Scalar-type kernels for block Toeplitz operators

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Abstract

It is shown that the kernel of a Toeplitz operator with 2×2 symbol G can be described exactly in terms of any given function in a very wide class, its image under multiplication by G, and their left inverses, if the latter exist. As a consequence, under many circumstances the kernel of a block Toeplitz operator may be described as the product of a space of scalar complex-valued functions by a fixed column vector of functions. Such kernels are said to be of scalar type, and in this paper they are studied and described explicitly in many concrete situations. Applications are given to the determination of kernels of truncated Toeplitz operators for several new classes of symbols.

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1 Introduction

Kernels of Toeplitz operators (also called Toeplitz kernels) have generated an enormous interest for various reasons, among which is the fact that they have fascinating properties and a rich structure, they are important in many applications, and several relevant classes of analytic functions can be presented as kernels of Toeplitz operators. For instance, model spaces (defined below) are Toeplitz kernels. Two recent surveys of this area are [21] and [11].

It is natural to expect that kernels of block Toeplitz operators, whose study provides a clear example of the fruitful interplay between operator

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theory, complex analysis and linear algebra, will have an even richer and more involved structure. The case of Toeplitz operators with 2×2 symbols is particularly interesting for its connections to truncated Toeplitz operators [9, 8] and the corona theorem [6], and because their study leads to various surprising results. One of these unexpected results is that, as we prove in Theorem 3.1, one can explicitly describe the kernel of a Toeplitz operator with 2×2 symbol G, not necessarily bounded, in terms of any given function f in a very wide class, its image under multiplication by G, and their left inverses, assuming that the latter exist. As a result of this, we show that, although those kernels consist of vector functions, in many cases they behave as having a scalar nature, since they can be expressed as the product of a space of scalar functions by a fixed vector function. These kernels will be called *scalar-type Toeplitz kernels*.

A natural question arising in this case regards which properties of scalar Toeplitz kernels remain valid for scalar-type Toeplitz kernels. While Coburn's Lemma, stating that for $\varphi \in L_\infty(\mathbb{R})$, either $\ker T_\varphi$ or $\ker T_\varphi^*$ is zero, cannot be extended in the same form to the case of a general 2×2 bounded symbol G, even if $\ker T_G$ and $\ker T_G^*$ are both of scalar type, Theorem 3.7 may be seen as a version of Coburn's Lemma for 2×2 symbols. Moreover we show that any scalar-type Toeplitz kernel is the product of a fixed vector function by a scalar nearly S^*-invariant space $\mathcal{K}$, which is closed if $G \in L_\infty^{2\times 2}$ and therefore, using a well known result by Hitt [22], can be characterized as the product of a scalar Toeplitz kernel, in fact a model space, by a fixed 2×1 function (Theorem 3.16). Although this model space is not known in general, by using the corona theorem we obtain sufficient conditions for $\mathcal{K}$ to be a model space, explicitly described in terms of the functions f and g, leading to conditions for injectivity and invertibility for T_G (Theorem 3.13 and its corollaries). We note that some related results can be found in [18, Prop. 4.6]. We show moreover that, as in the case of scalar symbols, every scalar-type Toeplitz kernel has a maximal function (Theorem 3.17).

The results of Section 3 are applied in Section 4 to study and describe the kernel of truncated Toeplitz operators in two different classes which extend previously studied ones. In the first case we show that the kernels are given by the product of a model space, which is explicitly determined, by a fixed vector function in H_∞^+, and we establish necessary and sufficient conditions for injectivity and invertibility of the truncated Toeplitz operators. In the second case we also obtain an explicit description of the kernel as a product of a scalar Toeplitz kernel by a fixed function.

We write $\mathbb{C}^+$ and $\mathbb{C}^-$ for the upper and lower complex half-planes, and $H_p^\pm$ $(1 \leq p \leq \infty)$ for the associated Hardy spaces of analytic functions on $\mathbb{C}^\pm$. The operators $P^\pm$ are the standard Riesz projections from $L_p = L_p(\mathbb{R})$ onto the subspaces $H_p^\pm$. It will be recalled that functions in H_p^+ have inner/outer factorizations. For two inner functions θ, φ we write $\theta \preceq \varphi$ or $\varphi \succeq \theta$ to mean that θ is a divisor of φ in H_∞^+. We may also use the strict versions of these relations, written $\prec$ and $\succ$.

The Smirnov class $\mathcal{N}_+$ consists of all analytic functions $f = g_+/h_+$, where $g_+ \in H_1^+$ and $h_+ \in H_2^+$ with h_+ outer. We may instead take $g_+, h_+ \in H_\infty^+$ (see, e.g. [25]).

These notions can be found in standard texts on Hardy spaces, such as [23] and [25].

For $G \in L_\infty^{n\times n}$, with $n = 1, 2, \ldots$, the Toeplitz operator T_G on $(H_2^+)^n$ is the composition $P^+ M_G$, where M_G denotes multiplication by G. For $\theta \in H_\infty^+$ inner, the model space K_θ is $\ker T_{\overline{\theta}}$, which equals $H_2^+ \ominus \theta H_2^+ = H_2^+ \cap \theta H_2^-$.

For a unital algebra $\mathcal{A}$, we write $\mathcal{G}(\mathcal{A})$ for the group of invertible elements.

We use the notation (f, g) interchangeably with $[f\ g]^T = \begin{bmatrix} f \\ g \end{bmatrix}$.

2 Motivation: matrix symbols with a bounded factorization

Let $G \in (L^\infty)^{2\times 2}$ admit a bounded (Wiener–Hopf) factorization ([13, 24]) on the real line, of the form

$$G = G_- \operatorname{diag}(r^{k_1}, r^{k_2})\, G_+^{-1}, \tag{2.1}$$

where $G_\pm \in \mathcal{G}(H_\infty^\pm)^{2\times 2}$, $k_1, k_2 \in \mathbb{Z}$, and

$$r(\xi) = \frac{\xi - i}{\xi + i}, \qquad \text{for} \quad \xi \in \mathbb{R}. \tag{2.2}$$

The class of matrix functions admitting such a factorization includes, in particular, all 2×2 matrix functions G with elements in the algebra $C^\mu(\dot{\mathbb{R}})$ of Hölder-continuous functions in $\dot{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ with exponent $\mu \in (0, 1)$, or in the Wiener algebra $W(\dot{\mathbb{R}})$ ([13, 24]), as long as $\det G \in \mathcal{G}L^\infty$. If $k_1 = k_2 \geq 0$ in (2.1), then $\ker T_G = \{0\}$; if at least one of the integers k_1, k_2 is negative, then $\ker T_G \neq \{0\}$. Let us assume, for simplicity, that $\operatorname{ind}(\det G) = 0$, in which case $k_1 = -k_2 = -k$, say, and (2.1) takes the form

$$G = G_- \operatorname{diag}(r^{-k}, r^k) G_+^{-1}, \qquad k \in \mathbb{Z}_0^+ = \{0, 1, 2, \ldots\}, \tag{2.3}$$

and let $G_\pm = \left[g_{ij}^\pm\right]_{i,j=1,2}$. Rewriting the equation (2.3) as

$$GG_+ \operatorname{diag}(r^k, r^{-k}) = G_- \tag{2.4}$$

and taking the first columns of the matrices on the left and right-hand side of (2.4), we obtain

$$Gr^k g_+ = g_-, \qquad \text{with} \quad g_\pm = \left(g_{11}^\pm, g_{21}^\pm\right). \tag{2.5}$$

On the other hand, $\ker T_G$ consists of all functions $\varphi_+ \in (H_2^+)^2$ such that

$$G\varphi_+ = \varphi_- \qquad \text{with} \quad \varphi_- \in (H_2^-)^2. \tag{2.6}$$

From (2.5) and (2.6) we have then

$$G[r^k g_+ \quad \varphi_+] = [g_- \quad \varphi_-] \tag{2.7}$$

and, on taking determinants on both sides and noting that $\det G = d_- d_+^{-1}$ where $d_\pm = \det G_\pm \in \mathcal{G}H_\infty^\pm$, it follows that

$$d_+^{-1} \det[r^k g_+ \quad \varphi_+] = d_-^{-1} \det[g_- \quad \varphi_-]. \tag{2.8}$$

The left-hand side of this identity is in H_2^+, while the right-hand side is in H_2^-. Consequently, they are both equal to zero and we have that

$$\varphi_+ = \Lambda r^k g_+, \qquad \varphi_- = \tilde{\Lambda} g_-, \tag{2.9}$$

where Λ and $\tilde{\Lambda}$ are scalar functions defined a.e. on $\mathbb{R}$. To show that $\Lambda = \tilde{\Lambda}$, we now take into account the fact that the invertibility of $G_\pm$ in $(H_\infty)^{2\times 2}$ means that the first column of $G_\pm$ is a corona pair in $\mathbb{C}^\pm$ ([27, 6]) and therefore left-invertible in $H_\infty^\pm$, with left inverse given by $\tilde{g}_\pm^T$, where $\tilde{g}_\pm = (g_{22}^\pm, -g_{12}^\pm)/d_\pm$, and

$$\tilde{g}_\pm^T g_\pm = 1 \qquad \text{in} \quad \mathbb{C}^\pm. \tag{2.10}$$

In fact, from (2.6) and (2.9) we have

$$G(\Lambda r^k g_+) = \tilde{\Lambda} g_-,$$

and it follows from (2.5) that $\Lambda g_- = \tilde{\Lambda} g_-$. Multiplying both sides by $\tilde{g}_-^T$ on the left we conclude that $\Lambda = \tilde{\Lambda}$. Then multiplying both sides of the two equations in (2.9) by $\tilde{g}_+^T$ and $\tilde{g}_-^T$ respectively, we obtain moreover that

$$\Lambda = r^{-k}(\tilde{g}_+^T \varphi_+) = \tilde{g}_-^T \varphi_- \qquad \text{with} \quad \tilde{g}_\pm^T \varphi_\pm \in H_2^\pm. \tag{2.11}$$

Therefore $r^k\Lambda = \tilde{g}_+^T\varphi_+ \in \ker T_{r^{-k}} = K_{r^k}$, where K_{r^k} is the model space $H_2^+ \ominus r^k H_2^+$, and it follows from (2.9) and (2.11) that

$$\ker T_G \subset K_{r^k} g_+.$$

Conversely, $K_{r^k} g_+ \subset \ker T_G$ because if $P \in K_{r^k}$ then

$$\begin{aligned} G(Pg_+) &= G_- \operatorname{diag}(r^{-k}, r^k) G_+^{-1}(Pg_+) \\ &= G_- \operatorname{diag}(r^{-k}P, r^k P)(1,0) \\ &= G_-(r^{-k}P, 0) \in (H_2^-)^2. \end{aligned}$$

Thus

$$\ker T_G = \mathcal{K} g_+, \tag{2.12}$$

where $\mathcal{K}$ is a scalar model space, associated with the inner function r^k, and g_+ is a fixed vector function. So we see that for a wide class of Toeplitz operators with 2×2 matrix symbols, including for instance all invertible 2×2 Hölder-continuous matrices G with $\operatorname{ind}(\det G) = 0$, the corresponding kernels are spaces of vector functions which can nonetheless be described as the product of a certain space $\mathcal{K}$ of scalar functions by a fixed vector function. We say in this case that $\ker T_G$ is a *scalar-type Toeplitz kernel*.

The same result would hold in the case of any 2×2 matrix symbol G for which one can find a solution to

$$Gf = g \tag{2.13}$$

with $f = r^k g_+$ and $g = g_-$ where $k \in \mathbb{Z}_0^+$, $g_\pm \in (H_\infty^\pm)^2$, such that $g_\pm$ satisfy the condition of Carleson's corona theorem in $\mathbb{C}^\pm$ ([19, 6]). These conditions can be seen in terms of left invertibility of f and g by saying that there exist vector functions $\tilde{g}_\pm \in (H_\infty)^2$ such that

$$\tilde{g}_\pm^T g_\pm = 1 \qquad \text{in} \quad \mathbb{C}^\pm. \tag{2.14}$$

The main difficulty in applying these results consists in the fact that it is in general very difficult, or even impossible, to find solutions to (2.13) satisfying the above-mentioned conditions. We may however find other solutions to (2.13), satisfying less restrictive conditions; it is natural to ask then whether such a relation would still allow us to describe the kernel of T_G, and whether the kernel would be of scalar type.

In the next section we shall show that it is indeed possible to describe the kernel of a Toeplitz operator with 2×2 symbol G and give conditions for it to be a scalar-type kernel, for a very general set of symbols, in terms of a solution to $Gf = g$ where f and g are assumed to be left-invertible vector functions in a very general class. In particular, we shall not assume any analyticity conditions on the functions f and g.

3 Scalar-type kernels for Toeplitz operators with 2×2 symbols

Let $\mathcal{F}$ denote the space of all complex-valued functions defined almost everywhere on $\mathbb{R}$, where as usual we identify two functions if they are equal almost everywhere. Let $G \in \mathcal{F}^{2\times 2}$ and let

$$\mathcal{D} = \{f_+ \in (H_2^+)^2 : Gf_+ \in (L_2)^2\}. \tag{3.1}$$

The operator $T_G : \mathcal{D} \to (H_2^+)^2$ defined by

$$T_G\, f_+ = P^+(Gf_+), \quad f_+ \in \mathcal{D}, \tag{3.2}$$

where $P^+ : (L_2)^2 \to (H_2^+)^2$ denotes the orthogonal projection, is called the *Toeplitz operator* with symbol G. If $G \in (L_\infty)^{2\times 2}$, then T_G is a bounded operator on $(H_2^+)^2$. Another class of symbols of interest arises on taking $G \in \lambda_+(L_2)^{2\times 2}$, where $\lambda_+(\xi) = \xi + i$; then T_G is densely defined on $(H_2^+)^2$.

In what follows f and g denote left-invertible functions in $\mathcal{F}^{2\times 1}$ with left inverses $\tilde{f}^T$ and $\tilde{g}^T$, where $\tilde{f}, \tilde{g} \in \mathcal{F}^{2\times 1}$. We assume moreover that $G \in \mathcal{F}^{2\times 2}$ and, unless said otherwise, $\det G \in \mathcal{GF}$.

We write J for the matrix $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

The following result shows that, surprisingly, given a Toeplitz operator T_G, one can describe its kernel in terms of any given function f, its image under multiplication by G, $g = Gf$, and their left inverses if they exist. Although, necessarily, with a somewhat technical appearance, it leads on to explicit characterizations of Toeplitz kernels, as we shall show here.

Theorem 3.1. *If f and g are left-invertible functions in $\mathcal{F}^{2\times 1}$ such that*

$$Gf = g \tag{3.3}$$

and we define

$$\mathcal{S} = \left(\det G . f^T (H_2^+)^2\right) \cap \left(g^T (H_2^-)^2\right), \tag{3.4}$$

and, for each $s \in \mathcal{S}$,

$$\mathcal{H}_+^s = \left\{\psi_+ \in (H_2^+)^2 : f\tilde{f}^T\psi_+ + \frac{s}{\det G} J\tilde{f} \in (H_2^+)^2\right\} \tag{3.5}$$

and

$$\mathcal{H}_-^s = \left\{\psi_- \in (H_2^-)^2 : g\tilde{g}^T\psi_- + sJ\tilde{g} \in (H_2^-)^2\right\}, \tag{3.6}$$

then $\varphi_+ \in \ker T_G$ *if and only if*

$$\varphi_+ = \Lambda f + \frac{s}{\det G} J\tilde{f}, \tag{3.7}$$

where

$$s \in \mathcal{S} \text{ with } \mathcal{H}^s_\pm \neq \emptyset, \quad \Lambda \in \tilde{f}^T \mathcal{H}^s_+, \quad \Lambda + \frac{s}{\det G}\tilde{g}^T(GJ\tilde{f}) \in \tilde{g}^T \mathcal{H}^s_-. \tag{3.8}$$

As we shall see later, it is possible for the space $\mathcal{S}$ in the statement of the theorem to reduce to $\{0\}$, in which case $\mathcal{H}^s_\pm$ may be empty.

Naturally, although f and g in (3.3) can be chosen in a very general class, it is important that they are such that the description given by Theorem 3.1 for the kernel of T_G is useful, in the sense that it allows for a good understanding of the kernel. There is no general method to obtain good solutions, in that sense, and the choice of f and g must be made on a case by case basis. However, as we show in Example 3.3 below, the great degree of freedom that we are allowed in that choice can permit us to obtain a clear description of $\ker T_G$ in terms of natural solutions to $Gf = g$ with no particular analytic properties.

Proof. (i) Let $\varphi_+ \in \ker T_G$, i.e., $\varphi_+ \in (H_2^+)^2$ and $G\varphi_+ = \varphi_- \in (H_2^-)^2$. From this identity and (3.3) we have that

$$G[f \;\; \varphi_+] = [g \;\; \varphi_-]. \tag{3.9}$$

Let

$$s = \det G . \det[f \;\; \varphi_+] = \det[g \;\; \varphi_-]. \tag{3.10}$$

Now, since $\det[f \;\; J\tilde{f}] = 1$, we have

$$\det\left[f \;\; \frac{s}{\det G} J\tilde{f}\right] = \frac{s}{\det G} = \det[f \;\; \varphi_+]$$

and analogously

$$\det[g \;\; sJ\tilde{g}] = s = \det[g \;\; \varphi_-].$$

Therefore,

$$\det\left[f \quad \left(\varphi_+ - \frac{s}{\det G} J\tilde{f}\right)\right] = 0 = \det[g \quad (\varphi_- - sJ\tilde{g})],$$

and, since f and g are left invertible, it follows that there are scalar functions $\Lambda, \tilde{\Lambda} \in \mathcal{F}$ such that

$$\varphi_+ \;\; = \;\; \Lambda f + \frac{s}{\det G} J\tilde{f}, \tag{3.11}$$

and

$$\varphi_- \quad = \quad \tilde{\Lambda} g + sJ\tilde{g}. \tag{3.12}$$

Multiplying (3.11) and (3.12) on the left by $\tilde{f}^T$ and $\tilde{g}^T$, respectively, and taking into account the fact that $\tilde{f}^T J \tilde{f} = \tilde{g}^T J \tilde{g} = 0$, we get

$$\Lambda = \tilde{f}^T \varphi_+, \qquad \tilde{\Lambda} = \tilde{g}^T \varphi_-. \tag{3.13}$$

Moreover, from (3.11), (3.12), and the assumption that $Gf = g$, we have

$$G\varphi_+ = \varphi_- \implies \Lambda g + \frac{s}{\det G} GJ\tilde{f} = \tilde{\Lambda} g + sJ\tilde{g}, \tag{3.14}$$

and, multiplying the last equation on the left by $\tilde{g}^T$, we get

$$\Lambda + \frac{s}{\det G}(\tilde{g}^T GJ\tilde{f}) = \tilde{\Lambda}. \tag{3.15}$$

On the other hand, multiplying (3.11) and (3.12) on the left by $f^T J$ and $g^T J$ respectively, we have

$$f^T J\varphi_+ \quad = \quad \frac{s}{\det G} f^T JJ\tilde{f} = -\frac{s}{\det G} f^T \tilde{f} = -\frac{s}{\det G}\,, \tag{3.16}$$

and

$$g^T J\varphi_- \quad = \quad s g^T JJ\tilde{g} = -s, \tag{3.17}$$

taking into account the fact that $f^T \tilde{f} = g^T \tilde{g} = 1$. Therefore,

$$s = -\det G . f^T J\varphi_+ = -g^T J \varphi_-,$$

and we conclude that $s \in \mathcal{S}$ and, since $\varphi_\pm \in \mathcal{H}^s_\pm$ by (3.11)–(3.13), that $\mathcal{H}^s_\pm \neq \emptyset$. From (3.11)–(3.15) we see that (3.8) holds.

(ii) Conversely, suppose that $\varphi_+ = \Lambda f + \frac{s}{\det G} J\tilde{f}$, where s, Λ and $\tilde{\Lambda}$ satisfy (3.8). Then, since $\Lambda \in \tilde{f}^T \mathcal{H}^s_+$, we have, for some $\psi_+ \in (H_2^+)^2$,

$$\Lambda = \tilde{f}^T \psi_+, \quad \text{where } f\tilde{f}^T \psi_+ = -\frac{s}{\det G} J\tilde{f} + F_+, \quad F_+ \in (H_2^+)^2,$$

and therefore

$$\begin{aligned} \varphi_+ \quad &= \quad (\tilde{f}^T \psi_+) f + \frac{s}{\det G} J\tilde{f} = f\tilde{f}^T \psi_+ + \frac{s}{\det G} J\tilde{f} \\ &= \quad -\frac{s}{\det G} J\tilde{f} + F_+ + \frac{s}{\det G} J\tilde{f} = F_+ \in (H_2^+)^2. \end{aligned}$$

On the other hand, using Lemma 3.2 below, we have

$$\begin{aligned} G\varphi_+ &= \Lambda g + \frac{s}{\det G} GJ\tilde{f} = \Lambda g + \frac{s}{\det G}[(\tilde{g}^T GJ\tilde{f})g + \det G.J\tilde{g}] \\ &= \left(\Lambda + \frac{s}{\det G}(\tilde{g}^T GJ\tilde{f})\right) g + sJ\tilde{g} = (\tilde{g}^T \psi_-)g + sJ\tilde{g} = g\tilde{g}^T \psi_- + sJ\tilde{g} \end{aligned}$$

with $\psi_- \in \mathcal{H}^s_-$; therefore, $G\varphi_+ \in (H_2^-)^2$ and it follows that $\varphi_+ \in \ker T_G$. □

Lemma 3.2. *Let $Gf = g$; then $GJ\tilde{f} = \tilde{g}^T(GJ\tilde{f})g + \det G.J\tilde{g}$.*

Proof. We have $G[f \quad J\tilde{f}] = [g \quad GJ\tilde{f}]$, thus $\det G = \det[g \quad GJ\tilde{f}]$. On the other hand, $\det[g \quad J\tilde{g}] = 1$, so we also have $\det G = \det[g \quad \det G.J\tilde{g}]$. It follows that $\det[g \quad (GJ\tilde{f} - \det G.J\tilde{g})] = 0$, and therefore, for some $\beta \in \mathcal{F}$,

$$GJ\tilde{f} = \beta g + \det G.J\tilde{g}.$$

Multiplying this equation on the left by $\tilde{g}^T$, we get $\beta = \tilde{g}^T GJ\tilde{f}$, since $\tilde{g}^T J\tilde{g} = 0$. □

Note that any function g belonging to $(H_\infty^\pm)^2$ or to $(H_2^\pm)^2$ is left invertible in $\mathcal{F}^{2\times 1}$ if it is not identically zero. Indeed if, for instance, the first component $g_{1\pm}$ of $g_\pm$ is not identically zero, then we can take $\tilde{g}_\pm = (g_{1+}^{-1}, 0)$.

Example 3.3. Let $G = \begin{bmatrix} \overline{\theta} & 0 \\ h & \overline{r} \end{bmatrix}$, where $h \in L_\infty$, θ is an inner function, and $r(\xi) = \dfrac{\xi - i}{\xi + i}$ for $\xi \in \mathbb{R}$. Note that in this case the first component of any element in $\ker T_G$ belongs to the model space K_θ.

We have $Gf = g$ with $f = (\theta, -h\theta r)$ and $g = (1, 0)$ and we can take as their left inverses the functions $\tilde{f}^T$ and $\tilde{g}^T$ with $\tilde{f} = (\overline{\theta}, 0)$ and $\tilde{g} = (1, 0)$. We shall now use Theorem 3.1 to describe $\ker T_G$. We have

$$\mathcal{S} = \{P^-(h\psi_+) + \frac{k}{\xi - i} : \psi_+ \in H_2^+, k \in \mathbb{C}\} \subset H_2^-$$

because from (3.4) we have, for $\psi_{1\pm}, \psi_{2\pm} \in H_2^\pm$,

$$\begin{aligned} & \overline{r}\overline{\theta}[\theta \ \ -h\theta r] \begin{bmatrix} \psi_{1+} \\ \psi_{2+} \end{bmatrix} = [1 \ \ 0] \begin{bmatrix} \psi_{1-} \\ \psi_{2-} \end{bmatrix} \\ \iff & \overline{r}\psi_{1+} - h\psi_{2+} = \psi_{1-} \\ \iff & \overline{r}\psi_{1+} - 2i\frac{\psi_{1+}(i)}{\xi - i} - P^+(h\psi_{2+}) = \psi_{1-} - 2i\frac{\psi_{1+}(i)}{\xi - i} + P^-(h\psi_{2+}). \end{aligned} \tag{3.18}$$

Since the left-hand side of this equation is in H_2^+ while the left-hand side is in H_2^-, both sides must be equal to 0, so we have from the right-hand side of (3.18) that

$$\psi_{1-} = P^-(h\psi_{2+}) + \frac{k}{\xi - i} \qquad \text{with} \quad \psi_{2+} \in H_2^+, k \in \mathbb{C}. \tag{3.19}$$

Conversely, if ψ_{1-} takes the form (3.19), then $\psi_{1-} \in \mathcal{S}$ because (3.18) holds with $\psi_{1+} = \dfrac{k}{\xi + i} + rP^+(h\psi_{2+})$.

We see that $\mathcal{S} \neq \{0\}$, since $\dfrac{1}{\xi - i} \in \mathcal{S}$. Given any $s \in \mathcal{S}$ of the form given by the right-hand side of (3.19), we have

$$\begin{aligned} \mathcal{H}_+^s &= \{(\varphi_{1+}, \varphi_{2+}) \in (H_2^+)^2 : -hr\varphi_{1+} + sr \in H_2^+\} \\ &= \{(\varphi_{1+}, \varphi_{2+}) \in (H_2^+)^2 : P^-(hr\varphi_{1+}) = P^-(sr)\} \end{aligned}$$

and $\mathcal{H}_-^s = (H_2^-)^2$. Since in this case $\tilde{g}^T(GJ\tilde{f}) = 0$ we have from (3.8) that $\Lambda \in \tilde{f}^T\mathcal{H}_+^s \cap \tilde{g}^T\mathcal{H}_-^s$, which is equivalent to

$$\Lambda = \overline{\theta}\varphi_{1+} \qquad \text{with} \quad \varphi_{1+} \in K_\theta, \quad P^-(hr\varphi_{1+}) = P^-(sr).$$

Therefore, from Theorem 3.1, specifically (3.7),

$$\varphi_+ = \left(\varphi_{1+},\ -P^+(hr\varphi_{1+}) + \frac{\tilde{k}}{\xi + i}\right),$$

with $\varphi_{1+} \in K_\theta$ and $\tilde{k} \in \mathbb{C}$, where we took into account the fact that $P^-(sr) = P^-(hr\varphi_{1+})$ and $P^+(sr) = P^+(rP^-(h\psi_+)) + \frac{k}{\xi+i} = \frac{\tilde{k}}{\xi+i}$ with $\tilde{k} \in \mathbb{C}$. Thus we have

$$\ker T_G = \{(\varphi_{1+},\ -P^+(hr\varphi_{1+})) : \varphi_{1+} \in K_\theta\} + \operatorname{span}\left\{\left(0, \frac{1}{\xi + i}\right)\right\}.$$

If θ is a finite Blaschke product of degree n, then $\dim\ker T_G = n + 1$; otherwise $\dim\ker T_G = \infty$.

As a consequence of Theorem 3.1, we have the following.

Corollary 3.4. *If $Gf = g$ and*

$$\left(\det G.f^T(H_2^+)^2\right) \cap \left(g^T(H_2^-)^2\right) = \{0\}, \tag{3.20}$$

and we define

$$\begin{aligned}\mathcal{H}_+ &= \{\psi_+ \in (H_2^+)^2 : f\tilde{f}^T\psi_+ \in (H_2^+)^2\}, && (3.21)\\ \mathcal{H}_- &= \{\psi_- \in (H_2^-)^2 : g\tilde{g}^T\psi_- \in (H_2^-)^2\}, && (3.22)\\ \mathcal{K} &= \tilde{f}^T\mathcal{H}_+ \cap \tilde{g}^T\mathcal{H}_-, && (3.23)\end{aligned}$$

then

$$\ker T_G = \mathcal{K}f. \qquad (3.24)$$

Since $0 \in \mathcal{S}$, with $\mathcal{S}$ defined in (3.4), we also have the following consequence of Theorem 3.1, which can be understood as establishing a lower bound for $\ker T_G$.

Corollary 3.5. *If $Gf = g$ then, with the same notation as in Corollary 3.4, we have*

$$\mathcal{K}f \subset \ker T_G.$$

By Coburn's Lemma [14], for any Toeplitz operator with scalar symbol $\varphi \in L_\infty$, either $\ker T_\varphi$ or $\ker T_\varphi^* = \ker T_{\overline{\varphi}}$ is zero. It is well known that this property no longer holds when we consider Toeplitz operators with matrix symbol, since T_G and $T_G^* = T_{\bar{G}^T}$ may both have a non-zero kernel. However, using the result of Corollary 3.4, we can state what may be seen as a version of Coburn's Lemma for 2×2 block Toeplitz operators with symbol G. We shall need the following, which can easily be verified:

Lemma 3.6. *Let G be a 2×2 matrix. Then*

$$\det G.I = -GJG^TJ. \qquad (3.25)$$

Theorem 3.7. *Let $\det G \in \mathcal{F} \setminus \{0\}$. Then either $\ker T_G$ or $\ker T_{\overline{G}^T}$ is of scalar type.*

Proof. Assume that $\ker T_{\overline{G}^T} \neq \{0\}$ and let $\psi_+ \in \ker T_{\overline{G}^T}$, $\psi_+ \neq 0$. Then we have $\overline{G}^T\psi_+ = \psi_- \in (H_2^-)^2$ and

$$\begin{aligned}\overline{G}^T\psi_+ = \psi_- \quad &\Longleftrightarrow \quad G^T\,\overline{\psi_+} = \overline{\psi_-} \iff GJG^TJ(J\overline{\psi_+}) = -GJ\overline{\psi_-}\\ &\Longleftrightarrow \quad \det G.(J\overline{\psi_+}) = GJ\overline{\psi_-}. \qquad (3.26)\end{aligned}$$

Therefore, $GF_+ = \det G.F_-$ with $F_\pm = J\overline{\psi_\mp} \in (H_2^\pm)^2$. For any $\varphi_+ \in \ker T_G$, we have $G\varphi_+ = \varphi_- \in (H_2^-)^2$, so $G[\varphi_+ \;\; F_+] = [\varphi_- \;\; \det G.F_-]$, and it follows that

$$\det G.\det[\varphi_+ \;\; F_+] = \det G.\det[\varphi_- \;\; F_-],$$

i.e., on a set of positive measure in $\mathbb{R}$,

$$\det[\varphi_+ \quad F_+] = \det[\varphi_- \quad F_-]. \tag{3.27}$$

Since the left-hand side of (3.27) represents a function in H_1^+ while the right-hand side represents a function in H_1^-, both are equal to zero. Since F_+ and $F_- = GF_+$ admit left inverses because neither is identically equal to zero, it follows that every $\varphi_+ \in \ker T_G$ is a scalar multiple of F_+. $\square$

Corollary 3.8. *For every $G \in (L_\infty^{2\times 2})$ with $\det G \in L_\infty \backslash \{0\}$, either $\ker T_G = \{0\}$, or $\ker T_G^* = \{0\}$, or both kernels are of scalar type.*

Corollary 3.9. *If $\det G$ admits a (canonical) bounded factorization [24] $\det G = d_- d_+$ with $d_\pm \in \mathcal{G}H_\infty^\pm$, then both $\ker T_G$ and $\ker T_{\overline{G}^T}$ are of scalar type. In particular, $\ker T_G$ and $\ker T_{\overline{G}^T}$ are of scalar type whenever $\det G = 1$.*

Proof. From (3.26) we have that $\overline{G}^T \psi_+ = \psi_- \iff G(d_+^{-1} J\overline{\psi_-}) = d_- J\overline{\psi_+}$. Since any $\varphi_+ \in (H_2^+)^2$ can be written in the form $\varphi_+ = d_+^{-1} J\overline{\psi_-}$ for some $\psi_- \in (H_2^-)^2$, and any $\varphi_- \in (H_2^-)^2$ can be written in the form $\varphi_- = d_- J\overline{\psi_+}$ for some $\psi_+ \in (H_2^+)^2$, it follows that $\ker T_G = \{0\}$ if and only if $\ker T_{\overline{G}^T} = \{0\}$. The result now follows from Corollary 3.8. $\square$

Note that, for block Toeplitz operators, it is not always the case that a non-zero kernel can be given by a symbol of determinant 1, as the following simple example shows: let $G = \begin{bmatrix} \overline{r} & 0 \\ \overline{r} & 0 \end{bmatrix}$. The kernel of T_G is (k, h) where $k \in K_r$ and $h \in H_2^+$. The symbol can only have rows of the form $(p \quad 0)$, for if $p, q \in L_\infty$ and $pk + qh \in H_2^-$ for all $k \in K_r$ and $h \in H_2^+$, then, taking $k = 0$ we see that $q = 0$.

While Corollary 3.4 provides sufficient conditions for the kernel of a Toeplitz operator with 2×2 symbol to be of scalar type, condition (3.20) is not a necessary one. To see this, let us consider the solution to $Gf = g$ that we obtain from (2.1) if we take the second columns of the matrix functions on the left and on the right hand sides of (2.3), instead of the first columns as was done in Section 2. We get, using the same notation,

$$Gf = g, \qquad \text{with} \qquad f = \begin{bmatrix} g_{12}^+ \\ g_{22}^+ \end{bmatrix}, \qquad g = r^k \begin{bmatrix} g_{12}^- \\ g_{22}^- \end{bmatrix}.$$

Assuming, for simplicity, that $\det G = 1$, we can choose $G_\pm$ such that $\det G_\pm = 1$, and thus as left inverses for f and g we can take $\tilde{f}^T$ and

$\tilde{g}^T$ given by

$$\tilde{f} = J \begin{bmatrix} g_{11}^+ \\ g_{21}^+ \end{bmatrix} = \begin{bmatrix} -g_{21}^+ \\ g_{11}^+ \end{bmatrix}, \qquad \tilde{g} = r^{-k} J \begin{bmatrix} g_{11}^- \\ g_{21}^- \end{bmatrix} = r^{-k} \begin{bmatrix} -g_{21}^- \\ g_{11}^- \end{bmatrix}.$$

Applying Theorem 3.1, we have $\mathcal{S} = f^T(H_2^+)^2 \cap g^T(H_2^-)^2 = K_{r^k} \neq \{0\}$; for each $s \in \mathcal{S} = K_{r^k}$, we have $\mathcal{H}_+^s = (H_2^+)^2$ and

$$\mathcal{H}_-^s = \left\{ \psi_- \in (H_2^-)^2 : sr^{-k} \begin{bmatrix} -g_{21}^- \\ g_{11}^- \end{bmatrix} \in (H_2^-)^2 \right\} = (H_2^-)^2,$$

because $sr^{-k} \in H_\infty^-$ for $s \in K_{r^k}$. Therefore, from (3.8),

$$\Lambda \in \tilde{f}^T (H_2^+)^2 = H_2^+, \tag{3.28}$$

and, since in this case $GJ\tilde{f} = J\tilde{g}$, which implies that $\tilde{g}^T GJ\tilde{f} = 0$, we must also have

$$\Lambda \in \tilde{g}^T (H_2^-)^2 = H_2^-. \tag{3.29}$$

From (3.28) and (3.29) we get $\Lambda = 0$ and it follows that $\ker T_G = K_{r^k} J\tilde{f} = K_{r^k}(g_{11}^+, g_{21}^+)$ as in Section 2.

The next result shows that every scalar-type Toeplitz kernel, for a 2×2 matrix symbol G, is of the form (3.24) with $\mathcal{K}$ given by (3.21)–(3.23), if f and $g = Gf$ have left inverses.

Theorem 3.10. *If $\ker T_G = \mathcal{K} f$, where f is a fixed function in $\mathcal{F}^{2\times 1}$ such that f and $g = Gf$ possess left inverses $\tilde{f}^T$ and $\tilde{g}^T$, respectively, and $\mathcal{K} \subset \mathcal{F}$, then*

$$\mathcal{K} = \tilde{f}^T \mathcal{H}_+ \cap \tilde{g}^T \mathcal{H}_-,$$

where $\mathcal{H}_\pm$ are defined as in (3.21)–(3.22).

Proof. Let k be any element of $\mathcal{K}$. Then $kf \in \ker T_G$ and we have

$$kf = \psi_+ \in (H_2^+)^2, \qquad G(kf) = \psi_- \in (H_2^-)^2. \tag{3.30}$$

From the first equation we get that $k = \tilde{f}^T \psi_+$, so $f\tilde{f}^T \psi_+ = fk = \psi_+ \in (H_2^+)^2$; therefore $\psi_+ \in \mathcal{H}_+$.

Analogously, from the second equation in (3.30), we have $kg = \psi_- \in (H_2^-)^2$. Therefore $k = \tilde{g}^T \psi_-$ and ψ_- is such that $g\tilde{g}^T \psi_- = gk = \psi_- \in (H_2^-)^2$; thus $\psi_- \in \mathcal{H}_-$. We conclude that $\mathcal{K} \subset \tilde{f}^T \mathcal{H}_+ \cap \tilde{g}^T \mathcal{H}_-$.

Conversely, if $k \in f^T \mathcal{H}_+ \cap \tilde{g}^T \mathcal{H}_-$, then $kf \in \ker T_G$ (as in the last part of the proof of Theorem 3.1, with $s = 0$) and, since $\ker T_G = \mathcal{K} f$, we have $kf = k_0 f$ with $k_0 \in \mathcal{K}$. Multiplying on the left by $\tilde{f}^T$, we conclude that $k = k_0 \in \mathcal{K}$, so $f^T \mathcal{H}_+ \cap \tilde{g}^T \mathcal{H}_- \subset \mathcal{K}$. $\square$

Naturally, one can say more about the space $\mathcal{K}$ if further assumptions are made on f and g in (3.3).

Theorem 3.11. *If $f = \theta f_+$, where θ is an inner function and $f_+ \in (H_\infty^+)^2$, and $g = f_- \in (H_\infty^-)^2$, where $f_\pm$ possess left inverses $\tilde{f}_\pm^T$ with $\tilde{f}_\pm \in (H_\infty^\pm)^2$, then*

$$\ker T_G = K_\theta f_+.$$

Proof. In this case we have $\mathcal{H}_\pm = (H_2^\pm)^2$ and, if $\tilde{f}_\pm^T$ are left inverses for $f_\pm$, then $\tilde{f} = \overline{\theta}\tilde{f}_+$ and $\tilde{g} = \tilde{f}_-$ provide left inverses for f and g respectively; the result now follows from (3.23) and (3.24). □

We say that $f_\pm = (f_{1\pm}, f_{2\pm}) \in (H_\infty^\pm)^2$ is a *corona pair* in $\mathbb{C}^\pm$ if and only if there exists $\tilde{f}_\pm \in (H_\infty^\pm)^2$ such that $\tilde{f}_\pm^T f_\pm = 1$. In this case we say that $f_\pm \in \mathrm{CP}^\pm$. By the Corona Theorem, $f_\pm \in \mathrm{CP}^\pm$ if and only if

$$\inf_{z\in\mathbb{C}^\pm} \left(|f_1^\pm(z)| + |f_2^\pm(z)|\right) > 0. \tag{3.31}$$

Thus, under the conditions of Theorem 3.11, we have $f_\pm \in \mathrm{CP}^\pm$.

The next theorem generalises Theorem 3.11, establishing sufficient conditions for $\mathcal{K}$, in Corollary 3.4, to be a model space or a shifted model space. We shall use the following well-known result, which follows easily from the observation that if α, β are coprime inner functions then $\alpha\varphi_+ \in \beta H_2^+$ if and only if $\varphi_+ \in \beta H_2^+$ (a consequence of the uniqueness of the inner–outer factorization).

Lemma 3.12. *If $\varphi_+ \in H_2^+$ and α_1 and α_2 are inner functions, then $\alpha_1\varphi_+ \in \alpha_2 H_2^+$ if and only if $\varphi_+ \in \dfrac{\alpha_2}{\gamma_\alpha} H_2^+$, where $\gamma_\alpha = \gcd\{\alpha_1, \alpha_2\}$.*

Theorem 3.13. *Let $\det G$ have a canonical bounded factorization $\det G = d_- d_+^{-1}$ (as in Section 2), and let $Gf = g$ with componentwise inner–outer factorizations*

$$f = (\alpha_1 f_{1+}, \alpha_2 f_{2+}), \qquad g = (\overline{\beta_1} f_{1-}, \overline{\beta_2} f_{2-}), \tag{3.32}$$

where $\alpha_1, \alpha_2, \beta_1$ and β_2 are inner functions, $f_{1\pm}, f_{2\pm}$ are outer functions in $H_\infty^\pm$, and $(f_{1\pm}, f_{2\pm}) \in \mathrm{CP}^\pm$. Then

$$\ker T_G = K_{\gamma_\alpha\gamma_\beta} \begin{bmatrix} \dfrac{\alpha_1}{\gamma_\alpha} f_{1+} \\ \dfrac{\alpha_2}{\gamma_\alpha} f_{2+} \end{bmatrix} = K_{\gamma_\alpha\gamma_\beta}\, \overline{\gamma_\alpha} f, \tag{3.33}$$

where

$$\begin{cases} \gamma_\alpha &= \gcd\{\alpha_1, \alpha_2\}, \\ \gamma_\beta &= \gcd\{\beta_1, \beta_2\}. \end{cases} \tag{3.34}$$

Proof. In this case we have $\mathcal{S} = \{0\}$ and $\ker T_G$ is given by Corollary 3.4. Let $[\tilde{f}_{1\pm}\ \tilde{f}_{2\pm}]$, with $(\tilde{f}_{1\pm}, \tilde{f}_{2\pm}) \in (H_\infty^\pm)^2$, be left inverses for $(f_{1\pm}, f_{2\pm})$, respectively, so that $\tilde{f} = (\overline{\alpha_1}\tilde{f}_{1+}, \overline{\alpha_2}\tilde{f}_{2+})$ and $\tilde{g} = (\beta_1\tilde{f}_{1-}, \beta_2\tilde{f}_{2-})$ are left inverses for f and g, respectively.

For any $\psi_+ = (\psi_{1+}, \psi_{2+}) \in (H_2^+)^2$ we have then

$$f\tilde{f}^T\psi_+ \quad = \quad \begin{bmatrix} f_{1+}\tilde{f}_{1+}\psi_{1+} + \alpha_1\overline{\alpha_2}\tilde{f}_{2+}f_{1+}\psi_{2+} \\ \overline{\alpha_1}\alpha_2\tilde{f}_{1+}f_{2+}\psi_{1+} + f_{2+}\tilde{f}_{2+}\psi_{2+} \end{bmatrix}$$

so $f\tilde{f}^T\psi_+ \in (H_2^+)^2$ if and only if

$$\begin{aligned} &\begin{cases} \overline{\alpha_2}\alpha_1\tilde{f}_{2+}f_{1+}\psi_{2+} & \in \quad H_2^+, \\ \overline{\alpha_1}\alpha_2\tilde{f}_{1+}f_{2+}\psi_{1+} & \in \quad H_2^+, \end{cases} \\ \iff &\begin{cases} \alpha_1\tilde{f}_{2+}f_{1+}\psi_{2+} & \in \quad \alpha_2 H_2^+, \\ \alpha_2\tilde{f}_{1+}f_{2+}\psi_{1+} & \in \quad \alpha_1 H_2^+, \end{cases} \\ \iff &\begin{cases} \tilde{f}_{2+}f_{1+}\psi_{2+} & \in \quad \frac{\alpha_2}{\gamma_\alpha}H_2^+, \\ \tilde{f}_{1+}f_{2+}\psi_{1+} & \in \quad \frac{\alpha_1}{\gamma_\alpha}H_2^+, \end{cases} \end{aligned} \tag{3.35}$$

with γ_α defined by (3.34). Thus $(\psi_{1+}, \psi_{2+}) \in \mathcal{H}_+$ if and only if $\psi_{1+}, \psi_{2+} \in H_2^+$ and

$$\psi_{1+} = \frac{\alpha_1}{\gamma_\alpha}\frac{\varphi_{1+}}{\tilde{f}_{1+}f_{2+}}, \qquad \psi_{2+} = \frac{\alpha_2}{\gamma_\alpha}\frac{\varphi_{2+}}{\tilde{f}_{2+}f_{1+}}, \qquad \text{with} \quad \varphi_{1+}, \varphi_{2+} \in H_2^+,$$

where, since $\frac{\varphi_{1+}}{f_{2+}} = \psi_{1+}\frac{\gamma_\alpha}{\alpha_1}\tilde{f}_{1+} \in \mathcal{N}^+ \cap L_2$ and $\frac{\varphi_{2+}}{f_{1+}} = \psi_{2+}\frac{\gamma_\alpha}{\alpha_2}\tilde{f}_{2+} \in \mathcal{N}^+ \cap L_2$, we have

$$\frac{\varphi_{1+}}{f_{2+}}, \frac{\varphi_{2+}}{f_{1+}} \in H_2^+. \tag{3.36}$$

Analogously, we get that $(\psi_{1-}, \psi_{2-}) \in \mathcal{H}_-$ if and only if $\psi_{1-}, \psi_{2-} \in H_2^-$ with

$$\psi_{1-} = \overline{\left(\frac{\beta_1}{\gamma_\beta}\right)}\frac{\varphi_{1-}}{\tilde{f}_{1-}f_{2-}}, \qquad \psi_{2+} = \overline{\left(\frac{\beta_2}{\gamma_\beta}\right)}\frac{\varphi_{2-}}{\tilde{f}_{2-}f_{1-}}, \qquad \varphi_{1-}, \varphi_{2-} \in H_2^-,$$

where

$$\frac{\varphi_{1-}}{f_{2-}}, \frac{\varphi_{2-}}{f_{1-}} \in H_2^-. \tag{3.37}$$

Thus $\mathcal{K} = \tilde{f}^T\mathcal{H}_+ \cap \tilde{g}^T\mathcal{H}_-$ consists of the functions k such that

$$k = [\overline{\alpha_1}\tilde{f}_{1+} \quad \overline{\alpha_2}\tilde{f}_{2+}] \begin{bmatrix} \frac{\alpha_1}{\gamma_\alpha}\frac{\varphi_{1+}}{\tilde{f}_{1+}f_{2+}} \\ \frac{\alpha_2}{\gamma_\alpha}\frac{\varphi_{2+}}{\tilde{f}_{2+}f_{1+}} \end{bmatrix} = [\beta_1\tilde{f}_{1-} \quad \beta_2\tilde{f}_{2-}] \begin{bmatrix} \overline{\left(\frac{\beta_1}{\gamma_\beta}\right)}\frac{\varphi_{1-}}{\tilde{f}_{1-}f_{2-}} \\ \overline{\left(\frac{\beta_2}{\gamma_\beta}\right)}\frac{\varphi_{2-}}{\tilde{f}_{2-}f_{1-}} \end{bmatrix}, \tag{3.38}$$

i.e.,

$$k = \overline{\gamma_\alpha}\left(\frac{\varphi_{1+}}{f_{2+}} + \frac{\varphi_{2+}}{f_{1+}}\right) = \gamma_\beta\left(\frac{\varphi_{1-}}{f_{2-}} + \frac{\varphi_{2-}}{f_{1-}}\right).$$

Taking (3.36) and (3.37) into account, it follows that $\mathcal{K} \subset \overline{\gamma_\alpha} H_2^+ \cap \gamma_\beta H_2^-$.

Conversely, if $k = \overline{\gamma_\alpha}\varphi_+ = \gamma_\beta\varphi_-$, with $\varphi_\pm \in H_2^\pm$, we can write

$$k = \overline{\gamma_\alpha}\varphi_+ = \overline{\gamma_\alpha}\tilde{f}^T f\varphi_+ = \tilde{f}^T(\overline{\gamma_\alpha} f\varphi_+),$$

where $\overline{\gamma_\alpha} f\varphi_+ \in \mathcal{H}_+$ because $\overline{\gamma_\alpha} f\varphi_+ \in (H_2^+)^2$ and $f\tilde{f}^T(\overline{\gamma_\alpha} f\varphi_+) = f\overline{\gamma_\alpha}\varphi_+$.

Thus $k \in \tilde{f}^T\mathcal{H}_+$ and, analogously, we can show that $k \in \tilde{g}^T\mathcal{H}_-$, so that $k \in \tilde{f}^T\mathcal{H}_+ \cap \tilde{g}\mathcal{H}_- = \mathcal{K}$. We conclude that $\mathcal{K} = \overline{\gamma_\alpha} H_2^+ \cap \gamma_\beta H_2^- = \overline{\gamma_\alpha} K_{\gamma_\alpha\gamma_\beta}$. □

Corollary 3.14. *With the same assumptions as in Theorem 3.13, T_G is injective if and only if γ_α and γ_β are constant.*

Corollary 3.15. *If the assumptions of Theorem 3.13 are satisfied, with*

$$f = \begin{bmatrix} \alpha \\ O_+ \end{bmatrix}, \qquad g = \begin{bmatrix} \overline{\beta} \\ O_- \end{bmatrix},$$

where α, β are inner functions and $O_+, \overline{O_-} \in H_\infty^+$ are outer functions, then T_G is injective.

Note that, if $\det G$ has a canonical bounded factorization and $f \in \mathrm{CP}^+$ and $g \in \mathrm{CP}^-$, then T_G is invertible ([6]). Roughly speaking, Corollary 3.15 means that if we have $Gf_+ = f_-$ with $f_\pm \in (H_\infty^\pm)^2$, and the two components of $f_\pm$ "approach zero simultaneously at some point" in $\mathbb{C}^\pm \cup \mathbb{R}$ (so that they do not satisfy the corona conditions (3.31)),we may still have an injective Toeplitz operator as long as they do not "approach zero simultaneously" through a common inner factor.

Suppose now that $\ker T_G$ is of scalar type, $\ker T_G = \mathcal{K}f$ as in Theorem 3.10. If $\ker T_G \neq \{0\}$, we may now ask, in view of the results of Theorems 3.11 and 3.13, whether a scalar type $\ker T_G$ can always be described as the product of some fixed 2×1 function by a scalar Toeplitz kernel, and in particular a model space.

Toeplitz kernels constitute an important subset of the class of nearly S^*-invariant subspaces of H_2^+. Here by S^* we denote the backward shift operator $S^* = P^+\bar{r}P^+_{|H_2^+}$; a subspace M of H_2^+ is nearly S^*-invariant if and only if $S^*\varphi_+ \in M$ for all $\varphi_+ \in M$ such that $\varphi_+(i) = 0$. Hitt proved (in the unit disk setting) that any nontrivial closed nearly S^*-invariant subspace of H_2^+ has the form $M = hK_\theta$ where θ is an inner function vanishing at i, $\frac{h}{\xi - i} \in M$ is the element of unit norm with positive value at i which is orthogonal to all elements of M vanishing at i, and h is an isometric multiplier from K_θ into H_2^+ ([22, 27]).

In the next theorem we show that, if $\ker T_G$ is of scalar type, then it is the product of a scalar nearly S^*-invariant subspace (which can be explicitly described) with a fixed 2×1 function.

Note that, when $\ker T_G$ is of scalar type and $\ker T_G \neq \{0\}$, then there exist $F_\pm \in (H_2^\pm)^2 \setminus \{0\}$ such that $GF_+ = F_-$ and, if $F_\pm = (F_{1\pm}, F_{2\pm})$, then either $F_{1\pm}$ or $F_{2\pm}$ is not identically zero, and thus $F_\pm$ has a left inverse defined a.e. on $\mathbb{R}$. We shall assume, without loss of generality, that $F_{1\pm} \neq 0$, so that $F_{1\pm}^{-1}$ are defined a.e. on $\mathbb{R}$. If $\ker T_G = \mathcal{K}f$ with $f = (f_1, f_2)$ and $Gf = g = (g_1, g_2)$, as in Theorem 3.10, then

$$F_{j+} = k_0 f_j, \qquad F_{j-} = k_0 g_j \qquad (j = 1, 2),$$

where $k_0 \in \mathcal{K}$ and k_0, f_1, g_1 are different from zero a.e. on $\mathbb{R}$.

We shall also use the following notation: if $F_\pm \in H_2^\pm \setminus \{0\}$, then we write $F_\pm = I_\pm O_\pm$, where $I_+, \overline{I_-}$ are inner functions and $O_+, \overline{O_-}$ are outer functions in H_2^+. If $F_\pm = 0$ then we write $F_\pm = I_\pm O_\pm$ with $I_\pm, O_\pm = 0$. If α is an inner function, then $\gcd\{\alpha, 0\} = \alpha$.

Theorem 3.16. *Let $\ker T_G$ be of scalar type, $\ker T_G \neq \{0\}$, with $\ker T_G = \mathcal{K}f$ as in Theorem 3.10. Then there exist an $F \in \mathcal{F}^{2\times 1}$ and a nearly S^*-invariant subspace $\tilde{\mathcal{K}} \subset H_2^+$, which is closed if $G \in L_\infty^{2\times 2}$, such that $\ker T_G = \tilde{\mathcal{K}}F$. Moreover, if $F_+ = (F_{1+}, F_{2+}) \neq 0$ is a given element of $\ker T_G$ and $GF_+ = F_- = (F_{1-}, F_{2-})$ with $F_{j\pm} = I_{j\pm} O_{j\pm}$ $(j = 1, 2)$, using the notation above, and we suppose that $F_{1\pm} \neq 0$, then*

$$\ker T_G = \tilde{\mathcal{K}} \frac{F_+}{\gamma_+ O_{1+}} = \tilde{\mathcal{K}} \left(\frac{I_{1+}}{\gamma_+}, \frac{F_{2+}}{\gamma_+ O_{1+}} \right), \tag{3.39}$$

where

$$\tilde{\mathcal{K}} = \left\{ \tilde{\psi}_+ \in \ker T_{\gamma_- \overline{\gamma_+} O_{1-}/O_{1+}} \cap \ker T_{\gamma_- \overline{\gamma_+} O_{2-}/O_{1+}} : \frac{O_{2+}}{O_{1+}} \tilde{\psi}_+ \in H_2^+ \right\}, \tag{3.40}$$

$$\gamma_+ = \gcd(I_{1+}, I_{2+}), \qquad \overline{\gamma_-} = \gcd(\overline{I_{1-}}, \overline{I_{2-}}). \tag{3.41}$$

Proof. Let $F_+ = (F_{1+}, F_{2+}) \neq 0$ belong to $\ker T_G$ and let $GF_+ = F_- = (F_{1-}, F_{2-})$, where we assume that $F_{1\pm} \neq 0$. Since we can write $f = k_0^{-1} F_+$ with $k_0 \in \mathcal{K}$, we have $\ker T_G = (\mathcal{K} k_0^{-1}) F_+$, and we can then assume that $\ker T_G = \mathcal{K} F_+$ and apply Theorem 3.10.

Defining $\mathcal{H}_\pm$ as in (3.21)–(3.22), we have, for $f = F_+$ and $\tilde{f} = (F_{1+}^{-1}, 0)$,

$$\begin{aligned}
\psi_+ \in \mathcal{H}_+ \quad &\Longleftrightarrow \quad f\tilde{f}^T\psi_+ \in (H_2^+)^2, \quad \psi_+ \in (H_2^+)^2 \\
&\Longleftrightarrow \quad (\tilde{f}^T\psi_+)f \in (H_2^+)^2, \quad \psi_+ \in (H_2^+)^2 \\
&\Longleftrightarrow \quad F_{1+}^{-1}\psi_{1+}(F_{1+}, F_{2+}) \in (H_2^+)^2, \quad \psi_{1+} \in H_2^+, \quad \psi_{2+} \in H_2^+ \\
&\Longleftrightarrow \quad \psi_{1+}F_{2+} \in F_{1+}H_2^+, \quad \psi_{1+} \in H_2^+, \quad \psi_{2+} \in H_2^+.
\end{aligned}$$

We have

$$\psi_{1+}F_{2+} \in F_{1+}H_2^+ \iff \psi_{1+}I_{2+}\frac{O_{2+}}{O_{1+}} \in I_{1+}H_2^+. \tag{3.42}$$

In this case $\psi_{1+}O_{2+}/O_{1+} \in \mathcal{N}_+ \cap L_2 = H_2^+$ and it follows from the second relation in (3.42) and Lemma 3.12 that the right hand side of (3.42) is equivalent to

$$\psi_{1+}\frac{O_{2+}}{O_{1+}} = \frac{I_{1+}}{\gamma_+}\varphi_+ \quad \text{with} \quad \varphi_+ \in H_2^+ \quad \text{and} \quad \gamma_+ = \gcd\{I_{1+}, I_{2+}\}.$$

Since $\psi_{1+} \in H_2^+$ and

$$\frac{O_{1+}}{O_{2+}}\frac{I_{1+}}{\gamma_+}\varphi_+ \in H_2^+ \iff \frac{O_{1+}}{O_{2+}}\varphi_+ \in H_2^+$$

we conclude that

$$\begin{aligned}
&\psi_{1+}F_{2+} \in F_{1+}H_2^+, \qquad \psi_{1+} \in H_2^+ \\
\Longleftrightarrow \quad &\psi_{1+} = \tilde{\psi}_+\frac{I_{1+}}{\gamma_+}, \quad \text{with} \quad \tilde{\psi}_+ \in H_2^+, \quad \frac{O_{2+}}{O_{1+}}\tilde{\psi}_+ \in H_2^+.
\end{aligned} \tag{3.43}$$

So, $\psi \in \tilde{f}^T\mathcal{H}_+$ if and only if $\psi = F_{1+}^{-1}\psi_{1+}$, where ψ_{1+} satisfies (3.43), i.e.,

$$\tilde{f}^T\mathcal{H}_+ = \left\{\psi \in \mathcal{F} : \psi = \frac{\tilde{\psi}_+}{\gamma_+O_{1+}} \text{ with } \tilde{\psi}_+ \in H_2^+, \ \frac{O_{2+}}{O_{1+}}\tilde{\psi}_+ \in H_2^+\right\}. \tag{3.44}$$

Analogously we get, for $F_{2-} \neq 0$ and γ_- defined in (3.41),

$$\tilde{g}^T\mathcal{H}_- = \left\{\psi \in \mathcal{F} : \psi = \frac{\tilde{\psi}_-}{\gamma_-O_{1-}} \text{ with } \tilde{\psi}_- \in H_2^-, \ \frac{O_{2-}}{O_{1-}}\tilde{\psi}_- \in H_2^-\right\}. \tag{3.45}$$

Therefore, for ψ to belong to $\mathcal{K} = \tilde{f}^T\mathcal{H}_+ \cap \tilde{g}^T\mathcal{H}_-$, the functions $\tilde{\psi}_\pm \in H_2^\pm$ in (3.44)–(3.45) must be such that

$$\frac{\tilde{\psi}_+}{\gamma_+O_{1+}} = \frac{\tilde{\psi}_-}{\gamma_-O_{1-}},$$

i.e., $\tilde{\psi}_+ \in \ker T_{\gamma_- \overline{\gamma_+} O_{1-}/O_{1+}}$, and the condition $\tilde{\psi}_- O_{2-}/O_{1-} \in H_2^-$ in (3.45) can be expressed by

$$\gamma_- \overline{\gamma_+} \frac{O_{2-}}{O_{1+}} \tilde{\psi}_+ \in H_2^-, \quad \text{i.e.,} \quad \tilde{\psi}_+ \in \ker T_{\gamma_- \overline{\gamma_+} O_{2-}/O_{1+}}.$$

Finally, taking into account the last condition in (3.43), we have

$$\mathcal{K} = \tilde{f}^T \mathcal{H}_+ \cap \tilde{g}^T \mathcal{H}_- = \frac{1}{\gamma_+ O_{1+}} \tilde{\mathcal{K}},$$

where, for $F_{2+}, F_{1+} \neq 0$,

$$\tilde{\mathcal{K}} = \left\{ \tilde{\psi}_+ \in \ker T_{\gamma_- \overline{\gamma_+} O_{1-}/O_{1+}} \cap \ker T_{\gamma_- \overline{\gamma_+} O_{2-}/O_{1+}} : \frac{O_{2+}}{O_{1+}} \tilde{\psi}_+ \in H_2^+ \right\}.$$

It is easy to see that $\tilde{\mathcal{K}}$ is nearly S^*-invariant, because Toeplitz kernels are nearly S^*-invariant subspaces and if $\frac{O_{2+}}{O_{1+}} \tilde{\psi} \in H_2^+$ and $r^{-1} \tilde{\psi}_+ \in H_2^+$, then $\frac{O_{2+}}{O_{1+}} \tilde{\psi} r^{-1} \in \mathcal{N}^+ \cap L_2 = H_2^+$.

If $F_{2\pm} = 0$, then $\mathcal{H}_\pm = (H_2^\pm)^2$ and we again find that $\mathcal{K} = \tilde{\mathcal{K}} \dfrac{F_+}{\gamma_+ O_{1+}}$, where

$$\begin{aligned} \tilde{\mathcal{K}} &= \ker T_{\gamma_- \overline{\gamma_+} O_{1-}/O_{1+}} \cap \ker T_{\gamma_- \overline{\gamma_+} O_{2-}/O_{1+}} \\ &= \ker T_{\gamma_- O_{1-}/F_{1+}} \cap \ker T_{\gamma_- \frac{O_{2-}}{F_{1+}}} \quad \text{if} \quad F_{2+} = 0, \quad F_{2-} \neq 0, \end{aligned}$$

and

$$\tilde{\mathcal{K}} = \ker T_{\gamma_- \overline{\gamma_+} O_{1-}/O_{1+}} = \ker T_{F_{1-}/F_{1+}} \quad \text{if} \quad F_{2+} = 0, \quad F_{2-} = 0.$$

Finally, if $G \in L_\infty^{2\times 2}$ then $\ker T_G$ is closed, and it follows from (3.39) that $\tilde{\mathcal{K}} \frac{I_{1+}}{\gamma_+}$ is closed, so $\tilde{\mathcal{K}}$ is closed. □

Since $\tilde{\mathcal{K}}$ is nearly-invariant, it follows from Hitt's theorem that it can be written as $\tilde{K} = K_\theta \, g_+$, where K_θ is a model space and g_+ is a scalar function (an isometric multiplier); however, there is no reason to suppose that $\tilde{K}$ is a Toeplitz kernel.

Related to these results, a very natural question regarding scalar type Toeplitz kernels is whether they have a maximal function. It was proved in [7] that for every $\varphi_+ \in H_2^+$ there exists a so called minimal kernel $K_m(\varphi_+)$ such that very other kernel K with $\varphi_+ \in K$ contains $K_m(\varphi_+)$. We say that φ_+ is a maximal function for K if $K = K_m(\varphi_+)$; every scalar Toeplitz kernel has a maximal function. For scalar type Toeplitz kernels we have the following, taking the result of Theorem 3.16 into account.

Theorem 3.17. *If* $\ker T_G$*, with* $G \in L_\infty^{2\times 2}$*, is of scalar type, then there exists a maximal function for* $\ker T_G$*.*

Proof. Let $\ker T_G = \tilde{\mathcal{K}} F$ and $\tilde{\mathcal{K}} = K_\theta\, g_+$ as above. Thus $\ker T_G = K_\theta\, g_+\, F$, so let us write $f = g_+ F$. Now K_θ is a model space with maximal function φ_+, say; let $\varphi_+ = IO$, where I is inner and O is an outer function in H_2^+. Then $K_\theta = \ker T_{(\bar{I}\bar{O}/O)}$ (by [7]) and every element in K has the form $[IO/\bar{O}]\psi_-$, where $\psi_- \in H_2^-$.

Obviously $\varphi_+ f$ belongs to $\ker T_G$. Suppose that $\varphi_+ f = IOf$ belongs to the kernel of some T_H with $H \in L_\infty^{2\times 2}$; then $H(IOf) = \varphi_- \in (H_2^-)^2$. We want to prove that $\ker T_G$ is contained in $\ker T_H$. Take any element $[IO/\bar{O}]\psi_- f \in K_\theta\, f$; we have $H[IO/\bar{O}]\psi_- f = (\psi_-/\bar{O})H(IOf) = (\psi_-/\bar{O})\varphi_-$. Since this is (componentwise) in the Smirnov class $\mathcal{N}_-$ and in L_2, it is in $(H_2^-)^2$. Therefore $[IO/\bar{O}]\psi_- f$ is in $\ker T_H$.

□

Note that, as remarked by R. O'Loughlin, not all block Toeplitz kernels have a maximal function. This is an immediate consequence of Theorem 5.5 and Corollary 5.3 in [7].

4 Applications to truncated Toeplitz operators

Let $h \in \mathcal{F}$ and, for any inner function θ, let

$$\mathcal{D}_\theta = \{f_\theta \in K_\theta : hf_\theta \in L_2\}. \tag{4.1}$$

The operator $A_h^\theta : \mathcal{D}_\theta \to K_\theta$ defined by

$$A_h^\theta f_\theta = P_\theta(hf_\theta), \quad f_\theta \in \mathcal{D}_\theta, \tag{4.2}$$

where $P_\theta : L_2 \to K_\theta$ denotes the orthogonal projection, is called the *truncated Toeplitz operator* (in K_θ) with symbol h. If h belongs to the Sobolev space $\lambda_+ L_2$, where $\lambda_+(\xi) = \xi + i$, then A_h^θ is densely defined on K_θ; if $h \in L_\infty$, then A_h^θ is a bounded operator on K_θ.

It is clear that $\varphi_{1+} \in \ker A_h^\theta$ if and only if $\varphi_{1+} \in H_2^+$ and the following two conditions hold:

$$\begin{cases} \bar{\theta}\varphi_{1+} &= \varphi_{1-}, \\ h\varphi_1+ &= \varphi_{2-} - \theta\varphi_{2+}, \quad \text{with} \quad \varphi_{1-}, \varphi_{2-} \in H_2^-, \quad \varphi_{2+} \in H_2^+. \end{cases} \tag{4.3}$$

Therefore, $\ker A_h^\theta$ consists of the first components of the elements in the kernel of the Toeplitz operator T_G with

$$G = \begin{bmatrix} \overline{\theta} & 0 \\ h & \theta \end{bmatrix}, \tag{4.4}$$

defined on $\mathcal{D} = \{\Phi_+ \in (H_2^+)^2 : G\Phi_+ \in (L_2)^2\}$. In particular, we have that $\ker A_h^\theta = \{0\}$ if and only if $\ker T_G = \{0\}$. Thus we can apply the results of Section 3, for G of the form (4.4), to study the kernels of truncated Toeplitz operators. Note that we have $Gf = g$ with $f = (f_1, f_2)$ and $g = (g_1, g_2)$ if and only if $g_1 = \overline{\theta} f_1$ and $h = \dfrac{g_2 - \theta f_2}{f_1}$.

The following is a consequence of Theorem 3.16 and Corollary 3.9.

Theorem 4.1. *The kernel of any truncated Toeplitz operator is the product of a nearly S^*-invariant subspace of H_2^+, given in* (3.40) *and* (3.41)*, by an inner function.*

Proof. Let h be the symbol of the truncated Toeplitz operator A_h^θ, and let G be defined by (4.4). By Corollary 3.9, $\ker T_G$ is of scalar type and, if $\ker T_G \neq \{0\}$, then by Theorem 3.16 we have $\ker T_G = \tilde{\mathcal{K}} \dfrac{F_+}{\gamma_+ O_{1+}}$, where $F_+ = (F_{1+}, F_{2+})$ is a given function in $(H_2^+)^2$ with $F_{1+} = I_{1+} O_{1+} \in K_\theta$, $F_{1+} \neq 0$, $\gamma_+ \preceq I_{1+}$, and $\tilde{\mathcal{K}}$ is given by (3.40) and (3.41). Therefore $\ker T_G = \tilde{\mathcal{K}} \left(\dfrac{I_{1+}}{\gamma_+}, \dfrac{F_{2+}}{\gamma_+ O_{1+}} \right)$, and $\ker A_h^\theta = \dfrac{I_{1+}}{\gamma_+} \tilde{\mathcal{K}}$. □

Remark 4.2. Recently, Ryan O'Loughlin [26] has arrived at a similar result by a different route. Namely, it follows from [12, Cor. 4.5] that the kernel of a 2×2 block Toeplitz operator T_G can be written as $F_0((H_2^+)^r \ominus \Theta (H_2^+)^{r'})$, where r and r' are integers with $1 \leq r' \leq r \leq 2$, $\Theta \in (H_\infty^+)^{r \times r'}$ is inner, and $F_0 \in (H_2^+)^{2 \times r}$; in the case $G = \begin{bmatrix} \overline{\theta} & 0 \\ g & \theta \end{bmatrix}$, it is possible to take $r' = r = 1$, although no explicit formula for Θ is given.
Note that if $\mathcal{K}_1 \varphi_+ = \mathcal{K}_2 \psi +$, where $\mathcal{K}_1$ and K_2 are model spaces and $\varphi_+ = (\varphi_{1+}, \varphi_{2+})$, $\psi_+ = (\psi_{1+}, \psi_{2+})$ are analytic in $\mathbb{C}^+$, then we have multipliers ψ_{+1}/φ_{1+} and ψ_{+2}/φ_{2+} from one model space onto another, so that the work in [15], [17] and [10] can be applied. Indeed, if $wK_\alpha = K_\beta$, then $\beta = c\alpha w/\overline{w}$, where c is a unimodular constant.

Regarding the formula (3.40) for $\tilde{\mathcal{K}}$ in Theorem 3.16, note that, for G of the form (4.4), if $\ker T_G \neq \{0\}$ then $F_{1\pm} \neq 0$ and we have $\overline{\theta} F_{1+} = F_{1-}$

if and only if $\frac{O_{1-}}{O_{1+}} = \overline{\theta} I_{1+} \overline{I_{1-}}$. Therefore the symbol $\gamma_- \overline{\gamma_+} \dfrac{O_{1-}}{O_{1+}}$ in (3.40) is bounded. Moreover, from (3.40) we have:

$$\tilde{\mathcal{K}} = \left\{ \tilde{\psi}_+ \in \ker T_{\overline{\theta}(I_{1+}/\gamma_+)\overline{(I_{1-}/\gamma_-)}} \cap \ker T_{\gamma_- \overline{\gamma_+} \frac{O_{2-}}{O_{1+}}} \; : \; \frac{O_{2+}}{O_{1+}} \tilde{\psi}_+ \in H_2^+ \right\},$$

which takes the form

$$\tilde{\mathcal{K}} = \ker T_{\overline{\theta}\overline{(I_{1-}/\gamma_-)}} \cap \ker T_{\gamma_- \frac{O_{2-}}{F_{1+}}} \quad \text{if} \quad F_{2+} = 0, F_{2-} \neq 0, \tag{4.5}$$

and

$$\tilde{\mathcal{K}} = \ker T_{\overline{\theta}} = K_\theta \quad \text{if} \quad F_{2+} = F_{2-} = 0.$$

It may happen that the inner function mentioned in Theorem 4.1 is necessarily a constant, as it happens if the symbol h of the truncated Toeplitz operator is in H_∞^-. In that case, for $\gamma = \gcd(\theta, (\bar{h})_i)$, it is easy to see that $F_+ = (F_{1+}, F_{2+}) = (\frac{\gamma - \gamma(i)}{\xi - i}, 0) \in \ker T_G$, with G given by (4.4), and $GF_+ = (F_{1-}, F_{2-}) = \frac{1 - \bar{\gamma}\gamma(i)}{\xi - i}(\bar{\theta}\gamma, h\gamma)$. With the notation of Theorem 3.16, we have $\gamma_- = 1$, $I_{1-} = \bar{\theta}\gamma$, $O_{2-} = \frac{1 - \bar{\gamma}\gamma(i)}{\xi - i}$, and it follows from (4.5) that $\ker A_h^\theta = \ker T_{\bar{\theta}} = K_\gamma$ (as may also be verified by direct calculation).

We now apply the previous results to studying the kernels of some classes of TTO. Our motivation for the examples that we shall consider is the following. The kernels of TTO with so-called θ-separated symbols, of the form

$$h = \overline{\alpha} h_1 + \beta h_2, \qquad (h_1 \in H_\infty^-, \quad h_2 \in H_\infty^+), \tag{4.6}$$

where

$$\alpha\beta \succeq \theta, \tag{4.7}$$

were studied in [8]. We consider here two cases where $h_1 \in H_\infty^+$ and $h_2 \in H_\infty^-$. In the first case we assume that h_1 and $\overline{h_2}$ are inner, with $h_1 \prec \alpha$ and $\overline{h_2} \prec \beta$. In the second case we assume that h_1 and h_2 are rational functions, $h_1 = R_1^+ \in \mathcal{R}^+$, and $h_2 = R_2^- \in \mathcal{R}^-$ with $\mathcal{R}^\pm = \mathcal{R} \cap H_\infty^\pm$, where $\mathcal{R}$ is the set of rational functions, which generalizes the study of truncated Toeplitz operators with θ-separated symbols to the case where h_1 and h_2 admit poles in the lower and upper half planes, respectively.

4.1 The first case: A_h^θ with $h = C_1\overline{\alpha} + C_2\beta$,

Toeplitz operators with almost-periodic symbols of the form (4.4) where $\theta(\xi) = e^{i\lambda\xi}$, with $\lambda \in \mathbb{R}$, and

$$h(\xi) = C_1 \exp(-ia\xi) + C_2 \exp(ib\xi), \qquad \text{with} \quad a, b \in \mathbb{R}^+ \quad \text{and} \quad a, b < \lambda,$$

have been studied by several authors (see, e.g., [5, 16]). In this section we generalize this class by studying symbols of the form (4.4) with

$$h = C_1\overline{\alpha} + C_2\beta, \qquad (C_1, C_2 \in \mathbb{C} \setminus \{0\}),$$

where α, β are non-constant inner functions satisfying the following conditions:

$$\alpha, \beta \prec \theta, \tag{4.8}$$

$$\text{for some} \quad n \geq 1, \qquad (\alpha\beta)^{n-1} \prec \theta, \qquad (\alpha\beta)^n \succeq \theta, \tag{4.9}$$

$$\text{either} \quad \alpha^n\beta^{n-1} \preceq \theta \qquad \text{or} \quad \alpha^n\beta^{n-1} \succeq \theta, \tag{4.10}$$

$$\text{either} \quad \alpha^{n-1}\beta^n \preceq \theta \qquad \text{or} \quad \alpha^{n-1}\beta^n \succeq \theta. \tag{4.11}$$

If γ, δ are inner functions such that either $\gamma \preceq \delta$ or $\delta \preceq \gamma$, we say that $\min\{\gamma, \delta\} = \gamma$ if $\gamma \preceq \delta$ and $\min\{\gamma, \delta\} = \delta$ if $\delta \preceq \gamma$. With this notation, if (4.10) holds then either $\alpha^n \preceq \theta\overline{\beta}^{n-1}$ or $\theta\overline{\beta}^{n-1} \preceq \alpha^n$, so in the first case $\min\{\theta\overline{\beta}^{n-1}, \alpha^n\} = \alpha^n$, and in the second case $\min\{\theta\overline{\beta}^{n-1}, \alpha^n\} = \theta\overline{\beta}^{n-1}$.

Analogously, if (4.11) holds, then either $\alpha^{n-1}\beta^n\overline{\theta} \preceq \alpha$ or $\alpha \preceq \alpha^{n-1}\beta^n\overline{\theta}$, and $\min\{\alpha^n\beta^n\overline{\theta}, \alpha\}$, $\min\{\beta, \theta\overline{\alpha}^{n-1}\overline{\beta}^{n-1}\}$ also exist.

Note that, if ϵ is a singular inner function (for instance, an exponential $\exp(i\lambda\xi)$ for a given positive real λ) and λ, a, b are real positive numbers with $a, b < \lambda$, then $\theta = \epsilon^\lambda$, $\alpha = \epsilon^a$, $\beta = \epsilon^b$ always satisfy (4.8)-(4.11) if we take n to be the smallest integer such that $\frac{\lambda}{a+b} \leq n$.

More ambitiously, we may take θ_1, θ_2 as coprime singular inner functions and consider $\alpha = \theta_1^a\theta_2^b$, $\beta = \theta_1^c\theta_2^d$. There is then a set of inequalities that a, b, c, d must satisfy, namely, all lie in $[0, 1)$, $(n-1)(a+c) \leq 1$, $(n-1)(b+d) \leq 1$ with at least one inequality strict, $n(a+c) \geq 1$, $n(b+d) \geq 1$, and similar inequalities for (4.10) and (4.11).

The solution to $Gf = g$ obtained in Proposition 4.3 below is analogous to the one obtained in [16] for a particular class of symbols G with $\theta = \exp(i\xi)$.

Proposition 4.3. *(i) A solution to $Gf = g$ with $f = (f_1^+, f_2^+)^T \in (H_\infty^+)^2$ and $g = (g_1^-, g_2^-) \in (H_\infty^-)^2$ is given by*

$$f_1^+ = \mu(C_1^{n-1}\overline{\alpha}^{n-1} - C_1^{n-2}C_2\overline{\alpha}^{n-2}\beta + \ldots + (-1)^{n-1}C_2^{n-1}\beta^{n-1}), \tag{4.12}$$

$$f_2^+ = (-1)^n C_2^n \mu\beta^n\overline{\theta}, \tag{4.13}$$

$$g_1^- = \overline{\theta} f_1^+, \tag{4.14}$$

$$g_2^- = C_1^n \overline{\alpha}^n \mu, \tag{4.15}$$

where

$$\mu = \min\{\theta\overline{\beta}^{n-1}, \alpha\}. \tag{4.16}$$

(ii) If $\mu = \alpha^n$, then $f = \lambda_1 F_+$, where

$$\lambda_1 = \min\{\alpha^n\beta^n\overline{\theta}, \alpha\}, \tag{4.17}$$

$F_+ = (F_1^+, F_2^+)$ is a corona pair in $(H_\infty^+)^2$ (written $F_+ \in \mathrm{CP}^+$, see(3.31)*), and g is a corona pair in $(H_\infty^-)^2$ (written $g \in \mathrm{CP}^-$).*

(iii) If $\mu = \theta\,\overline{\beta}^{n-1}$, then $f = \lambda_2 F_+$, where

$$F_+ \in \mathrm{CP}^+, \quad \lambda_2 = \min\{\beta, \theta\overline{\alpha}^{n-1}\overline{\beta}^{n-1}\}, \tag{4.18}$$

and $g \in \mathrm{CP}^-$.

Proof. (i) It is easy to see that $Gf = g$ and $f_2^+ \in H_\infty^+$, $g_2^- \in H_\infty^-$. It remains to prove that

$$f_1^+ \in H_\infty^+, \qquad \overline{\theta} f_1^+ \in H_\infty^-,$$

for which it is enough to see that $\mu\overline{\alpha}^{n-1} \in H_\infty^+$ (from (4.16) and (4.9)) and $\overline{\theta}\mu\beta^{n-1} \in H_\infty^-$ (from (4.16) and the fact that, if $\mu = \alpha^n$ then $\alpha^n \preceq \theta\overline{\beta}^{n-1}$).

(ii) If $\mu = \alpha^n$ then

$$\alpha^n\beta^{n-1} \preceq \theta \tag{4.19}$$

and

$$f_1^+ = \alpha(C_1^{n-1} - C_1^{n-2}C_2(\alpha\beta) + \ldots + (-1)^{n-1}C_2^{n-1}(\alpha\beta)^{n-1}), \tag{4.20}$$

$$f_2^+ = (-1)^n C_2^n \alpha^n\beta^n\overline{\theta}, \tag{4.21}$$

$$g_1^- = \overline{\theta} f_1^+, \tag{4.22}$$

$$g_2^- = C_1^n. \tag{4.23}$$

Clearly $g = (g_1^-, g_2^-) \in \mathrm{CP}^-$. On the other hand,

$$f = \lambda_1(F_1^+, F_2^+), \tag{4.24}$$

where λ_1 is defined by (4.17) and $F_1^+, F_2^+ \in H_\infty^+$. If $\lambda_1 = \alpha^n\beta^n\overline{\theta}$, then it is clear that $(F_1^+, F_2^+) \in \mathrm{CP}^+$ because $F_2^+ = (-1)^n C_2^n$; if $\lambda_1 = \alpha$, then

$$\begin{aligned} F_1^+ &= C_1^{n-1} - C_1^{n-2}C_2(\alpha\beta) + \ldots + (-1)^{n-1}C_2^{n-1}(\alpha\beta)^{n-1}, &(4.25)\\ F_2^+ &= (-1)^n(\alpha^{n-1}\beta^n\overline{\theta}) = (-1)^n C_2^n h^+ &(4.26)\end{aligned}$$

with $h^+ \in H_\infty^+$ because in this case $\alpha \preceq \alpha^n\beta^n\overline{\theta}$. We can write

$$\alpha\beta = (\alpha^{n-1}\beta^n\overline{\theta})(\theta\overline{\alpha}^n\overline{\beta}^{n-1})\alpha^2 = h^+(\underbrace{\theta\overline{\alpha}^n\overline{\beta}^{n-1}}_{\in H_\infty^+ \text{ by } (4.19)})\underbrace{\alpha^2}_{\in H_\infty^+},$$

and comparing the expressions (4.25) and (4.26) for F_1^+ and F_2^+ respectively, we see that $(F_1^+, F_2^+) \in \mathrm{CP}^+$.

(iii) If $\mu = \theta\overline{\beta}^{n-1}$, then

$$\theta \preceq \alpha^n\beta^{n-1} \tag{4.27}$$

and

$$\begin{aligned} f_1^+ &= \theta(C_1^{n-1}(\overline{\alpha\beta})^{n-1} - C_1^{n-2}C_2(\overline{\alpha\beta})^{n-2} + \ldots + (-1)^{n-1}C_2^{n-1}), &(4.28)\\ f_2^+ &= (-1)^n C_2^n\beta, &(4.29)\\ g_1^- &= C_1^n(\overline{\alpha\beta})^{n-1} - C_1^{n-2}C_2(\overline{\alpha\beta})^{n-2} + \ldots + (-1)^{n-1}C_2^{n-1}, &(4.30)\\ g_2^- &= C_1^n\overline{\alpha}^n\overline{\beta}^{n-1}\theta. &(4.31)\end{aligned}$$

We have $(g_1^-, g_2^-) \in \mathrm{CP}^-$, using (4.27) as above. On the other hand,

$$(f_1^+, f_2^+) = \lambda_2(F_1^+, F_2^+) \qquad \text{with} \quad F_1^+, F_2^+ \in H_\infty^+,$$

where λ_2 is defined in (4.18). If $\lambda_2 = \beta$, then it is clear that $(F_1^+, F_2^+) \in \mathrm{CP}^+$ since $F_2^+ = (-1)^n C_2^n$; if $\lambda_2 = \theta\overline{\alpha}^{n-1}\overline{\beta}^{n-1}$, then $\theta\bar{\alpha}^{n-1}\bar{\beta}^{n-1} \preceq \beta$, which implies that $\bar{\theta}\alpha^{n-1}\beta^n \in H_\infty^+$ and

$$\begin{aligned} F_1^+ &= C_1^{n-1} - C_1^{n-2}C_2(\alpha\beta) + \ldots + (-1)^{n-1}C_2^{n-1}(\alpha\beta)^{n-1},\\ F_2^+ &= (-1)^n C_2^n\,\overline{\theta}\alpha^{n-1}\beta^n.\end{aligned}$$

Using the relation $\alpha\beta = (\overline{\theta}\alpha^{n-1}\beta^n)(\theta\overline{\alpha}^{n-1}\overline{\beta}^{n-1})\alpha$, where $\theta\overline{\alpha}^{n-1}\overline{\beta}^{n-1} \in H_\infty^+$ by (4.9), we see as above that $(F_1^+, F_2^+) \in \mathrm{CP}^+$. □

As a consequence of Theorem 3.13 and Proposition 4.3, we have then:

Theorem 4.4. *If*

$$G = \begin{bmatrix} \overline{\theta} & 0 \\ C_1\overline{\alpha} + C_2\beta & \theta \end{bmatrix},$$

where $C_1, C_2 \in \mathbb{C} \setminus \{0\}$ *and* α, β *satisfy* (4.8)–(4.11), *then*

$$\ker T_G = \bar{\lambda} K_\lambda f,$$

where $f = (f_1^+, f_2^+)$ *is defined by* (4.12)–(4.13) *and* (4.16), *and*

$$\lambda = \begin{cases} \min\{\alpha^n \beta^n \overline{\theta}, \alpha\}, & \text{if } \alpha^n \beta^{n-1} \preceq \theta, \\ \min\{\beta, \theta \overline{\alpha}^{n-1} \overline{\beta}^{n-1}\}, & \text{if } \alpha^n \beta^{n-1} \succeq \theta. \end{cases}$$

The cases $C_1 = 0$ and $C_2 = 0$ are rather easier, and we omit them.

Corollary 4.5. *With the same assumptions as in Theorem 4.4*

$$\ker A^\theta_{C_1\bar{\alpha}+C_2\beta} = \bar{\lambda} K_\lambda f_1^+.$$

Corollary 4.6. *With the same assumptions as in Theorem 4.4,* T_G *(respectively,* A_h^θ*) is injective if and only if* λ *is a constant and, in that case,* T_G *(respectively,* A_h^θ*) is invertible.*

Proof. The injectivity is a direct consequence of Corollary 4.5. On the other hand, the operator A_h^θ is equivalent after extension to T_G [3, 9]; therefore both operators are simultaneously invertible or not. In this case $\det G = 1$ and $Gf_+ = f_-$ has a solution $f_\pm \in (H_\infty^\pm)^2$ with $f^\pm \in \mathrm{CP}^\pm$ and therefore the operator T_G is invertible [6]. □

Example 4.7. Take $\theta(\xi) = e^{i\xi}$, $\alpha(\xi) = e^{ia\xi}$, $\beta(\xi) = e^{ib\xi}$, $(0 < a, b < 1)$ and write $h = C_1\overline{\alpha} + C_2\beta \quad (C_1, C_2 \in \mathbb{C} \setminus \{0\})$. We also write $\mathcal{K}_\lambda = K_{e_\lambda}$ for $\lambda > 0$, where $e_\lambda(\xi) = e^{i\lambda\xi}$.

Depending on α and β there are various possibilities for $\ker A_h^\theta$, some of which we indicate in Figure 1, where we have:

A: $\mathcal{K}_{a+b-1} e^{i(1-b)\xi}$.
B: $\mathcal{K}_a (C_1 - C_2 e^{i(a+b)\xi})$.
C: $\mathcal{K}_{1-a-b} (C_1 - C_2 e^{i(a+b)\xi})$.
D: $\mathcal{K}_b (C_1 e^{i(1-2b-a)\xi} - C_2 e^{i(1-b)\xi})$.
E: $\mathcal{K}_{2a+2b-1} (C_1 e^{i(1-a)\xi} - C_2 e^{i(1-b)\xi})$.

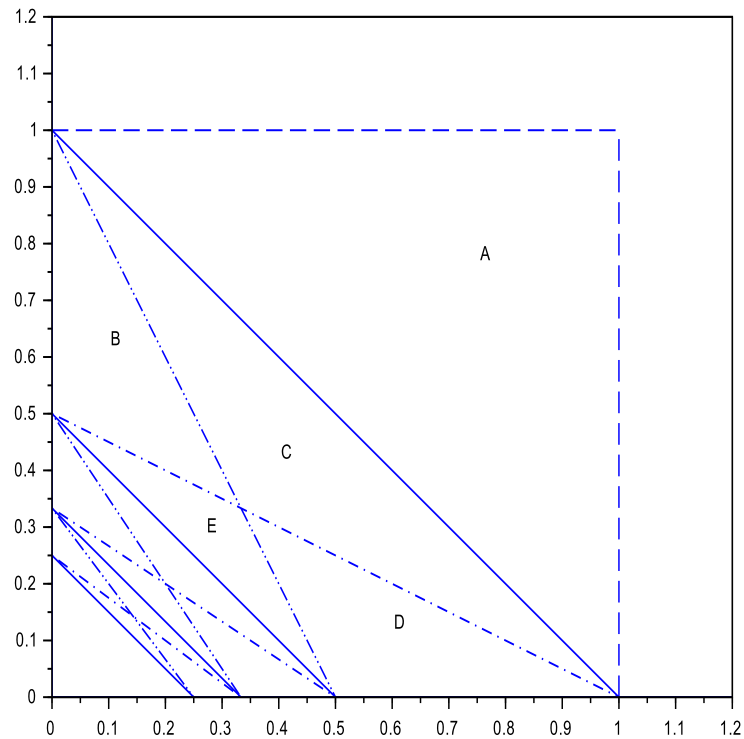

Figure 1: Dependence of $\ker A_h^\theta$ on α and β

Figure 1 provides a better understanding of the dependence of $\ker A_h^\theta$ on the parameters a and b. For instance, one can see that on the lines $a + b = 1/n$ the operator is invertible. On the other hand it can easily be verified that the expressions for $\ker A_h^\theta$ "on the left hand side" and "on the right hand side" of the dotted lines coincide, thus making apparent the continuous dependency of the kernel on the parameters α and β across those lines.

4.2 The second case: A_h^θ with $h = \overline{\alpha}R_1^+ + \beta R_2^-$

Let $h = \overline{\alpha}R_1^+ + \beta R_2^-$, with

$$\alpha, \beta \preceq \theta, \qquad \alpha\beta \succ \theta, \qquad R_1^+ \in \mathcal{R}^+, \qquad R_2^- \in \mathcal{R}^-, \tag{4.32}$$

where $\mathcal{R}^\pm = \mathcal{R} \cap H_\infty^\pm$. We shall exclude the degenerate cases $R_1^+ = 0$ and $R_2^- = 0$.

If $\alpha = \beta = \theta$, then the operators A_h^θ are the so-called finite-rank truncated Toeplitz operators of Type I [20, 4].

We assume here that θ is an inner function that is not a finite Blaschke product (written $\theta \notin$ FBP), otherwise the matrix G would be rational, and also that $\alpha, \beta, \alpha\beta/\theta \notin$ FBP. Note that if $\alpha, \beta \in$ FBP, then $h \in \mathcal{R}$; this case was studied in [8].

We have:

$$G = \begin{bmatrix} \overline{\theta} & 0 \\ \overline{\alpha}R_1^+ + \beta R_2^- & \theta \end{bmatrix}, \tag{4.33}$$

and $Gf = g$ holds with

$$f = \begin{bmatrix} \alpha \\ -\frac{\alpha\beta}{\theta}R_2^- \end{bmatrix}, \qquad g = \begin{bmatrix} \overline{\theta}\alpha \\ R_1^+ \end{bmatrix}. \tag{4.34}$$

Let

$$R_1^+ = \frac{N_1}{D_1^+}, \tag{4.35}$$

where N_1 and D_1^+ are polynomials without common zeroes, with $\deg N_1 \leq \deg D_1^+ = n_1$, such that all zeroes of D_1^+ are in $\mathbb{C}^-$, and

$$F_2^- = \frac{N_2}{D_2^-}, \tag{4.36}$$

where N_2 and D_2^- are polynomials without common zeroes with $\deg N_2 \leq \deg D_2^- = n_2$, such that all zeroes of D_2^- are in $\mathbb{C}^+$. Condition (3.20) is satisfied in this case, by Lemma 4.11 below. In fact we have

$$\begin{aligned}
& \begin{bmatrix} \alpha & -\frac{\alpha\beta}{\theta}R_2^- \end{bmatrix} \begin{bmatrix} \varphi_{1+} \\ \varphi_{2+} \end{bmatrix} = [\overline{\theta}\alpha \quad R_1^+] \begin{bmatrix} \varphi_{1-} \\ \varphi_{2-} \end{bmatrix} \\
\iff \quad & \alpha\varphi_{1+} - \frac{\alpha\beta}{\theta}R_2^-\varphi_{2+} = \overline{\theta}\alpha\varphi_{1-} + R_1^+\varphi_{2-} \\
\iff \quad & \underbrace{\frac{\alpha\beta}{\theta}}_{\notin \text{FBP}} \underbrace{D_1^+\overline{D_2^-}}_{p_1} \underbrace{\left(\frac{\theta}{\beta}\frac{D_2^-}{\overline{D_2^-}}\varphi_{1+} - \frac{N_2}{\overline{D_2^-}}\varphi_{2+}\right)}_{\in H_2^+} = \underbrace{D_2^-\overline{D_1^+}}_{p_2} \underbrace{\left(\frac{\overline{\theta}\alpha D_1^+}{\overline{D_1^+}}\varphi_{1-} + \frac{N_1}{\overline{D_1^+}}\varphi_{2-}\right)}_{\in H_2^-},
\end{aligned}$$

so both sides of the equation must be equal to zero.

We have the following left inverses for f and g:

$$\tilde{f} = (\overline{\alpha}, 0), \qquad \tilde{g} = (\theta\overline{\alpha}, \quad 0), \tag{4.37}$$

so $\mathcal{H}_\pm$ are defined by

$$\begin{aligned} \mathcal{H}_+ &= \{(\psi_{1+}, \psi_{2+}) \in (H_2^+)^2 : R_2^- \beta\overline{\theta}\psi_{1+} \in H_2^+\}, \\ \mathcal{H}_- &= \{(\psi_{1-}, \psi_{2-}) \in (H_2^-)^2 : \theta\overline{\alpha}R_1^+\psi_{1-} \in H_2^-\}. \end{aligned} \tag{4.38}$$

From (4.38), we have that, for $\varphi_+ \in H_2^+$:

$$R_2^-\overline{\theta}\beta\psi_{1+} = \varphi_+ \iff \frac{N_2}{D_2^-}\psi_{1+} = \frac{\theta}{\beta}\varphi_+ \iff \frac{N_2}{\overline{\overline{D_2^-}}}\psi_{1+} = \frac{\theta}{\beta}\frac{D_2^-}{\overline{\overline{D_2^-}}}\varphi_+. \tag{4.39}$$

Now let $\left(\frac{N_2}{\overline{\overline{D_2^-}}}\right)_i$ denote the inner factor in an inner–outer factorization of $\frac{N_2}{\overline{\overline{D_2^-}}}$. Since $\frac{\theta}{\beta}\frac{D_2^-}{\overline{\overline{D_2^-}}}$ is inner and there are no common zeroes for N_2 and D_2^- we have

$$\gamma_2 := \gcd\left\{\left(\frac{N_2}{\overline{\overline{D_2^-}}}\right)_i, \frac{\theta}{\beta}\frac{D_2^-}{\overline{\overline{D_2^-}}}\right\} = \gcd\left\{\left(\frac{N_2}{\overline{\overline{D_2^-}}}\right)_i, \frac{\theta}{\beta}\right\}, \tag{4.40}$$

and it follows from (4.38) and (4.39) that $(\psi_{1+}, \psi_{2+}) \in \mathcal{H}_+$ if and only if $\psi_{1+}, \psi_{2+} \in H_2^+$ and

$$\psi_{1+} \in \frac{\theta}{\beta\gamma_2}\frac{D_2^-}{\overline{\overline{D_2^-}}}H_2^+. \tag{4.41}$$

Analogously, defining

$$\gamma_1 = \gcd\left\{\left(\frac{\overline{N_1}}{D_1^+}\right)_i, \frac{\theta}{\alpha}\frac{\overline{D_1^+}}{D_1^+}\right\} = \gcd\left\{\left(\frac{\overline{N_1}}{D_1^+}\right)_i, \frac{\theta}{\alpha}\right\}, \tag{4.42}$$

we have that $(\psi_{1-}, \psi_{2-}) \in \mathcal{H}_-$ if and only if

$$\psi_{1-}, \psi_{2-} \in H_2^- \quad \text{and} \quad \psi_{1-} \in \frac{\alpha}{\theta}\gamma_1\frac{D_1^+}{\overline{\overline{D_1^+}}}H_2^-. \tag{4.43}$$

Therefore, from (3.24), $\mathcal{K}$ is defined by the equation

$$[\overline{\alpha} \quad 0]\begin{bmatrix} \frac{\theta}{\beta\gamma_2}\frac{D_2^-}{\overline{\overline{D_2^-}}}\varphi_+ \\ \psi_{2+} \end{bmatrix} = [\theta\overline{\alpha} \quad 0]\begin{bmatrix} \frac{\alpha\gamma_1}{\theta}\frac{D_1^+}{\overline{\overline{D_1^+}}}\varphi_- \\ \psi_{2-} \end{bmatrix} \tag{4.44}$$

with $\varphi_\pm, \psi_{2\pm} \in H_2^\pm$, i.e.,

$$\mathcal{K} = \left(\frac{\theta}{\alpha\beta} \overline{\gamma_2} \frac{D_2^-}{\overline{D_2^-}} H_2^+ \right) \cap \left(\gamma_1 \frac{D_1^+}{\overline{D_1^+}} H_2^- \right). \tag{4.45}$$

Consequently,

$$\begin{aligned} \ker T_G &= \mathcal{K} f = \mathcal{K} \begin{bmatrix} \alpha \\ -\frac{\alpha\beta}{\theta} R_2^- \end{bmatrix} = K \frac{\alpha\beta}{\theta} \begin{bmatrix} \theta/\beta \\ -R_2^- \end{bmatrix} \\ &= \underbrace{H_2^+ \cap \left(\gamma_1\gamma_2 \frac{D_{1+}}{\overline{D_1^+}} \frac{\overline{D_2^-}}{D_2^-} \frac{\alpha\beta}{\theta} H_2^- \right)}_{\ker T_{\eta^{-1}}} \qquad \underbrace{\begin{bmatrix} \overline{\gamma_2} \frac{\theta}{\beta} \frac{D_2^-}{\overline{D_2^-}} \\ -\overline{\gamma_2} \frac{N_2}{\overline{D_2^-}} \end{bmatrix}}_{\in (H^2_\infty)^2 \text{ by (4.40) and (4.42)}} \end{aligned}$$

with

$$\eta = \gamma_1\gamma_2 \frac{D_{1+}}{\overline{D_1^+}} \frac{\overline{D_2^-}}{D_2^-} \frac{\alpha\beta}{\theta}. \tag{4.46}$$

We have thus shown:

Theorem 4.8. *If $g = \overline{\alpha} R_1^+ + \beta R_2^-$, where α, β are inner functions and* (4.32) *is satisfied, then, for G defined by* (4.33)*, we have*

$$\ker T_G = \ker T_{\eta^{-1}} \begin{bmatrix} \overline{\gamma_2} \frac{\theta}{\beta} \frac{D_2^-}{\overline{D_2^-}} \\ -\overline{\gamma_2} \frac{N_2}{\overline{D_2^-}} \end{bmatrix} \quad \text{and} \quad \ker A_g^\theta = \overline{\gamma_2} \frac{\theta}{\beta} \frac{D_2^-}{\overline{D_2^-}} \ker T_{\eta^{-1}}, \tag{4.47}$$

with the notation above and η given by (4.46)*.*

Remark. Note that $\overline{\gamma_2} \frac{\theta}{\beta} \frac{D_2^-}{\overline{D_2^-}}$ is an inner function. Also note that if

$$\left(\overline{\gamma}_1 \frac{\theta}{\alpha} \frac{\overline{D_1^+}}{D_1^+} \right) \left(\overline{\gamma}_2 \frac{\theta}{\beta} \frac{D_2^-}{\overline{D_2^-}} \right) \preceq \theta$$

then $\ker T_{\eta^{-1}}$ is a model space, and in that case we have $\ker T_G = \{0\}$ (and $\ker A_g^\theta = \{0\}$) if and only if η is a constant.

Some auxiliary results

Lemma 4.9. *Let α be an inner function, $\alpha \notin$ FBP, and let p_1, p_2 be polynomials. Then $\alpha p_1 H_2^+ \cap p_2 H_2^- = \{0\}$.*

Proof. If there are $\varphi_\pm \in H_2^\pm$ such that $\alpha p_1 \varphi_+ = p_2 \varphi_-$, then both sides of this equation must be equal to a polynomial p, by an easy generalization of Liouville's Theorem, and we have $\alpha\varphi_+ = \frac{p}{p_1} \in H_2^+$, i.e., $\overline{\alpha}\frac{p}{p_1} = \varphi_+ \in H_2^+$. Therefore, $\overline{\alpha}p/p_1 = P^+\left(\overline{\alpha}p/p_1\right)$ must be rational, which is impossible since $\alpha \notin \mathrm{FBP}$. □

Corollary 4.10. *$\alpha\mathcal{H}_2^+ \cap \mathcal{H}_2^- = \{0\}$, where $\mathcal{H}_2^\pm = (\xi \pm i)H_2^\pm$.*

As a consequence of Lemma 4.9 we can obtain the following generalization:

Lemma 4.11. *Let α be an inner function, $\alpha \notin \mathrm{FBP}$, and let $R_1^+ = \dfrac{N_1}{D_1^+}$, $R_2^- = \dfrac{N_2}{D_2^-}$, where N_1, D_1^+, N_2, D_2^- are polynomials such that N_1 and D_1^+ (and similarly N_2 and D_2^-) have no common zeroes, all zeroes of D_1^+ (respectively, D_2^-) are in $\mathbb{C}^-$ (respectively, $\mathbb{C}^+$) and $\deg N_1 \leq \deg D_1^+ = n_1$ and $\deg N_2 \leq \deg D_2^- = n_2$. Then*

$$\alpha R_2^- H_2^+ \cap R_1^+ H_2^- = \{0\}. \tag{4.48}$$

Proof. If $\varphi_\pm \in H_2^\pm$ and

$$\alpha \frac{N_2}{D_2^-}\varphi_+ = \frac{N_1}{D_1^+}\varphi_-,$$

then

$$\alpha \underbrace{D_1^+ \overline{D_2^-}}_{p_1} \underbrace{\left(\frac{N_2}{\overline{\overline{D_2^-}}}\varphi_+\right)}_{\in H_2^+} = \underbrace{D_2^- \overline{D_1^+}}_{p_2} \underbrace{\left(\frac{N_1}{\overline{\overline{D_1^+}}}\varphi_-\right)}_{\in H_2^-} = 0$$

by Lemma 4.9, and (4.48) follows. □

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