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The stability of parallel-propagating circularly polarized Alfvén waves revisited

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Abstract. The parametric instability of parallel-propagating circularly polarized Alfvén waves (pump waves) is revisited. The stability of these waves is determined by the linearized system of magnetohydrodynamic equations with periodic coefficients. The variable substitution that reduces this system of equations to a system with constant coefficients is suggested. The system with constant coefficients is used to derive the dispersion equation that was previously derived by many authors with the use of different approaches. The dependences of general stability properties on the dimensionless amplitude of the pump wave a and the ratio of the sound and Alfvén speed b are studied analytically. It is shown that, for any a and b, there are such quantities k_1 and k_2 that a perturbation with the dimensionless wavenumber k is unstable if $k_1^2 < k^2 < k_2^2$, and stable otherwise. It is proved that, for any fixed b, k_2 is a monotonically growing function of a. The dependence of k_1 on a is different for different values of b. When $b^2 < 1/3$, k_1 is a monotonically decreasing function of a. When $1/3 < b^2 < 1$, k_1 monotonically decreases when a varies from zero to $a_c(b)$, takes its minimum value at $a = a_c$ and then monotonically increases when a increases from a_c to infinity. When $b > 1$, k_1 is a monotonically increasing function of a. For any b, k_1 tends to a limiting value approximately equal to 1.18 as $a \to \infty$.

1. Introduction

The parametric instabilities of finite-amplitude circularly polarized Alfvén waves have been studied for the last four decades. They are interesting both from a purely theoretical point of view and from the point of view of applications to laboratory and space plasmas. The first analysis of the stability of a finite-amplitude circularly polarized Alfvén wave (the 'pump wave') carried out by Galeev and Oraevskii (1963) and Sagdeev and Galeev (1969) showed that this wave decays into a backward-propagating Alfvén wave and a forward-propagating sound wave. The analysis was based on the ideal magnetohydrodynamic (MHD) description and assumed that the plasma β and the pump wave amplitude are small. Derby (1978) and Goldstein (1978) also used the ideal MHD description to study the stability of circularly polarized Alfvén waves; however, with arbitrary amplitudes and in finite β plasmas. The stability analysis was further extended in different directions. Mio et al. (1976a, b), Mjølhus (1976), Ovenden et al. (1983), and Spangler and Sheerin (1982, 1983) used the derivative nonlinear Schrödinger equation (DNLS) to study the stability of small-amplitude circularly polarized Alfvén waves in a dispersive plasma. Sakai and Sonnerup (1983), Longtin and Sonnerup (1986) and Wong and Goldstein (1986) used the two fluid description to take dispersion into account. Viñas and Goldstein (1991) studied the stability of a circularly polarized Alfvén wave with respect to non-one-dimensional perturbations. Hollweg et al. (1993) investigated the stability of a circularly polarized Alfvén wave in a three-component plasma consisting of electrons, protons and He^{++} ions. Ling and Abraham-Shrauner (1979), Spangler (1989) and Inhester (1990) used the kinetic description to study the stability of circularly polarized Alfvén waves. An excellent comparison of theory and observations is given in a review paper by Spangler (1997).

In the traditional fluid treatment, the density perturbation is assumed to vary as $\exp[i(kz - \omega t)]$. With this ansatz for the density perturbation, the linearized MHD equations dictate how other quantities must vary. Jayanti and Hollweg (1993a) noticed that the linearized system of ideal MHD equations is a system with periodic coefficients and used Floquet's theorem to derive the dispersion equation determining the stability of a circularly polarized Alfvén wave in a Hall plasma. They obtained the same dispersion equation as Wong and Goldstein (1986) and Longtin and Sonnerup (1986). In the non-dispersive approximation corresponding to wavelengths much larger than the ion-inertia length, this dispersion equation coincides with the one obtained by Goldstein (1978) and Derby (1978). The derivation of the dispersion equation given by Jayanti and Hollweg (1993a) is fairly complicated. They first derived an infinite series of dispersion equations and then showed that, in fact, all of them are the same. In this paper we present a new, more transparent method of deriving the dispersion equation. This method is also based on Floquet's theorem and consists of making the variable substitution that reduces the system of differential equations with periodic coefficients to the system with constant coefficients. As a result, similar to Jayanti and Hollweg (1993a), we also derive the dispersion equation without any *ad hoc* assumptions about the variation of the density perturbation, but in a much simpler way.

The dispersion equation for parametric instabilities of circularly polarized Alfvén waves is a complicated algebraic equation relating the frequency and wavenumber of the density perturbation. This equation was mainly studied numerically (e.g. Wong and Goldstein (1986)). The only extensive analytical study was carried out by Jayanti and Hollweg (1993b), who used either the amplitude of the pump-wave or the plasma beta, or the difference between the plasma beta and unity as a small parameter. In this paper we present a qualitative analytical analysis of the dispersion equation which is valid for arbitrary values of the pump-wave amplitude. The paper is organized as follows. Section 2 contains the derivation of the dispersion equation using the variable substitution that reduces the linearized system of MHD equations with periodic coefficients to a system with constant coefficients. In Sec. 3 we carry out the qualitative analysis of the dispersion equation. In Sec. 4 we study the dependences of the boundaries of the interval of unstable wavenumbers on the wave amplitude. We give a summary and conclusions in Sec. 5.

2. Derivation of the dispersion equation

The starting point of our analysis is the system of linearized ideal MHD equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v} + \rho \mathbf{v}_0) = 0, \tag{2.1a}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}_0 \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}_0 = -\frac{1}{\rho_0}\nabla p + \frac{\rho}{\rho_0^2}\nabla p_0$$

$$+ \frac{1}{\mu\rho_0}\left[(\nabla \times \mathbf{B}) \times \mathbf{B}_0 + (\nabla \times \mathbf{B}_0) \times \mathbf{B} - \frac{\rho}{\rho_0}(\nabla \times \mathbf{B}_0) \times \mathbf{B}_0\right], \quad (2.1\text{b})$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}_0 + \mathbf{v}_0 \times \mathbf{B}), \quad (2.1\text{c})$$

$$p = c_\mathrm{s}^2 \rho. \quad (2.1\text{d})$$

Here ρ_0, p_0, $\mathbf{v}_0$ and $\mathbf{B}_0$ are the unperturbed density, pressure, velocity and magnetic field, and the same quantities without the subscript '0' indicate perturbations; μ is magnetic permeability of empty space, and c_s^2 is the square of the sound speed given by

$$c_\mathrm{s}^2 = \frac{\gamma p_0}{\rho_0}, \quad (2.2)$$

where γ is the ratio of specific heats.

In what follows we assume that all quantities depend on the x-variable in Cartesian coordinates x, y, z, and the unperturbed state is a parallel-propagating circularly polarized Alfvén wave given by

$$\begin{gathered} u_0 = 0, \quad \rho_0 = \text{constant}, \quad p_0 = \text{constant}, \quad B_{ox} = \text{constant}, \\ v_{0y} = V_0 \cos\phi, \quad v_{0z} = V_0 \sin\phi, \\ B_{0y} = A_0 \cos\phi, \quad B_{0z} = A_0 \sin\phi. \end{gathered} \quad (2.3)$$

Here $\phi = k_0 x - \omega_0 t$, and the quantities k_0, ω_0, A_0 and V_0 are related by

$$V_0 = -\frac{A_0 \omega_0}{B_{0x} k_0}, \quad \omega_0^2 = v_\mathrm{A}^2 k_0^2, \quad v_\mathrm{A}^2 = \frac{B_{0x}^2}{\mu \rho_0}. \quad (2.4)$$

Then the system of equations (2.1) is rewritten as

$$\frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial u}{\partial x} = 0, \quad (2.5\text{a})$$

$$\frac{\partial u}{\partial t} = -\frac{c_\mathrm{s}^2}{\rho_0}\frac{\partial \rho}{\partial x} - \frac{A_0}{\mu\rho_0}\frac{\partial}{\partial x}(B_y \cos\phi + B_z \sin\phi), \quad (2.5\text{b})$$

$$\frac{\partial \mathbf{v}_\perp}{\partial t} + u\frac{\partial \mathbf{v}_0}{\partial x} = \frac{B_{0x}}{\mu\rho_0}\frac{\partial \mathbf{B}_\perp}{\partial x} - \frac{B_{0x}\rho}{\mu\rho_0^2}\frac{\partial \mathbf{B}_0}{\partial x}, \quad (2.5\text{c})$$

$$\frac{\partial \mathbf{B}_\perp}{\partial t} = B_{0x}\frac{\partial \mathbf{v}_\perp}{\partial x} - \frac{\partial (u\mathbf{B}_0)}{\partial x}. \quad (2.5\text{d})$$

Here

$$\mathbf{v}_\perp = (0, v_y, v_z), \quad \mathbf{B}_\perp = (0, B_y, B_z). \quad (2.6)$$

When deriving the system of equations (2.5) we have taken into account that $B_x = 0$ and used (2.1d) to eliminate p.

The traditional approach to solving the system of equations (2.5) is to assume that ρ varies as $\exp[i(kx - \omega t)]$ (see, e.g., Galeev and Oraevskii (1963); Sagdeev and Galeev (1969); Goldstein (1978); Derby (1978)). Jayanti and Hollweg (1993a) used a different approach. They noticed that if the variables t and ϕ are used instead of t and x then the coefficients of the linear system of equations (2.5) depend only on ϕ.

This observation enabled them to look for the solution proportional to $\exp[-i\omega t]$. After that, (2.5) becomes a system of ordinary differential equations with periodic coefficients, so that Floquet's theory can be used. Jayanti and Hollweg (1993a) have used the first part of Floquet's theorem which prescribes the form of solutions to a linear system of ordinary differential equations with periodic coefficients. However, there is also the second part of Floquet's theorem which states that for any system of ordinary differential equations with periodic coefficients, there exists a linear transformation of dependent variables reducing this system to a system with constant coefficients. In general, finding this variable transformation is not simpler than solving the system using the first part of Floquet's theorem. However, in a particular case of system (2.5) this variable transformation is trivial. It is given by

$$B_+ = B_y \cos\phi + B_z \sin\phi, \quad B_- = B_y \sin\phi - B_z \cos\phi, \tag{2.7a}$$

$$v_+ = v_y \cos\phi + v_z \sin\phi, \quad v_- = v_y \sin\phi - v_z \cos\phi. \tag{2.7b}$$

In the new variables, (2.5) is rewritten as

$$\frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial u}{\partial x} = 0, \tag{2.8a}$$

$$\frac{\partial u}{\partial t} = -\frac{c_{\mathrm{s}}^2}{\rho_0}\frac{\partial \rho}{\partial x} - \frac{A_0}{\mu\rho_0}\frac{\partial B_+}{\partial x}, \tag{2.8b}$$

$$\frac{\partial v_+}{\partial t} - \omega_0 v_- = \frac{B_{0x}}{\mu\rho_0}\left(\frac{\partial B_+}{\partial x} + k_0 B_-\right), \tag{2.8c}$$

$$\frac{\partial v_-}{\partial t} + \omega_0 v_+ - V_0 k_0 u = \frac{B_{0x}}{\mu\rho_0}\left(\frac{\partial B_-}{\partial x} - k_0 B_+ + \frac{A_0 k_0 \rho}{\rho_0}\right), \tag{2.8d}$$

$$\frac{\partial B_+}{\partial t} - \omega_0 B_- = B_{0x}\left(\frac{\partial v_+}{\partial x} + k_0 v_-\right) - A_0 \frac{\partial u}{\partial x}, \tag{2.8e}$$

$$\frac{\partial B_-}{\partial t} + \omega_0 B_+ = B_{0x}\left(\frac{\partial v_-}{\partial x} - k_0 v_+\right) + A_0 k_0 u. \tag{2.8f}$$

This is the system of equations with constant coefficients, so that we can look for solutions proportional to $\exp[i(Kx - \Omega t)]$. As a result, after straightforward calculations, we arrive at the dispersion equation

$$\begin{aligned} &\big\{\big(\Omega^2 - c_{\mathrm{s}}^2 K^2\big)(\Omega k_0 - \omega_0 K)\big[(\Omega k_0 + \omega_0 K)^2 - 4\omega_0^2 k_0^2\big] \\ &\quad - a^2\omega_0^2 K^2 \big(\Omega^3 k_0 + \Omega^2 K\omega_0 - 3\Omega\omega_0^2 k_0 + \omega_0^3 K\big)\big\}(\Omega k_0 - \omega_0 K) = 0, \end{aligned} \tag{2.9}$$

where $a = A_0/B_{0x}$ is the dimensionless amplitude of the circularly polarized Alfvén wave. This equation coincides with the accuracy up to notation with the dispersion equation obtained previously by, e.g., Jayanti and Hollweg (1993a).

3. Stability analysis

In what follows we are only interested in unstable modes. Since the second multiplier in the dispersion equation (2.9) gives the solution $\Omega = \omega_0 K/k_0$ which does not lead to instability, we can disregard this multiplier. Then, introducing the dimensionless variables $b = c_{\mathrm{s}}/v_{\mathrm{A}}$, $k = K/k_0$ and $\omega = \Omega/\omega_0$, we re-write (2.9) as

$$(\omega^2 - b^2k^2)(\omega - k)[(\omega + k)^2 - 4] - a^2k^2(\omega^3 + \omega^2 k - 3\omega + k) = 0. \tag{3.1}$$

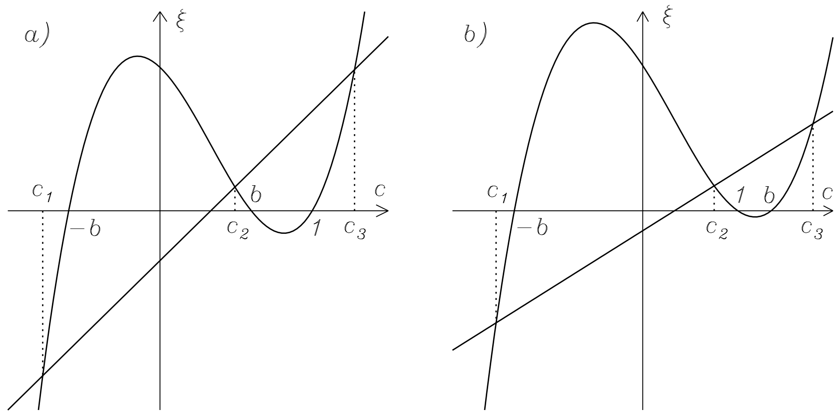

Figure 1. The graphical investigation of the cubic equation $G(c)=0$, where $G(c)$ is given by (3.3): (a) $1/3<b<1$; (b) $b>1$. The picture for $b<1/3$ is qualitatively the same as (a), however with $c_2>b$.

For any fixed k (3.1) is the fifth-order polynomial equation for ω. If all roots of this equation are real, then the perturbation with the fixed k is neutrally stable. However, if $\tilde{\omega}$ is a non-real root of (3.1), then $\tilde{\omega}^*$ is also a root, where the asterisk indicates a complex conjugate quantity. The imaginary part of either $\tilde{\omega}$ or $\tilde{\omega}^*$ is positive, which implies that the perturbation with this particular fixed k is unstable. Hence, to find the interval of k corresponding to unstable perturbations we have to find all values of k such that (3.1) does not have five real roots with respect to ω.

Let us introduce the dimensionless phase velocity $c = \omega/k$. Then (3.1) can be re-written as

$$k^2 = F(c) \equiv \frac{G(c)}{(c+1)[c^4 - (1+a^2+b^2)c^2 + b^2]}, \tag{3.2}$$

where the function $G(c)$ is given by,

$$G(c) = 4(c-1)(c^2 - b^2) - a^2(3c-1). \tag{3.3}$$

It is obvious that (3.1) has five real roots for a particular k if and only if the horizontal line $\xi = k^2$ in the $c\xi$-plane has five intersections with the graph of the function $\xi = F(c)$.

The zeros of the numerator are the roots of the cubic equation $G(c)=0$. They are given by the intersections of the graphs of two functions, $\xi = a^2(3c-1)$ and $\xi = 4(c-1)(c^2-b^2)$. These graphs are shown in Fig. 1(a) for $b<1$ and 1(b) for $b>1$. It is straightforward to see that these figures are qualitatively the same for any a and, hence, there are three zeros of the numerator. We denote these zeros as $c_1 < c_2 < c_3$. Note that

$$c_2 > b \quad \text{for } b<1/3, \qquad c_2 < b \quad \text{for } b>1/3, \tag{3.4}$$

so that Fig. 1a corresponds to $b>1/3$. It follows from Fig. 1 that

$$c_3 > \max(1, b). \tag{3.5}$$

Since $G(-1) = 4[a^2 - 2(1-b^2)]$, it follows that

$$-1 < c_1 < -b \quad \text{for } a^2 < 2(1-b^2), \qquad c_1 < -1 \quad \text{for } a^2 > 2(1-b^2). \tag{3.6}$$

The zeros of the denominator are $c = -1$ and $c = \pm c_{\pm}$, where

$$c_{\pm} = \tfrac{1}{2}\big\{[a^2 + (1+b)^2]^{1/2} \pm [a^2 + (1-b)^2]^{1/2}\big\}. \tag{3.7}$$

It follows from this expression that

$$\begin{aligned} c_+ &> \tfrac{1}{2}\big\{[(1+b)^2]^{1/2} + [(1-b)^2]^{1/2}\big\} \\ &= \tfrac{1}{2}(1 + b + |1-b|) = \max(1, b). \end{aligned} \tag{3.8}$$

Since $c_+ c_- = b$, it follows from (3.8) that

$$c_- < \min(1, b). \tag{3.9}$$

The mutual positions of the zeros of the numerator and denominator are of great importance in what follows. To find these mutual positions we write the function $G(c)$ as

$$G(c) = \frac{4[c^4 - (1 + a^2 + b^2)c^2 + b^2] + a^2(c-1)^2}{c+1}. \tag{3.10}$$

It immediately follows from this expression and the inequalities (3.8) and (3.9) that

$$G(\pm c_-) > 0, \quad G(c_+) > 0, \quad G(-c_+) < 0. \tag{3.11}$$

It follows from Fig. 1 and the inequalities (3.8), (3.9) and (3.11) that $c_+ > c_3$, $-c_+ < c_1$ and $c_1 < -c_- < c_- < c_2$. Summarizing the analysis we conclude that the graph of the function $F(c)$ has the form shown in Fig. 2(a) when $a^2 < 2(1-b^2)$ and the form in Fig. 2(b) when $a^2 > 2(1-b^2)$ (note that this inequality is always satisfied when $b > 1$).

Figure 2(a) corresponds to $a^2 = 1.4$ and $b = 0.5$, and Fig. 2(b) to $a^2 = 1.6$ and $b = 0.5$. For these particular values of a and b the maximum value of $F(\xi)$ in the interval (c_-, c_+), ξ_1, and the minimum value of $F(\xi)$ in the interval $(-c_-, c_-)$, ξ_2, satisfy the inequality $\xi_1 < \xi_2$. As a result the horizontal line $\xi = k^2$ intersects the graph of $F(\xi)$ in five points when either $k^2 < \xi_1$ or $k^2 > \xi_2$, and in three points when $\xi_1 < k^2 < \xi_2$ (see Figs 2(a) and 2(b)). Hence, (3.1) has five roots if and only if k^2 is not in the interval (ξ_1, ξ_2). This implies that the perturbation corresponding to a particular k is unstable if $k \in (-k_2, -k_1) \cup (k_1, k_2)$, where $k_{1,2} = \xi_{1,2}^{1/2}$, and stable otherwise.

Let us prove that the inequality $\xi_1 < \xi_2$ is satisfied for any values of a and b. To do this we use (3.10) to rewrite the expression for $F(c)$ as

$$F(c) = \frac{4}{(c+1)^2} + \frac{a^2(1-c)^2}{4(c+1)^2[c^4 - (1 + a^2 + b^2)c^2 + b^2]} \tag{3.12}$$

Let $\tilde{c}_2 \in (-c_-, c_-)$ and $\tilde{c}_1 \in (c_-, c_+)$. Since $0 < 1 + \tilde{c}_2 < 1 + \tilde{c}_1$, we obtain that

$$\frac{4}{(\tilde{c}_2 + 1)^2} > \frac{4}{(\tilde{c}_1 + 1)^2}. \tag{3.13}$$

The second term on the right-hand side of (3.12) is positive for $c = \tilde{c}_2$ and negative for $c = \tilde{c}_1$. It follows from this fact and (3.13) that $F(\tilde{c}_2) > F(\tilde{c}_1)$. Taking $\tilde{c}_2$ being the point where $F(c)$ takes its minimum value on the interval $(-c_-, c_-)$, and $\tilde{c}_1$ being the point where $F(c)$ takes its maximum value on the interval (c_-, c_+), we conclude that $\xi_1 < \xi_2$. Hence, for any values of a and b there are such values of k, k_1 and k_2 that $0 < k_1 < k_2$ and the perturbation is unstable if and only if $k \in (-k_2, -k_1) \cup (k_1, k_2)$.

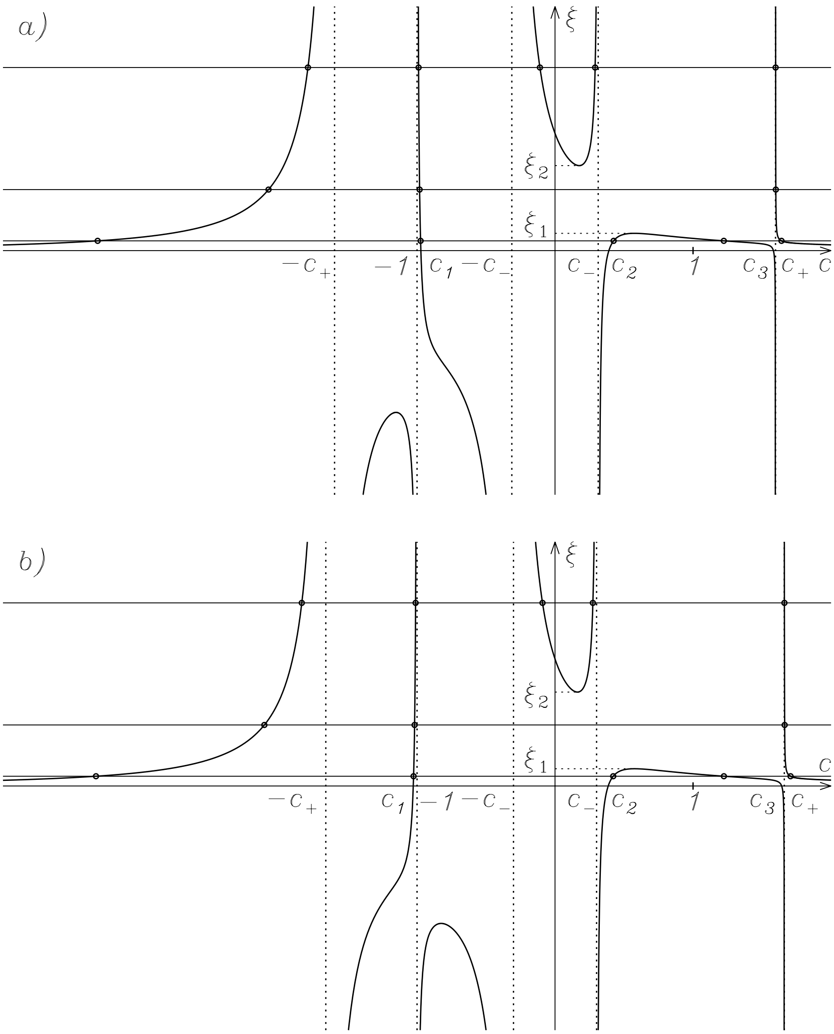

Figure 2. The graphs of the function $F(c)$ given by (3.2): (a) $a^2=1.4$ and $b=0.5$; (b) $a^2=1.6$ and $b=0.5$. The graph of the function $F(c)$ is qualitatively the same as in (a) when $a^2<2(1-b^2)$, and as in (b) when $a^2>2(1-b^2)$. The bold dots show the intersections of the graph of the function $F(c)$ with horizontal lines $\xi=\text{constant}$. The points of intersection correspond to the real roots of the equation $F(c)=\xi$. We see that there are five points of intersection and, consequently, five roots, when either $\xi<\xi_1$ or $\xi>\xi_2$. When $\xi_1<\xi<\xi_2$, there are only three points of intersection and, consequently, three roots.

4. The dependence of the instability boundaries on the wave amplitude

In this section we study the dependence of the instability boundaries, k_1 and k_2, on the amplitude a. It is straightforward to show that

$$\lim_{a\to 0} k_1 = \lim_{a\to 0} k_2 = \begin{cases} \dfrac{2}{1+b}, & b<1, \\ 1, & b>1. \end{cases} \tag{4.1}$$

The expansions of k_1 and k_2 for small a are given by Jayanti and Hollweg (1993b). Let $k_1^2 = F(\tilde{c}_1)$ and $k_2^2 = F(\tilde{c}_2)$. Then

$$\frac{dk_j^2}{da} = \frac{\partial F}{\partial c}\frac{\partial \tilde{c}_j}{\partial a} + \frac{\partial F}{\partial a} = \frac{\partial F}{\partial a}, \tag{4.2}$$

where $j = 1, 2$ and we have taken into account that $\partial F/\partial c = 0$ for $c = \tilde{c}_j$. Using (3.12) and (4.2) we easily obtain

$$\frac{dk_j^2}{da} = \frac{a(\tilde{c}_j - 1)^3\left(\tilde{c}_j^2 - b^2\right)}{2(\tilde{c}_j + 1)\left[\tilde{c}_j^4 - (1 + a^2 + b^2)\tilde{c}_j^2 + b^2\right]^2}. \tag{4.3}$$

Since $|\tilde{c}_2| < c_- < \min(1, b)$, it follows from (4.3) that

$$\frac{dk_2}{da} > 0, \tag{4.4}$$

i.e. the upper boundary of the interval of unstable wavenumbers increases together with a. It is easy to show that $k_2 \approx a/(2b)$ for $a \gg 1$.

Now we study the dependence of k_1 on a. The quantities $\tilde{c}_1$ and $\tilde{c}_2$ are determined by the equation $\partial F/\partial c = 0$, which can be reduced to

$$a^4 c(3c^2 - 1) + a^2(c - 1)[c(c - 1)(c^2 - 1) - (c^2 - b^2)(7c^2 + 3c - 2)] + 4(c - 1)^2(c + 1)(c^2 - b^2)^2 = 0. \tag{4.5}$$

It is straightforward to obtain that, for $a \ll 1$, $\tilde{c}_1$ is given by

$$\tilde{c}_1 = \begin{cases} b + \dfrac{a(1 - b)^{1/2}}{4b^{1/2}} + O(a^2), & b < 1, \\ 1 - \dfrac{a^2}{2(b^2 - 1)} + O(a^3), & b > 1. \end{cases} \tag{4.6}$$

Hence $\tilde{c}_1 < 1$ for $a \ll 1$. If $\tilde{c}_1 > 1$ for some values of a, then, since $\tilde{c}_1$ is a continuous function of a, there should be such a value a_0 that $\tilde{c}_1 = 1$ at $a = a_0$. However, $c = 1$ is a root of (4.5) only when $a = 0$, so that $c \neq 1$ for $a > 0$. Then it follows that $\tilde{c}_1 < 1$ for any $a > 0$ and the sign of the right-hand side of (4.3) with $j = 1$ is opposite to the sign of the quantity $\tilde{c}_1^2 - b^2$.

Let us first study the case where $b < 1$. Then it follows from (4.6) that $\tilde{c}_1 > b$ for $a \ll 1$. The direct substitution in (4.5) shows that $\tilde{c}_1 = b$ when

$$a^2 = a_c^2 = \frac{(1 + b)(1 - b)^3}{3b^2 - 1}. \tag{4.7}$$

It is obvious that (4.7) cannot be satisfied for any real a if $b^2 < 1/3$. Hence $\tilde{c}_1 \neq b$ for any a if $b^2 < 1/3$. Since $\tilde{c}_1(a)$ is a continuous function, we conclude that $\tilde{c}_1 > b$ for any a and

$$\frac{dk_1}{da} < 0 \quad \text{for } b^2 < \frac{1}{3}. \tag{4.8}$$

In the upper panel of Fig. 3 the dependences of k_1 and k_2 on a are shown for $b = 0.1$. These dependences are qualitatively the same for any $b^2 < 1/3$.

Let us now consider b satisfying $1/3 < b^2 < 1$. Then $\tilde{c}_1 = b$ for $a = a_c$. It is straightforward to verify that $d\tilde{c}_1/da < 0$ at $a = a_c$, so that $\tilde{c}_1 < b$ for a slightly larger than a_c. Since $\tilde{c}_1$ is a continuous function of a and $\tilde{c}_1 \neq b$ for $a > a_c$, we conclude that

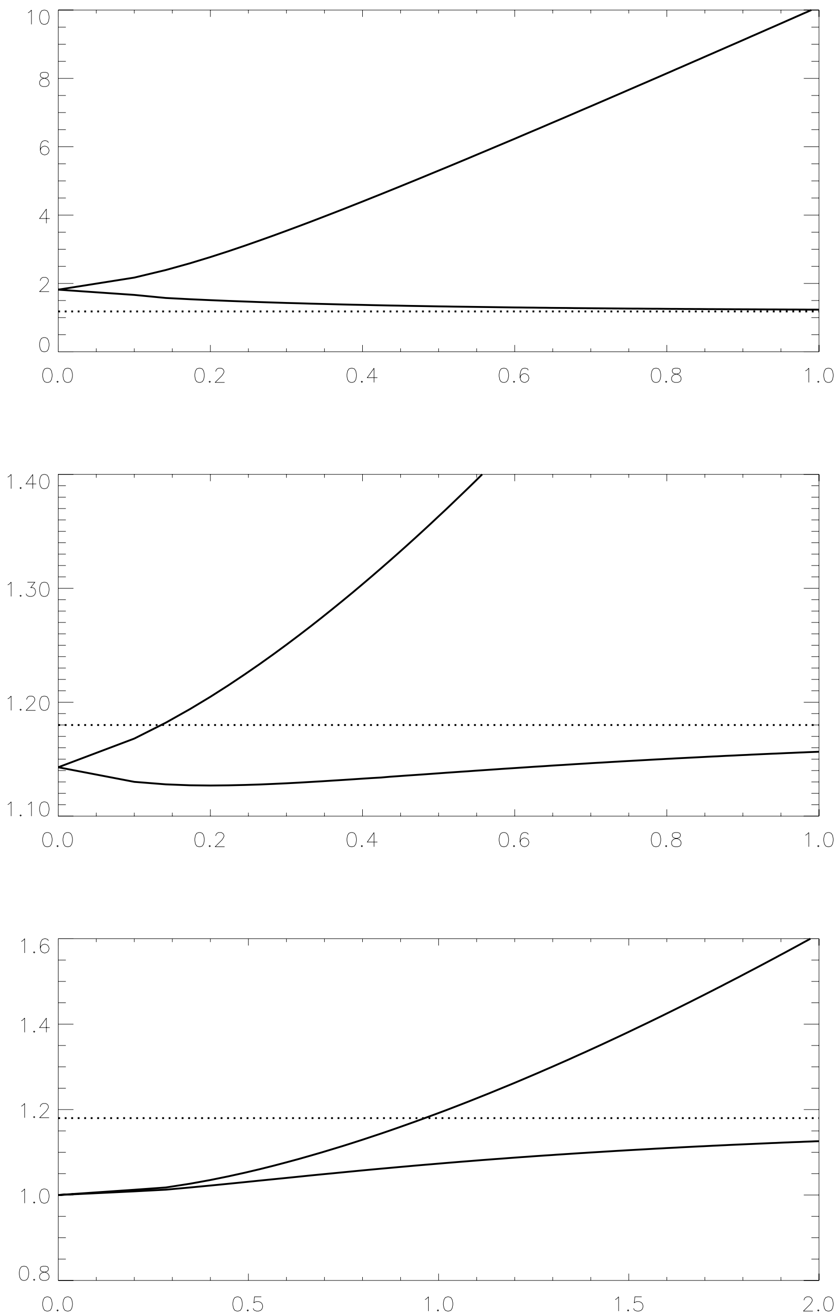

Figure 3. The dependences of the boundaries of the instability interval with respect to the wavenumber, k_1 and k_2, on a. The upper panel corresponds to $b = 0.1$, the middle panel to $b = 0.75$, and the lower panel to $b = 1.5$. The dependences of k_1 and k_2 on a are qualitatively the same as in the upper panel when $b^2 < 1/3$, as in the middle panel when $1/3 < b^2 < 1$, and as in the lower panel when $b > 1$. The horizontal dotted lines are the asymptotes for $k_1(a)$ as $a \to \infty$.

$\tilde{c}_1 < b$ for any $a > a_c$. Hence, for $1/3 < b^2 < 1$,

$$\frac{dk_1}{da} < 0 \quad \text{for } a < a_c, \quad \frac{dk_1}{da} > 0 \quad \text{for } a > a_c. \tag{4.9}$$

When $a = a_c$, k_1 takes its minimum value k_{1m} given by

$$k_{1m} = [F(b)]^{1/2} = \frac{(3b-1)^{1/2}}{b(b+1)^{1/2}}. \tag{4.10}$$

Note that $k_{1m} \geqslant 1$. In the middle panel of Fig. 3 the dependences of k_1 and k_2 on a are shown for $b = 0.75$. These dependences are qualitatively the same for any $1/3 < b^2 < 1$.

Now we study the case $b > 1$. Since $\tilde{c}_1 < 1 < b$, we conclude that

$$\frac{dk_1}{da} > 0 \quad \text{for } b > 1. \tag{4.11}$$

In the lower panel of Fig. 3 the dependences of k_1 and k_2 on a are shown for $b = 1.5$. They remain qualitatively the same for any $b > 1$.

It is not difficult to calculate the limit of k_1 when $a \to \infty$. It follows from (4.5) that $\tilde{c}_1 \to 3^{-1/2}$ as $a \to \infty$, so that

$$\lim_{a\to\infty} k_1 = \lim_{a\to\infty} \left[F\left(a, c = 3^{-1/2}\right)\right]^{1/2} = 2^{-1/2} 3^{3/4} \left(3^{1/2} - 1\right) \approx 1.18. \tag{4.12}$$

5. Conclusions

In this paper we have revisited the parametric instability of circularly polarized Alfvén waves (pump waves) propagating along the background magnetic field in a plasma described by the system of ideal MHD equations. We present a variable substitution that reduces the linearized system of MHD equations determining the stability of a circularly polarized Alfvén wave to a system with constant coefficients. Using this system we have derived the dispersion equation determining the Alfvén wave stability. This dispersion equation was previously obtained by other authors with the use of different approaches.

We have studied the properties of this dispersion equation. Written in the dimensionless form, it contains two parameters, the ratio of the sound and the Alfvén speed $b = c_\mathrm{s}/v_\mathrm{A}$, and the dimensionless amplitude of the pump wave a. We have proved that, for any values of a and b, there are such numbers k_1 and k_2 that the pump wave is unstable with respect to a harmonic perturbation with the dimensionless wavenumber k if $k_1^2 < k^2 < k_2^2$ and stable otherwise.

We have studied the dependences of the instability boundaries, k_1 and k_2 on a. We have shown that k_2 is a monotonically growing function of a for any value of b. The function $k_1(a)$ is monotonically decreasing for $b^2 < 1/3$. When $1/3 < b^2 < 1$, k_1 monotonically decreases when a increases from zero to a_c given by (4.7), takes its minimum value at $a = a_c$, and then increases when a increases from a_c to infinity. The function $k_1(a)$ is monotonically increasing for $b > 1$.

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