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Liang, Y and Partington, JR orcid.org/0000-0002-6738-3216 (2020) Representing Kernels of Perturbations of Toeplitz Operators by Backward Shift-Invariant Subspaces. Integral Equations and Operator Theory, 92 (4). 35. ISSN 0378-620X

https://doi.org/10.1007/s00020-020-02592-7

REPRESENTING KERNELS OF PERTURBATIONS OF TOEPLITZ OPERATORS BY BACKWARD SHIFT-INVARIANT SUBSPACES

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Abstract. It is well known the kernel of a Toeplitz operator is nearly invariant under the backward shift S^*. This paper shows that kernels of finite rank perturbations of Toeplitz operators are nearly S^*-invariant with finite defect. This enables us to apply a recent theorem by Chalendar–Gallardo–Partington to represent the kernel in terms of backward shift-invariant subspaces, which we identify in several important cases.

1. Introduction

Let $H(\mathbb{D})$ be the space of all analytic functions on the open unit disc $\mathbb{D}$. The Hardy space $H^2 := H^2(\mathbb{D})$ is defined by

$$H^2 = \{f \in H(\mathbb{D}) : \ f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ with } \|f\|^2 := \sum_{n=0}^{\infty} |a_n|^2 < +\infty\}.$$

The limit $\lim\limits_{r\to 1^-} f(re^{it})$ exists almost everywhere, which gives the values of f on the unit circle $\mathbb{T}$. Since the H^2 norm of f and the $L^2(\mathbb{T})$ norm of its boundary function coincide, H^2 embeds isometrically as a closed subspace of $L^2(\mathbb{T})$ via

$$\sum_{n=0}^{\infty} a_n z^n \mapsto \sum_{n=0}^{\infty} a_n e^{int}.$$

This indicates a natural orthogonal decomposition $L^2(\mathbb{T}) = H^2 \oplus \overline{H_0^2}$, where H^2 is identified with the subspace spanned by $\{e^{int} : \ n \geq 0\}$ and $\overline{H_0^2}$ is the subspace spanned by $\{e^{int} : \ n < 0\}$, respectively.

Let $L^\infty := L^\infty(\mathbb{T})$ be the space containing all essentially bounded functions on $\mathbb{T}$. And $H^\infty := H^\infty(\mathbb{D})$ is the Banach algebra of bounded

2010 *Mathematics Subject Classification.* Primary: 46E22, 47B38; Secondary 47A15.

Key words and phrases. Shift-invariant subspace, nearly S^*-invariant subspace, Toeplitz operator.

analytic functions on $\mathbb{D}$ with the norm defined

$$\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|.$$

Similarly, the radial boundary function of an H^∞ function belongs to L^∞, and then H^∞ can be viewed as a Banach subalgebra of L^∞.

We recall an inner function is an H^∞ function that has unit modulus almost everywhere on $\mathbb{T}$. An outer function is a function $f \in H^1$ which can be written in the form

$$f(re^{i\eta}) = \alpha \exp\Big(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+re^{i\eta}}{e^{it}-re^{i\eta}} k(e^{it})dt\Big)$$

for $re^{i\eta} \in \mathbb{D}$, where k is a real-valued integrable function and $|\alpha| = 1$. It is known that each $f \in H^1 \setminus \{0\}$ has a factorization $f = \theta \cdot u$, where θ is inner and u is outer. This factorization is unique up to a constant of modulus 1 (cf. [9]).

Let $P: L^2(\mathbb{T}) \to H^2$ be the orthogonal projection on H^2 defined by a Cauchy integral

$$(Pf)(z) = \int \frac{f(\zeta)}{1-\bar{\zeta}z} dm(\zeta), \; |z| < 1.$$

Given $g \in L^\infty$, the Toeplitz operator $T_g: H^2 \to H^2$ is defined by

$$T_g f = P(gf)$$

for any $f \in H^2$. If θ is an inner function, then $\operatorname{Ker} T_{\overline{\theta}}$ is the model space $K_\theta = H^2 \ominus \theta H^2 = H^2 \cap \theta\overline{H_0^2}$ (cf. [10, 11]). It has also been proved that $\|T_g\| = \|g\|_\infty$ and $T_g^* = T_{\overline{g}}$ (cf. [3]). For more investigations into Toeplitz operators, the reader can refer to [7, 4, 14] and so on.

Beurling's theorem states that the subspaces θH^2 with inner function θ constitute the nontrivial invariant subspaces for the unilateral shift $S: H^2 \to H^2$ defined by $[Sf](z) = zf(z)$. Also the model space K_θ is invariant under the backward shift $S^*: H^2 \to H^2$ (cf. [10, Proposition 5.2]) defined by

$$S^*f(z) = \frac{f(z)-f(0)}{z} \;\; (f \in H^2, \; z \in \mathbb{D}).$$

The invariant subspace problem is still an unresolved problem in operator theory and there are various related investigations (cf. [6, 5]). Moreover, the study of nearly S^*-invariant subspaces has attracted a lot of attention (cf. [12, 13, 5]).

Definition 1.1. A subspace $M \subset H^2$ is called nearly S^*-invariant if $S^*f \in M$ whenever $f \in M$ and $f(0) = 0$. Furthermore, a subspace $M \subset H^2$ is said to be nearly S^*-invariant with defect m if there is an

m-dimensional subspace F such that $S^*f \in M + F$ whenever $f \in M$ with $f(0) = 0$; we call F the defect space.

If $f \in \operatorname{Ker} T_g$ with $f(0) = 0$, so $gf \in \overline{H_0^2}$ and then $g(\overline{z}f) \in \overline{H_0^2}$. Since $\overline{z}f \in H^2$, this implies $S^*f = \overline{z}f \in \operatorname{Ker} T_g$, which shows the kernel of a Toeplitz operator is nearly S^*-invariant. Motivated by this well-known result, we continue to examine a question which has a close link with the invariant subspace problem:

Given a Toeplitz operator T_g acting on Hardy space H^2, is the kernel of a rank n perturbation of T_g nearly S^*-invariant with finite defect?

We recall that an operator $T : \mathcal{H} \to \mathcal{H}$ of rank n on a Hilbert space $\mathcal{H}$ takes the form

$$Th = \sum_{i=1}^{n} \langle h, u_i \rangle v_i \text{ for all } h \in \mathcal{H},$$

where $\{u_i\}$ and $\{v_i\}$ are orthogonal sets in $\mathcal{H}$ (we may also suppose that $\{u_i\}$ is orthonormal). For simplicity, write $A_n := \{1, 2, \cdots, n\}$ and let $|\Lambda|$ stand for the number of integers in a set Λ.

A rank n perturbation of the Toeplitz operator $T_g : H^2 \to H^2$ denoted by $R_n : H^2 \to H^2$ is defined by

$$R_n(h) = T_g h + Th = T_g h + \sum_{i=1}^{n} \langle h, u_i \rangle v_i \tag{1.1}$$

with orthonormal set $\{u_i\}$ and orthogonal set $\{v_i\}$ in H^2.

The rest of the paper is organized as follows. In Section 2, we discuss the nearly S^*-invariant subspace $\operatorname{Ker} R_n$ with finite defect for several important classes of symbols and present the corresponding defect space in each case. Then we apply a recent theorem by Chalendar–Gallardo–Partington to represent the kernel of the operator R_1 in terms of backward shift-invariant subspaces in Section 3. The challenging task here is to identify the subspaces in question, which we do in various important cases. Note that even in the nearly S^*-invariant (defect 0) case, this is known to be a difficult question in general.

2. NEARLY S^*-INVARIANT $\operatorname{Ker} R_n$ WITH FINITE DEFECT

In this section, we prove that the kernel of the operator R_n in (1.1) is nearly S^*-invariant with finite defect for various important cases, especially identify the finite-dimensional defect spaces. First of all, we recall a useful theorem for later use.

Theorem 2.1. [10, Theorem 4.22] *For $\psi,\ \varphi \in L^\infty$, the operator $T_\psi T_\varphi$ is a Toeplitz operator if and only if either $\overline{\psi} \in H^\infty$ or $\varphi \in H^\infty$. In both cases, $T_\psi T_\varphi = T_{\psi\varphi}$.*

So for all $g \in L^\infty$, it holds that

$$T_{\overline{z}} T_g = T_{\overline{z}g} = T_{g\overline{z}}. \tag{2.1}$$

For every $h \in \operatorname{Ker} R_n$, it follows that

$$T_g h + \sum_{i=1}^{n} \langle h, u_i \rangle v_i = 0. \tag{2.2}$$

Letting $S^* = T_{\overline{z}}$ act on both sides of (2.2) and using (2.1), we have

$$T_{g\overline{z}} h + \sum_{i=1}^{n} \langle h, u_i \rangle S^* v_i = 0.$$

Now let $h \in \operatorname{Ker} R_n$ satisfy $h(0) = 0$, and then the above equation implies the following equivalent expressions.

$$T_g(\frac{h}{z}) + \sum_{i=1}^{n} \langle h, u_i \rangle S^* v_i = 0 \tag{2.3}$$

$$\Leftrightarrow \quad g\frac{h}{z} + \sum_{i=1}^{n} \langle h, u_i \rangle S^* v_i \in \overline{H_0^2}. \tag{2.4}$$

So the question of nearly S^*-invariant $\operatorname{Ker} R_n$ with finite defect is that: *for each $h \in \operatorname{Ker} R_n$ with $h(0) = 0$, find a vector w in some suitable finite-dimensional space F such that*

$$S^* h + w = \frac{h}{z} + w \in \operatorname{Ker} R_n,$$

which is equivalent to the following equations.

$$T_g(\frac{h}{z} + w) + \sum_{i=1}^{n} \langle \frac{h}{z} + w, u_i \rangle v_i = 0 \tag{2.5}$$

$$\Leftrightarrow \quad g(\frac{h}{z} + w) + \sum_{i=1}^{n} \langle \frac{h}{z} + w, u_i \rangle v_i \in \overline{H_0^2}. \tag{2.6}$$

Next we will construct the defect space F in several important cases.

2.1. $g = 0$ **a.e. on** $\mathbb{T}$. In this case, R_n is a rank-n operator and equation (2.5) with $g = 0$ implies

$$\operatorname{Ker} R_n = \bigcap_{i=1}^{n} (\bigvee\{u_i\})^{\perp} = H^2 \ominus (\bigvee\{u_i, i \in A_n\}),$$

where $\bigvee$ denotes the closed linear span in H^2.

For any $h \in \operatorname{Ker} R_n$ with $h(0) = 0$, it always holds that

$$S^* h \in \operatorname{Ker} R_n \oplus (\bigvee\{u_i, i \in A_n\}) = H^2,$$

which gives the following elementary observation on the nearly S^*-invariant subspace $\operatorname{Ker} R_n$ with finite defect.

Proposition 2.2. *Suppose $g = 0$ almost everywhere on $\mathbb{T}$. Then the subspace* $\operatorname{Ker} R_n$ *is nearly S^*-invariant with defect n and defect space*

$$F = \bigvee\{u_i,\ i \in A_n\}.$$

2.2. $g = \theta$ **an inner function.** In this case $T_\theta f = \theta f$ is an isometric multiplication operator on H^2. For each $h \in \operatorname{Ker} R_n$ with $h(0) = 0$, the relation (2.4) becomes

$$\theta \frac{h}{z} + \sum_{i=1}^{n} \langle h, u_i \rangle S^* v_i = 0. \tag{2.7}$$

The required relation (2.6) turns into

$$\theta(\frac{h}{z} + w) + \sum_{i=1}^{n} \langle \frac{h}{z} + w, u_i \rangle v_i = 0.$$

Combining it with (2.7), the above equation is equivalent to

$$(\theta w - \sum_{k=1}^{n} \langle h, u_k \rangle S^* v_k) + \sum_{i=1}^{n} \langle (\theta w - \sum_{k=1}^{n} \langle h, u_k \rangle S^* v_k), \theta u_i \rangle v_i = 0. \tag{2.8}$$

Now choosing

$$w = \overline{\theta}(\sum_{k=1}^{n} \langle h, u_k \rangle S^* v_k) = \sum_{k=1}^{n} \langle h, u_k \rangle T_{\overline{\theta}}(S^* v_k) \in H^2,$$

the required equation (2.8) holds. So we can obtain a theorem on the nearly S^*-invariant $\operatorname{Ker} R_n$ with finite defect.

Theorem 2.3. *Suppose $g = \theta$ an inner function. Then the subspace* $\operatorname{Ker} R_n$ *is nearly S^*-invariant with defect at most n and defect space*

$$F = \bigvee\{T_{\overline{\theta}}(S^* v_i),\ i \in A_n\}.$$

Example 2.4. For $g(z) = z^m$ ($m \in \mathbb{N}$), $\operatorname{Ker} R_n$ is nearly S^*-invariant with defect at most n and defect space $F = \bigvee\{(S^*)^{m+1}(v_i),\ i \in A_n\}$.

2.3. $g = f_1\overline{f_2}$ **with** $f_j \in \mathcal{G}H^\infty$ **for** $j = 1, 2$**.** Here $\mathcal{G}H^\infty$ denotes the set of all invertible elements in H^∞. In [2], Bourgain proved: *If g is a bounded measurable function on $\mathbb{T}$, then the condition $\int_\pi \log|g|dm > -\infty$ (m is the normalized invariant measure on $\mathbb{T}$) is the necessary and sufficient condition for $g \neq 0$ to be of the form $g = f_1 \cdot \overline{f_2}$ where $f_1, f_2 \in H^\infty$.* The interested reader can also refer to [1, Theorem 4.1] for a matricial version with norm estimates. In this subsection, we suppose $f_j \in \mathcal{G}H^\infty$ for $j = 1, 2$, and then Theorem 2.1 ensures that $T_{f_1\overline{f_2}} = T_{\overline{f_2}}T_{f_1}$.

For each $h \in \operatorname{Ker} R_n$ with $h(0) = 0$, (2.3) can be rewritten as

$$T_{\overline{f_2}}T_{f_1}(\frac{h}{z}) + \sum_{i=1}^{n}\langle h, u_i\rangle S^* v_i = 0, \tag{2.9}$$

which together with Theorem 2.1 imply

$$\frac{h}{z} + \sum_{i=1}^{n}\langle h, u_i\rangle T_{f_1^{-1}}T_{\overline{f_2}^{-1}}(S^* v_i) = 0. \tag{2.10}$$

The required equation (2.5) is changed into

$$T_{\overline{f_2}}T_{f_1}(\frac{h}{z} + w) + \sum_{i=1}^{n}\langle\frac{h}{z} + w, u_i\rangle v_i = 0,$$

which, by (2.9), is equivalent to

$$T_{\overline{f_2}}T_{f_1}w - \sum_{k=1}^{n}\langle h, u_k\rangle S^* v_k + \sum_{i=1}^{n}\langle\frac{h}{z} + w, u_i\rangle v_i = 0.$$

Now choosing

$$w = \sum_{k=1}^{n}\langle h, u_k\rangle T_{f_1^{-1}}T_{\overline{f_2}^{-1}}(S^* v_k)$$

and using (2.10), the result follows. Hence we can present a theorem on the nearly S^*-invariant $\operatorname{Ker} R_n$ with finite defect.

Theorem 2.5. *Suppose $g = f_1\overline{f_2}$ with $f_j \in \mathcal{G}H^\infty$ for $j = 1, 2$. Then the subspace* $\operatorname{Ker} R_n$ *is nearly S^*-invariant with defect at most n and defect space*

$$F = \bigvee\{T_{f_1^{-1}}T_{\overline{f_2}^{-1}}(S^* v_i),\ i \in A_n\}.$$

The following is a remark on two special cases of Theorem 2.5.

Remark 2.6. (i) For the operator R_n in (1.1) with $\overline{g} \in \mathcal{G}H^\infty$, $\operatorname{Ker} R_n$ is nearly S^*-invariant with defect at most n and defect space

$$F = \bigvee\{T_{g^{-1}}(S^* v_i),\ i \in A_n\}.$$

(ii) For the operator R_n in (1.1) with $g \in \mathcal{G}H^\infty$, $\operatorname{Ker} R_n$ is nearly S^*-invariant with defect at most n and defect space

$$F = \bigvee\{T_{g^{-1}}(S^* v_i),\ i \in A_n\}.$$

2.4. $g(z) = \overline{\theta(z)}$ **with** θ **a nonconstant inner function.** In this case, $T_{\overline{\theta}}$ is a special conjugate analytic Toeplitz operator with kernel K_θ. And then the relation (2.4) becomes

$$\psi := \overline{\theta}\frac{h}{z} + \sum_{k=1}^{n}\langle h, u_k\rangle S^* v_k \in \overline{H_0^2},$$

with

$$\theta\psi = \frac{h}{z} + \sum_{k=1}^{n}\langle h, u_k\rangle \theta S^* v_k \in H^2. \tag{2.11}$$

The desired relation (2.6) now takes the form

$$\overline{\theta}(\frac{h}{z} + w) + \sum_{i=1}^{n}\langle \frac{h}{z} + w, u_i\rangle v_i \in \overline{H_0^2}, \tag{2.12}$$

which, by (2.11), is equivalent to

$$\begin{aligned} &\psi - \sum_{k=1}^{n}\langle h, u_k\rangle S^* v_k + \overline{\theta}w \\ &\quad + \sum_{i=1}^{n}\langle \psi - \sum_{k=1}^{n}\langle h, u_k\rangle S^* v_k + \overline{\theta}w, \overline{\theta}u_i\rangle v_i \in \overline{H_0^2}. \end{aligned} \tag{2.13}$$

We denote the decompositions of u_i and ψ as below: $u_i = u_{i1} + \theta u_{i2}$ with $u_{i1} = P_{K_\theta} u_i \in K_\theta,\ u_{i2} \in H^2$, and $\psi = \psi_1 + \psi_2$ with $\psi_1 \in B := \bigvee\{\overline{\theta}u_{i1},\ i \in A_n\} \subset \overline{H_0^2}$ and $\psi_2 \in \overline{H_0^2} \ominus B$. So it is clear that

$$\langle \psi_2, \overline{\theta}u_i\rangle = 0 \text{ and } \langle \psi_2, u_{i2}\rangle = 0 \text{ for all } i \in A_n.$$

The above indicates that (2.13) is equivalent to

$$\begin{aligned} &(\psi_1 - \sum_{k=1}^{n}\langle h, u_k\rangle S^* v_k + \overline{\theta}w) \\ &\quad + \sum_{i=1}^{n}\langle (\psi_1 - \sum_{k=1}^{n}\langle h, u_k\rangle S^* v_k + \overline{\theta}w), \overline{\theta}u_{i1} + u_{i2}\rangle v_i \in \overline{H_0^2}. \end{aligned}$$

Choosing

$$w = \sum_{k=1}^{n} \langle h, u_k \rangle \theta S^* v_k - \theta \psi_1,$$

the above desired relation is true and the defect space F is

$$F = \bigvee\{\theta S^* v_i, P_{K_\theta} u_i, i \in A_n\} = \bigvee\{\theta S^* v_i, P_{K_\theta} u_k, i \in A_n,\ k \in \Lambda\},$$

where Λ denotes the subset of A_n consisting of all $k \in A_n$ such that $P_{K_\theta} u_k \neq 0$, i.e. $\theta \nmid u_k$. So in conclusion we have the following theorem.

Theorem 2.7. *Suppose $g(z) = \overline{\theta(z)}$ with θ an inner function. Then the subspace* $\operatorname{Ker} R_n$ *is nearly S^*-invariant with defect at most $n + |\Lambda|$ and defect space*

$$F = \bigvee\{\theta S^* v_i,\ P_{K_\theta} u_k,\ i \in A_n,\ k \in \Lambda\},$$

with $\Lambda \subset A_n$ consisting of all $k \in A_n$ such that $\theta \nmid u_k$.

3. The application of the C-G-P theorem

In this section, we apply a recent theorem (for short the C-G-P Theorem) by Chalendar–Gallardo–Partington to represent the kernels of rank one perturbations of Toeplitz operators in terms of backward shift-invariant subspaces. We shall take $n = 1$ and denote the operator

$$R_1 f = T_g f + \langle f, u \rangle v$$

with $\|u\| = 1$ and $S^* v \neq 0$. First we cite the C-G-P Theorem on nearly S^*-invariant subspaces with defect m from [8].

Theorem 3.1. [8, Theorem 3.2] *Let M be a closed subspace that is nearly S^*-invariant with defect m. Then*

(1) *in the case where there are functions in M that do not vanish at* 0, *then*

$$M = \{f :\ f(z) = k_0(z) f_0(z) + z \sum_{j=1}^{m} k_j(z) e_j(z) :\ (k_0, \cdots, k_m) \in K\},$$

where f_0 is the normalized reproducing kernel for M at 0, $\{e_1, \cdots, e_m\}$ *is any orthonormal basis for the defect space F, and K is a closed $S^* \oplus \cdots \oplus S^*$-invariant subspace of the vector-valued Hardy space* $H^2(\mathbb{D}, \mathbb{C}^{m+1})$, *and* $\|f\|^2 = \sum_{j=0}^{m} \|k_j\|^2$.

(2) *in the case where all functions in M vanish at* 0, *then*

$$M = \{f :\ f(z) = z \sum_{j=1}^{m} k_j(z) e_j(z) :\ (k_1, \cdots, k_m) \in K\},$$

with the same notation as in (1), *except that* K *is now a closed* $S^* \oplus \cdots \oplus S^*$*-invariant subspace of the vector-valued Hardy space* $H^2(\mathbb{D}, \mathbb{C}^m)$, *and* $\|f\|^2 = \sum_{j=1}^m \|k_j\|^2$.

The following proposition asserts that the kernels of some Toeplitz operators with special symbols are model spaces.

Proposition 3.2. [10, Proposition 5.8] *Let* $\varphi \in H^\infty \setminus \{0\}$ *and let* η *be the inner factor of* φ, *then*

$$\operatorname{Ker} T_{\overline{\varphi}} = K_\eta.$$

Now we apply the C-G-P Theorem to represent $\operatorname{Ker} R_1$ by backward shift-invariant subspaces in several important cases. Note that we can find K as the largest S^*-invariant subspace such that

$$S^{*n} k_0 f_0 + z \sum_{j=1}^m S^{*n} k_j e_j \in M \quad \text{or} \quad z \sum_{j=1}^m S^{*n} k_j e_j \in M$$

for all $n \in \mathbb{N}$.

3.1. $g = 0$ **a.e. on** $\mathbb{T}$. In this case $M = \operatorname{Ker} R_1 = H^2 \ominus \bigvee\{u\}$, which is a vector hyperplane. It is clear that such a hyperplane is the solution of a single linear equation. Also Proposition 2.2 showed that $\operatorname{Ker} R_1$ is nearly S^*-invariant with a 1-dimensional defect space $F = \bigvee\{u\}$. Using Theorem 3.1, we deduce a corollary on the representations of $\operatorname{Ker} R_1$.

Corollary 3.3. *Given a nearly* S^**-invariant subspace* $M = H^2 \ominus \bigvee\{u\}$ *with defect* 1, *let* $f_0 = P_M 1 = 1 - \overline{u(0)}u$, $v_0 = P(u - u(0)|u|^2)$ *and* $v_1 = P(\overline{z}|u|^2)$. *Then*

(1) *in the case* $P_M 1 \neq 0$, *we have*

$$M = \{f : \ f = k_0 f_0 + k_1 z u : \ (k_0, k_1) \in K\},$$

with an $S^* \oplus S^*$*-invariant subspace* $K = \{(k_0, k_1) : \langle k_0, z^n v_0\rangle + \langle k_1, z^n v_1\rangle = 0$ *for* $n \in \mathbb{N}\}$.

(2) *in the case* $P_M 1 = 0$, *we have*

$$M = \{f : \ f = k_1 z u : \ k_1 \in K\},$$

with an S^**-invariant subspace* $K = \{k_1 : \ \langle k_1, z^n v_1\rangle = 0$ *for* $n \in \mathbb{N}\}$.

Here we show some examples illustrating the variety of subspaces K that can occur.

Example 3.4. (*i*) Suppose $u = 1$, then $M = zH^2$, $f_0 = 0$ and $v_0 = v_1 = 0$. So Corollary 3.3 (2) implies M has the representation

$$M = \{f : \ f = z k_1 : \ k_1 \in K\}$$

with $K = H^2$ a trivial S^*-invariant subspace.

(ii) Suppose u is a nonconstant inner function, then $M = K_u \oplus zuH^2$, $f_0 = 1 - \overline{u(0)}u \neq 0$ and $v_0 = u - u(0)$, $v_1 = 0$. So Corollary 3.3 (1) implies M has the representation

$$M = \{f : \ f = k_0(1 - \overline{u(0)}u) + k_1 zu : \ (k_0, k_1) \in K\}$$

with an $S^* \oplus S^*$-invariant subspace $K = K_\eta \times H^2$, where η is the inner factor of v_0. Besides, Proposition 3.2 is used to show that K is backward shift-invariant.

(iii) Suppose u is a normalized reproducing kernel of H^2, that is $u(z) = \sqrt{1 - |\alpha|^2}(1 - \overline{\alpha}z)^{-1}$, $\alpha \in \mathbb{D} \setminus \{0\}$, then $M = \{f : \ f(\alpha) = 0\}$, $f_0 = \overline{\alpha}(\alpha - z)(1 - \overline{\alpha}z)^{-1} \neq 0$ and $v_0 = 0$, $v_1 = \overline{\alpha}(1 - \overline{\alpha}z)^{-1}$. So Corollary 3.3 (1) implies M has the representation

$$M = \{f : \ f = \overline{\alpha}k_0 \frac{\alpha - z}{1 - \overline{\alpha}z} + zk_1 \frac{\sqrt{1 - |\alpha|^2}}{1 - \overline{\alpha}z} : \ (k_0, k_1) \in K\},$$

with an $S^* \oplus S^*$-invariant subspace $K = H^2 \times \{0\}$.

(iv) Suppose $u(z) = (1 + z^k)/\sqrt{2}$ with $k \geq 1$, then $M = \bigvee\{1 - z^k, z, \cdots, z^{k-1}, z^{k+1}, z^{k+2}, \cdots\}$, $f_0 = 2^{-1}(1 - z^k) \neq 0$ and $v_0 = z^k/(2\sqrt{2})$, $v_1 = 2^{-1}z^{k-1}$. So Corollary 3.3 (1) implies M has the representation

$$M = \{f : \ f = k_0 \frac{1 - z^k}{2} + zk_1 \frac{1 + z^k}{\sqrt{2}} : \ (k_0, k_1) \in K\}$$

with an $S^* \oplus S^*$-invariant subspace $K = \{(k_0, k_1) : \sqrt{2}(S^*)^{k-1}k_1 = -(S^*)^k k_0, \ k_0 \in H^2\}$.

3.2. $g = \theta$ **an inner function.** In this case, $M = \operatorname{Ker} R_1 \subset \bigvee\{\overline{\theta}v\}$. Take any vector $f = \lambda\overline{\theta}v \in M$ satisfying $R_1 f = 0$, which is equivalent to $\lambda(1 + \langle\overline{\theta}v, u\rangle) = 0$. If $1 + \langle\overline{\theta}v, u\rangle \neq 0$, then $\lambda = 0$, meaning $M = \{0\}$ a trivial S^*-invariant subspace. So suppose $1 + \langle\overline{\theta}v, u\rangle = 0$, and then $M = \bigvee\{\overline{\theta}v\}$, which is nearly S^*-invariant with a 1-dimensional defect space $F = \bigvee\{S^*(\overline{\theta}v)\}$ from Theorem 2.3. So Theorem 3.1 implies a corollary on the representations of $\operatorname{Ker} R_1$.

Corollary 3.5. *Given a nearly S^*-invariant subspace $M = \bigvee\{\overline{\theta}v\}$ with defect* 1*, then*

(1) *in the case $a_0 := \langle\overline{\theta}v, 1\rangle \neq 0$, let $f_0 = P_M 1 = \overline{a_0}\|v\|^{-2}\overline{\theta}v$, we have*

$$M = \{f : \ f = k_0 f_0 : \ (k_0, 0) \in K\},$$

with an $S^* \oplus S^*$-invariant subspace $K = \mathbb{C} \times \{0\}$.

(2) *in the case $a_0 := \langle\overline{\theta}v, 1\rangle = 0$, we have*

$$M = \{f : \ f = \|S^*(\overline{\theta}v)\|^{-1}k_1\overline{\theta}v : \ k_1 \in K\},$$

with an S^*-invariant subspace $K=\mathbb{C}$.

Proof. (1) in this case, using Theorem 3.1 (1), we represent M by

$$\begin{aligned} M &= \{f:\ f=k_0f_0+zk_1\frac{S^*(\overline{\theta}v)}{\|S^*(\overline{\theta}v)\|}:\ (k_0,k_1)\in K\} \\ &= \{f:\ f=\frac{\overline{a_0}}{\|v\|^2}k_0\overline{\theta}v+\|S^*(\overline{\theta}v)\|^{-1}k_1(\overline{\theta}v-a_0):\ (k_0,k_1)\in K\}. \end{aligned}$$

Since $M=\bigvee\{\overline{\theta}v\}$, it yields

$$k_0\in\mathbb{C} \text{ and } \|S^*(\overline{\theta}v)\|^{-1}k_1(\overline{\theta}v-a_0)=\mu\overline{\theta}v \text{ with } \mu\in\mathbb{C},$$

which is equivalent to $k_0\in\mathbb{C}$ and $k_1=0$ due to $a_0\neq 0$. So the statement (1) is true. The statement (2) can be similarly shown by Theorem 3.1 (2). □

3.3. $g=f_1\overline{f_2}$ **with** $f_j\in\mathcal{G}H^\infty$ **for** $j=1,2$**.** In this case, $M=\operatorname{Ker}R_1\subset\bigvee\{f_1^{-1}(T_{\overline{f_2}^{-1}}v)\}$. Take any vector $f=\lambda f_1^{-1}(T_{\overline{f_2}^{-1}}v)\in M$ such that $R_1f=0$, which is equivalent to $\lambda(1+\langle f_1^{-1}(T_{\overline{f_2}^{-1}}v),u\rangle)=0$. It is clear $M=\{0\}$ is a trivial S^*-invariant subspace for $1+\langle f_1^{-1}T_{\overline{f_2}^{-1}}v,u\rangle\neq 0$. Now we always assume $1+\langle f_1^{-1}(T_{\overline{f_2}^{-1}}v),u\rangle=0$, and obtain $M=\bigvee\{f_1^{-1}(T_{\overline{f_2}^{-1}}v)\}$, which is nearly S^*-invariant with a 1-dimensional defect space $F=\bigvee\{f_1^{-1}T_{\overline{f_2}^{-1}}(S^*v)\}$ from Theorem 2.5. Denote the Taylor coefficients of $T_{\overline{f_2}^{-1}}v$ and f_1^{-1} by $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$, respectively. So $\langle f_1^{-1}T_{\overline{f_2}^{-1}}v,1\rangle=a_0b_0$, and using Theorem 3.1, we deduce a corollary on the representations of $\operatorname{Ker}R_1$.

Corollary 3.6. *Given a nearly S^*-invariant subspace $M=\bigvee\{f_1^{-1}(T_{\overline{f_2}^{-1}}v)\}$ with defect 1, then*

(1) *in the case $a_0b_0\neq 0$, let $f_0=P_M1=\overline{a_0b_0}\|f_1^{-1}T_{\overline{f_2}^{-1}}v\|^{-2}f_1^{-1}T_{\overline{f_2}^{-1}}v$; then we have*

$$M=\{f:\ f=k_0f_0:\ (k_0,0)\in K\},$$

with an $S^*\oplus S^*$-invariant subspace $K=\mathbb{C}\times\{0\}$.

(2) *in the case $a_0b_0=0$, we have*

$$M=\{f:\ f=k_1\frac{f_1^{-1}T_{\overline{f_2}^{-1}}v}{\|f_1^{-1}T_{\overline{f_2}^{-1}}(S^*v)\|}:\ k_1\in K\}$$

with $K=\mathbb{C}$ an S^*-invariant subspace.

Proof. (1) in this case, Theorem 3.1 (1) gives

$$M=\{f:\ f=k_0f_0+k_1\frac{f_1^{-1}(T_{\overline{f_2}^{-1}}v-a_0)}{\|f_1^{-1}T_{\overline{f_2}^{-1}}(S^*v)\|}:\ (k_0,k_1)\in K\},$$

due to $zf_1^{-1}T_{\overline{f_2}^{-1}}(S^*v)]=f_1^{-1}z[S^*(T_{\overline{f_2}^{-1}}v)]=f_1^{-1}(T_{\overline{f_2}^{-1}}v-a_0)$. Further by $M=\bigvee\{f_1^{-1}(T_{\overline{f_2}^{-1}}v)\}$, it follows that

$$k_0\in\mathbb{C}\ \text{ and }\ k_1\frac{f_1^{-1}(T_{\overline{f_2}^{-1}}v-a_0)}{\|f_1^{-1}T_{\overline{f_2}^{-1}}(S^*v)\|}=\mu f_1^{-1}(T_{\overline{f_2}^{-1}}v)\text{ with }\mu\in\mathbb{C},$$

which is equivalent to $k_0\in\mathbb{C}$ and $k_1=0$ by $a_0\neq 0$.

(2) in this case, it follows either $a_0=0$ or $b_0=0$ and $f_0=P_M1=0$. If $b_0=0$, Theorem 3.1 (2) implies that

$$M=\{f:\ f=k_1\frac{f_1^{-1}(T_{\overline{f_2}^{-1}}v-a_0)}{\|f_1^{-1}T_{\overline{f_2}^{-1}}(S^*v)\|}:\ k_1\in K\},$$

which is valid if and only if $a_0=0$ and $k_1\in\mathbb{C}$. □

3.4. $g=\overline{\theta}$ **with** θ **nonconstant inner function.** Because of its link with model spaces, this case is of particular interest. For every $h\in\operatorname{Ker}R_1$, the equation (2.2) is equivalent to $h+\langle h,u\rangle\theta v\in\theta\overline{H_0^2}$. So

$$M=\operatorname{Ker}R_1\subset(H^2\cap\theta\overline{H_0^2})\oplus\bigvee\{\theta v\}=K_\theta\oplus\bigvee\{\theta v\}.$$

Take any vector $h=h_1+\lambda\theta v\in M$ with $h_1\in K_\theta$ and $\lambda\in\mathbb{C}$, such that $R_1h=0$, which is equivalent to

$$\lambda(1+\langle\theta v,u\rangle)=-\langle h_1,u\rangle. \tag{3.1}$$

Now we divide this into two subsections to represent $M=\operatorname{Ker}R_1$ in terms of backward shift-invariant subspaces.

3.4.1. $\theta|u$. In this case, the equation (3.1) now is changed into $\lambda(1+\langle\theta v,u\rangle)=0$. If $1+\langle\theta v,u\rangle\neq 0$, then $\lambda=0$ and $M=K_\theta$ a nearly S^*-invariant subspace. So we suppose $1+\langle\theta v,u\rangle=0$, and then $M=K_\theta\oplus\bigvee\{\theta v\}$ is nearly S^*-invariant with a 1-dimensional defect space $F=\bigvee\{\theta S^*v\}$ from Theorem 2.7. Using Theorem 3.1, we obtain a corollary on the representation of $\operatorname{Ker}R_1$.

Corollary 3.7. *Given a nearly S^*-invariant subspace $M=K_\theta\oplus\bigvee\{\theta v\}$ with defect 1, and let $f_0=P_M1=1-\overline{\theta(0)}\theta+\overline{\theta(0)v(0)}\|v\|^{-2}\theta v$, we have*

$$\begin{aligned}M\ =\ \{f:\ f=k_0-(k_0\overline{\theta(0)}+k_1\frac{v(0)}{\|S^*v\|})\theta+(k_0\frac{\overline{\theta(0)v(0)}}{\|v\|^2}\\ +\|S^*v\|^{-1}k_1)\theta v:\ (k_0,k_1)\in K\},\end{aligned} \tag{3.2}$$

with an $S^*\oplus S^*$*-invariant subspace* $K=\{(k_0,k_1):\ k_i$ *satisfies* (3.3) *for* $i=0,1\}$, *where*

$$k_0-(k_0\overline{\theta(0)}+k_1\frac{v(0)}{\|S^*v\|})\theta\in K_\theta \ \ and\ \ k_0\frac{\overline{\theta(0)v(0)}}{\|v\|^2}+\|S^*v\|^{-1}k_1\in\mathbb{C}. \quad (3.3)$$

Proof. By Theorem 3.1 (1), we obtain

$$M=\{f:\ f=k_0f_0+k_1\frac{\theta(v-v(0))}{\|S^*v\|}:\ (k_0,k_1)\in K\},$$

which equals $K_\theta\oplus\bigvee\{\theta v\}$ implying the desired representation in (3.2). It is clear the second relation in (3.3) holds for S^*k_i, $i=1,2$. At the same time, the first relation in (3.3) together with the fact K_θ is an S^*-invariant subspace verify that

$$Y_\theta:=S^*k_0-S^*(k_0\overline{\theta(0)}\theta+\frac{v(0)k_1}{\|S^*v\|}\theta)\in K_\theta.$$

Then it turns out that

$$\begin{aligned} & S^*k_0-(S^*k_0\overline{\theta(0)}+\frac{v(0)}{\|S^*v\|}S^*k_1)\theta \\ = \ & Y_\theta+\overline{\theta(0)}k_0(0)\frac{\theta-\theta(0)}{z}+\frac{v(0)k_1(0)}{\|S^*v\|}\frac{\theta-\theta(0)}{z} \\ = \ & Y_\theta+(\overline{\theta(0)}k_0(0)+\frac{v(0)k_1(0)}{\|S^*v\|})T_{\overline{z}}\theta\in K_\theta, \end{aligned}$$

since $\langle T_{\overline{z}}\theta,\theta\rangle=\langle 1,z\rangle=0$ holds. This means the first relation in (3.3) also holds for S^*k_i, $i=1,2$. So K is an $S^*\oplus S^*$-invariant subspace. □

3.4.2. $\theta\nmid u$. In this case, we decompose u into $u=u_1+u_\theta$ with nonzero $u_1\in K_\theta$ and $u_\theta\in\theta H^2$. Then the identity (3.1) becomes

$$\lambda(1+\langle\theta v,u_\theta\rangle)=-\langle h_1,u_1\rangle. \quad (3.4)$$

Especially Theorem 2.7 implies $\operatorname{Ker}R_1$ is nearly S^*-invariant with a 2-dimensional defect space $F=\bigvee\{\theta S^*v,u_1\}$. For later use we present a remark concerning the projection P_M1.

Remark 3.8. Let $M=\operatorname{Ker}R_1\subset N:=K_\theta\oplus\bigvee\{\theta v\}$, and denote $N=M\oplus\bigvee\{G\}$ with $G=g+\mu\theta v$, where $g\in K_\theta$ and $\mu\in\mathbb{C}$. Then

$$\begin{aligned} P_M1 \ &= \ 1-\overline{\theta(0)}\theta+\frac{\overline{\theta(0)v(0)}}{\|v\|^2}\theta v \\ &\quad -\frac{\langle 1-\overline{\theta(0)}\theta,g\rangle+\overline{\theta(0)v(0)\mu}}{\|g\|^2+|\mu|^2\|v\|^2}(g+\mu\theta v). \end{aligned} \quad (3.5)$$

For simplicity, we denote $w_\theta := 1 + \langle \theta v, u_\theta \rangle$ and

$$\rho_\theta := \frac{\overline{u_1(0)} + \overline{\theta(0)v(0)}w_\theta}{\|u_1\|^2 + |w_\theta|^2\|v\|^2}.$$

Applying Theorem 3.1, we present a corollary on $\operatorname{Ker} R_1$.

Corollary 3.9. (1) *In the case $w_\theta \neq 0$, $M = N \ominus \bigvee\{u_1 + \overline{w_\theta}\theta v\}$ is nearly S^*-invariant with defect 2, and letting*

$$\begin{aligned} f_0 &= P_M 1 = 1 - \overline{\theta(0)}\theta + \overline{\theta(0)v(0)}\|v\|^{-2}\theta v - \rho_\theta(u_1 + \overline{w_\theta}\theta v), \qquad (3.6)\\ v_0 &= P(u_1 + \overline{w_\theta}\theta v - \theta(0)\overline{w_\theta}v + \theta(0)v(0)\|v\|^{-2}\overline{w_\theta}|v|^2 - \overline{\rho_\theta}|u_1 + \overline{w_\theta}\theta v|^2),\\ v_1 &= \|S^*v\|^{-1}P(\overline{w_\theta}v(\overline{v - v(0)})) \text{ and } v_2 = \|u_1\|^{-1}P(\overline{z}|u_1|^2), \end{aligned}$$

we have

$$\begin{aligned} M &= \{f : f = k_0 - (k_0\overline{\theta(0)} + k_1\frac{v(0)}{\|S^*v\|})\theta + (k_0\frac{\overline{\theta(0)v(0)}}{\|v\|^2} + \frac{k_1}{\|S^*v\|} - k_2\frac{z\overline{w_\theta}}{\|u_1\|})\theta v \\ &\qquad + (-k_0\rho_\theta + k_2\frac{z}{\|u_1\|})(u_1 + \overline{w_\theta}\theta v) : \ (k_0, k_1, k_2) \in K\}, \qquad (3.7) \end{aligned}$$

with an $S^* \oplus S^* \oplus S^*$-invariant subspace $K = \{(k_0, k_1, k_2) : \ k_i$ satisfies (3.8) for $i = 0, 1, 2$ \}, where

$$\begin{cases} k_0 - (k_0\overline{\theta(0)} + k_1\frac{v(0)}{\|S^*v\|})\theta \in K_\theta, \\ k_0\frac{\overline{\theta(0)v(0)}}{\|v\|^2} + \frac{k_1}{\|S^*v\|} - k_2\frac{z\overline{w_\theta}}{\|u_1\|} \in \mathbb{C}, \\ \langle k_0, \ z^n v_0\rangle + \langle k_1, \ z^n v_1\rangle + \langle k_2, \ z^n v_2\rangle = 0 \ \text{ for all } n \in \mathbb{N}. \end{cases} \qquad (3.8)$$

(2) *In the case $w_\theta = 0$, $M = N \ominus \bigvee\{u_1\}$ is nearly S^*-invariant with defect 2, and letting*

$$\begin{aligned} f_0 &= P_M 1 = 1 - \overline{\theta(0)}\theta - \overline{u_1(0)}\|u_1\|^{-2}u_1 + \overline{\theta(0)v(0)}\|v\|^{-2}\theta v,\\ v_0 &= P(u_1 - u_1(0)\|u_1\|^{-2}|u_1|^2) \quad \text{and} \quad v_2 = P(\|u_1\|^{-1}\overline{z}|u_1|^2), \end{aligned}$$

we have

$$\begin{aligned} M &= \{f : f = k_0 - (k_0\overline{\theta(0)} + k_1\frac{v(0)}{\|S^*v\|})\theta + (k_0\frac{\overline{\theta(0)v(0)}}{\|v\|^2} + \frac{k_1}{\|S^*v\|})\theta v \\ &\qquad + (-k_0\frac{\overline{u_1(0)}}{\|u_1\|^2} + k_2\frac{z}{\|u_1\|})u_1 : \ (k_0, k_1, k_2) \in K\}, \end{aligned}$$

with an $S^*\oplus S^*\oplus S^*$*-invariant subspace* $K=\{(k_0,k_1,k_2):\ k_i$ *satisfies* (3.9) *for* $i=0,1,2$ $\}$, *where*

$$\begin{cases} k_0-(k_0\overline{\theta(0)}+k_1\frac{v(0)}{\|S^*v\|})\theta\in K_\theta, \\ k_0\frac{\overline{\theta(0)v(0)}}{\|v\|^2}+\|S^*v\|^{-1}k_1\in\mathbb{C}, \\ \langle k_0,z^nv_0\rangle+\langle k_2,z^nv_2\rangle=0\ \ for\ n\in\mathbb{N}. \end{cases} \tag{3.9}$$

Proof. For the case $w_\theta\neq 0$, the equation (3.4) implies $\lambda=-w_\theta^{-1}\langle h_1,u_1\rangle$ and then $M=\operatorname{Ker}R_1=\{f:\ f=k-w_\theta^{-1}\langle k,u_1\rangle\theta v,\ k\in K_\theta\}$. By some calculations, it follows

$$M=N\ominus\bigvee\{u_1+\overline{w_\theta}\theta v\}. \tag{3.10}$$

Letting $g=u_1$ and $\mu=\overline{w_\theta}$ in (3.5), we obtain f_0 in (3.6). By Theorem 3.1 (1), it follows

$$M=\{f:\ f=k_0f_0+k_1\frac{\theta(v-v(0))}{\|S^*v\|}+k_2\frac{zu_1}{\|u_1\|}:\ (k_0,k_1,k_2)\in K\}$$

which together with (3.10) imply the representation of M in (3.7). Note the third formula in (3.8) holds for S^*k_i, $i=0,1,2$. Following the similar lines for proving (3.3) is S^*-invariant, it is easy to check the first two relations of (3.8) are valid for S^*k_i, $i=0,1,2$. So K is an $S^*\oplus S^*\oplus S^*$-invariant subspace. In particular, if $w_\theta=0$, the equation (3.4) implies

$$M=N\ominus\bigvee\{u_1\},$$

which is a special case of (3.10) with $w_\theta=0$. Hence we can prove the statement (2) from the similar proof of statement (1) with $w_\theta=0$. □

In order to help understand the case $g=\overline{\theta}$ with θ an inner function, we present an example for Corollary 3.9.

Example 3.10. Let $\theta=z^m$ $(m\geq 1)$ and $u=u_1+u_2$ with $u_1=z^{m-1}/4$ and $u_2\in z^mH^2$. It easy to check the kernel of R_1 is

$$M=\bigvee\{1,z,\cdots,z^{m-1}\rangle\oplus\bigvee\{z^mv\}\ominus\bigvee\{z^{m-1}/4+\overline{w_\theta}z^mv\},$$

which is nearly S^*-invariant with 2-dimensional defect space $F=\bigvee\{z^mS^*v,z^{m-1}\}$ from Theorem 2.7. If $w_\theta=1+\langle z^mv,u_\theta\rangle\neq 0$, then Corollary 3.9 (2) indicates the following representation for M:

$$\begin{aligned} M\ &=\ \{f:\ f=k_0-k_1\frac{v(0)}{\|S^*v\|}z^m+(k_1\|S^*v\|^{-1}-4k_2z\overline{w_\theta})z^mv \\ &\quad +k_2z(z^{m-1}+4\overline{w_\theta}z^mv):\ (k_0,k_1,k_2)\in K\}, \end{aligned}$$

with an $S^* \oplus S^* \oplus S^*$-invariant subspace $K = \{(k_0, k_1, k_2) : \ k_i$ satisfies (3.11) for $i = 0, 1, 2\}$ where

$$\begin{cases} k_0 - k_1 \frac{v(0)}{\|S^* v\|} z^m \in K_{z^m}, \\ k_1 \|S^* v\|^{-1} - 4k_2 z \overline{w_\theta} \in \mathbb{C}, \\ \langle k_0,\ z^n v_0 \rangle + \langle k_1,\ z^n v_1 \rangle = 0 \text{ for } n \in \mathbb{N}, \end{cases} \tag{3.11}$$

with

$$v_0 = 4^{-1} z^{m-1} + \overline{w_\theta} z^m v, \quad v_1 = \|S^* v\|^{-1} P(\overline{w_\theta} v (\overline{v - v(0)})).$$

Acknowledgments. The authors thank the referee for many useful comments which improve the presentation considerably. This work was done while the first author was visiting the University of Leeds. She is grateful to the School of Mathematics at the University of Leeds for its hospitality. The first author is supported by the National Natural Science Foundation of China (Grant No. 11701422).

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