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ON THE CUMULATIVE PARISIAN RUIN OF MULTI-DIMENSIONAL BROWNIAN MOTION RISK MODELS

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Abstract: Consider a multi-dimensional Brownian motion which models the surplus processes of multiple lines of business of an insurance company. Our main result gives exact asymptotics for the cumulative Parisian ruin probability as the initial capital tends to infinity. An asymptotic distribution for the conditional cumulative Parisian ruin time is also derived. The obtained results on the cumulative Parisian ruin can be seen as generalizations of some of the results derived in [5]. As a particular interesting case, the two-dimensional Brownian motion risk model is discussed in detail.

Key Words: multi-dimensional Brownian motion; cumulative Parisian ruin; exact asymptotics; ruin probability; quadratic programming problem.

AMS Classification: 91B30, 60G15, 60G70

1. Introduction

Consider an insurance company which operates simultaneously d ($d \geq 1$) lines of business. It is assumed that the surplus processes of these lines of business are described by a multi-dimensional risk model:

$$\boldsymbol{U}(t) = \boldsymbol{u} + \boldsymbol{\mu} t - \boldsymbol{X}(t), \quad t \geq 0, \tag{1}$$

where $\boldsymbol{u} = (u_1, u_2, \ldots, u_d)^\top$, with $u_i \geq 0$, is a (column) vector of initial capitals of these business lines, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d)^\top$, with $\mu_i > 0$, is a vector of net premium income rates, and $\boldsymbol{X}(t) = (X_1(t), X_2(t), \cdots, X_d(t))^\top, t \geq 0$ is a vector of net loss processes by time t.

In recent years, there has been an increasing interest in risk theory in the study of multi-dimensional risk models with different stochastic processes modeling $\boldsymbol{X}(t), t \geq 0$; see, e.g., [1] for an overview. In comparison with the well-understood 1-dimensional risk models, the study of multi-dimensional risk models is more challenging.

We consider in this paper the multi-dimensional Brownian motion risk model, i.e.,

$$\boldsymbol{X}(t) = A\boldsymbol{B}(t), \quad t \geq 0, \tag{2}$$

where $A \in \mathbb{R}^{d\times d}$ is a non-singular matrix, and $\boldsymbol{B}(t) = (B_1(t), \ldots, B_d(t))^\top, t \geq 0$ is a standard d-dimensional Brownian motion with independent coordinates. Multi-dimensional Brownian motion risk models have drawn a lot of attention due to its tractability; see, e.g., [6, 10] and references therein.

We shall investigate the *cumulative Parisian ruin* problem of the multi-dimensional Brownian motion risk model (1) with $\boldsymbol{X}$ defined by (2). The cumulative Parisian ruin was first introduced by [11] based on the occupation (or sojourn) times of the 1-dimensional risk process. In the multi-dimensional setup the cumulative Parisian

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ruin time (at level $r > 0$) is defined as

$$\tau_r(\boldsymbol{u}) := \inf\left\{t > 0 : \int_0^t \mathbb{I}(\boldsymbol{U}(s) < \boldsymbol{0})ds > r\right\}, \tag{3}$$

where $\mathbb{I}(\cdot)$ is the indicator function, and the inequality for vectors $\boldsymbol{U}(s) < \boldsymbol{0}$ is meant component-wise. As remarked in [11] "*the parameter r could be interpreted as the length of a clock started at the beginning of the first excursion, paused when the process returns above zero, and resumed at the beginning of the next excursion, and so on.*". Clearly, if r is set to be 0 one obtains the *simultaneous ruin time* $\tau_0(\boldsymbol{u})$ for the multi-dimensional Brownian motion risk model, i.e.,

$$\tau_0(\boldsymbol{u}) := \inf\{t > 0 : \boldsymbol{U}(t) < \boldsymbol{0}\} = \inf\{t > 0 : U_i(t) < 0, \ \ \forall 1 \le i \le d\},$$

which has been discussed recently in [5] under a different context.

In this paper our primary focus is on the infinite-time cumulative Parisian ruin probability defined as

$$\mathbb{P}\{\tau_r(\boldsymbol{u}) < \infty\}.$$

Note that in the 1-dimensional setup, we have from (5) in [7] (see also [11]) that

$$\begin{aligned}\mathbb{P}\{\tau_r(u) < \infty\} &= \mathbb{P}\left\{\int_0^\infty \mathbb{I}(B_1(t) - \mu_1 t > u)dt > r\right\} \\ &= \left(2(1+\mu_1^2 r)\Psi(\mu_1\sqrt{r}) - \frac{\mu_1\sqrt{2r}}{\sqrt{\pi}}e^{-\frac{\mu_1^2 r}{2}}\right)e^{-2\mu_1 u} \end{aligned} \tag{4}$$

for all $u \in \mathbb{R}$, where $\Psi(s)$ is the standard normal survival function.

It turns out that explicit formula for the cumulative Parisian ruin probability in the multi-dimensional setup is difficult to obtain. In this case, it is of interest to derive some asymptotic results by letting the initial capitals tend to infinity. We shall assume that

$$\boldsymbol{u} = \boldsymbol{\alpha} u = (\alpha_1 u, \alpha_2 u, \ldots, \alpha_d u), \qquad \alpha_i > 0, \ u \ge 0,$$

and consider the asymptotics of the cumulative Parisian ruin probability as $u \to \infty$. For simplicity, hereafter we denote

$$\tau_r(u) := \tau_r(\boldsymbol{u}), \qquad u \ge 0. \tag{5}$$

Define

$$g(t) = \frac{1}{t}\inf_{\boldsymbol{x} \ge \boldsymbol{\alpha} + \boldsymbol{\mu} t} \boldsymbol{x}^\top \Sigma^{-1}\boldsymbol{x}, \quad t \ge 0, \qquad \text{with} \ \ \Sigma = AA^\top, \tag{6}$$

where $1/0$ is understood as ∞. Our principal result presented in Theorem 3.1 shows that, for any $r \ge 0$,

$$\begin{aligned}\mathbb{P}\{\tau_r(u) < \infty\} &= \mathbb{P}\left\{\int_0^\infty \mathbb{I}(\boldsymbol{X}(t) - \boldsymbol{\mu} t > \boldsymbol{\alpha} u)dt > r\right\} \\ &\sim C_I \mathcal{H}_I(r) u^{\frac{1-m}{2}} e^{-\frac{\inf_{t\ge 0} g(t)}{2} u}, \qquad u \to \infty, \end{aligned} \tag{7}$$

where $C_I > 0$, $m \in \mathbb{N}$ are known constants and $\mathcal{H}_I(r)$ is a counterpart of the celebrated Pickands constant; explicit expressions of these constants will be displayed in Section 3.

As a by-product, we also derive in Theorem 3.1 the asymptotic distribution of

$$\tau_{r_2}(u)|\tau_{r_1}(u) < \infty, \quad u \to \infty$$

for any $0 \le r_1 \le r_2 < \infty$. The approximation of the above quantity is of interest in risk theory; it will provide us with some idea of when cumulative Parisian ruin actually occurred at level r_2 knowing that it has occurred at some level r_1. We refer to [1, 9, 13] and references therein for related discussions on ruin times.

It is worth mentioning that there are some related interesting studies on the asymptotic properties of sojourn times above a high level of 1-dimensional (real-valued) stochastic processes; see, e.g., [2–4]. We refer to [7, 8] for recent developments. The multi-dimensional counterparts of this problem are more challenging, and to the best knowledge of the author there has been no result in this direction. Our study on the cumulative Parisian ruin probability for the multi-dimensional Brownian motion risk models covers this gap in a sense by deriving some asymptotic properties of the sojourn times.

As an important illustration, the two-dimensional Brownian motion risk model is discussed in detail. Asymptotic results for the cumulative Parisian ruin probabilities and the conditional cumulative Parisian ruin times are obtained for the full range of the parameters involved in the model.

The rest of this paper is organised as follows. In Section 2 we introduce some notation and present some preliminaries, which are extracted from [5]. The main results are presented in Section 3, followed by a discussion on the two-dimensional Brownian motion risk model in Section 4. The technical proofs are displayed in Section 5 and Section 6.

2. Notation and Preliminaries

We assume that all vectors are d-dimensional column vectors written in bold letters with $d \ge 2$. Operations with vectors are meant component-wise, e.g., $\lambda \boldsymbol{x} = \boldsymbol{x}\lambda = (\lambda x_1, \ldots, \lambda x_d)^\top$ for any $\lambda \in \mathbb{R}, \boldsymbol{x} \in \mathbb{R}^d$. Further, we denote $\mathbf{0} = (0, \ldots, 0)^\top \in \mathbb{R}^d$. If $I \subset \{1, \ldots, d\}$, then for a vector $\boldsymbol{a} \in \mathbb{R}^d$ we denote by $\boldsymbol{a}_I = (a_i, i \in I)$ a sub-block vector of $\boldsymbol{a}$. Similarly, if further $J \subset \{1, \ldots, d\}$, for a matrix $M = (m_{ij})_{i,j\in\{1,\ldots,d\}} \in \mathbb{R}^{d\times d}$ we denote by $M_{IJ}= M_{I,J} = (m_{ij})_{i\in I, j\in J}$ the sub-block matrix of M determined by I and J. Moreover, write $M^{-1}_{II} = (M_{II})^{-1}$ for the inverse matrix of M_{II} whenever it exists.

As we will see, the solution to the quadratic programming problem involved in (6) is the key to our discussions. We introduce the next lemma stated in [12] (see also [5]), which is important for several definitions in the sequel.

Lemma 2.1. *Let $M \in \mathbb{R}^{d\times d}, d \ge 2$ be a positive definite matrix. If $\boldsymbol{b} \in \mathbb{R}^d \setminus (-\infty, 0]^d$, then the quadratic programming problem*

$$P_M(\boldsymbol{b}) : \textit{Minimise } \boldsymbol{x}^\top M^{-1}\boldsymbol{x} \textit{ under the linear constraint } \boldsymbol{x} \ge \boldsymbol{b}$$

has a unique solution $\widetilde{\boldsymbol{b}}$ and there exists a unique non-empty index set $I \subseteq \{1, \ldots, d\}$ such that

$$\widetilde{\boldsymbol{b}}_I = \boldsymbol{b}_I \neq \mathbf{0}_I, \quad M^{-1}_{II}\boldsymbol{b}_I > \mathbf{0}_I, \tag{8}$$

$$\textit{and} \qquad \textit{if } I^c = \{1, \ldots, d\} \setminus I \neq \emptyset, \textit{ then } \widetilde{\boldsymbol{b}}_{I^c} = M_{I^cI}M^{-1}_{II}\boldsymbol{b}_I \ge \boldsymbol{b}_{I^c}. \tag{9}$$

Furthermore,

$$\min_{\boldsymbol{x}\geq\boldsymbol{b}} \boldsymbol{x}^\top M^{-1}\boldsymbol{x} = \widetilde{\boldsymbol{b}}^\top M^{-1}\widetilde{\boldsymbol{b}} = \boldsymbol{b}_I^\top M_{II}^{-1}\boldsymbol{b}_I > 0.$$

Definition 2.2. *The unique index set I that defines the solution of the quadratic programming problem in question will be referred to as the essential index set.*

Consider the minimisation problem involved in (6), i.e., $\inf_{\boldsymbol{x}\geq\boldsymbol{\alpha}+\boldsymbol{\mu}t} \boldsymbol{x}^\top\Sigma^{-1}\boldsymbol{x}$. For any fixed $t\geq 0$, we define $\boldsymbol{b}(t) = \boldsymbol{\alpha}+\boldsymbol{\mu}t$, and let $I(t)\subseteq\{1,\dots,d\}$ be the essential index set of the quadratic programming problem $P_\Sigma(\boldsymbol{b}(t))$.

Note that the two-layer minimisation problem in the exponent of (7), i.e.,

$$\inf_{t\geq 0} g(t) = \inf_{t\geq 0}\frac{1}{t}\inf_{\boldsymbol{x}\geq\boldsymbol{\alpha}+\boldsymbol{\mu}t}\boldsymbol{x}^\top\Sigma^{-1}\boldsymbol{x}$$

has been solved in Lemma 2.2 in [5], with the aid of Lemma 2.1. More precisely, it is proved therein the function $g(t), t\geq 0$ is convex and attains its unique minimum at some t_0. Let $I = I(t_0)$ be the essential index set of the quadratic programming problem $P_\Sigma(\boldsymbol{b})$ with $\boldsymbol{b} = \boldsymbol{b}(t_0) = \boldsymbol{\alpha}+\boldsymbol{\mu}t_0$. Then

$$t_0 = \sqrt{\frac{\boldsymbol{\alpha}_I^\top\Sigma_{II}^{-1}\boldsymbol{\alpha}_I}{\boldsymbol{\mu}_I^\top\Sigma_{II}^{-1}\boldsymbol{\mu}_I}} > 0, \tag{10}$$

and

$$g(t_0) = \inf_{t\geq 0} g(t) = \frac{1}{t_0}\boldsymbol{b}_I^\top\Sigma_{II}^{-1}\boldsymbol{b}_I. \tag{11}$$

Hereafter, we shall use the notation $\boldsymbol{b} = \boldsymbol{b}(t_0)$, and use $I = I(t_0)$ for the *essential index set* of the quadratic programming problem $P_\Sigma(\boldsymbol{b})$. Furthermore, let $\widetilde{\boldsymbol{b}}$ be the unique solution of $P_\Sigma(\boldsymbol{b})$. If $I^c = \{1,\dots,d\}\setminus I\neq\emptyset$, we define (cf. (9)) *weakly essential index set* and *unessential index set* by

$$K = \{j\in I^c : \widetilde{\boldsymbol{b}}_j = \Sigma_{jI}\Sigma_{II}^{-1}\boldsymbol{b}_I = \boldsymbol{b}_j\}, \quad \text{and } J = \{j\in I^c : \widetilde{\boldsymbol{b}}_j = \Sigma_{jI}\Sigma_{II}^{-1}\boldsymbol{b}_I > \boldsymbol{b}_j\}. \tag{12}$$

As we shall see, the index set I determines m, $\inf_{t\geq 0} g(t)$ and $\mathcal{H}_I(r)$ in the asymptotics (7), whereas both I and K determine the constant C_I. Moreover, the set J, whenever non-empty, contains indices that do not play any role in the asymptotic result, but it does appear in the proof (see (33)).

Next, define for $t > 0$

$$g_I(t) := \frac{1}{t}\boldsymbol{b}(t)_I^\top\Sigma_{II}^{-1}\boldsymbol{b}(t)_I = \frac{1}{t}\boldsymbol{\alpha}_I^\top\Sigma_{II}^{-1}\boldsymbol{\alpha}_I + 2\boldsymbol{\alpha}_I^\top\Sigma_{II}^{-1}\boldsymbol{\mu}_I + \boldsymbol{\mu}_I^\top\Sigma_{II}^{-1}\boldsymbol{\mu}_I t.$$

Clearly, by (11) we have

$$\widehat{g} := g(t_0) = g_I(t_0).$$

Furthermore, we denote

$$\widetilde{g} := g_I''(t_0) = 2t_0^{-3}(\boldsymbol{\alpha}_I^\top\Sigma_{II}^{-1}\boldsymbol{\alpha}_I),$$

which will appear in the definition of the constant C_I in Section 3.

3. Main Results

We introduce some constants that will appear in the main results. First we write

$$m := \sharp\{i : i \in I\} \ \geq 1$$

for the number of elements of the essential index set I. Further, define the following constant (existence is confirmed in Theorem 3.1)

$$\mathcal{H}_I(r) = \lim_{T\to\infty} \frac{1}{T}\mathcal{H}_I(r,T), \tag{13}$$

with

$$\mathcal{H}_I(r,T) = \int_{\mathbb{R}^m} e^{\frac{1}{t_0}\boldsymbol{x}_I^\top \Sigma_{II}^{-1}\boldsymbol{b}_I}\mathbb{P}\left\{\int_{t\in[0,T]} \mathbb{I}((\boldsymbol{X}(t)-\boldsymbol{\mu}t)_I > \boldsymbol{x}_I)dt > r\right\} d\boldsymbol{x}_I, \quad r<T. \tag{14}$$

Moreover, set

$$C_I := \frac{1}{\sqrt{(2\pi t_0)^m\,|\Sigma_{II}|}}\int_{\mathbb{R}} e^{-\widetilde{g}\frac{x^2}{4}}\psi(x)\,dx,$$

where $|\Sigma_{II}|$ denotes the determinant of the matrix Σ_{II}, and for $x\in\mathbb{R}$

$$\psi(x) = \begin{cases} 1, & \text{if } K=\emptyset,\\ \mathbb{P}\left\{\boldsymbol{Y}_K > \frac{1}{\sqrt{t_0}}(\boldsymbol{\mu}_K - \Sigma_{KI}\Sigma_{II}^{-1}\boldsymbol{\mu}_I)x\right\}, & \text{if } K\neq\emptyset. \end{cases} \tag{15}$$

Here the index set K is defined in (12), $\boldsymbol{Y}_K$ is a Gaussian random vector with mean vector $\mathbf{0}_K$ and covariance matrix D_{KK} given by

$$D_{KK} = \Sigma_{KK} - \Sigma_{KI}\Sigma_{II}^{-1}\Sigma_{IK}.$$

The next theorem constitutes our main results. Its proof is demonstrated in Section 5.

Theorem 3.1. *Let $\tau_r(u)$ be defined in (5) (see also (3)). We have, for any $r\geq 0$,*

$$\mathbb{P}\{\tau_r(u)<\infty\} \sim C_I\mathcal{H}_I(r)u^{\frac{1-m}{2}}e^{-\frac{\widehat{g}}{2}u}, \qquad u\to\infty, \tag{16}$$

where

$$0<\mathcal{H}_I(r)<\infty, \qquad \forall r\geq 0. \tag{17}$$

Moreover, we have, for any $0\leq r_1\leq r_2<\infty$ and any $s\in\mathbb{R}$,

$$\lim_{u\to\infty}\mathbb{P}\left\{\frac{\tau_{r_2}(u)-t_0u}{\sqrt{2u/\widetilde{g}}}\leq s\middle|\tau_{r_1}(u)<\infty\right\} = \frac{\mathcal{H}_I(r_2)\int_{-\infty}^{s}e^{-\frac{x^2}{2}}\psi(\sqrt{2/\widetilde{g}}x)\,dx}{\mathcal{H}_I(r_1)\int_{-\infty}^{\infty}e^{-\frac{x^2}{2}}\psi(\sqrt{2/\widetilde{g}}x)\,dx}. \tag{18}$$

Remarks 3.2. *(a). If $d=1$, we have from (16) that*

$$\mathbb{P}\{\tau_r(u)<\infty\}\sim\frac{1}{\mu_1}\mathcal{H}_{\{1\}}(r)e^{-2\alpha_1\mu_1 u}, \qquad u\to\infty.$$

This together with (4) yields that

$$\mathcal{H}_{\{1\}}(r) = \mu_1\left(2(1+\mu_1^2 r)\Psi(\mu_1\sqrt{r}) - \frac{\mu_1\sqrt{2r}}{\sqrt{\pi}}e^{-\frac{\mu_1^2 r}{2}}\right). \tag{19}$$

(b). As in [5] we can check that the results in Theorem 3.1 still hold for general $\boldsymbol{\alpha},\boldsymbol{\mu}\in\mathbb{R}^d$ such that $\alpha_i>0,\mu_i>0$ for some $1\leq i\leq d$.

4. Two-dimensional Brownian motion risk models

In this section, we focus on the two-dimensional Brownian motion risk models given by

$$\boldsymbol{U}(t) = \boldsymbol{\alpha} v + \boldsymbol{\mu} t - A\boldsymbol{B}(t), \quad t \geq 0, \tag{20}$$

with $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)^\top > \mathbf{0}$ and $\boldsymbol{\mu} = (\mu_1, \mu_2)^\top > \mathbf{0}$ and

$$\Sigma = AA^\top = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \qquad \rho \in (-1, 1).$$

We aim to find the asymptotics of the cumulative Parisian ruin probability and the asymptotic distribution of the conditional cumulative Parisian ruin as u tends to infinity, for all the possible values of $\rho \in (-1, 1)$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)^\top > \mathbf{0}$ and $\boldsymbol{\mu} = (\mu_1, \mu_2)^\top > \mathbf{0}$.

In order to simplify the analysis, we first do some variable changes. Consider $\mu_1 = ab$, with $a = 1/\mu_1, b = \mu_1^2$. By the self-similarity of Brownian motion we derive that

$$\begin{aligned}
\tau_r(u) &= \inf\left\{ t \geq 0 : \int_0^t \mathbb{I}\begin{pmatrix} X_1(s) - \mu_1 s > \alpha_1 u \\ X_2(s) - \mu_2 s > \alpha_2 u \end{pmatrix} ds > r \right\} \\
&= \inf\left\{ t \geq 0 : \int_0^t \mathbb{I}\begin{pmatrix} X_1(b^{-1}(bs)) - a(bs) > \alpha_1 u \\ X_2(b^{-1}(bs)) - \mu_2 b^{-1}(bs) > \alpha_2 u \end{pmatrix} ds > r \right\} \\
&\overset{d}{=} \inf\left\{ t \geq 0 : \int_0^{\mu_1^2 t} \mathbb{I}\begin{pmatrix} X_1(s) - s > \alpha_1 \mu_1 u \\ X_2(s) - \mu_2/\mu_1 s > \alpha_2 \mu_1 u \end{pmatrix} ds > \mu_1^2 r \right\},
\end{aligned}$$

where $\overset{d}{=}$ denotes equivalence in distribution. Next we denote

$$v = \alpha_1 \mu_1 u, \quad \mu = \mu_2/\mu_1, \quad \alpha = \alpha_2/\alpha_1, \quad \widetilde{r} = \mu_1^2 r, \tag{21}$$

and define

$$\tau_{\widetilde{r}}(v) := \inf\left\{ t \geq 0 : \int_0^t \mathbb{I}\begin{pmatrix} X_1(s) - s > v \\ X_2(s) - \mu s > \alpha v \end{pmatrix} ds > \widetilde{r} \right\}.$$

Clearly, we have

$$\tau_{\widetilde{r}}(v) \overset{d}{=} \mu_1^2 \cdot \tau_r(u). \tag{22}$$

Thanks to this equivalence, we can derive the results for the cumulative Parisian ruin time $\tau_r(u)$, by applying Theorem 3.1 to the cumulative Parisian ruin time $\tau_{\widetilde{r}}(v)$ of the auxiliary risk model

$$\widetilde{\boldsymbol{U}}(t) = \widetilde{\boldsymbol{\alpha}} v + \widetilde{\boldsymbol{\mu}} t - A\boldsymbol{B}(t), \quad t \geq 0, \tag{23}$$

with (recall also (21))

$$\widetilde{\boldsymbol{\alpha}} = (1, \alpha)^\top > \mathbf{0}, \quad \widetilde{\boldsymbol{\mu}} = (1, \mu)^\top > \mathbf{0}.$$

Note that the auxiliary risk model defined in (23) is easier to analyse as it involves a smaller number of parameters (namely, ρ, α, μ) than the original risk model (20).

The main results of this section are displayed in Theorem 4.1 below. From these results we can observe how different values of ρ yield different scenarios of the asymptotic behaviour, which shows an interesting reduction of dimension phenomenon; see also [5] for discussions on this phenomenon. The proof of Theorem 4.1 is deferred to Section 6.

Theorem 4.1. *Consider the original two-dimensional Brownian motion models described in (20). Recall the notation in (21).*

(i). *Suppose α and μ satisfy one of the following conditions:*

(i.C1) $\mu < 1$ *and* $\alpha < 1$,

(i.C2) $\mu < 1$, $\alpha \geq 1$ *and* $\mu \leq 1/\alpha$,

(i.C3) $\mu \geq 1$, $\alpha < 1$ *and* $\mu \leq 1/\alpha$.

We have, for any $r \geq 0$,

(i.R1). *If* $-1 < \rho < \frac{\alpha+\mu}{2}$, *then, as* $u \to \infty$

$$\mathbb{P}\left\{\tau_r(u) < \infty\right\} \sim \frac{\mathcal{H}_{\{1,2\}}(\mu_1^2 r)}{\sqrt{\alpha_1\mu_1 t_0^2 \pi(1-\rho^2)\widetilde{g}}} u^{-\frac{1}{2}} e^{-\frac{\widehat{g}}{2}\alpha_1\mu_1 u}, \tag{24}$$

where

$$t_0 = \sqrt{\frac{1+\alpha^2-2\alpha\rho}{1+\mu^2-2\mu\rho}}, \quad \widehat{g} = \frac{2}{t_0}\frac{1+\alpha^2-2\alpha\rho}{1-\rho^2} + \frac{2(1+\alpha\mu-\mu\rho-\alpha\rho)}{1-\rho^2}, \quad \widetilde{g} = 2t_0^{-3}\frac{1+\alpha^2-2\alpha\rho}{1-\rho^2},$$

and, for any $\widetilde{r} \geq 0$

$$\mathcal{H}_{\{1,2\}}(\widetilde{r}) = \lim_{T\to\infty}\frac{1}{T}\int_{\mathbb{R}^2} e^{\left(\frac{1-\rho\mu}{1-\rho^2}+\frac{1=\rho\alpha}{t_0(1-\rho^2)}\right)x_1+\left(\frac{\mu-\rho}{1-\rho^2}+\frac{\alpha-\rho}{t_0(1-\rho^2)}\right)x_2}\mathbb{P}\left\{\int_{t\in[0,T]}\mathbb{I}\left(\begin{array}{c} X_1(t)-t > x_1 \\ X_2(t)-\mu t > x_2\end{array}\right)dt > \widetilde{r}\right\} d\boldsymbol{x}.$$

Furthermore, for any $0 \leq r_1 \leq r_2 < \infty$ *and any* $s \in \mathbb{R}$

$$\lim_{u\to\infty}\mathbb{P}\left\{\frac{\tau_{r_2}(u)-t_0\alpha_1/\mu_1 u}{\sqrt{2\alpha_1/(\mu_1^3\widetilde{g})u}} \leq s \Big| \tau_{r_1}(u) < \infty\right\} = \frac{\mathcal{H}_{\{1,2\}}(\mu_1^2 r_2)}{\mathcal{H}_{\{1,2\}}(\mu_1^2 r_1)}\Phi(s), \tag{25}$$

where $\Phi(s)$ *is the standard normal distribution function.*

(i.R2). *If* $\rho = \frac{\alpha+\mu}{2}$, *then, as* $u \to \infty$

$$\mathbb{P}\left\{\tau_r(u) < \infty\right\} \sim \frac{1}{2}\mathcal{H}_{\{1\}}(\mu_1^2 r)\ e^{-2\alpha_1\mu_1 u},$$

where (cf. (19))

$$\mathcal{H}_{\{1\}}(\widetilde{r}) = 2(1+\widetilde{r})\Psi(\sqrt{\widetilde{r}}) - \frac{\sqrt{2\widetilde{r}}}{\sqrt{\pi}}e^{-\frac{\widetilde{r}}{2}}.$$

Furthermore, for any $0 \leq r_1 \leq r_2 < \infty$ *and any* $s \in \mathbb{R}$

$$\lim_{u\to\infty}\mathbb{P}\left\{\frac{\tau_{r_2}(u)-\alpha_1/\mu_1 u}{\sqrt{\alpha_1/\mu_1^3 u}} \leq s \Big| \tau_{r_1}(u) < \infty\right\} = \frac{\sqrt{2}\ \mathcal{H}_{\{1\}}(\mu_1^2 r_2)}{\sqrt{\pi}\ \mathcal{H}_{\{1\}}(\mu_1^2 r_1)}\int_{-\infty}^{s} e^{-\frac{x^2}{2}}\Psi\left(\frac{\mu-\rho}{\sqrt{(1-\rho^2)}}x\right)dx,$$

where $\Psi(s) = 1-\Phi(s)$ *is the standard normal survival function.*

(i.R3). *If* $\frac{\alpha+\mu}{2} < \rho < 1$, *then, as* $u \to \infty$

$$\mathbb{P}\left\{\tau_r(u) < \infty\right\} \sim \mathcal{H}_{\{1\}}(\mu_1^2 r)\ e^{-2\alpha_1\mu_1 u},$$

and for any $0 \leq r_1 \leq r_2 < \infty$ *and any* $s \in \mathbb{R}$

$$\lim_{u\to\infty}\mathbb{P}\left\{\frac{\tau_{r_2}(u)-\alpha_1/\mu_1 u}{\sqrt{\alpha_1/\mu_1^3 u}} \leq s \Big| \tau_{r_1}(u) < \infty\right\} = \frac{\mathcal{H}_{\{1\}}(\mu_1^2 r_2)}{\mathcal{H}_{\{1\}}(\mu_1^2 r_1)}\Phi(s).$$

(ii). *Suppose* α *and* μ *satisfy one of the following conditions:*

(ii.C1) $\mu \ge 1$ *and* $\alpha \ge 1$,

(ii.C2) $\mu < 1$, $\alpha \ge 1$ *and* $\mu > 1/\alpha$,

(ii.C3) $\mu \ge 1$, $\alpha < 1$ *and* $\mu > 1/\alpha$.

We have, for any $r \ge 0$,

(ii.R1). *If* $-1 < \rho < \frac{\alpha+\mu}{2\alpha\mu}$, *then* (24) *and* (25) *hold.*

(ii.R2). *If* $\rho = \frac{\alpha+\mu}{2\alpha\mu}$, *then, as* $u \to \infty$

$$\mathbb{P}\{\tau_r(u) < \infty\} \sim \frac{1}{2\mu}\mathcal{H}_{\{2\}}(\mu_1^2 r)\ e^{-2\alpha_2\mu_2 u},$$

where (cf. (19)*)*

$$\mathcal{H}_{\{2\}}(\widetilde{r}) = \mu\left(2(1+\mu^2\widetilde{r})\Psi(\mu\sqrt{\widetilde{r}}) - \frac{\mu\sqrt{2\widetilde{r}}}{\sqrt{\pi}}e^{-\frac{\mu^2\widetilde{r}}{2}}\right).$$

Furthermore, for any $0 \le r_1 \le r_2 < \infty$ *and any* $s \in \mathbb{R}$

$$\lim_{u\to\infty}\mathbb{P}\left\{\frac{\tau_{r_2}(u) - \alpha_2/\mu_2 u}{\sqrt{\alpha_2/\mu_2^3 u}} \le s \Big| \tau_{r_1}(u) < \infty\right\} = \frac{\sqrt{2}\ \mathcal{H}_{\{2\}}(\mu_1^2 r_2)}{\sqrt{\pi}\ \mathcal{H}_{\{2\}}(\mu_1^2 r_1)} \int_{-\infty}^{s} e^{-\frac{x^2}{2}}\Psi\left(\frac{\mu_1 - \rho\mu_2}{\sqrt{(1-\rho^2)}\mu_2}x\right)dx,$$

(i.R3). *If* $\frac{\alpha+\mu}{2\alpha\mu} < \rho < 1$, *then, as* $u \to \infty$

$$\mathbb{P}\{\tau_r(u) < \infty\} \sim \frac{1}{\mu}\mathcal{H}_{\{2\}}(\mu_1^2 r)\ e^{-2\alpha_2\mu_2 u},$$

and for any $0 \le r_1 \le r_2 < \infty$ *and any* $s \in \mathbb{R}$

$$\lim_{u\to\infty}\mathbb{P}\left\{\frac{\tau_{r_2}(u) - \alpha_2/\mu_2 u}{\sqrt{\alpha_2/\mu_2^3 u}} \le s \Big| \tau_{r_1}(u) < \infty\right\} = \frac{\mathcal{H}_{\{2\}}(\mu_1^2 r_2)}{\mathcal{H}_{\{2\}}(\mu_1^2 r_1)}\Phi(s).$$

5. Proofs of Theorem 3.1

In this section we present the proof of Theorems 3.1. We shall focus on the case where $r > 0$, since the case where $r = 0$ has been included in [5].

First, by the self-similarity of Brownian motion we have, for any $u > 0$,

$$\mathbb{P}\{\tau_r(u) < \infty\} = \mathbb{P}\left\{\int_0^\infty \mathbb{I}(\boldsymbol{X}(t) - \boldsymbol{\mu}t > \boldsymbol{\alpha}u)dt > r\right\} = \mathbb{P}\left\{u\ \int_0^\infty \mathbb{I}(\boldsymbol{X}(t) > \sqrt{u}(\boldsymbol{\alpha} + \boldsymbol{\mu}t))dt > r\right\}.$$

Next, we have the following sandwich bounds

$$p_r(u) \le \mathbb{P}\{\tau_r(u) < \infty\} \le p_r(u) + r_0(u), \tag{26}$$

where

$$p_r(u) := \mathbb{P}\left\{u\ \int_{t\in\triangle_u} \mathbb{I}(\boldsymbol{X}(t) > \sqrt{u}(\boldsymbol{\alpha} + \boldsymbol{\mu}t))dt > r\right\},\quad r_0(u) := \mathbb{P}\left\{u\ \int_{t\in\widetilde{\triangle}_u} \mathbb{I}(\boldsymbol{X}(t) > \sqrt{u}(\boldsymbol{\alpha} + \boldsymbol{\mu}t))dt > 0\right\},$$

with (recall the definition of t_0 in (10))

$$\triangle_u = \left[t_0 - \frac{\ln(u)}{\sqrt{u}}, t_0 + \frac{\ln(u)}{\sqrt{u}}\right],\quad \widetilde{\triangle}_u = \left[0, t_0 - \frac{\ln(u)}{\sqrt{u}}\right] \cup \left[t_0 + \frac{\ln(u)}{\sqrt{u}}, \infty\right).$$

In order to convey the main ideas and to reduce the complexity, we shall prove the theorem in several steps and finally we complete the proof by putting all the arguments together.

5.1. **Analysis of $r_0(u)$.** This step is concerned with sharp upper bound for $r_0(u)$ when u is large. Note that

$$r_0(u) = \mathbb{P}\left\{\exists_{t\in\widetilde{\triangle}_u} \boldsymbol{X}(t) > \sqrt{u}(\boldsymbol{\alpha}+\boldsymbol{\mu}t)\right\}.$$

The following result is Lemma 4.1 in [5] (there was a misprint with $\sqrt{u}$ missing, and in eq.(30) therein u should be $\sqrt{u}$).

Lemma 5.1. *For all large u we have*

$$r_0(u) \le C\sqrt{u}e^{-\frac{\widehat{g}u}{2}-\left(\frac{\min(g''(t_0+),g''(t_0-))}{2}-\varepsilon\right)(\ln(u))^2} \tag{27}$$

[1] *holds for some constant $C>0$ and some sufficiently small $\varepsilon>0$ which do not depend on u.*

5.2. **Analysis of $p_r(u)$.** Denote, for any fixed $T>0$ and $u>0$

$$\triangle_{j;u} = \triangle_{j;u}(T) = [t_0 + jTu^{-1}, t_0+(j+1)Tu^{-1}], \quad -N_u \le j \le N_u,$$

where $N_u = \lceil T^{-1}\ln(u)\sqrt{u}\rceil$ (here $\lceil x\rceil$ denotes the smallest integer larger than x).
Denote

$$\mathcal{A}_{j,u} = u\int_{t\in\Delta_{j;u}} \mathbb{I}(\boldsymbol{X}(t) > \sqrt{u}(\boldsymbol{\alpha}+t\boldsymbol{\mu}))dt,$$

and define

$$p_{r,j;u} = \mathbb{P}\left\{\mathcal{A}_{j,u} > r\right\}, \quad p_{r,i,j;u} = \mathbb{P}\left\{\mathcal{A}_{i,u} > r,\ \mathcal{A}_{j,u} > r\right\}.$$

It follows, using a similar idea as in [7], that

$$\begin{aligned} p_r(u) &\le \mathbb{P}\left\{\sum_{j=-N_u}^{N_u}\mathcal{A}_{j,u} > r\right\}\\ &= \mathbb{P}\left\{\sum_{j=-N_u}^{N_u}\mathcal{A}_{j,u} > r, \quad \text{there exists exactly one } j \text{ such that } \mathcal{A}_{j,u}>0\right\}\\ &\quad +\mathbb{P}\left\{\sum_{j=-N_u}^{N_u}\mathcal{A}_{j,u} > r, \quad \text{there exist } i\neq j \text{ such that } \mathcal{A}_{i,u}>0 \ \& \ \mathcal{A}_{j,u}>0\right\}\\ &\le p_{1,r}(u) + \Pi_0(u), \end{aligned} \tag{28}$$

and by Bonferroni's inequality

$$\begin{aligned} p_r(u) &\ge \mathbb{P}\left\{\sum_{j=-N_u+1}^{N_u-1}\mathcal{A}_{j,u} > r\right\}\\ &\ge \mathbb{P}\left\{\exists -N_u+1 \le j \le N_u-1 \text{ such that } \mathcal{A}_{j,u} > r\right\}\\ &\ge p_{2,r}(u) - \Pi_0(u), \end{aligned} \tag{29}$$

where

$$p_{1,r}(u) = \sum_{j=-N_u}^{N_u} p_{r,j;u}, \quad p_{2,r}(u) = \sum_{j=-N_u+1}^{N_u-1} p_{r,j;u}, \quad \Pi_0(u) = \sum_{-N_u\le i<j\le N_u} p_{0,i,j;u}.$$

In the following two subsetions we shall focus on the analysis of $p_{i,r}(u), i=1,2$, and $\Pi_0(u)$, respectively.

---

[1]Note that in general $g''(t_0+)\neq g''(t_0-)$; see Remark A.7 in [5] for an example.

5.2.1. **Analysis of the single sum** $p_{i,r}(u), i=1,2$**.** First, with the aid of Lemma 4.2 in [5] we can check that

$$0<\mathcal{H}_I(r,T)\le \mathcal{H}_I(0,T)<\infty. \tag{30}$$

Lemma 5.2. *For any $T>0$ and $r\in(0,T)$, we have as $u\to\infty$*

$$p_{1,r}(u)\sim p_{2,r}(u)\sim \frac{1}{\sqrt{(2\pi t_0)^m|\Sigma_{II}|}}\frac{\mathcal{H}_I(r,T)}{T}u^{\frac{1-m}{2}}e^{-\frac{\hat{g}u}{2}}\int_{\mathbb{R}}e^{-\frac{\tilde{g}x^2}{4}}\psi(x)\,dx, \tag{31}$$

where $\psi(x)$ is given in (15).

Proof: We shall focus on the asymptotics of $p_{1,r}(u)$, which is easily seen to be asymptotically equivalent to $p_{2,r}(u)$ as $u\to\infty$.

Fix $T>0$. We shall prove the lemma in two steps. In Step I we derive that (31) holds for any $r\in(0,T)$ at which $\mathcal{H}_I(r,T)$, as a function of r, is continuous, and then in Step II we show that $\mathcal{H}_I(r,T), r\in(0,T)$ is actually continuous everywhere, implying that (31) holds for all $r\in(0,T)$.

Step I: The claim follows from similar arguments as in the proof of Lemma 4.3 in [5]. By the independence and stationary increments property and the self-similarity of Brownian motion we derive that

$$\begin{aligned}p_{r,j;u} &= \mathbb{P}\left\{u\int_{t\in[t_0+\frac{jT}{u},t_0+\frac{(j+1)T}{u}]}\mathbb{I}\Big(\boldsymbol{X}(t_0+\frac{jT}{u})+\boldsymbol{X}(t)-\boldsymbol{X}(t_0+\frac{jT}{u})>\sqrt{u}(\boldsymbol{\alpha}+t\boldsymbol{\mu})\Big)dt>r\right\}\\ &= \mathbb{P}\left\{\int_{t\in[0,T]}\mathbb{I}\Big(\boldsymbol{Z}_{j;u}+\frac{1}{\sqrt{u}}(\boldsymbol{X}(t)-t\boldsymbol{\mu})>\sqrt{u}\boldsymbol{b}_{j;u}\Big)dt>r\right\},\end{aligned}$$

where $\boldsymbol{Z}_{j;u}$ is an independent of $\boldsymbol{B}$ Gaussian random vector with mean vector $\mathbf{0}$ and covariance matrix $\Sigma_{j;u}=c_{j;u}\Sigma$ with $c_{j;u}=c_{j;u}(T)=t_0+jT/u$, and

$$\boldsymbol{b}_{j;u}=\boldsymbol{b}_{j;u}(T)=\boldsymbol{b}(t_0+\frac{jT}{u})=\boldsymbol{b}+\frac{jT}{u}\boldsymbol{\mu}.$$

Denote (recall $I^c=K\cup J$ in (12))

$$\begin{aligned}\boldsymbol{Z}_K(t,\boldsymbol{x}_I) &= (\boldsymbol{X}(t)-t\boldsymbol{\mu})_K-\Sigma_{KI}\Sigma_{II}^{-1}\boldsymbol{x}_I,\\ \boldsymbol{Z}_J(t,\boldsymbol{x}_I) &= (\boldsymbol{X}(t)-t\boldsymbol{\mu})_J-\Sigma_{JI}\Sigma_{II}^{-1}\boldsymbol{x}_I,\end{aligned}$$

and define $\boldsymbol{Y}_{I^c}$ to be an independent of $\boldsymbol{B}$ Gaussian random vector with mean vector $\mathbf{0}_{I^c}$ and covariance matrix $D_{I^cI^c}=\Sigma_{I^cI^c}-\Sigma_{I^cI}\Sigma_{II}^{-1}\Sigma_{II^c}$. Using the same arguments as in [5] gives

$$\begin{aligned}p_{1,r}(u) &= \frac{u^{-m/2}}{\sqrt{(2\pi)^m|\Sigma_{II}|}}\sum_{-N_u\le j\le N_u}\frac{1}{c_{j;u}^{m/2}}\exp\left(-\frac{1}{2}ug_I(t_0+\frac{jT}{u})\right)\int_{\mathbb{R}^m}f_{j;u}(T,\boldsymbol{x}_I)P_{j;u}(r,T,\boldsymbol{x}_I)\,d\boldsymbol{x}_I\\ &=: \frac{1}{T}\frac{1}{\sqrt{(2\pi)^m|\Sigma_{II}|}}u^{(1-m)/2}e^{-\frac{\hat{g}u}{2}}R_{r,T}(u), \end{aligned} \tag{32}$$

where

$$\begin{aligned}R_{r,T}(u) &= \exp\left(\frac{\hat{g}u}{2}\right)\frac{T}{\sqrt{u}}\sum_{-N_u\le j\le N_u}\frac{1}{c_{j;u}^{m/2}}\exp\left(-\frac{1}{2}ug_I(t_0+\frac{jT}{u})\right)\\ &\quad\times\int_{\mathbb{R}^m}f_{j;u}(T,\boldsymbol{x}_I)P_{j;u}(r,T,\boldsymbol{x}_I)\,d\boldsymbol{x}_I,\end{aligned}$$

and

$$\begin{aligned} f_{j;u}(T,\boldsymbol{x}_I) &= \exp\left(\frac{1}{c_{j;u}}\boldsymbol{x}_I^\top\Sigma_{II}^{-1}(\boldsymbol{b}_{j;u})_I-\frac{1}{2uc_{j;u}}\boldsymbol{x}_I^\top\Sigma_{II}^{-1}\boldsymbol{x}_I\right),\\ P_{j;u}(r,T,\boldsymbol{x}_I) &= \mathbb{P}\left\{\int_{t\in[0,T]}\mathbb{I}(E(j,u,r,\boldsymbol{x}_I,t))dt\right\}, \end{aligned}$$

with the event $E(j,u,r,\boldsymbol{x}_I,t)$ defined as

$$E(j,u,r,\boldsymbol{x}_I,t)=\left(\begin{array}{l}(\boldsymbol{X}(t)-t\boldsymbol{\mu})_I>\boldsymbol{x}_I\\ \sqrt{c_{j;u}}\boldsymbol{Y}_K+\frac{1}{\sqrt{u}}\boldsymbol{Z}_K(t,\boldsymbol{x}_I)>\frac{jT}{\sqrt{u}}(\boldsymbol{\mu}_K-\Sigma_{KI}\Sigma_{II}^{-1}\boldsymbol{\mu}_I)\\ \sqrt{c_{j;u}}\boldsymbol{Y}_J+\frac{1}{\sqrt{u}}\boldsymbol{Z}_J(t,\boldsymbol{x}_I)>\sqrt{u}(\boldsymbol{b}_J-\Sigma_{JI}\Sigma_{II}^{-1}\boldsymbol{b}_I+(\boldsymbol{\mu}_J-\Sigma_{JI}\Sigma_{II}^{-1}\boldsymbol{\mu}_I)\frac{jT}{u})\end{array}\right). \tag{33}$$

Using similar argument as in Section 5.4 in [5] we can prove that

$$\lim_{u\to\infty}R_{r,T}(u) = t_0^{-m/2}\mathcal{H}_I(r,T)\int_{-\infty}^{\infty}e^{-\frac{g_I''(t_0)x^2}{4}}\psi(x)\,dx$$

holds for any $r\in(0,T)$ at which $\mathcal{H}_I(r,T)$ is continuous. This together with (32) yields that (31) holds for any $r\in(0,T)$ at which $\mathcal{H}_I(r,T)$ is continuous.

Step II: We show that $\mathcal{H}_I(r,T)$ is continuous at any point $r\in(0,T)$. Hereafter, let $r\in(0,T)$ be arbitrarily chosen and fixed. We shall adopt an idea of [7]. Recall

$$\mathcal{H}_I(r,T)=\int_{\mathbb{R}^m}e^{\frac{1}{t_0}\boldsymbol{x}_I^\top\Sigma_{II}^{-1}\boldsymbol{b}_I}\mathbb{P}\left\{\int_{t\in[0,T]}\mathbb{I}((\boldsymbol{X}(t)-\boldsymbol{\mu}t)_I>\boldsymbol{x}_I)dt>r\right\}d\boldsymbol{x}_I.$$

We first show that

$$\int_{\mathbb{R}^m}e^{\frac{1}{t_0}\boldsymbol{x}_I^\top\Sigma_{II}^{-1}\boldsymbol{b}_I}\mathbb{P}\left\{\int_{t\in[0,T]}\mathbb{I}((\boldsymbol{X}(t)-\boldsymbol{\mu}t)_I>\boldsymbol{x}_I)dt=r\right\}d\boldsymbol{x}_I=0. \tag{34}$$

To this end, we consider the probability space $(C_d([0,T]),\mathcal{F},\mathbb{P}^*)$ which is induced by the multi-dimensional Brownian motion with drift $\{\boldsymbol{B}(t)-A^{-1}\boldsymbol{\mu}t,t\in[0,T]\}$, where $C_d([0,T])$ is the Banach space of all d-dimensional continuous vector functions over $[0,T]$, and $\mathcal{F}$ is the Borel σ-field of $C_d([0,T])$. With the above notation, (34) becomes

$$\int_{\mathbb{R}^m}e^{\frac{1}{t_0}\boldsymbol{x}_I^\top\Sigma_{II}^{-1}\boldsymbol{b}_I}\mathbb{P}^*\left\{\boldsymbol{w}\in C_d([0,T]):\ \int_{t\in[0,T]}\mathbb{I}((A\boldsymbol{w}(t))_I>\boldsymbol{x}_I)dt=r\right\}d\boldsymbol{x}_I=0. \tag{35}$$

Denote, for the fixed r,

$$D_{\boldsymbol{x}_I}^{(r)}=\left\{\boldsymbol{w}\in C_d([0,T]:\int_{t\in[0,T]}\mathbb{I}((A\boldsymbol{w}(t))_I>\boldsymbol{x}_I)dt=r\right\},\quad \boldsymbol{x}_I\in\mathbb{R}^m.$$

By the continuity of $\boldsymbol{w}$, one can see that

$$D_{\boldsymbol{x}_I}^{(r)}\cap D_{\boldsymbol{x}_I'}^{(r)}=\emptyset,\quad \boldsymbol{x}_I\neq\boldsymbol{x}_I'\in\mathbb{R}^m.$$

Thus, for any finite number of distinct points $\boldsymbol{x}_I^{(1)},\ldots,\boldsymbol{x}_I^{(N)}\in\mathbb{R}^m$ we have

$$\sum_{i=1}^N\mathbb{P}^*\{D_{\boldsymbol{x}_I^{(i)}}^{(r)}\}=\mathbb{P}^*\{\cup_{i=1}^N D_{\boldsymbol{x}_I^{(i)}}^{(r)}\}\le 1.$$

This means that the set defined by

$$A_n^{(r)}=\{\boldsymbol{x}_I:\boldsymbol{x}_I\in\mathbb{R}^m\text{ such that }\mathbb{P}^*\{D_{\boldsymbol{x}_I}^{(r)}\}>1/n\}$$

consists of at most $n-1$ distinct $\boldsymbol{x}_I$'s. Therefore,

$$\{\boldsymbol{x}_I : \boldsymbol{x}_I \in \mathbb{R}^m \text{ such that } \mathbb{P}^*\{D^{(r)}_{\boldsymbol{x}_I}\} > 0\} = \cup_{n=1}^{\infty} A_n^{(r)}$$

must be a countable set. Consequently,

$$\int_{\mathbb{R}^m} e^{\frac{1}{t_0}\boldsymbol{x}_I^\top \Sigma_{II}^{-1}\boldsymbol{b}_I}\mathbb{P}^*\left\{D^{(r)}_{\boldsymbol{x}_I}\right\} d\boldsymbol{x}_I = 0,$$

which means that (34) holds for the fixed $r \in (0,T)$. Next, for any small $\varepsilon \in (0, r/2)$ we have

$$\mathbb{P}\left\{\int_{t\in[0,T]} \mathbb{I}((\boldsymbol{X}(t)-\boldsymbol{\mu}t)_I > \boldsymbol{x}_I)dt > r \pm \varepsilon\right\} \le \mathbb{P}\left\{\int_{t\in[0,T]} \mathbb{I}((\boldsymbol{X}(t)-\boldsymbol{\mu}t)_I > \boldsymbol{x}_I)dt > r/2\right\}.$$

Note that $\mathcal{H}_I(r/2,T) < \infty$; see (30). Thus, by the dominated convergence theorem we derive that

$$\lim_{\varepsilon\downarrow 0} \mathcal{H}_I(r+\varepsilon, T) = \mathcal{H}_I(r,T),$$

and

$$\lim_{\varepsilon\downarrow 0} \mathcal{H}_I(r-\varepsilon, T) = \mathcal{H}_I(r,T) + \int_{\mathbb{R}^m} e^{\frac{1}{t_0}\boldsymbol{x}_I^\top \Sigma_{II}^{-1}\boldsymbol{b}_I}\mathbb{P}\left\{\int_{t\in[0,T]} \mathbb{I}((\boldsymbol{X}(t)-\boldsymbol{\mu}t)_I > \boldsymbol{x}_I)dt = r\right\} d\boldsymbol{x}_I,$$

which together with (34) conclude the continuity of $\mathcal{H}_I(r,T)$ at this r. Since such r was arbitrarily chosen in $(0,T)$, we conclude that $\mathcal{H}_I(r,T), r \in (0,T)$ is a continuous function. This completes the proof. □

5.2.2. **Estimation of the double-sum $\Pi_0(u)$.** In this subsection we shall focus on asymptotic upper bounds of $\Pi_0(u)$, as $u \to \infty$. Note that

$$\Pi_0(u) = \sum_{-N_u \le i<j\le N_u} p_{0,i,j;u} = \sum_{\substack{-N_u \le i<j\le N_u\\ j=i+1}} p_{0,i,j;u} + \sum_{\substack{-N_u \le i<j\le N_u\\ j>i+1}} p_{0,i,j;u} =: \Pi_{0,1}(u) + \Pi_{0,2}(u). \tag{36}$$

Since

$$p_{0,i,j;u} = \mathbb{P}\left\{\exists_{t\in\triangle_{i;u}} \boldsymbol{X}(t) > \sqrt{u}(\boldsymbol{\alpha}+\boldsymbol{\mu}t),\ \exists_{t\in\triangle_{j;u}} \boldsymbol{X}(t) > \sqrt{u}(\boldsymbol{\alpha}+\boldsymbol{\mu}t)\right\},$$

we obtain from (52) in [5] that

$$\lim_{u\to\infty} \frac{\Pi_{0,1}(u)}{u^{(1-m)/2}\exp\left(-\frac{\widehat{g}}{2}u\right)} = Q_1\left(\frac{2\mathcal{H}_I(0,T)}{T} - \frac{\mathcal{H}_I(0,2T)}{T}\right) \tag{37}$$

for some constant $Q_1 > 0$ which does not depend on T. Similarly,

$$\lim_{u\to\infty} \frac{\Pi_{0,2}(u)}{u^{(1-m)/2}\exp\left(-\frac{\widehat{g}}{2}u\right)} \le Q_2 T \sum_{j\ge 1} \exp\Big(-\frac{\widehat{g}}{8t_0}(jT)\Big) \tag{38}$$

holds with some constant $Q_2 > 0$ which does not depend on T.

Now we are ready to present the proof of Theorem 3.1.

Proof of (16) and (17). We have from (26)-(31) and (36)-(38) that, for any $T_1, T_2 > 0$

$$\limsup_{u\to\infty} \frac{\mathbb{P}\{\tau_r(u) < \infty\}}{C_I u^{\frac{1-m}{2}} e^{-\frac{\widehat{g}}{2}u}} \le \frac{\mathcal{H}_I(r,T_1)}{T_1} + Q_1\left(\frac{2\mathcal{H}_I(0,T_1)}{T_1} - \frac{\mathcal{H}_I(0,2T_1)}{T_1}\right) + Q_2 T_1 \sum_{j\ge 1} \exp\Big(-\frac{\widehat{g}}{8t_0}(jT_1)\Big), \tag{39}$$

$$\liminf_{u\to\infty} \frac{\mathbb{P}\{\tau_r(u) < \infty\}}{C_I u^{\frac{1-m}{2}} e^{-\frac{\widehat{g}}{2}u}} \ge \frac{\mathcal{H}_I(r,T_2)}{T_2} - Q_1\left(\frac{2\mathcal{H}_I(0,T_2)}{T_2} - \frac{\mathcal{H}_I(0,2T_2)}{T_2}\right) - Q_2 T_2 \sum_{j\ge 1} \exp\Big(-\frac{\widehat{g}}{8t_0}(jT_2)\Big). \tag{40}$$

Note that it has been shown in [5] that

$$\mathcal{H}_I(0) = \lim_{T\to\infty} \frac{\mathcal{H}_I(0,T)}{T} < \infty.$$

Letting $T_2 \to \infty$ in (40), with T_1 in (39) fixed, we have

$$\limsup_{T\to\infty} \frac{\mathcal{H}_I(r,T)}{T} < \infty.$$

Furthermore, letting $T_1 \to \infty$ we conclude that

$$\liminf_{T\to\infty} \frac{\mathcal{H}_I(r,T)}{T} = \limsup_{T\to\infty} \frac{\mathcal{H}_I(r,T)}{T} < \infty.$$

Therefore, it remains to prove that

$$\liminf_{T\to\infty} \frac{\mathcal{H}_I(r,T)}{T} > 0 \tag{41}$$

holds. To this end, first note that

$$\begin{aligned}\mathbb{P}\{\tau_r(u) < \infty\} &\geq p_r(u) \geq \mathbb{P}\left\{\sum_{j=-N_u+1;j\in\{2k:k\in\mathbb{Z}\}}^{N_u-1} \mathcal{A}_{j,u} > r\right\}\\ &\geq \mathbb{P}\{\exists -N_u+1 \leq j \leq N_u-1, j\in\{2k:k\in\mathbb{Z}\} \text{ such that } \mathcal{A}_{j,u} > r\}\\ &\geq p_{3,r}(u) - \widetilde{\Pi}(u),\end{aligned}$$

where

$$p_{3,r}(u) = \sum_{j=-N_u+1;j\in\{2k:k\in\mathbb{Z}\}}^{N_u-1} p_{r,j;u}, \quad \widetilde{\Pi}(u) = \sum_{-N_u\leq i<j\leq N_u;i,j\in\{2k:k\in\mathbb{Z}\}} p_{0,i,j;u}.$$

Similar augments as in the derivation of (40) gives that, for some $T_3 > 0$,

$$\liminf_{u\to\infty} \frac{\mathbb{P}\{\tau_r(u)<\infty\}}{C_I u^{\frac{1-m}{2}} e^{-\frac{\widehat{g}}{2}u}} \geq \frac{\mathcal{H}_I(r,T_3)}{2T_3} - Q_3T_3\sum_{j\geq1}\exp\Big(-\frac{\widehat{g}}{8t_0}(jT_3)\Big)$$

holds with some constant $Q_3 > 0$ which does not dependent on T_3. This together with (39) yields that

$$\begin{aligned}\liminf_{T_1\to\infty} \frac{\mathcal{H}_I(r,T_1)}{T_1} &\geq \frac{\mathcal{H}_I(r,T_3)}{2T_3} - Q_3T_3\sum_{j\geq1}\exp\Big(-\frac{\widehat{g}}{8t_0}(jT_3)\Big)\\ &\geq \frac{\mathcal{H}_I(r,r+1)}{2T_3} - Q_3T_3\sum_{j\geq1}\exp\Big(-\frac{\widehat{g}}{8t_0}(jT_3)\Big),\end{aligned}$$

holds for all $T_3 \geq r+1$, where the last inequality follows since $\mathcal{H}_I(r,T)$ as a function of T is non-decreasing. Since for sufficiently large T_3 the right-hand side of the above formula is positive, we conclude that (41) is valid. Thus, the proof of (16) and (17) is complete. □

Proof of (18). We have, for any $s \in \mathbb{R}$

$$\begin{aligned}\mathbb{P}\left\{\frac{\tau_{r_2}(u)-t_0u}{\sqrt{u}} \leq s \Big| \tau_{r_1}(u)<\infty\right\} &= \frac{\mathbb{P}\left\{\frac{\tau_{r_2}(u)-t_0u}{\sqrt{u}} \leq s, \tau_{r_1}(u)<\infty\right\}}{\mathbb{P}\{\tau_{r_1}(u)<\infty\}}\\ &= \frac{\mathbb{P}\{\tau_{r_2}(u) \leq ut_0+\sqrt{u}s\}}{\mathbb{P}\{\tau_{r_1}(u)<\infty\}}\\ &= \frac{\mathbb{P}\left\{u\int_0^{t_0+s/\sqrt{u}} \mathbb{I}(\boldsymbol{X}(t) > (\boldsymbol{\alpha}+\boldsymbol{\mu}t)\sqrt{u})dt > r_2\right\}}{\mathbb{P}\{\tau_{r_1}(u)<\infty\}}.\end{aligned} \tag{42}$$

Furthermore, using the same arguments as in the proof of (16) we can show, as $u \to \infty$

$$\begin{aligned}
&\mathbb{P}\left\{u \int_0^{t_0+s/\sqrt{u}} \mathbb{I}(\boldsymbol{X}(t) > (\boldsymbol{\alpha}+\boldsymbol{\mu}t)\sqrt{u})dt > r_2\right\} \\
&\sim \mathbb{P}\left\{u \int_{t_0-\ln(u)/\sqrt{u}}^{t_0+s/\sqrt{u}} \mathbb{I}(\boldsymbol{X}(t) > (\boldsymbol{\alpha}+\boldsymbol{\mu}t)\sqrt{u})dt > r_2\right\} \\
&\sim \frac{\mathcal{H}_I(r_2)}{\sqrt{(2\pi t_0)^m |\Sigma_{II}|}} \int_{-\infty}^{s} e^{-\widetilde{g}\frac{x^2}{4}} \psi(x)\, dx\, u^{\frac{1-m}{2}} e^{-\frac{\widehat{g}}{2}u}.
\end{aligned}$$

Consequently, by plugging the above asymptotics and (16) into (42) and rearranging, we obtain (18). Thus, the proof is complete. □

6. Proof of Theorem 4.1

The proof of Theorem 4.1 will be done by first deriving the corresponding results for the cumulative Parisian ruin problem of the auxiliary risk model (23), as $v \to \infty$, and then using the equivalence described in (22).

In order to apply Theorem 3.1 to the auxiliary risk model (23), a crucial step is to find the minimiser of the g-function given by

$$g(t) = \frac{1}{t} \inf_{\boldsymbol{x} \geq \widetilde{\boldsymbol{\alpha}}+\widetilde{\boldsymbol{\mu}}t} \boldsymbol{x}^\top \Sigma^{-1} \boldsymbol{x},$$

for which we must first solve the quadratic programming problem $P_\Sigma(\widetilde{\boldsymbol{\alpha}}+\widetilde{\boldsymbol{\mu}}t)$ involved. To this end, we adopt a direct approach, which is different from that in [5]. It follows from Lemma 2.1 that the g-function has different expressions on different sets of t defined below:

(S1). On the set $E_1 = \{t \geq 0 : \rho(\alpha+\mu t) \geq (1+t)\}$, $g(t) = g_2(t) := g_{\{2\}}(t) = \frac{1}{t}(\alpha+\mu t)^2$;

(S2). On the set $E_2 = \{t \geq 0 : \rho(1+t) \geq (\alpha+\mu t)\}$, $g(t) = g_1(t) := g_{\{1\}}(t) = \frac{1}{t}(1+t)^2$;

(S3). On the set $E_3 = [0,\infty) \setminus (E_1 \cup E_2)$, $g(t) = g_0(t)$,

where

$$\begin{aligned}
g_0(t) := g_{\{1,2\}}(t) &= \frac{1}{t}(\widetilde{\boldsymbol{\alpha}}+\widetilde{\boldsymbol{\mu}}t)^\top \Sigma^{-1} (\widetilde{\boldsymbol{\alpha}}+\widetilde{\boldsymbol{\mu}}t) \\
&= \frac{1+\alpha^2-2\alpha\rho}{1-\rho^2}\frac{1}{t} + \frac{2(1+\alpha\mu-\rho\alpha-\rho\mu)}{1-\rho^2} + \frac{1+\mu^2-2\rho\mu}{1-\rho^2}t.
\end{aligned}$$

Moreover, it is easy to see that the unique minimiser of $g_0(t), t \geq 0$ is $t_0^{(0)} = \sqrt{\frac{1+\alpha^2-2\alpha\rho}{1+\mu^2-2\mu\rho}}$, the unique minimiser of $g_1(t), t \geq 0$ is $t_0^{(1)} = 1$, and the unique minimiser of $g_2(t), t \geq 0$ is $t_0^{(2)} = \alpha/\mu$. Note that all functions $g_i(t), t \geq 0$, $i = 0,1,2$, are decreasing to the left of their own minimiser and then increasing to infinity.

In order to find the global minimiser of the g-function and the exact form of the sets $E_i, i = 1,2,3$, we shall discuss the following four cases, separately.

(1). $\mu < 1$ and $\alpha < 1$, (2). $\mu \geq 1$ and $\alpha \geq 1$,
(3). $\mu < 1$ and $\alpha \geq 1$, (4). $\mu \geq 1$ and $\alpha < 1$.

6.1. Case (1) $\mu < 1$ and $\alpha < 1$.

Clearly, we have $E_1 = \emptyset$ for any $\rho \in (-1,1)$. To analyse E_2 we distinguish the following three sub-cases:

(1.1). $\alpha < \mu$, (1.2). $\alpha > \mu$, (1.3). $\alpha = \mu$.

6.1.1. Case (1.1) $\alpha < \mu$. In this case, we have

$$E_2 = \begin{cases} \emptyset, & \text{if } -1 < \rho \le \alpha; \\ \{t \ge 0 : t \le w\}, & \text{if } \alpha < \rho < \mu ; \\ [0,\infty), & \text{if } \mu \le \rho < 1, \end{cases} \qquad \text{with } w = \frac{\rho-\alpha}{\mu-\rho}.$$

This combined with the fact that $E_1 = \emptyset$ for any $\rho \in (-1,1)$ yields that, if $-1 < \rho \le \alpha$ then $g(t) \equiv g_0(t), t \ge 0$ where the minimum is attained at the unique point $t_0^{(0)}$, if $\mu \le \rho < 1$ then $g(t) \equiv g_1(t), t \ge 0$ where the minimum is attained at the unique point $t_0^{(1)}$, and if $\alpha < \rho < \mu$ then

$$g(t) = \begin{cases} g_0(t), & \text{if } t > w; \\ g_1(t), & \text{if } t \le w, \end{cases} \qquad \text{with } g_0(w) = g_1(w),$$

and thus

$$\inf_{t\ge 0} g(t) = \min\left(\inf_{t\le w} g_1(t),\ \inf_{t>w} g_0(t)\right).$$

In order to derive $\inf_{t\le w} g_1(t)$ and $\inf_{t>w} g_0(t)$ we need to check if $t_0^{(1)} < w$ and if $t_0^{(0)} > w$. We can show that

$$t_0^{(1)} < w \ \Leftrightarrow\ \rho > \frac{\alpha+\mu}{2}, \quad t_0^{(0)} > w \ \Leftrightarrow\ \rho < \frac{\alpha+\mu}{2}, \quad t_0^{(0)} = t_0^{(1)} = w \ \Leftrightarrow\ \rho = \frac{\alpha+\mu}{2}. \tag{43}$$

Note that $\alpha < \frac{\alpha+\mu}{2} < \mu$. Thus, we have for $\alpha < \rho < \mu$

$$\inf_{t\ge 0} g(t) = \begin{cases} g_0(t_0^{(0)}), & \text{if } \alpha < \rho < \frac{\alpha+\mu}{2}; \\ g_0(t_0^{(0)}) = g_1(t_0^{(1)}), & \text{if } \rho = \frac{\alpha+\mu}{2} ; \\ g_1(t_0^{(1)}), & \text{if } \frac{\alpha+\mu}{2} < \rho < \mu, \end{cases}$$

and in each of the above three cases the minimiser of the function $g(t), t \ge 0$ is unique. Using the notation in Theorem 3.1, the above findings for Case (1.1) $\alpha < \mu$ are summarized in the following lemma:

Lemma 6.1. *(1). If $-1 < \rho < \frac{\alpha+\mu}{2}$, then*

$$t_0 = t_0^{(0)}, \quad I = \{1,2\}, \quad K = \emptyset, \quad \widehat{g} = g_0(t_0^{(0)}), \quad \widetilde{g} = g_0''(t_0^{(0)}).$$

(2). If $\rho = \frac{\alpha+\mu}{2}$, then

$$t_0 = t_0^{(0)} = t_0^{(1)} = w, \quad I = \{1\}, \quad K = \{2\}, \quad \widehat{g} = g_0(t_0^{(0)}) = g_1(t_0^{(1)}) = 4, \quad \widetilde{g} = g_1''(t_0^{(1)}) = 2.$$

(3). If $\frac{\alpha+\mu}{2} < \rho < 1$, then

$$t_0 = t_0^{(1)}, \quad I = \{1\}, \quad K = \emptyset, \quad \widehat{g} = g_1(t_0^{(1)}) = 4, \quad \widetilde{g} = g_1''(t_0^{(1)}) = 2.$$

6.1.2. *Case (1.2)* $\alpha > \mu$. In this case, we have (recall $w = \frac{\alpha-\rho}{\rho-\mu}$)

$$E_2 = \begin{cases} \emptyset, & \text{if } -1 < \rho \le \mu; \\ \{t \ge 0 : t \ge w\}, & \text{if } \mu < \rho < \alpha; \\ [0, \infty), & \text{if } \alpha \le \rho < 1. \end{cases}$$

Thus, we have that, if $-1 < \rho \le \mu$ then $g(t) \equiv g_0(t), t \ge 0$ where the minimum is attained at the unique point $t_0^{(0)}$, if $\alpha \le \rho < 1$ then $g(t) \equiv g_1(t), t \ge 0$ where the minimum is attained at the unique point $t_0^{(1)}$, and if $\mu < \rho < \alpha$ then

$$g(t) = \begin{cases} g_0(t), & \text{if } t < w; \\ g_1(t), & \text{if } t \ge w. \end{cases}$$

Similarly as in Case (1.1), we have for $\mu < \rho < \alpha$

$$\inf_{t \ge 0} g(t) = \begin{cases} g_0(t_0^{(0)}), & \text{if } \mu < \rho < \frac{\alpha+\mu}{2}; \\ g_0(t_0^{(0)}) = g_1(t_0^{(1)}), & \text{if } \rho = \frac{\alpha+\mu}{2}; \\ g_1(t_0^{(1)}), & \text{if } \frac{\alpha+\mu}{2} < \rho < \alpha, \end{cases}$$

and in each of the above three cases the minimiser of the function $g(t), t \ge 0$ is unique. Summarizing the above we conclude that the results in Lemma 6.1 still hold for Case (1.2) $\alpha > \mu$.

6.1.3. *Case (1.3)* $\alpha = \mu$. In this case, we have

$$E_2 = \begin{cases} \emptyset, & \text{if } -1 < \rho < \mu; \\ [0, \infty), & \text{if } \mu \le \rho < 1. \end{cases}$$

Using similar arguments as in Case (1.1) and noting that $\frac{\alpha+\mu}{2} = \alpha = \mu$, we could prove that the results in Lemma 6.1 still hold for Case (1.3) $\alpha = \mu$.

Consequently, we conclude that Lemma 6.1 holds for Case (1) $\mu < 1$ and $\alpha < 1$.

6.2. **Case (2)** $\mu \ge 1$ **and** $\alpha \ge 1$**.** Clearly, we have $E_2 = \emptyset$ for any $\rho \in (-1, 1)$. To analyse E_1 we distinguish the following three sub-cases:

$$(2.1).\ \alpha < \mu, \quad (2.2).\ \alpha > \mu, \quad (2.3).\ \alpha = \mu.$$

6.2.1. *Case (2.1)* $\alpha < \mu$. In this case, we have

$$E_1 = \begin{cases} \emptyset, & \text{if } -1 < \rho \le 1/\mu; \\ \{t \ge 0 : t \ge Q\}, & \text{if } 1/\mu < \rho < 1/\alpha; \\ [0, \infty), & \text{if } 1/\alpha \le \rho < 1, \end{cases} \qquad \text{with } Q = \frac{1-\rho\alpha}{\mu\rho - 1}.$$

This combined with the fact that $E_2 = \emptyset$ for any $\rho \in (-1, 1)$ yields that, if $-1 < \rho \le 1/\mu$ then $g(t) \equiv g_0(t), t \ge 0$ where the minimum is attained at the unique point $t_0^{(0)}$, if $1/\alpha \le \rho < 1$ then $g(t) \equiv g_2(t), t \ge 0$ where the minimum is attained at the unique point $t_0^{(2)}$, and if $1/\mu < \rho < 1/\alpha$ then

$$g(t) = \begin{cases} g_0(t), & \text{if } t < Q; \\ g_2(t), & \text{if } t \ge Q, \end{cases} \qquad \text{with } g_0(Q) = g_2(Q),$$

and thus

$$\inf_{t\ge 0} g(t) = \min\left(\inf_{t\ge Q} g_2(t),\ \inf_{t<Q} g_0(t)\right).$$

In order to derive $\inf_{t\ge Q} g_2(t)$ and $\inf_{t<Q} g_0(t)$ we need to check if $t_0^{(2)} > Q$ and if $t_0^{(0)} < Q$. We can show that

$$t_0^{(2)} > Q \quad\Leftrightarrow\quad \rho > \frac{\alpha+\mu}{2\alpha\mu},\quad t_0^{(0)} < Q \quad\Leftrightarrow\quad \rho < \frac{\alpha+\mu}{2\alpha\mu},\quad t_0^{(0)} = t_0^{(2)} = Q \quad\Leftrightarrow\quad \rho = \frac{\alpha+\mu}{2\alpha\mu}. \tag{44}$$

Note that $1/\mu < \frac{\alpha+\mu}{2\alpha\mu} < 1/\alpha$. Thus, we have for $1/\mu < \rho < 1/\alpha$

$$\inf_{t\ge 0} g(t) = \begin{cases} g_0(t_0^{(0)}), & \text{if } 1/\mu < \rho < \frac{\alpha+\mu}{2\alpha\mu};\\ g_0(t_0^{(0)}) = g_2(t_0^{(2)}), & \text{if } \rho = \frac{\alpha+\mu}{2\alpha\mu};\\ g_2(t_0^{(2)}), & \text{if } \frac{\alpha+\mu}{2\alpha\mu} < \rho < 1/\alpha, \end{cases}$$

and in each of the above three cases the minimiser of the function $g(t), t\ge 0$ is unique. Using the notation in Theorem 3.1, the above findings for Case (2.1) $\alpha < \mu$ are summarized in the following lemma:

Lemma 6.2. *(1). If $-1 < \rho < \frac{\alpha+\mu}{2\alpha\mu}$, then*

$$t_0 = t_0^{(0)},\quad I = \{1,2\},\quad K = \emptyset,\quad \widehat{g} = g_0(t_0^{(0)}),\quad \widetilde{g} = g_0''(t_0^{(0)}).$$

(2). If $\rho = \frac{\alpha+\mu}{2\alpha\mu}$, then

$$t_0 = t_0^{(0)} = t_0^{(2)} = Q,\quad I = \{2\},\quad K = \{1\},\quad \widehat{g} = g_0(t_0^{(0)}) = g_2(t_0^{(2)}) = 4\alpha\mu,\quad \widetilde{g} = g_2''(t_0^{(2)}) = 2\alpha^{-1}\mu^3.$$

(3). If $\frac{\alpha+\mu}{2\alpha\mu} < \rho < 1$, then

$$t_0 = t_0^{(2)},\quad I = \{2\},\quad K = \emptyset,\quad \widehat{g} = g_2(t_0^{(2)}) = 4\alpha\mu,\quad \widetilde{g} = g_2''(t_0^{(2)}) = 2\alpha^{-1}\mu^3.$$

Case (2.1) and Case (2.2) can be analysed similarly, and we can conclude that Lemma 6.2 holds for Case (2) $\mu \ge 1$ and $\alpha \ge 1$.

6.3. **Case (3)** $\mu < 1$ **and** $\alpha \ge 1$**.** To analyse E_1, E_2 we distinguish the following three sub-cases:

$$(3.1).\ \mu < 1/\alpha,\quad (3.2).\ \mu > 1/\alpha,\quad (3.3).\ \mu = 1/\alpha.$$

6.3.1. *Case (3.1) $\mu < 1/\alpha$.* In this case, we have

$$E_1 = \begin{cases} \emptyset, & \text{if } -1 < \rho \le 1/\alpha;\\ \{t\ge 0 : t \le Q\}, & \text{if } 1/\alpha < \rho < 1, \end{cases}\qquad E_2 = \begin{cases} \emptyset, & \text{if } -1 < \rho \le \mu;\\ \{t \ge 0 : t \ge w\}, & \text{if } \mu < \rho < 1. \end{cases} \tag{45}$$

This implies that, if $-1 < \rho \le \mu$ then $g(t) \equiv g_0(t), t\ge 0$, where the minimum is attained at the unique point $t_0^{(0)}$, if $\mu < \rho \le 1/\alpha$ then

$$g(t) = \begin{cases} g_0(t), & \text{if } t < w;\\ g_1(t), & \text{if } t \ge w, \end{cases}$$

implying that

$$\inf_{t\ge 0} g(t) = \min\left(\inf_{t\ge w} g_1(t),\ \inf_{t<w} g_0(t)\right),$$

and if $1/\alpha < \rho < 1$ then

$$g(t) = \begin{cases} g_2(t), & \text{if } \ t \le Q; \\ g_0(t), & \text{if } \ Q < t < w \ ; \\ g_1(t), & \text{if } \ t \ge w. \end{cases}$$

implying that

$$\inf_{t\ge 0} g(t) = \min\left(\inf_{t\le Q} g_2(t), \ \inf_{Q<t<w} g_0(t), \ \inf_{t\ge w} g_1(t) \right). \tag{46}$$

Note that $Q < w$ for any $\mu < \rho < 1$. Similarly as (43) and (44) we have, for any $\mu < \rho < 1(\le \alpha)$

$$t_0^{(1)} < w \ \Leftrightarrow \ \rho < \frac{\alpha+\mu}{2}, \quad t_0^{(0)} > w \ \Leftrightarrow \ \rho > \frac{\alpha+\mu}{2}, \quad t_0^{(0)} = t_0^{(1)} = w \ \Leftrightarrow \ \rho = \frac{\alpha+\mu}{2}, \tag{47}$$

and, for any $1/\alpha < \rho < 1$

$$t_0^{(2)} < Q \ \Leftrightarrow \ \rho > \frac{\alpha+\mu}{2\alpha\mu}, \quad t_0^{(0)} > Q \ \Leftrightarrow \ \rho < \frac{\alpha+\mu}{2\alpha\mu}, \quad t_0^{(0)} = t_0^{(2)} = Q \ \Leftrightarrow \ \rho = \frac{\alpha+\mu}{2\alpha\mu}. \tag{48}$$

Furthermore, it follows that $\frac{\alpha+\mu}{2\alpha\mu} > \frac{1}{2}(1/\mu+\mu) > 1$ in the considered case $\mu < 1/\alpha \le 1$, then from (48) we conclude that $t_0^{(2)} > Q, t_0^{(0)} > Q$ hold for any $1/\alpha < \rho < 1$, which helps to further simplify (46) as follows,

$$\inf_{t\ge 0} g(t) = \min\left(\inf_{t\le Q} g_2(t), \ \inf_{Q<t<w} g_0(t), \ \inf_{t\ge w} g_1(t) \right) = \min\left(\inf_{Q<t<w} g_0(t), \ \inf_{t\ge w} g_1(t) \right). \tag{49}$$

After some simple calculations as before, we can show that for $\mu < \rho < 1$

$$\inf_{t\ge 0} g(t) = \begin{cases} g_0(t_0^{(0)}), & \text{if } \ \mu < \rho < \frac{\alpha+\mu}{2}; \\ g_0(t_0^{(0)}) = g_1(t_0^{(1)}), & \text{if } \ \rho = \frac{\alpha+\mu}{2} \ ; \\ g_1(t_0^{(1)}), & \text{if } \ \frac{\alpha+\mu}{2} < \rho < 1, \end{cases}$$

and in each of the above three cases the minimiser of the function $g(t), t \ge 0$ is unique. Summarizing the above findings we can conclude that Lemma 6.1 holds for Case (3.1).

6.3.2. Case (3.2) $\mu > 1/\alpha$. In this case, we have (45) still holds. It follows that, if $-1 < \rho \le 1/\alpha$ then $g(t) \equiv g_0(t), t \ge 0$, where the minimum is attained at the unique point $t_0^{(0)}$, if $1/\alpha < \rho \le \mu$ then

$$g(t) = \begin{cases} g_0(t), & \text{if } \ t > Q; \\ g_2(t), & \text{if } \ t \le Q, \end{cases}$$

implying that

$$\inf_{t\ge 0} g(t) = \min\left(\inf_{t\le Q} g_2(t), \ \inf_{t>Q} g_0(t) \right),$$

and if $\mu < \rho < 1$ then

$$g(t) = \begin{cases} g_2(t), & \text{if } \ t \le Q; \\ g_0(t), & \text{if } \ Q < t < w \ ; \\ g_1(t), & \text{if } \ t \ge w. \end{cases}$$

implying that

$$\inf_{t\ge 0} g(t) = \min\left(\inf_{t\le Q} g_2(t), \ \inf_{Q<t<w} g_0(t), \ \inf_{t\ge w} g_1(t) \right). \tag{50}$$

Note that in this case both (47) and (48) are still valid for the corresponding values of ρ mentioned therein. Furthermore, it follows that $\frac{\alpha+\mu}{2} \geq \sqrt{\alpha\mu} > 1$ in the considered case $\mu > 1/\alpha$, then from (47) we conclude that $t_0^{(1)} < w, t_0^{(0)} < w$ hold for any $\mu < \rho < 1$, which helps to further simplify (50) as follows,

$$\inf_{t\geq 0} g(t) = \min\left(\inf_{t\leq Q} g_2(t),\ \inf_{Q<t<w} g_0(t),\ \inf_{t\geq w} g_1(t)\right) = \min\left(\inf_{t\leq Q} g_2(t), \inf_{Q<t<w} g_0(t)\right). \tag{51}$$

Thus, we can show that for $1/\alpha < \rho < 1$

$$\inf_{t\geq 0} g(t) = \begin{cases} g_0(t_0^{(0)}), & \text{if } 1/\alpha < \rho < \frac{\alpha+\mu}{2\alpha\mu}; \\ g_0(t_0^{(0)}) = g_2(t_0^{(2)}), & \text{if } \rho = \frac{\alpha+\mu}{2\alpha\mu}\ ; \\ g_2(t_0^{(2)}), & \text{if } \frac{\alpha+\mu}{2\alpha\mu} < \rho < 1, \end{cases}$$

and in each of the above three cases the minimiser of the function $g(t), t \geq 0$ is unique. Summarizing the above findings we can conclude that Lemma 6.2 holds for Case (3.2).

6.3.3. <u>*Case (3.3)* $\mu = 1/\alpha$.</u> In this case, we have that (45) still holds. As now $\frac{\alpha+\mu}{2} = \frac{\alpha+\mu}{2\alpha\mu} \geq 1$, we obtain that (i) in Lemma 6.1 (the same to (i) in Lemma 6.2) is valid for any $-1 < \rho < 1$.

Consequently, we can conclude that for Case (3) $\mu < 1$ and $\alpha \geq 1$, if further $\mu \leq 1/\alpha$ then Lemma 6.1 holds, and if further $\mu > 1/\alpha$ then Lemma 6.2 holds.

6.4. **Case (4)** $\mu \geq 1$ **and** $\alpha < 1$**.** To analyse E_1, E_2 we distinguish the following three sub-cases:

$$(4.1).\ 1/\mu < \alpha, \quad (4.2).\ 1/\mu > \alpha, \quad (4.3).\ 1/\mu = \alpha.$$

6.4.1. <u>*Case (4.1)* $1/\mu < \alpha$.</u> In this case, we have

$$E_1 = \begin{cases} \emptyset, & \text{if } -1 < \rho \leq 1/\mu; \\ \{t \geq 0 : t \geq Q\}, & \text{if } 1/\mu < \rho < 1, \end{cases} \quad E_2 = \begin{cases} \emptyset, & \text{if } -1 < \rho \leq \alpha; \\ \{t \geq 0 : t \leq w\}, & \text{if } \alpha < \rho < 1. \end{cases} \tag{52}$$

This implies that, if $-1 < \rho \leq 1/\mu$ then $g(t) \equiv g_0(t), t \geq 0$, where the minimum is attained at the unique point $t_0^{(0)}$, if $1/\mu < \rho \leq \alpha$ then

$$g(t) = \begin{cases} g_0(t), & \text{if } t < Q; \\ g_2(t), & \text{if } t \geq Q, \end{cases}$$

implying that

$$\inf_{t\geq 0} g(t) = \min\left(\inf_{t\geq Q} g_2(t),\ \inf_{t<Q} g_0(t)\right),$$

and if $\alpha < \rho < 1$ then

$$g(t) = \begin{cases} g_2(t), & \text{if } t \geq Q; \\ g_0(t), & \text{if } w < t < Q\ ; \\ g_1(t), & \text{if } t \leq w. \end{cases}$$

implying that

$$\inf_{t\geq 0} g(t) = \min\left(\inf_{t\geq Q} g_2(t),\ \inf_{w<t<Q} g_0(t),\ \inf_{t\leq w} g_1(t)\right). \tag{53}$$

Note that $Q > w$ for any $1/\mu < \rho < 1$. Similarly as in Case (3.1) we have for $1/\mu < \rho < 1$

$$\inf_{t\ge 0} g(t) = \begin{cases} g_0(t_0^{(0)}), & \text{if } 1/\mu < \rho < \frac{\alpha+\mu}{2\alpha\mu}; \\ g_0(t_0^{(0)}) = g_2(t_0^{(2)}), & \text{if } \rho = \frac{\alpha+\mu}{2\alpha\mu}\ ; \\ g_2(t_0^{(2)}), & \text{if } \frac{\alpha+\mu}{2\alpha\mu} < \rho < 1, \end{cases}$$

and in each of the above three cases the minimiser of the function $g(t), t \ge 0$ is unique. Summarizing the above findings we can conclude that Lemma 6.2 holds for Case (4.1).

6.4.2. *Case (4.2)* $1/\mu > \alpha$. In this case, we have (52) still holds. It follows that, if $-1 < \rho \le \alpha$ then $g(t) \equiv g_0(t), t \ge 0$, where the minimum is attained at the unique point $t_0^{(0)}$, if $\alpha < \rho \le 1/\mu$ then

$$g(t) = \begin{cases} g_0(t), & \text{if } t > w; \\ g_1(t), & \text{if } t \le w, \end{cases}$$

implying that

$$\inf_{t\ge 0} g(t) = \min\left(\inf_{t\le w} g_1(t),\ \inf_{t>w} g_0(t)\right),$$

and if $1/\mu < \rho < 1$ then

$$g(t) = \begin{cases} g_2(t), & \text{if } t \ge Q; \\ g_0(t), & \text{if } w < t < Q\ ; \\ g_1(t), & \text{if } t \le w. \end{cases}$$

implying that

$$\inf_{t\ge 0} g(t) = \min\left(\inf_{t\ge Q} g_2(t),\ \inf_{w<t<Q} g_0(t),\ \inf_{t\le w} g_1(t)\right). \tag{54}$$

Similarly as before, we can show that for $\alpha < \rho < 1$

$$\inf_{t\ge 0} g(t) = \begin{cases} g_0(t_0^{(0)}), & \text{if } \alpha < \rho < \frac{\alpha+\mu}{2}; \\ g_0(t_0^{(0)}) = g_1(t_0^{(1)}), & \text{if } \rho = \frac{\alpha+\mu}{2}\ ; \\ g_1(t_0^{(1)}), & \text{if } \frac{\alpha+\mu}{2} < \rho < 1, \end{cases}$$

and in each of the above three case the minimiser of the function $g(t), t \ge 0$ is unique. Summarizing the above findings we can conclude that Lemma 6.1 holds for Case (4.2).

6.4.3. *Case (4.3)* $1/\mu = \alpha$. In this case, we have that (52) still holds. As now $\frac{\alpha+\mu}{2} = \frac{\alpha+\mu}{2\alpha\mu} \ge 1$, we obtain that (i) in Lemma 6.2 (the same to (i) in Lemma 6.1) is valid for any $-1 < \rho < 1$.

Consequently, we can conclude that for Case (4) $\mu \ge 1$ and $\alpha < 1$, if further $1/\mu < \alpha$ then Lemma 6.2 holds, and if further $1/\mu \ge \alpha$ then Lemma 6.1 holds.

The following corollary is a collection of the main findings in Sections 6.1, 6.2, 6.3, 6.4.

Corollary 6.3. *Consider the auxiliary two-dimensional Brownian motion models described in* (23).

(i). *Suppose α and μ satisfy one of the following conditions:*

(i.C1) $\mu < 1$ *and* $\alpha < 1$,

(i.C2) $\mu < 1$, $\alpha \ge 1$ *and* $\mu \le 1/\alpha$, (i.C3) $\mu \ge 1$, $\alpha < 1$ *and* $\mu \le 1/\alpha$.

We have

(i.R1). *If* $-1 < \rho < \frac{\alpha+\mu}{2}$, *then*

$$t_0 = t_0^{(0)}, \quad I = \{1, 2\}, \quad K = \emptyset, \quad \widehat{g} = g_0(t_0^{(0)}), \quad \widetilde{g} = g_0''(t_0^{(0)}).$$

(i.R2). *If* $\rho = \frac{\alpha+\mu}{2}$, *then*

$$t_0 = t_0^{(0)} = t_0^{(1)}, \quad I = \{1\}, \quad K = \{2\}, \quad \widehat{g} = g_0(t_0^{(0)}) = g_1(t_0^{(1)}) = 4, \quad \widetilde{g} = g_1''(t_0^{(1)}) = 2.$$

(i.R3). *If* $\frac{\alpha+\mu}{2} < \rho < 1$, *then*

$$t_0 = t_0^{(1)}, \quad I = \{1\}, \quad K = \emptyset, \quad \widehat{g} = g_1(t_0^{(1)}) = 4, \quad \widetilde{g} = g_1''(t_0^{(1)}) = 2.$$

(ii). *Suppose* α *and* μ *satisfy one of the following conditions:*

(ii.C1) $\mu \ge 1$ *and* $\alpha \ge 1$,

(ii.C2) $\mu < 1$, $\alpha \ge 1$ *and* $\mu > 1/\alpha$,

(ii.C3) $\mu \ge 1$, $\alpha < 1$ *and* $\mu > 1/\alpha$.

We have

(ii.R1). *If* $-1 < \rho < \frac{\alpha+\mu}{2\alpha\mu}$, *then*

$$t_0 = t_0^{(0)}, \quad I = \{1, 2\}, \quad K = \emptyset, \quad \widehat{g} = g_0(t_0^{(0)}), \quad \widetilde{g} = g_0''(t_0^{(0)}).$$

(ii.R2). *If* $\rho = \frac{\alpha+\mu}{2\alpha\mu}$, *then*

$$t_0 = t_0^{(0)} = t_0^{(2)}, \quad I = \{2\}, \quad K = \{1\}, \quad \widehat{g} = g_0(t_0^{(0)}) = g_2(t_0^{(2)}) = 4\alpha\mu, \quad \widetilde{g} = g_2''(t_0^{(2)}) = 2\alpha^{-1}\mu^3.$$

(i.R3). *If* $\frac{\alpha+\mu}{2} < \rho < 1$, *then*

$$t_0 = t_0^{(2)}, \quad I = \{2\}, \quad K = \emptyset, \quad \widehat{g} = g_2(t_0^{(2)}) = 4\alpha\mu, \quad \widetilde{g} = g_2''(t_0^{(2)}) = 2\alpha^{-1}\mu^3.$$

Consequently, by using results in Corollary 6.3 and applying Theorem 3.1 we can obtain asymptotic results for the cumulative Parisian ruin probability and the conditional cumulative Parisian ruin time of the auxiliary risk model (23), as $v \to \infty$. Finally, Theorem 4.1 follows by directly using the equivalence described in (22).

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