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On the strict value of the non-linear optimal stopping problem

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Abstract

We address the non-linear strict value problem in the case of a general filtration and a completely irregular pay-off process (ξ_t). While the value process (V_t) of the non-linear problem is only right-uppersemicontinuous, we show that the strict value process (V_t^+) is necessarily right-continuous. Moreover, the strict value process (V_t^+) coincides with the process of right-limits (V_{t+}) of the value process. As an auxiliary result, we obtain that a strong non-linear f-supermartingale is right-continuous if and only if it is right-continuous along stopping times in conditional f-expectation.

Keywords: optimal stopping; non-linear expectation; strict value process; general filtration; irregular payoff; strong $\mathcal{E}^f$-supermartingale.
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1 Introduction

This note provides a useful complement on the non-linear optimal stopping problem with f-evaluations.

Let $T > 0$ be a fixed terminal horizon and let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a right-continuous complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \in [0, T]\}$. We will denote by $\mathcal{T}_0$ the set of stopping times with values a.s. in $[0, T]$. The notation $\mathcal{S}^2$ stands for the space of optional processes (X_t) such that $E[\operatorname{ess\,sup}_{\tau \in \mathcal{T}_0} X_\tau^2] < \infty$. Let $\xi = (\xi_t)$ be a given optional pay-off process in $\mathcal{S}^2$.

Let us recall that, for a stopping time $S \in \mathcal{T}_0$, the non-linear optimal stopping problem is defined by

$$V(S) := \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_S} \mathcal{E}^f_{S,\tau}(\xi_\tau), \tag{1.1}$$

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where $\mathcal{T}_S$ denotes the set of stopping times valued a.s. in $[S,T]$ and $\mathcal{E}^f_{S,\tau}(\cdot)$ denotes the (conditional) f-evaluation at time S when the terminal time is τ, induced by a BSDE with Lipschitz driver f.

Problem (1.1) has been studied in [7] (in the case of a Brownian filtration and a continuous pay-off process ξ), in [17] (in the case of a Brownian-Poisson filtration and a cadlag pay-off), in [2] under an assumption of convexity on the non-linear operator, in [8] in the case of right-uppersemicontinuos pay-off ξ and a Brownian-Poisson filtration, in [9] in the case of completely irregular optional pay-off and a general filtration. Applications of this problem to the pricing and hedging of American options in *non-linear complete* market models have been studied in [7] (the case of continuous pay-off), [9] (the case of completely irregular pay-off). We mention also that connections to Reflected BSDEs have been provided in [7], [17], [8], [9].

In this short note, we are interested in the corresponding non-linear *strict value* problem, defined by

$$V^+(S) := \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{S+}} \mathcal{E}^f_{S,\tau}(\xi_\tau), \tag{1.2}$$

where $\mathcal{T}_{S+}$ is the set of stopping times $\tau \in \mathcal{T}_0$ with $\tau > S$ a.s. on $\{S < T\}$ and $\tau = T$ a.s. on $\{S = T\}$.

We note that $V^+(T) = \xi_T$. Moreover, $V(S) \geq V^+(S)$ a.s., for all $S \in \mathcal{T}_0$. The purpose of the note is to establish a non-trivial connection between the two problems (1.1) and (1.2). This is done in Theorem 3.1, which is the main result of this note. The result is useful in the treatment of more complex non-linear mixed control-stopping problems such as the one appearing in the study of the superhedging price of an American option in a *non-linear incomplete* market model (cf. [11]).

In Section 2 we give some preliminaries on the value family $V(S)$ and the strict value family $V^+(S)$. The main result is stated and proved in Section 3.

2 Preliminaries

The following properties of the (conditional) f-evaluations (which we also call f-expectations) $\mathcal{E}^f$ are used for the results of this note[1]. For $S \in \mathcal{T}_0$, $S' \in \mathcal{T}_0$, $\tau \in \mathcal{T}_S$, for η, η_1, and η_2 in $L^2(\mathcal{F}_\tau)$, for $(\xi_t) \in \mathcal{S}^2$:

(i) $\mathcal{E}^f_{S,\tau} : L^2(\mathcal{F}_\tau) \longrightarrow L^2(\mathcal{F}_S)$

(ii) (admissibility) $\mathcal{E}^f_{S,\tau}(\eta) = \mathcal{E}^f_{S',\tau}(\eta)$ a.s. on $\{S = S'\}$.

(iii) $\mathcal{E}^f_{S,S}(\eta) = \eta$, for all $\eta \in L^2(\mathcal{F}_S)$,

(iv) *(monotonicity)* $\mathcal{E}^f_{S,\tau}(\eta_1) \leq \mathcal{E}^f_{S,\tau}(\eta_2)$ a.s., if $\eta_1 \leq \eta_2$ a.s.

(v) *(consistency)* $\mathcal{E}^f_{S,\theta}(\mathcal{E}^f_{\theta,\tau}(\eta)) = \mathcal{E}^f_{S,\tau}(\eta)$, for all S, θ, τ in $\mathcal{T}_0$ such that $S \leq \theta \leq \tau$ a.s.

(vi) *("generalized zero-one law")* $I_A \mathcal{E}^f_{S,\tau}(\xi_\tau) = I_A \mathcal{E}^f_{S,\tau'}(\xi_{\tau'})$, for all $A \in \mathcal{F}_S$, $\tau \in \mathcal{T}_S$, $\tau' \in \mathcal{T}_S$ such that $\tau = \tau'$ on A.

(vii) *(continuity with respect to terminal time and terminal condition)* $\lim_{n\to\infty} \mathcal{E}^f_{S,\tau_n}(\eta_n) = \mathcal{E}^f_{S,\tau}(\eta)$, for $(\tau_n) \in \mathcal{T}_S^{\mathbb{N}}$, such that $\lim_{n\to\infty} \tau_n = \tau$ a.s. and for (η_n), η, such that $\eta_n \in L^2(\mathcal{F}_{\tau_n})$ and $\lim_{n\to\infty} \eta_n = \eta$ a.s.

Let us emphasize that no assumptions of convexity or translation invariance of the non-linear operators $\mathcal{E}^f$ are made.

Remark 2.1. It is well-known that the above properties (i)-(vii) of f-evaluations are satisfied, for instance: in the case of a Brownian filtration and a Lipschitz driver f; in the case of a Brownian-Poisson filtration (or a strictly bigger general filtration) and a Lipschitz driver f satisfying an additional assumption ensuring the monotonicity property

[1] Note that the notion of f-expectation of S. Peng [16] is a particular case.

of the non-linear operators; in the case of a filtration generated by a Brownian motion and a default martingale and a λ-admissible Lipschitz driver f (where λ is the default intensity) satisfying an additional assumption ensuring the monotonicity.

The notion of a strong $\mathcal{E}^f$-supermartingale will be often used in the sequel. The definition is recalled for the convenience of the reader.

Definition 2.2. *A process $(X_t) \in \mathcal{S}^2$ is said to be a strong $\mathcal{E}^f$-supermartingale, if $\mathcal{E}^f_{S,\tau}(X_\tau) \leq X_S$ a.s. on $S \leq \tau$, for all $S, \tau \in \mathcal{T}_0$.*

The following lemma has been established in [9] (Lemma 8.2) for the value family $(V(S))$. The proof for the strict value family $V^+(S)$ is similar and is given for the convenience of the reader.

Lemma 2.3. *There exists a maximizing sequence for $V(S)$ in problem* (1.1) *(resp. $V^+(S)$ in problem* (1.2)*).*

Proof: It is sufficient to show that the family $(\mathcal{E}^f_{S,\tau}(\xi_\tau))_{\tau\in\mathcal{T}_S}$ (resp. the family $(\mathcal{E}^f_{S,\tau}(\xi_\tau))_{\tau\in\mathcal{T}_{S^+}}$) is stable under pairwise maximization. Let $\tau \in \mathcal{T}_S$ and $\tau' \in \mathcal{T}_S$. Set $A := \{\mathcal{E}^f_{S,\tau'}(\xi_{\tau'}) \leq \mathcal{E}^f_{S,\tau}(\xi_\tau)\}$ and $\nu := \tau I_A + \tau' I_{A^c}$. We have $A \in \mathcal{F}_S$ and $\nu \in \mathcal{T}_S$; also, $\nu = \tau$ on A, $\nu = \tau'$ on A^c. By the "generalized zero-one law" for f-evaluations, we get

$$\begin{aligned} \mathcal{E}^f_{S,\nu}(\xi_\nu) = \mathcal{E}^f_{S,\nu}(\xi_\nu) I_A + \mathcal{E}^f_{S,\nu}(\xi_\nu) I_{A^c} &= \mathcal{E}^f_{S,\tau}(\xi_\tau) I_A + \mathcal{E}^f_{S,\tau'}(\xi_{\tau'}) I_{A^c} \\ &= \max\left(\mathcal{E}^f_{S,\tau}(\xi_\tau), \mathcal{E}^f_{S,\tau'}(\xi_{\tau'})\right). \end{aligned} \tag{2.1}$$

This shows the stability under pairwise maximization of the value family (indexed by $\mathcal{T}_S$). In the case where τ and τ' are moreover in $\mathcal{T}_{S^+}$, we have $\nu \in \mathcal{T}_{S^+}$, and equation (2.1) (which holds true as $\mathcal{T}_{S^+} \subset \mathcal{T}_S$) allows to conclude that the strict value family (indexed by $\mathcal{T}_{S^+}$) is stable. $\square$

We recall also the following result on the value family $(V(S))$, established in [9] (cf. Lemma 8.1, and Theorems 8.1 and 8.2). The proof uses properties of the f-evaluations and the maximizing sequence lemma (cf. Lemma 2.3), combined with a result of the general theory of processes (due to C. Dellacherie and E. Lenglart) and is based on a direct study of the value family (cf. the work by N. El Karoui [6] or [14] for the classical case of linear expectations).

Proposition 2.4. The value *family $(V(S))_{S\in\mathcal{T}_0}$ is a strong $\mathcal{E}^f$-supermartingale family*[2]*. Moreover, there exists a unique right-uppersemicontinuous optional process, denoted by $(V_t)_{t\in[0,T]}$, which aggregates the family $(V(S))_{S\in\mathcal{T}_0}$.*[3] *The process $(V_t)_{t\in[0,T]}$ is a strong $\mathcal{E}^f$-supermartingale.*

The process (V_t) is called *the value process*. It is useful to recall also that, as (V_t) is a strong $\mathcal{E}^f$-supermartingale, (V_t) has right (and also left) limits[4].

By using the same type of arguments as those used to prove the above proposition on the value family, the following proposition on the *strict value* family can be shown.

Proposition 2.5. The strict value *family $(V^+(S))_{S\in\mathcal{T}_0}$ is a strong $\mathcal{E}^f$-supermartingale family. There exists a unique right-uppersemicontinuous optional process, denoted by $(V^+_t)_{t\in[0,T]}$, which aggregates the family $(V^+(S))_{S\in\mathcal{T}_0}$. The process $(V^+_t)_{t\in[0,T]}$ is a strong $\mathcal{E}^f$-supermartingale.*

The process $(V^+_t)_{t\in[0,T]}$ will be called *the strict value process*.

[2]That is, an admissible square-integrable family, such that $\mathcal{E}^f_{S,\tau}(V(\tau)) \leq V(S)$ a.s. on $S \leq \tau$, for all $S, \tau \in \mathcal{T}_0$.

[3]That is, $V_S = V(S)$ a.s. for all $S \in \mathcal{T}_0$.

[4]The existence of right limits (and also left limits) of a strong $\mathcal{E}^f$-supermartingale has been observed in [8] and [9] as an immediate corollary of the non-linear $\mathcal{E}^f$-decomposition. This has been also noticed in [3] based on a down-crossing inequality (cf. Lemmas A.1 and A.2 in [3]).

Proof of Proposition 2.5: The strict value family $(V^+(S))_{S\in\mathcal{T}_0}$ is admissible, that is, for $S\in\mathcal{T}_0$, $S'\in\mathcal{T}_0$, $V(S)$ is $\mathcal{F}_S$-measurable and $V(S)=V(S')$ a.s. on the set $\{S=S'\}$. The latter property is based on the "generalized zero-one law", the admissibility property of f-evaluations, and on a property of concatenation of the set $\mathcal{T}_{S^+}$. Indeed, let $\tau\in\mathcal{T}_{S^+}$, and set $\tau_A := \tau I_A + T I_{A^c}$, where $A:=\{S=S'\}$. We have $\tau_A\in\mathcal{T}_{S^+}\cap\mathcal{T}_{(S')^+}$. By the "generalized zero-one law" and the property of admissibility of f-evaluations,

$$I_A\mathcal{E}^f_{S,\tau}(\xi_\tau) = I_A\mathcal{E}^f_{S,\tau_A}(\xi_{\tau_A}) = I_A\mathcal{E}^f_{S',\tau_A}(\xi_{\tau_A}) \le I_A V(S').$$

Hence, $I_A V(S)\le I_A V(S')$ (as $\tau\in\mathcal{T}_{S^+}$ is arbitrary). By exchanging the roles of S and S', we get the converse inequality, and hence the equality. The inequality $\mathcal{E}^f_{S,T}(\xi_T)\le V^+(S)\le V(S)$ a.s., implies that $V^+(S)$ is square-integrable. Let now $S\in\mathcal{T}_0$ (as before) and $\theta\in\mathcal{F}_S$. To show the $\mathcal{E}^f$-supermartingale property of the strict value family, it remains to show $\mathcal{E}^f_{S,\theta}(V^+(\theta))\le V^+(S)$ a.s. By the maximizing sequence lemma (Lemma 2.3), and the continuity and consistency properties of f-evaluations, there exists a sequence $(\theta_p)\in(\mathcal{T}_{\theta^+})^{\mathbb{N}}$ such that

$$\mathcal{E}^f_{S,\theta}(V^+(\theta)) = \mathcal{E}^f_{S,\theta}\big(\lim_{p\to\infty}\mathcal{E}^f_{\theta,\theta_p}(\xi_{\theta_p})\big) = \lim_{p\to\infty}\mathcal{E}^f_{S,\theta}(\mathcal{E}^f_{\theta,\theta_p}(\xi_{\theta_p}) = \lim_{p\to\infty}\mathcal{E}^f_{S,\theta_p}(\xi_{\theta_p})\le V^+(S),$$

the last inequality being due to $\mathcal{T}_{\theta^+}\subset\mathcal{T}_{S^+}$. By Lemma 3.1 in [9], any $\mathcal{E}^f$-supermartingale family is a right-uppersemicontinuous (r.u.s.c.) family. By Theorem 4 in Dellacherie-Lenglart [4], there exists a (unique) r.u.s.c. optional process (which we denote by (V_t^+)) aggregating the family. The aggregating process (V_t^+) is an (r.u.s.c.) $\mathcal{E}^f$-supermartingale (as the family $(V^+(S))$ is an $\mathcal{E}^f$-supermartingale family). □

3 The main result and its proof

The following theorem is the main result of this note. It establishes in particular the equality between the strict value process $(V_t^+)_{t\in[0,T]}$ and the process of right-limits $(V_{t+})_{t\in[0,T]}$, where (V_t) denotes as before the value process of the non-linear problem (1.1).

Theorem 3.1. (i) *The strict value process (V_t^+) is right-continuous.*

(ii) *For all $S\in\mathcal{T}_0$, $V_S^+ = V_{S+}$ a.s.*

(iii) *For all $S\in\mathcal{T}_0$, $V_S = V_S^+\vee\xi_S$ a.s.*

We first show an auxiliary result on (non-linear) $\mathcal{E}^f$-supermartingales, which will be used in the proof of the main theorem. This auxiliary result, interesting on its own right, allows also to emphasize some difficulties when dealing with non-linear supermartingales, compared to the usual linear case.

In the sequel, a process (X_t) in $\mathcal{S}^2$ will be called *right-continuous along stopping times in $\mathcal{E}^f$-conditional expectation* if for each $S\in\mathcal{T}_0$ and for each sequence of stopping times $(S_n)_{n\in\mathbb{N}}$ belonging to $\mathcal{T}_{S+}$ such that $S_n\downarrow S$, we have

$$\lim_{n\to\infty}\mathcal{E}^f_{S,S_n}(X_{S_n}) = X_S \quad \text{a.s.} \tag{3.1}$$

We will say that (X_t) in $\mathcal{S}^2$ is *right-continuous along stopping times in $\mathcal{E}^f$- expectation*, if for each $S\in\mathcal{T}_0$ and for each sequence of stopping times $(S_n)_{n\in\mathbb{N}}$ belonging to $\mathcal{T}_{S+}$ such that $S_n\downarrow S$, we have

$$\lim_{n\to\infty}\mathcal{E}^f_{0,S_n}(X_{S_n}) = \mathcal{E}^f_{0,S}(X_S) \quad \text{a.s.} \tag{3.2}$$

Remark 3.2. Property (3.1) implies property (3.2) (by applying the continuity property and the property of consistency of the non-linear operators). The converse statement is not true in general.

Proposition 3.3. *Let (X_t) be a strong $\mathcal{E}^f$-supermartingale.*

The following assertions are equivalent:

1. *(X_t) is right-continuous.*
2. *(X_t) is right-continuous along stopping times in $\mathcal{E}^f$-conditional expectation.*

If, moreover, we assume that the non-linear operators $\mathcal{E}^f$ are strictly monotone[5], *then each of the above assertions is equivalent to:*

3. *(X_t) is right-continuous along stopping times in $\mathcal{E}^f$- expectation.*

Proof of Proposition 3.3: We will use that, as (X_t) is a strong $\mathcal{E}^f$-supermartingale, (X_t) is right-upper-semicontinuous and right-limited. Hence, for all $S \in \mathcal{T}_0$, we have $X_S \geq X_{S+}$.

The implications $1. \Rightarrow 2. \Rightarrow 3.$ can be easily shown by using the continuity property of (conditional) f-expectations (with respect to the terminal condition and to the terminal time).

Let us now prove $2. \Rightarrow 1.$ Assume 2. Let $S \in \mathcal{T}_0$ and let (S_n) be a non-increasing sequence of stopping times in $\mathcal{T}_{S+}$ with $\lim \downarrow S_n = S$ a.s. Using the assumption of right-continuity of (X_t) along stopping times in $\mathcal{E}^f$-conditional expectation, and then the continuity property of conditional f-expectations (with respect to the terminal condition and to the terminal time), we get $X_S = \lim_{n\to\infty} \mathcal{E}^f_{S,S_n}(X_{S_n}) = \mathcal{E}^f_{S,S}(X_{S+}) = X_{S+}$, (where we have used that X_{S+} exists and is $\mathcal{F}_S$-measurable). Hence, $X_S = X_{S+}$. As S is arbitrary, we conclude (by a well-known property from the general theory of processes) that (X_t) coincides with (X_{t+}), and hence that (X_t) is right-continuous. Thus, 1. holds.

Let us now make the additional assumption of strict monotonicity of the non-linear operators. We show the implication $3. \Rightarrow 1.$ Assume 3. As above, let $S \in \mathcal{T}_0$ and let (S_n) be a non-increasing sequence of stopping times in $\mathcal{T}_{S+}$ with $\lim \downarrow S_n = S$ a.s. By the continuity property of f- expectations (with respect to terminal time and terminal condition), we have $\lim_{n\to\infty} \mathcal{E}^f_{0,S_n}(X_{S_n}) = \mathcal{E}^f_{0,S}(X_{S+})$. On the other hand, by the assumption of right-continuity along stopping times in $\mathcal{E}^f$-expectation, we have $\lim_{n\to\infty} \mathcal{E}^f_{0,S_n}(X_{S_n}) = \mathcal{E}^f_{0,S}(X_S)$. By uniqueness of the limit, we get $\mathcal{E}^f_{0,S}(X_{S+}) = \mathcal{E}^f_{0,S}(X_S)$. This, together with the fact $X_S \geq X_{S+}$ and the additional assumption of strict monotonicity of the non-linear operators, gives $X_S = X_{S+}$. We conclude as above that (X_t) is right-continuous, which is assertion 1. The proof is completed. $\square$

Remark 3.4. (The linear case) In the particular linear case (the case where $f = 0$), the non-linear strong supermartingales are reduced to the strong supermartingales (well-understood in the general theory of processes since the work of Mertens). In this case, the above proposition simplifies further to the following statement (cf., e.g., Dellacherie-Meyer [5, Appendix A]):

> *A strong (linear) supermartingale (X_t) is right-continuous if and only if it is right-continuous along stopping times in (linear) expectation.*

This result[6] is often used in the literature to show that that the *strict value process* of the *linear* optimal stopping problem is right-continuous (cf. [12] or [14, Proposition 1.12]). Note also that the continuity along stopping times in (linear) expectation of (X_t) is trivially equivalent to the continuity along stopping times in conditional (linear)

[5] That is, for $S \in \mathcal{T}_0$, for $\tau \in \mathcal{T}_S$, for $\eta_1 \in L^2(\mathcal{F}_\tau)$, $\eta_2 \in L^2(\mathcal{F}_\tau)$, if $\eta_1 \leq \eta_2$ and $\mathcal{E}^f_{S,\tau}(\eta_1) = \mathcal{E}^f_{S,\tau}(\eta_2)$, then $\eta_1 = \eta_2$.

[6] or the following more general result: for each strong (linear) supermartingale family $X = (X(\tau))_{\tau\in\mathcal{T}_0}$, which is right-continuous along stopping times in (classical) expectation, there exists an RCLL (linear) supermartingale (X_t) aggregating the family X (cf., e.g., [6] or [15, Proposition 4.1]).

expectation (the linear expectation being strictly monotone). As illustrated in the above proposition, the non-linear case is more involved and we cannot "reproduce" *mutatis mutandis* the proof from the linear case (unless the additional assumption of strict monotonicity is made, which is not the case in the present note).

In virtue of the above Proposition, in order to show that the strict value process (V_t^+) is right-continuous, it is sufficient to show that it is right-continuous along stopping times in conditional $\mathcal{E}^f$-expectation. This is the object of the following result.

Lemma 3.5. *The strict value process* (V_t^+) *is* right-continuous along stopping times in conditional $\mathcal{E}^f$-expectation*.*

For the proof, we recall the following classical statement:

Remark 3.6. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $A \in \mathcal{F}$. Let (X_n) be a sequence of real valued random variables. Suppose that (X_n) converges a.s. on A to a random variable X. Then, for each $\varepsilon > 0$, $\lim_{n\to+\infty} P(\{|X - X_n| < \varepsilon\} \cap A) = P(A)$.

From this property, it follows that for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $P(\{|X - X_n| < \varepsilon\} \cap A) \geq \frac{P(A)}{2}$.

Proof of Lemma 3.5: Let $n \in \mathbb{N}$. By the consistency property of $\mathcal{E}^f$, we have

$$\mathcal{E}^f_{\theta,\theta_n}(V^+_{\theta_n}) = \mathcal{E}^f_{\theta,\theta_{n+1}}\left(\mathcal{E}^f_{\theta_{n+1},\theta_n}(V^+_{\theta_n})\right) \quad \text{a.s.} \tag{3.3}$$

Now, since the process (V_t^+) is a strong $\mathcal{E}^f$- supermartingale, we have $\mathcal{E}^f_{\theta_{n+1},\theta_n}(V^+_{\theta_n}) \leq V^+_{\theta_{n+1}}$ a.s. Using this inequality, together with equality (3.3) and the monotonicity of $\mathcal{E}^f_{\theta,\theta_{n+1}}$, we obtain

$$\mathcal{E}^f_{\theta,\theta_n}(V^+_{\theta_n}) \leq \mathcal{E}^f_{\theta,\theta_{n+1}}(V^+_{\theta_{n+1}}) \quad \text{a.s.}$$

Since this inequality holds for each $n \in \mathbb{N}$, we derive that the sequence of random variables $\left(\mathcal{E}^f_{\theta,\theta_n}(V^+_{\theta_n})\right)_{n\in\mathbb{N}}$ is nondecreasing. Moreover, since the process (V_t^+) is a strong $\mathcal{E}^f$- supermartingale, we have $\mathcal{E}^f_{\theta,\theta_n}(V^+_{\theta_n}) \leq V^+_\theta$ a.s. for each $n \in \mathbb{N}$. By taking the limit as n tends to $+\infty$, we thus get

$$\lim_{n\to\infty} \uparrow \mathcal{E}^f_{\theta,\theta_n}(V^+_{\theta_n}) \leq V^+_\theta \quad \text{a.s.}$$

It remains to show the converse inequality:

$$\lim_{n\to\infty} \uparrow \mathcal{E}^f_{\theta,\theta_n}(V^+_{\theta_n}) \geq V^+_\theta \quad \text{a.s.} \tag{3.4}$$

Suppose, by way of contradiction, that this inequality does not hold. Then, there exists a constant $\alpha > 0$ such that the event A defined by

$$A := \{\lim_{n\to\infty} \uparrow \mathcal{E}^f_{\theta,\theta_n}(V^+_{\theta_n}) \leq V^+_\theta - \alpha\}$$

satisfies $P(A) > 0$. By definition of A, we have

$$\lim_{n\to\infty} \uparrow \mathcal{E}^f_{\theta,\theta_n}(V^+_{\theta_n}) + \alpha \leq V^+_\theta \quad \text{a.s. on } A. \tag{3.5}$$

By Lemma 2.3, there exists an optimizing sequence $(\tau_p)_{p\in\mathbb{N}}$ for the strict value V^+_θ, that is, such that, for each $p \in \mathbb{N}$, $\tau_p \in \mathcal{T}_{\theta^+}$, and such that

$$V^+_\theta = \lim_{p\to\infty} \uparrow \mathcal{E}^f_{\theta,\tau_p}(\xi_{\tau_p}) \quad \text{a.s.}$$

By Remark 3.6 (applied with $\varepsilon = \frac{\alpha}{2}$), we derive that there exists $p_0 \in \mathbb{N}$ such that the event B defined by

$$B := \{V^+_\theta \leq \mathcal{E}^f_{\theta,\tau_{p_0}}(\xi_{\tau_{p_0}}) + \frac{\alpha}{2}\} \cap A$$

satisfies $P(B) \geq \frac{P(A)}{2}$. Denoting τ_{p_0} by θ', we have

$$V_\theta^+ \leq \mathcal{E}^f_{\theta,\theta'}(\xi_{\theta'}) + \frac{\alpha}{2} \quad \text{a.s. on } B.$$

By the inequality (3.5), we derive that

$$\lim_{n\to\infty} \uparrow \mathcal{E}^f_{\theta,\theta_n}(V^+_{\theta_n}) + \frac{\alpha}{2} \leq \mathcal{E}^f_{\theta,\theta'}(\xi_{\theta'}) \quad \text{a.s. on } B. \tag{3.6}$$

Let us first consider the simpler case where $\theta < T$ a.s.

In this case, since $\theta' \in \mathcal{T}_{\theta^+}$, we have $\theta' > \theta$ a.s. Hence, we have $\Omega = \cup_{n\in\mathbb{N}} \uparrow \{\theta' > \theta_n\}$ a.s.

Define the stopping time $\overline{\theta}_n := \theta' \mathbf{1}_{\{\theta' > \theta_n\}} + T\mathbf{1}_{\{\theta' \leq \theta_n\}}$. We note that $\overline{\theta}_n \in \mathcal{T}_{\theta_n{}^+}$ for each $n \in \mathbb{N}$. Moreover, $\lim_{n\to\infty} \overline{\theta}_n = \theta'$ a.s. and $\lim_{n\to\infty} \xi_{\overline{\theta}_n} = \xi_{\theta'}$ a.s. By the continuity property of $\mathcal{E}^f$ (with respect to terminal condition and terminal time), we get

$$\lim_{n\to\infty} \mathcal{E}^f_{\theta,\overline{\theta}_n}(\xi_{\overline{\theta}_n}) = \mathcal{E}^f_{\theta,\theta'}(\xi_{\theta'}) \quad \text{a.s.}$$

By Remark 3.6, we derive that there exists $n_0 \in \mathbb{N}$ such that the event C defined by

$$C := \{|\mathcal{E}^f_{\theta,\theta'}(\xi_{\theta'}) - \mathcal{E}^f_{\theta,\overline{\theta}_{n_0}}(\xi_{\overline{\theta}_{n_0}})| \leq \frac{\alpha}{4}\} \cap B$$

satisfies $P(C) > 0$. By the inequality (3.6), we derive that

$$\lim_{n\to\infty} \uparrow \mathcal{E}^f_{\theta,\theta_n}(V^+_{\theta_n}) + \frac{\alpha}{4} \leq \mathcal{E}^f_{\theta,\overline{\theta}_{n_0}}(\xi_{\overline{\theta}_{n_0}}) \quad \text{a.s. on } C. \tag{3.7}$$

Now, by the consistency of $\mathcal{E}^f$, we have

$$\mathcal{E}^f_{\theta,\overline{\theta}_{n_0}}(\xi_{\overline{\theta}_{n_0}}) = \mathcal{E}^f_{\theta,\theta_{n_0}}\left(\mathcal{E}^f_{\theta_{n_0},\overline{\theta}_{n_0}}(\xi_{\overline{\theta}_{n_0}})\right) \leq \mathcal{E}^f_{\theta,\theta_{n_0}}(V^+_{\theta_{n_0}}) \quad \text{a.s.},$$

where the last inequality follows from the fact that $\overline{\theta}_{n_0} \in \mathcal{T}_{\theta_{n_0}{}^+}$ and from the definition of $V^+_{\theta_{n_0}}$. By (3.7), we thus derive that

$$\lim_{n\to\infty} \uparrow \mathcal{E}^f_{\theta,\theta_n}(V^+_{\theta_n}) + \frac{\alpha}{4} \leq \mathcal{E}^f_{\theta,\theta_{n_0}}(V^+_{\theta_{n_0}}) \quad \text{a.s. on } C,$$

which gives a contradiction. Hence, the desired inequality (3.4) holds.

Let us now consider a general $\theta \in \mathcal{T}_0$.

On the set $\{\theta = T\}$, we have $\theta_n = \theta$ a.s. for all n. Hence, on $\{\theta = T\}$, we have $\lim_{n\to\infty} \mathcal{E}^f_{\theta,\theta_n}(V^+_{\theta_n}) = V^+_\theta$ a.s. On the set $\{\theta < T\}$, using the same arguments as above with $\overline{\theta}_n = \theta' \mathbf{1}_{\{\theta' > \theta_n\} \cap \{T > \theta\}} + T\mathbf{1}_{\{\theta' \leq \theta_n\} \cup \{T = \theta\}}$, we show the inequality (3.4). The proof is thus complete. □

Remark 3.7. If the additional assumption of *strict monotonicity* of the non-linear operators is made, then, in virtue of Proposition 3.3, instead of showing the right-continuity along stopping times in $\mathcal{E}^f$*-conditional expectation*, it is enough to show only the right-continuity along stopping times in $\mathcal{E}^f$*-expectation*, that is, the property

$$\lim_{n\to\infty} \mathcal{E}^f_{0,\theta_n}(V^+_{\theta_n}) = \mathcal{E}^f_{0,\theta}(V^+_\theta). \tag{3.8}$$

In this case, the above proof will become simpler and will follow that for the strict value process associated with the classical *linear* optimal stopping problem (cf., e.g., the proof of Proposition 1.12 in [14]), but where the classical expectation is replaced with the f-expectation. However, in our general case, property (3.8) is not sufficient to ensure the right-continuity of the process (V_t^+) (cf. also Remark 3.4).

We are now ready to prove the theorem.

Proof of Theorem 3.1: By Proposition 2.5, the process (V_t^+) is a strong $\mathcal{E}^f$-supermartingale. By applying the above Lemma 3.5 and Proposition 3.3 to (V_t^+), we obtain assertion (i). We now show (ii). Let $S \in \mathcal{T}_0$. Let (S_n) be a non-increasing sequence of stopping times in $\mathcal{T}_{S^+}$ with $\lim \downarrow S_n = S$ a.s. We know that $V_\tau \geq V_\tau^+$ a.s., for all $\tau \in \mathcal{T}_0$. Hence, $V_{S_n} \geq V_{S_n}^+$ a.s., for all n. We derive that $\lim_{n\to\infty} V_{S_n} \geq \lim_{n\to\infty} V_{S_n}^+$ a.s. Using this and the right-continuity of V^+ established in (i), gives $V_{S+} \geq V_S^+$ a.s. In order to show the converse inequality, we first show

$$\mathcal{E}^f_{S,S^n}(V_{S_n}) \leq V_S^+ \text{ a.s. for all } n. \tag{3.9}$$

We fix n and we take $(\tau^p) \in \mathcal{T}_{S_n}$ an optimizing sequence for the problem with value V_{S_n}, i.e. $V_{S_n} = \lim_{p\to\infty} \mathcal{E}^f_{S_n,\tau_p}(\xi_{\tau_p})$. We have

$$\mathcal{E}^f_{S,S_n}(V_{S_n}) = \mathcal{E}^f_{S,S_n}(\lim_{p\to\infty} \mathcal{E}^f_{S_n,\tau_p}(\xi_{\tau_p})) = \lim_{p\to\infty} \mathcal{E}^f_{S,S_n}(\mathcal{E}^f_{S_n,\tau_p}(\xi_{\tau_p})) \text{ a.s.}, \tag{3.10}$$

where we have used the continuity property of $\mathcal{E}^f_{S,S^n}(\cdot)$ with respect to the terminal condition (recall that here n is fixed). Using the consistency property of $\mathcal{E}^f$-expectations, we get $\mathcal{E}^f_{S,S_n}(\mathcal{E}^f_{S_n,\tau_p}(\xi_{\tau_p})) = \mathcal{E}^f_{S,\tau_p}(\xi_{\tau_p}) \leq V_S^+$ a.s. (where for the inequality we have used that $\tau_p \in \mathcal{T}_{S^+}$). From this, together with equation (3.10), we derive the desired inequality (3.9). From inequality (3.9), together with the continuity of $\mathcal{E}^f$-expectations with respect to the terminal time and the terminal condition, we derive $V_S^+ \geq \lim_{n\to\infty} \mathcal{E}^f_{S,S^n}(V_{S_n}) = \mathcal{E}^f_{S,S}(V_{S+}) = V_{S+}$ a.s. Hence, $V_S^+ \geq V_{S+}$ a.s., which, together with the previously shown converse inequality, proves the equality $V_{S+} = V_S^+$ a.s.

Let us now show statement (iii). Let $S \in \mathcal{T}_0$. As $V_S \geq \xi_S$, there are two sub-cases:

(a) $V_S = \xi_S$ and (b) $V_S > \xi_S$. By the right-uppersemicontinuity of the value process (V_t) (Proposition 2.4) and by statement (ii), we have $V_S \geq V_{S+} = V_S^+$. Hence, on the set $\{V_S = \xi_S\}$, we have $V_S = \xi_S \geq V_{S+} = V_S^+$. Hence, on this set, $V_S = \xi_S \vee V_S^+$. Let us now consider sub-case (b). On the set $\{V_S > \xi_S\}$, we have $S < T$ and $V_S = \operatorname{ess\,sup}_{\tau\in\mathcal{T}_S} \mathcal{E}^f_{S,\tau}(\xi_\tau) > \xi_S = \mathcal{E}^f_{S,S}(\xi_S)$. Hence, on this set, $V_S = \operatorname{ess\,sup}_{\tau\in\mathcal{T}_S} \mathcal{E}^f_{S,\tau}(\xi_\tau) = \operatorname{ess\,sup}_{\tau\in\mathcal{T}_{S+}} \mathcal{E}^f_{S,\tau}(\xi_\tau) = V_S^+$ (where we have used the definition of V_S^+ for the last equality). Thus, $V_S = \xi_S \vee V_S^+$ in sub-case (b). Statement (iii) is thus shown, and the proof is completed. □

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