S^1) \times \widetilde{C}, \quad e^{s\frac{\partial}{\partial\theta}} e^{tZ}(\zeta_1) = \zeta_2\}.$$

Here, $\widetilde{C}$ is defined in (38). The canonical relation $\mathcal{C}$ is parametrized by

$$\mathbb{R}_t \times S_s^1 \times \mathbb{R}_w \times \mathrm{Char}(\widetilde{\Box}) \to \mathcal{C}, (t, s, w, \zeta) \to (t, s, \zeta_1(Z), (e^{s\frac{\partial}{\partial\theta}} e^{tZ} \widetilde{G}^w(\zeta), \zeta).$$

The principal symbol pulls back under the parametrization to,

$$\sigma_{E_{t,s}}|_{\mathcal{C}_\pm} = \mp\frac{i}{2}(2\pi)|dt|^{\frac{1}{2}} \otimes |ds|^{\frac{1}{2}} \otimes |dw|^{\frac{1}{2}} \times |d\mu_{\mathrm{Char}\widetilde{\Box}}|^{\frac{1}{2}}. \tag{64}$$

Proof. This follows by the composition calculus using that $\widetilde{E}_{t,s} = U(t,s)\widetilde{E}$. Composing with $\widetilde{E}$ is transversal, so $U \circ \widetilde{E} := \widetilde{E}_{t,s}$ is also a Fourier integral operator, whose canonical relation is the composition of the Hamiltonian flows of the three principal symbols in (32). The order of the composition is given by the sum rule in Section 7.1 with $e = 0$, using Theorem 4.1 and Lemma 9.3.

$\square$

Next, we consider the canonical relation and principal symbol of the operator

$$R_\Sigma \circ d_{\tilde{x}}(\widetilde{E}_{t,s}(\tilde{x}, \tilde{y}) + \widetilde{E}_{-t,-s}(\tilde{y}, \tilde{x})).$$

Lemma 9.6. *Let $\iota : \widetilde{\mathcal{N}} \to \widetilde{\Sigma}$ be the map (31). The order of $R_{\widetilde{\Sigma}} \circ \Delta^* \circ d_{\tilde{x}}\widetilde{E}_{t,s}(\tilde{x}, \tilde{y})$ is $-1 + \frac{n+2}{4}$ and its canonical relation is given by,*

$$\Lambda = \{(t, \tau, s, \sigma, \iota e^{s\frac{\partial}{\partial\theta}} e^{tZ} \iota^{-1}\eta - \eta)) \mid \eta \in T^*\widetilde{\Sigma} \smallsetminus 0, \tau = p_Z(\eta), \sigma = p_\theta(\eta)\}.$$

Proof. Except for the fact that we have two flows, $e^{tZ}, e^{s\frac{\partial}{\partial\theta}}$, the proof is essentially the same as for [SZ18, Lemma 8.4]. Therefore, we only sketch the proof and concentrate on the additional features. Λ is the composition $\Lambda_{\widetilde{\Sigma}} \circ \mathcal{C}$, namely,

$$\{(\zeta|_{T\widetilde{\Sigma}}, \zeta) \in \dot{T}^*\Sigma \times \dot{T}^*_{\widetilde{\Sigma}}\widetilde{M}\} \circ \{(t, \tau, s, \sigma, \zeta_1, \zeta_2) \in \dot{T}^*(\mathbb{R} \times \widetilde{M} \times \widetilde{M}) \mid \tau = \zeta_1(Z), (e^{s\frac{\partial}{\partial\theta}} e^{tZ}(\zeta_1), \zeta_2) \in C\}$$

$$= \{(t, \tau, s, \sigma, (e^{s\frac{\partial}{\partial\theta}} e^{tZ}\zeta_1)|_{T\widetilde{\Sigma}}, \zeta_2) \mid \tau = \zeta_1(Z), \sigma = p_\theta(\zeta_1), (\zeta_1, \zeta_2) \in C \cap T^*_{\widetilde{\Sigma}}\widetilde{M} \times T^*\widetilde{M}\}$$

We pull back to the diagonal using the definitions in Section 7.1 to determine the canonical relation. Regarding the order, we use the statement in Section 7.1 that under pullback the order increases by $\frac{k}{4}$ where k is the codimension. The composite pullback to the diagonal of $\widetilde{M}$ (increasing the order by $\frac{n+1}{4}$ and then by restriction to $\widetilde{\Sigma}$ (increasing the order by $\frac{1}{4}$) implies the order statement in the Lemma. $\square$

Finally, we consider the pushforward defined by integration over $\widetilde{\Sigma}$.

Lemma 9.7. *The order of $\pi_*(R_\Sigma\Delta^*d_{\tilde{x}}\widetilde{E}_{t,s}(\tilde{x}, \tilde{y})))$ is $-1 + \frac{n+2}{4} + 1 - \frac{n}{4} + \frac{2n-3}{2} = n - 1$ and its canonical relation is given by,*

$$\mathrm{WF}(\pi_*(R_\Sigma\Delta^*d_{\tilde{x}}E_{t,s}(\tilde{x}, \tilde{y}))) \subseteq \{(t, \tau, s, \sigma) \mid \exists\gamma, e^{s\frac{\partial}{\partial\theta}} e^{tZ}\gamma = \gamma \in \widetilde{\mathcal{N}}, \tau = p_Z(\gamma), \sigma = p_\theta(\gamma)\}. \tag{65}$$

Proof. The canonical relation is the pushforward of the one in the previous Lemma. As in the calculation of the order of the pushforward in Section 7.1, the order of the pushforward is $1 - \frac{\dim \widetilde{\Sigma}}{4}$ (since we have two time variables). The composition is not transversal, but has an excess $e = 2n - 2$ equal to the dimension of the fixed point set, which is all of $S\widetilde{\mathcal{N}}$ and has dimension $2n - 3$.

$\square$

This proves Proposition 9.1. $\square$

9.1.1. *Principal symbol of* $\mathrm{Tr}_{\mathcal{H}_{KG}} U(t, s)$ *at* $(t, s) = (0, 0)$. Proposition 9.1 implies the first statement of Theorem 1.1, namely that the pair of singular times are a subset of the periods of $(s, t) \to e^{s \frac{\partial}{\partial \theta}} e^{tZ} e^{tZ}$ acting on $\widetilde{\mathcal{N}}$. The calculation of the principal symbol of the trace at $(s, t) = (0, 0)$ is essentially the same as in [SZ18, Section 8.2.1].

The principal symbol is a homogeneous half-density on $T^*_{(0,0)} \mathbb{R}^2$ and therefore may be represented as a homogeneous multiple of $|\mathrm{d}\tau \wedge \mathrm{d}\sigma|^{\frac{1}{2}}$. The coefficient density is the volume of the fiber $\mathcal{P}_{\mathcal{N}}^{-1}(\tau, \sigma) \subset \widetilde{\mathcal{N}}$, where $\mathcal{P}_{\mathcal{N}} = (p_Z, p_\theta)$ with respect to the natural 'Leray form' $\frac{\Omega_{\mathcal{N}}}{\mathrm{d}p_Z \wedge \mathrm{d}p_\theta}$ (see e.g. [CdV79]). By definition, this is the Liouville measure on the codimension-two fiber. We denote it by $\mu_L(\mathcal{P}_{\mathcal{N}}^{-1}(s, \tau))$. Then,

Corollary 9.8. *The order of* $\mathrm{Tr}_{\mathcal{H}_{KG}} U(t, s)$ *at* $(t, s) = (0, 0)$ *equals* $n - 1$ *and the principal symbol at* $(s, t) = (0, 0)$ *is the half density on* $T^*_{(0,0)} \mathbb{R}^2$ *given by*

$$C_n \mu_{\mathcal{P}_{\mathcal{N}}^{-1}(s, \tau)} |\mathrm{d}\tau \wedge \mathrm{d}\sigma|^{\frac{1}{2}}.$$

Proof. By Lemma 9.5 and Lemma 64, the operator with kernel $\mathrm{d}_{\tilde{x}} E_{t,s}(\tilde{x}, \tilde{y})$ has order -1 and principal symbol equal to $\frac{1}{2}(2\pi)(\xi \wedge)|\mathrm{d}t|^{\frac{1}{2}} \otimes |\mathrm{d}s|^{\frac{1}{2}} \otimes |\mathrm{d}_C|^{\frac{1}{2}}$ on each component C_+ and C_-, where $(\xi \wedge)$ denotes the operator of exterior multiplication by ξ. Restriction to $\mathbb{R} \times \widetilde{\Sigma} \times \widetilde{\Sigma}$ gives an operator of order $-\frac{1}{2}$.

Restriction to the diagonal and integration over $\widetilde{\Sigma}$ gives an element in $I^{(n-1)}(\mathbb{R}^2)$ with principal symbol

$$C_n \mu_{\mathcal{P}_{\mathcal{N}}^{-1}(s, \tau)} |\mathrm{d}\tau \wedge \mathrm{d}\sigma|^{\frac{1}{2}}$$

at $s = t = 0$, where C_n is a dimensional constant. Note that $\dim \widetilde{\mathcal{N}} = 2(n + 1) - 2$ and the fiber has dimension $2n - 2$, so $\mu_{\mathcal{P}_{\mathcal{N}}^{-1}(s, \tau)}$ is homogeneous of degree $n - 1$. $\square$

9.2. **Completion of the proof of Theorem 1.1 by the second approach.** To complete the proof of Theorem 1.1, we further need to change variables with α in $\mathrm{Tr}_{\mathcal{H}_{KG}} U(t, s)$, to multiply by $\hat{\psi}$ and then integrate in s, i.e. we need to find the wave front set and principal symbol of

$$\int_{\mathbb{R}} \hat{\psi}(s) \left(\mathrm{Tr}_{\mathcal{H}_{KG}} U(s, t - \nu s) \right) \mathrm{d}s. \tag{66}$$

Note that $U(s, t - \nu s) = e^{\mathrm{i}(t - \nu s)D_\theta} e^{sD_Z}$ (44). As above, let $\alpha(t, s) = (s, t - \nu s)$.

The change of variables results in composing the canonical relation of Lemma 9.7 with the graph of the canonical transformation $(t, s, \tau, \sigma) \in T^*\mathbb{R}^2 \to (\alpha(t,s), (d\alpha)^{*-1}(\tau, \sigma))$, or equivalently, pulling it back under this map. Thus, the canonical relation of $\alpha^*\pi_*(R_\Sigma \Delta^* \nu_{\tilde{x}} E_{t,s}(\tilde{x}, \tilde{y}))$ is given by,

$$\mathrm{WF}(\alpha^*\pi_*(R_\Sigma \Delta^* \nu_{\tilde{x}} E_{t,s}(\tilde{x}, \tilde{y}))) \subseteq \begin{aligned} &\{(\alpha(t,s), -(d\alpha)^{*-1}(\tau, \sigma)) \mid \exists \gamma, e^{s\frac{\partial}{\partial\theta}} e^{tZ} \gamma = \gamma \in \widetilde{\mathcal{N}}, \\ &\tau = p_Z(\gamma), \sigma = p_\theta(\gamma)\}. \end{aligned} \tag{67}$$

We then pushforward under $\rho(s,t) = t$ to get,

$$\mathrm{WF}(\rho_* \alpha^* \pi_* (R_\Sigma \Delta^* \nu_{\tilde{x}} E_{t,s}(\tilde{x}, \tilde{y}))) \subseteq \begin{aligned} &\{(t, -\tau) \mid \exists (s, 0): \\ &(\alpha(t,s), (d\alpha)^{*-1}(\tau, \sigma)) \in \mathrm{WF}(\alpha^*\pi_*(R_\Sigma \Delta^* \nu_{\tilde{x}} E_{t,s}(\tilde{x}, \tilde{y})))\}. \end{aligned} \tag{68}$$

If $\alpha(t,s) = (s, t - \nu s)$, then $\alpha^{-1*}(\tau, \sigma) = (\nu\tau + \sigma, \tau)$, and the condition that

$$(\alpha(t,s), (d\alpha)^{*-1}(\tau, \sigma)) \in \mathrm{WF}(\alpha^*\pi_*(R_\Sigma \Delta^* \nu_{\tilde{x}} E_{t,s}(\tilde{x}, \tilde{y})))$$

is,

$$\exists \gamma, e^{s\frac{\partial}{\partial\theta}} e^{(t-\nu s)Z} \gamma = \gamma, \quad (p_Z(\gamma), p_\theta(\gamma)) = (\tau, \sigma).$$

The condition that $(t, \tau, s, 0) \in \mathrm{WF}(\alpha^*\pi_*(R_\Sigma \Delta^* \nu_{\tilde{x}} E_{t,s}(\tilde{x}, \tilde{y})))$ further adds the condition that the WF point is $(s, t - \nu s, \nu\tau, \tau)$. Note that $p_\nu(\nu\tau, \tau) = 0$. If we denote the equivalence class of $\gamma \in \widetilde{\mathcal{N}}$ with respect to the S^1 action by $[\gamma]$ the condition is that

$$\exp s H_{p_\nu}([\gamma]) = [\gamma], \quad p_\nu[\gamma] = 0,$$

as claimed in Theorem 1.1.

9.3. **Comparison to** [GU89]**.** The distribution $\Upsilon^{(1)}(t)$ is the analogue of

$$\Upsilon(s) = \sum_{j=0}^{\infty} \varphi(k^\perp \cdot \underline{\lambda}_j) e^{is\underline{\lambda}_j \cdot \underline{\ell}}.$$

[GU89, (4.2)]. In [GU89, Theorem 2.8] they calculate its order and principal symbol at $s = 0$. Let

$$\mathcal{F} = \{(s, t, \zeta) \in \mathbb{R} \times \mathbb{R} \times T^*\widetilde{M} \mid \zeta \in W, f_s g_t(\zeta) = \zeta\}$$

be the fixed point variety [GU89, (2.5)], where $W = X_1 \cap X_2$ is a codimension two coisotropic submanifold, with $X_1 = p^{-1}(k_1), X_2 = q^{-1}(k_2)$, where $p = \sigma_P, q = \sigma_Q$ and $\underline{k} = (k_1, k_2)$ generates the ladder (line) $\mathcal{L}$. $S : \mathcal{F} \to \mathbb{R}$ is [GU89, (2.8)], the map,

$$S(T_1, T_2, \zeta) = s = \underline{T} \cdot \underline{k}.$$

The dimension of $\mathcal{F}$ for a given t is the excess of the composition in [GU89, (3.10)] and the order of Υ at a singularity s is $\dim S^{-1}(s)/2 - 3/2$

At $s = 0$ its order (or "degree") is $\frac{d}{2} - 1$ where $d = \dim S^{-1}(0) \subset \mathcal{F}$. The half-density part of the symbol is defined as follows: first, define the density α on $\mathcal{F}$ [GU89, (4.11)] by

$$\alpha = |dt|^{\frac{1}{2}} \otimes |\alpha_1|^{\frac{1}{2}} \otimes |d\theta|^{\frac{1}{2}} \otimes |\alpha_2|^{\frac{1}{2}},$$

where α_1 is Liouville density on $\{p = 1\}$, α_2 is Liouville density on $\{q = 1\}$. The principal symbol of Υ by the integral over $S^{-1}(0)$ of the density

$$2\pi\hat{\psi}(t)\alpha$$

times a Maslov and sub-principal phase which is zero in our case.

As mentioned in the introduction, our result on $\Upsilon^{(1)}$ is not simply the Lorentzian analogue of theirs because $\mathcal{H}_{KG}$ is not a subspace of $L^2(\widetilde{M})$. But, at least intuitively, we may relate our result to theirs as follows: our $\Upsilon^{(1)}$, as in [GU89, (4.2)], corresponds to the case $\ell_1 = 0, \ell_2 = 1$, $k_1 = \nu, k_2 = 1$. The cotangent bundle T^*M of [GU89] should be replaced by $\mathrm{Char}(\widetilde{\Box})$ (or, more precisely, by its quotient $\widetilde{\mathcal{N}}$ by the geodesic flow.) The co-isotropic cone of the present article is $\sigma(\sigma_{D_Z} - \nu\sigma_{D_\theta}) = 0$ or $\langle \xi, Z \rangle = \nu\langle \xi, \partial_\theta \rangle$. For us, $\xi = (\xi', \sigma)$ and the equation is $\langle Z, \xi' \rangle = \nu\sigma$. The base of the cone may be fixed by setting $\sigma = 1$ and from the equation, our W is $\{\tau = \nu, \sigma = 1\}$. Thus, our ℓ is $\mathbb{R}_+(\nu, 1)$.

10.   PROOF OF THEOREM 1.2

In this section we study the asymptotics of the smooth Weyl functions (4).

$$N_{\nu,\psi}(m) := \sum_{j \in \mathbb{Z}} \psi(\lambda_j(m) - \nu m). \tag{69}$$

We determine the asymptotics as $m \to \infty$ by studying the singularity of (25) at $s = 0$ and applying a Hardy Tauberian theorem.

Thus, we study the Hardy distribution,

$$\mathrm{d}N_{\nu,\psi}^{(1)}(x) = \sum_{j \in \mathbb{Z}, m \geq 0} \psi(\lambda_j(m) - \nu m)\delta(x - m), \tag{70}$$

Its Fourier transform is the Hardy space distribution,

$$\Upsilon_{+,\nu,\psi}^{(1)}(s) := \sum_{m \in \mathbb{Z}_+, j \in \mathbb{Z}} \psi(\lambda_j(m) - m\nu)e^{ims}, \tag{71}$$

which may be regarded as a Hardy Fourier series on S^1. We now give background on Hardy Fourier series.

The space of Hardy distributions $H^+(S^1)$ is the space of distributions with only positive Fourier coefficients. Such distributions are boundary values of holomorphic functions in the upper half-plane. Hardy Lagrangian distributions are sums of Hardy homogeneous distributions, for which the Fourier coefficients are homogeneous in m. We define the space $H_r^+(S^1)$ of homogeneous Hardy distributions of degree s to be the one-dimensional space spanned by

$$\nu_r(t) = \sum_{m=1}^{\infty} m^r e^{imt}. \tag{72}$$

It has singularities only at $t \in 2\pi\mathbb{Z}$.

Proposition 10.1. *Assume that the fixed point sets of the flow of e^{tZ} on (26) are clean. Then, there exist coefficients A_k, A_{jk} such that:*

$$\Upsilon^{(1)}_{+,\nu,\psi}(s) \quad \sim \quad \sum_{k=0}^{\infty} A_k \nu_{n-2-k}(s)$$

$$+ \quad \sum_{j:\tau_j \neq 0} \sum_{k=0}^{\infty} A_{jk} \nu_{n_j-2-k}(s-\tau_j);$$

Proof. It is clear that $\Upsilon^{(1)}_{+,\nu,\psi}(s)$ is periodic and by definition has only positive frequencies. Moreover, it is a Lagrangian distribution, i.e. a sum of homogeneous distributions near 0. Since the distributions (72) span the periodic Lagrangian Hardy distributions, it follows that it may be expressed as a sum of these distributions. We only need to know the order of the terms, and these are given in Theorem 1.1. $\qquad\square$

To complete the proof of Theorem 1.2, we combine Proposition 10.1 with (72). We are only interested in the singularity at $s = 0$ so we ignore the second term, which has homogeneous singularities away from $s = 0$. We thus have,

$$\sum_{m \in \mathbb{Z}_+, j \in \mathbb{Z}} \psi(\lambda_j(m) - m\nu)e^{ims} \simeq \sum_{m \geq 0, k=0}^{\infty} A_k m^{n-2-k} e^{ims}$$

where $\simeq$ means that the difference of the two sides is smooth in a neighborhood of $s = 0$. By matching Hardy Fourier expansions, it follows that (cf. (4))

$$N_{\nu,\psi}(m) = \sum_{j \in \mathbb{Z}} \psi(\lambda_j(m) - m\nu) \simeq \sum_{k=0}^{\infty} A_k m^{n-2-k},$$

where $\simeq$ means here that the two sides differ by a rapidly decaying function of m. Comparing with (62) shows that

$$A_0 = a_0(\nu, \psi) = (2\pi)^{-n+1} \hat{\psi}(0) \mu_L(\mathcal{N}_1(\nu)).$$

This is precisely the statement of Theorem 1.2 .

11. Proof of Theorem 1.3

In this section we prove Theorem 1.3 for the sharp Weyl functions (3),

$$N_{\nu,C}(m) := \#\{j \mid \frac{\lambda_j(m)}{m} \in [\nu - \frac{C}{m}, \nu + \frac{C}{m}]\}, \tag{73}$$

where $C > 0$ is a given constant. The main point is to use ψ in Theorem 1.2 to approximate an indicator function.

First we will need a proof of Lemma 1.4. We now need to take into account the other periods τ_j in Proposition 10.1. Under a cleanliness assumption we can drop the assumption on the support of $\hat{\psi}$ and conclude that $\Upsilon_1(s)$ is a Fourier-Hardy distribution with isolated conormal singularities. A non-degeneracy condition then implies that the dimensions n_j of the fixed point set at periods other than 0 are strictly smaller than n. Hence, the additional terms for non-zero periods contribute $O(m^{n-3})$, under the assumption of cleanliness and non-degeneracy. This assumption is however not necessary and can be replaced by the stated

zero-measure condition. Indeed, in this case the remainder is of order $o(m^{n-2})$. This follows from the now-standard Duistermaat-Guillemin-Ivrii argument as in [HoIV, Theorem 29.1.5]. The argument is to introduce microlocal cutoffs b to an open set of volume $O(\epsilon)$ around the set of points on periodic orbits of period $\leq T$, and break up the trace using $I = B + b$ where $B = I - b$. In our setting this cut-off is introduced in the integral formula for the trace in Theorem 5.1. Both B and b are pseudo-differential operators on $\widetilde{\Sigma}$. The Tauberian theorem gives 'credit' $O(\epsilon)$ for the volume of the micro-support of B, and $O(\frac{1}{T})$ for the length of the interval around 0 where the trace composed with b is free of singularities for $t \neq 0$. Since ϵ, T are arbitrary, one obtains the remainder $o(m^{n-2})$. We then complete the proof using the argument of [DG75, BrU91].

Lemma 11.1. *For ν admissible, $C > 0$, there exists a constant $k = k(\nu, C)$ so that*

$$\#\{j \mid |\lambda_j(m) - m\nu| \leq C\} \leq k \, m^{n-2}.$$

This is an immediate corollary of Lemma 1.4, using a test function satisfying $\psi \geq 1$ on the indicated set.

Now we choose a non-negative even smooth function ψ with non-negative compactly supported Fourier transform $\hat{\psi}$ and $\hat{\psi}(0) = 1$. For $\delta > 0$ define the scaled $\psi_\delta(x) = \delta^{-1}\psi(\delta^{-1}x)$. Then define,

$$\chi_{C,\delta} := \psi_\delta * \mathbf{1}_{[-C,C]}.$$

Lemma 11.2. *For any $0 < \gamma < C$ and N sufficiently large, there exist constants k_N, K_N such that*

$$N_{\nu,C-\gamma}(m)\left(1 - k_N \left(\frac{\delta}{\gamma}\right)^N\right) \leq \sum_{j \geq 1} \chi_{C,\delta}(\lambda_j(m) - \nu m) \leq N_{\nu,C+\gamma}(m) + K_N \left(\frac{\delta}{\gamma}\right)^N m^{n-1}.$$

The proof is essentially identical to that of [DG75, Lemma 3.3] and [BrU91, Lemma 3.4], and is omitted.

To complete the proof of Theorem 1.3, we use Lemma 11.1-Lemma 11.2 to obtain,

$$\begin{cases} N_{\nu,C-\gamma}(m)(1 - k_N(\delta/\gamma)^N) \leq (2\pi)^{-n+1}2CVm^{n-2} + o_{\delta,\gamma}(m^{n-2}), \\[2ex] N_{\nu,C+\gamma}(m) + K_N(\delta/\gamma)^N m^{n-2} \geq (2\pi)^{-n+1}2CVm^{n-2} + o_{\delta,\gamma}(m^{n-2}), \end{cases}$$

where V is the Liouville volume in Theorem 1.3. Dividing by m^{n-2} and letting $m \to \infty$ first and then $\delta \to 0$ one gets,

$$\limsup_{m \to \infty} m^{-(n-2)} N_{\nu,C-\gamma}(m) \leq (2\pi)^{-n+1}2CV \leq \liminf_{m \to \infty} m^{-(n-2)} N_{\nu,C+\gamma}(m).$$

Letting $C \to C + \gamma$ and then $C \to C - \gamma$ and letting $\gamma \to 0$ concludes the proof.

12. PRODUCT ULTRA-STATIC SPACETIMES

As mentioned in the Introduction, the 'ladder' asymptotics of this article are non-standard even in the case of product spacetimes. To clarify the symplectic geometry in the simplest

setting, we briefly run through the ladder theory of this article in the case of product space-times $\mathbb{R} \times \Sigma$ where Σ is compact. In particular, we aim to clarify the relation between the massive geodesic flow on the unit mass hyperboloid and the space $\widetilde{\mathcal{N}}_\nu$ (26).

We first identity the function $N_{\nu,\psi}(m)$ of Theorem 1.2 and of $N_{\nu,C}(m)$ of Theorem 1.3 in the case of product (ultra-static) spacetimes $(M, g) = (\mathbb{R} \times \Sigma, -\mathrm{d}t^2 + h)$, where (Σ, h) is a compact Riemannian manifold of dimension $n-1$. Of course, the results in this case are the well-known Weyl asymptotics for Laplacians of compact Riemannian manifolds, but the calculations illuminate the nature of the asymptotics in the general stationary setting and given simple tests of the numerology.

Let $\Delta = \Delta_h$ denote the Laplacian of (Σ, h) and let $(\varphi_j, -\omega_j^2)$ denote its spectral data, i.e. $(\Delta + \omega_j^2)\varphi_j = 0$. The solutions of the system (2) obviously have the form, $u_j(t, x) = e^{\mathrm{i}t\lambda_j}\varphi_j(x)$ where $\lambda_j^2 = m^2 + \omega_j^2$. Thus, $\lambda_j(m) = \pm\sqrt{m^2 + \omega_j^2}$. We also have rather explicitly $Q(u_j, u_j) = \lambda_j^2\|u_j\|_{L^2(\Sigma)}^2$ which shows directly that the topology on the Cauchy data space induced by Q is $H^1(\Sigma) \oplus L^2(\Sigma)$. The quantum ladder $\mathcal{L}_\nu$ is defined, roughly speaking, by $\frac{\lambda_j(m)}{m} \simeq \nu$ or equivalently, $\omega_j \simeq m\sqrt{\nu^2 - 1}$.

We only count the positive eigenvalues of D_Z, where we have,

$$N_{\nu,C}(m) = \#\{j \mid \sqrt{m^2 + \omega_j^2} \in [\nu m - C, \nu m + C]\}.$$

The condition is equivalent to $(\nu m - C)^2 - m^2 \le \omega_j^2 \le (\nu m + C)^2 - m^2$ and has no solutions unless $(\nu - 1)m + C \ge 0$. For large m, there are no solutions unless $\nu \ge 1$. In that case, we have,

$$N_{\nu,C}(m) = \#\{j \mid \omega_j \in m\left[\sqrt{(\nu - \frac{C}{m})^2 - 1}, \sqrt{(\nu + \frac{C}{m})^2 - 1}\right]\}.$$

The interval is centered at the point $m\sqrt{\nu^2 - 1}$ and has diameter $\simeq 2C\frac{\nu}{\sqrt{\nu^2-1}}$. Hence,

$$N_{\nu,C}(m) \simeq \#\{j \mid \omega_j \in \left[m\sqrt{\nu^2 - 1} - C\frac{\nu}{\sqrt{\nu^2 - 1}}, m\sqrt{\nu^2 - 1} + C\frac{\nu}{\sqrt{\nu^2 - 1}}\right]\}.$$

The asymptotics depend on the periodicity properties of the geodesic flow G_h^t of (Σ, h). For instance, if $\Sigma_h = S^3$ is a standard S^3, then the eigenvalues of $\sqrt{-\Delta}$ occur only at the points $\sqrt{(N+1)^2 - 1} \simeq N + 1$ and have multiplicity N^2. On the other hand, if the set of closed geodesics of (Σ, h) has Liouville measure zero, then it follows from the two-term Weyl law with remainder estimates of [DG75] that

$$N_{\nu,C}(m) \simeq \alpha_{n-1}\mathrm{Vol}(\Sigma, h)(m\sqrt{\nu^2 - 1})^{n-2}[2C\frac{\nu}{\sqrt{\nu^2 - 1}}], \quad n - 1 = \dim \Sigma. \tag{74}$$

Here, α_{n-1} is $(2\pi)^{-n+1}$ times the Euclidean surface measure of the unit sphere in $\mathbb{R}^{n-1}$. Although the calculation pertains to a special and well-known case, the order of growth in m and the geometric coefficient are universal and corroborate the details in Theorem 1.3.

The smoothed counting function $N_{\nu,\psi}(m)$ has a complete asymptotic expansion in m irrespective of the periodicity properties of the geodesic flow. We can see this by studying the

simpler Weyl function,

$$N_{\nu,\psi}^+(m) = \sum_{j=1}^{\infty} \psi(\sqrt{m^2 + \omega_j^2} - \nu m) = \operatorname{Tr}\psi(\sqrt{m^2 - \Delta_h} - \nu m),$$

using that

$$\psi(\sqrt{m^2 - \Delta_h} - \nu m) = \int_{\mathbb{R}} \hat{\psi}(t) e^{itm\left(\sqrt{1 - m^{-2}\Delta_h} - \nu\right)} dt.$$

The operator $\sqrt{1 - m^{-2}\Delta_h}$ is a positive elliptic semi-classical pseudo-differential operator of order zero, and one may construct a semi-classical parametrix for it and apply stationary phase.

The leading singularity at $s = 0$ of $\Upsilon_{\nu,\psi}^{(1)}(s) := \sum_{m,j\in\mathbb{Z}} \psi(\lambda_j(m) - m\nu)e^{ims}$ is essentially the same as for the flat case of $\widetilde{M} = \mathbb{R}_t \times \mathbb{R}_x^{n-1} \times \mathbb{R}_\theta$, as long as we do not integrate in the space variable $(x,\theta) \in \mathbb{R}^n \times S^1$; for expository simplicity we only give the details in this model case. It is of interest because the leading coefficient in Theorem 1.1 is universal, and the details of the coefficients and order of singularity may be checked in this explicitly computable case. In fact, instead of homogenizing and summing in m, we work on $\Sigma = \mathbb{R}^{n-1}$ and compute the semi-classical asymptotics as $m \to \infty$ of

$$\sum_{j\in\mathbb{Z}} \psi(\lambda_j(m) - m\nu) = \int_{\mathbb{R}} \hat{\psi}(t) \operatorname{Tr} e^{it\sqrt{m^2 - \Delta_{\mathbb{R}^{n-1}}}} e^{-itm\nu} dt \tag{75}$$

directly from the semi-classical kernel,

$$e^{it\sqrt{m^2 - \Delta}}(x,x) = \int_{\mathbb{R}^{n-1}} e^{it\sqrt{m^2 + |\xi|^2}} d\xi.$$

(We repeat that we do not integrate in x, so the integrals are well-defined distributions with the same singularities as in the compact spacelike case). Then

$$(75) \simeq \int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} \hat{\psi}(t) e^{it\sqrt{m^2 + |\xi|^2}} e^{-itm\nu} dt d\xi \quad = \quad m^{n-1}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} \hat{\psi}(t) e^{itm\sqrt{1 + |\xi|^2}} e^{-itm\nu} dt d\xi$$

$$= \quad \alpha_{n-1} m^{n-1}\int_{\mathbb{R}}\int_0^{\infty} \hat{\psi}(t) e^{im\left(t\sqrt{1 + \rho^2} - t\nu\right)} \rho^{n-2} d\rho dt .$$

Here $\xi = \rho\omega$ in polar coordinates and α_{n-1} is the surface area of the sphere in $\mathbb{R}^{n-1}$. The phase is stationary when (and only when) $\sqrt{1 + \rho^2} = \nu, t = 0$ if supp $\hat{\psi}$ is sufficiently small. Stationary phase then gives the expansion,

$$(75) \simeq \alpha_{n-1}\,\hat{\psi}(0)(\nu^2 - 1)^{\frac{n-2}{2}}\frac{\nu}{\sqrt{\nu^2 - 1}} m^{n-2},$$

consistently with the asymptotics in Theorem 1.2. It follows that the principal singularity at $t = 0$ of (10) is given by,

$$\Upsilon^{(1)}(s) \sim \alpha_{n-1}\,(\nu^2 - 1)^{\frac{n-2}{2}}\frac{\nu}{\sqrt{\nu^2 - 1}}\hat{\psi}(0)\sum_{m=0}^{\infty} m^{n-2}e^{ims} = \alpha_{n-1}\,(\nu^2 - 1)^{\frac{n-2}{2}}\frac{\nu}{\sqrt{\nu^2 - 1}}\hat{\psi}(0)\,\mu_{n-2}(s).$$

For the trace in the compact case, we would integrate over Σ and pick up the extra factor of $\operatorname{Vol}_h(\Sigma)$. The result then agrees with Theorem 1.1, once we show that $\mu_L(\mathcal{N}_1(\nu)) = \alpha_{n-1}(\nu^2 - 1)^{\frac{n-2}{2}}\operatorname{Vol}_h(\Sigma)$ in this case (see (77) below.)

If instead we follow the homogenization procedure of this article by letting $m \in \mathbb{Z}$ be eigenvalues of S^1 and study ladders in $\widetilde{\Sigma} = \Sigma \times S^1$. We then define $\widetilde{M} = \mathbb{R}_t \times \Sigma \times S^1$ and

$$\widetilde{\square} = \frac{\partial^2}{\partial t^2} - \Delta - \frac{\partial^2}{\partial \theta^2},$$

where $\Delta = \Delta_h$. The null solutions of $\widetilde{\square}$ then have the form,

$$u_{m,j}(t,x,\theta) = e^{im\theta} e^{it\sqrt{\omega_j^2 + m^2}} \varphi_j(x).$$

We then consider

$$\psi(\sqrt{\Delta_{\widetilde{\Sigma}}} - \nu D_\theta) = \int_{\mathbb{R}} \hat{\psi}(t) e^{-it\nu D_\theta} e^{it\sqrt{\Delta_{\widetilde{\Sigma}}}} \mathrm{d}t.$$

$e^{it\sqrt{\Delta_{\widetilde{\Sigma}}}}$ is the half-wave operator $\widetilde{U}(t,(x,\theta),(x',\theta'))$ of $\widetilde{\Sigma}$ and has the parametrix used in [DG75]. The operator $e^{-it\nu D_\theta}$ is translation in s by t units. Then,

$$\Upsilon_{\nu,\psi}^{(1)}(s) = \mathrm{Tr}\, e^{isD_\theta} \psi(\sqrt{\Delta_{\widetilde{\Sigma}}} - \nu D_\theta) = \int_{\mathbb{R}} \int_{S^1} \int_{\Sigma} \hat{\psi}(t) \widetilde{U}(t\nu + s, (x,\theta),(x,\theta)) dV(x) \mathrm{d}\theta \mathrm{d}t.$$

The singularity at $s = 0$ is determined in by the same Fourier integral techniques as in [DG75, GU89] or as in previous sections. Note that in the product case,

$$\widetilde{E}(t,t') = (\widetilde{\Delta})^{-\frac{1}{2}} \sin((t-t')(\widetilde{\Delta})^{\frac{1}{2}}).$$

We now identify the set of admissible ν in the sense of Definition 0.3, the double characteristic variety $\mathrm{DChar}_\nu(s,\sigma)$ of Definition 0.2, the allowed region $\mathcal{A}_\nu$ where $\frac{\partial}{\partial t} - \nu \frac{\partial}{\partial \theta}$ is spacelike, the governing flow e^{tZ} on $\mathcal{N}_1(\nu)$, and the volume of $\mathcal{N}_1(\nu)$.

Lemma 12.1. *We have:*

(1) The set of admissible ν is the set $\{\nu > 1\}$.
(2) For $\nu > 1$, $\mathcal{A}_\nu = \widetilde{M}$, i.e. $\frac{\partial}{\partial t} - \nu \frac{\partial}{\partial \theta}$ is everywhere spacelike.
(3) For $\nu > 1$, $\mathrm{Dchar}_\nu(s,\sigma) = \{\tau = \sigma\nu,\ |\xi| = \sigma\sqrt{\nu^2 - 1}\}$.
(4) $\mathcal{N}_1(\nu) = \{\tau = 1,\ |\xi| = \sqrt{\nu^2 - 1}\}/\sim$ (divide by the Hamiltonian flow.)
(5) $a_0(\nu,\psi) = (2\pi)^{-n+1} \hat{\psi}(0) \mu_L(\mathcal{N}_1(\nu)) = \alpha_{n-1} \left(\sqrt{\nu^2 - 1}\right)^{n-2} \frac{\nu}{\sqrt{\nu^2-1}} \mathrm{vol}_h(\Sigma)$,
(6) The governing Hamiltonian flow is the geodesic flow of Σ (for any ν).

The characteristic variety of $\widetilde{\square}$ is defined by

$$\widetilde{\mathrm{Char}} = \bigcup_{\sigma \in \mathbb{R}} \mathcal{H}_\sigma,$$

where $\mathcal{H}_\sigma$ is the "mass σ hyperboloid bundle'

$$\mathcal{H}_\sigma = \{((t,x,\theta;\tau,\xi,\sigma) \in T^*(\widetilde{M}) \mid \tau^2 - (|\xi|^2 + \sigma^2) = 0\}.$$

Evidently, σ is a homogeneous coordinate playing the role of m. We then define $\widetilde{\mathcal{N}}$ by quotienting $\widetilde{\mathrm{Char}}$ by the Hamiltonian flow of $\sigma_{\widetilde{\square}} = -\tau^2 + (|\xi|^2 + \sigma^2)$. Since the flow preserves each hyperboloid,

$$\widetilde{\mathcal{N}} = \bigcup_{\sigma \in \mathbb{R}} \mathcal{H}_\sigma/\sim.$$

In Section 2.1 we defined a homogeneous symplectic isomorphism $\iota : \widetilde{\mathcal{N}} \to T^*\widetilde{\Sigma}$ (31), which amounts here to restricting the null covector (τ, ξ, σ) to $T_{t,x,\theta}\widetilde{\Sigma}$. Since $\widetilde{\Sigma}$ is a level set of the time variable t, $dt = 0$ (and so $\tau = 0$) on $T\widetilde{\Sigma}$, and $\iota(t, x, \theta; \tau, \xi, \sigma) = (t, x, \theta; 0, \xi, \sigma)$.

The Killing vector field in this context is $Z = \frac{\partial}{\partial t}$, and the principal symbol of D_Z is $\langle (\tau, \xi, \sigma), \frac{\partial}{\partial t} \rangle = \tau$. The ladder is non-empty if and only if $\nu \geq 1$; since the equations to not involve the base variables (t, x, θ), $\mathcal{A}_\nu = \widetilde{M}$ when $\nu \geq 1$ and is empty otherwise. We henceforth assume $\nu > 1$.

The classical ladder in $T^*\widetilde{M}$ (or in $\widetilde{\mathcal{N}}$) is defined by $p_\mu(t, x, \theta; \tau, \xi, \sigma) = \tau - \nu\sigma = 0$, and is given by

$$\mathcal{L}_\nu = \bigcup_{(\theta, \sigma) \in T^*S^1} \{\tau = \sigma\nu, \ |\xi| = \sigma\sqrt{\nu^2 - 1}\} \times \{(\theta, \sigma)\}. \tag{76}$$

Under ι (31), the ladder is taken to

$$\iota(\mathcal{L}_\nu) = \bigcup_{(\theta, \sigma) \in T^*S^1} \{|\xi| = \sigma\sqrt{\nu^2 - 1}\} \times \{(\theta, \sigma)\} = \{(|\xi| = \sigma\sqrt{\nu^2 - 1}\} \subset T^*\Sigma \times T^*S^1.$$

For fixed (θ, σ), the classical ladder is the τ-slice of a mass hyperboloid (bundle),

$$\{\tau = \sigma\nu, \ |\xi| = \sigma\sqrt{\nu^2 - 1}\}.$$

As above, we fix $\sigma = 1$ to de-homogenize and then obtain the spherical slice,

$$\{\tau = \nu, \ |\xi| = \sqrt{\nu^2 - 1}\} \subset \mathrm{Char}(\square) \subset T^*M.$$

We then quotient by time translation. The quotient is $\mathcal{N}_1(\nu)$ (23)

$$\mathcal{N}_1(\nu) \simeq \{(x, \xi) \in T^*\Sigma \mid |\xi| = \sqrt{\nu^2 - 1}\}.$$

The Liouville volume $\mu_L(\mathcal{N}_1(\nu))$ is by definition the usual Liouville volume of the right side, namely,

$$\mu_L(S^*_{\sqrt{\nu^2 - 1}}\Sigma) = \mathrm{Vol}(S_{n-2})\left(\sqrt{\nu^2 - 1}\right)^{n-2} \frac{\nu}{\sqrt{\nu^2 - 1}} \mathrm{Vol}_h(\Sigma). \tag{77}$$

Indeed, the Liouville measure satisfies $\mu_L(S^*_{\sqrt{\nu^2-1}}\Sigma) = \frac{d}{d\nu}\mathrm{Vol}_\Omega\{|\xi| \leq \sqrt{\nu^2 - 1}\}$. Compared to (74), the additional factor of $2C$ comes from the width of the interval (i.e. the test function).

The governing dynamical system underlying the ladder trace $\mathrm{Tr}\, e^{tZ}|_{\mathfrak{H}_\nu}$ is identified by ι with the null foliation of the sphere bundle $\{|\xi| = \sqrt{\nu^2 - 1}\} \subset T^*\Sigma$, i.e. with its geodesic flow. It follows that the zero-measure condition on periodic orbits in Theorem 1.3 is independent of ν, and is the well-known condition that the set of periodic orbits of G^t has measure zero for $t \not\equiv 0$.

References

[BGP] C. Bär, N. Ginoux and F. Pfäffle, Wave equations on Lorentzian manifolds and quantization. ESI Lectures in Mathematics and Physics. European Mathematical Society (EMS), Zürich, 2007.

[BW]   C. Bär, and R.T. Wafo, Math Phys. Anal. Geom. (2015) 18, 7.

[BrU91] Brummelhuis, R.; Uribe, A. A semi-classical trace formula for Schrödinger operators. Comm. Math. Phys. 136 (1991), no. 3, 567-584.

[CdV79] Colin de Verdière, Yves Spectre conjoint d'opérateurs pseudo-différentiels qui commutent. I. Le cas non intégrable. Duke Math. J. 46 (1979), no. 1, 169-182.

[CM16] J. Cortier and V. Minerbe, On complete stationary vacuum initial data, Journal of Geometry and Physics **99** (2016), 20-27.

[DeW75] B. DeWitt, Quantum Field theory on a curved spacetime, Phys. Reps. **19** (6) (1975), 295-357.

[D96] J.J. Duistermaat, *Fourier integral operators.* Reprint of the 1996 edition. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2011

[DG75] J.J.Duistermaat and V.Guillemin, The spectrum of positive elliptic operators and periodic bicharacteristics, Inv.Math. **24** (1975), 39-80.

[DH72] J. J. Duistermaat and L. Hörmander, Fourier integral operators. II. Acta Math. **128** (1972), no. 3-4, 183-269.

[GS77] V. Guillemin and S. Sternberg, *Geometric asymptotics.* Mathematical Surveys, No. 14. American Mathematical Society, Providence, R.I., 1977.

[GS79] V. Guillemin and S. Sternberg, Some problems in integral geometry and some related problems in microlocal analysis. Amer. J. Math. 101 (1979), no. 4, 915-955.

[GS82] V. Guillemin and S. Sternberg, Homogeneous quantization and multiplicities of group representations. J. Functional Analysis 47 (1982), no. 3, 344-380.

[GU89] V. Guillemin and A. Uribe, Circular symmetry and the trace formula. Invent. Math. 96 (1989), no. 2, 385-423.

[Gutz71] M. C. Gutzwiller, Periodic Orbits and Classical Quantization Conditions, J. Math. Phys. **12** (1971), 343–358]

[HoI] L. Hörmander, A remark on the characteristic Cauchy problem. J. Funct. Anal. **93** (1990), no. 2, 270–277.

[HoIV] L. Hörmander, *The analysis of linear partial differential operators. IV. Fourier integral operators.* Grundlehren der Mathematischen Wissenschaften 275. Springer-Verlag, Berlin, 1985, Springer-Verlag, New York (1985).

[RSII] M. Reed and B. Simon, Methods of modern mathematical physics. II. Fourier analysis, self-adjointness. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975.

[RSIII] M. Reed and B. Simon, Methods of modern mathematical physics. III. Scattering theory. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1979.

[R87] D. Robert,*Autour de l'approximation semi-classique.* Progress in Mathematics, 68. Birkhäuser Boston, Inc., Boston, MA, 1987.

[SZ18] A. Strohmaier and S. Zelditch, Gutzwiller trace formula for stationary spacetimes. (arXiv:1808.08425).

[SZ20] A. Strohmaier and S. Zelditch, Spectral asymptotics on stationary space-times. Reviews in Mathematical Physics, 2060007, 2020.

[TU92] M.E. Taylor and A. Uribe, Semiclassical spectra of gauge fields. J. Funct. Anal. 110 (1992), no. 1, 1-46.

[ToZ13] J. A. Toth and S. Zelditch, Quantum ergodic restriction theorems: manifolds without boundary. Geom. Funct. Anal. 23 (2013), no. 2, 715-775.

[W] R. M. Wald, *Quantum field theory in curved spacetime and black hole thermodynamics.* Chicago Lectures in Physics. University of Chicago Press, Chicago, IL, 1994.

SCHOOL OF MATHEMATICS, UNIVERSITY OF LEEDS, LEEDS, LS2 9JT, UK
Email address: `a.strohmaier@leeds.ac.uk`

DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, IL 60208-2370, USA
Email address: `zelditch@math.northwestern.edu`