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Research article

The point vortex model for the Euler equation

Carina Geldhauser[1] **and Marco Romito**[2,*]

[1] Faculty of Mathematics, Technische Universität Dresden, 01062 Dresden, Germany
[2] Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, I–56127 Pisa, Italia

* **Correspondence:** Email: marco.romito@unipi.it; Tel: +390502213305; Fax: +390502210678.

Abstract: In this article we describe the system of point vortices, derived by Helmholtz from the Euler equation, and their associated Gibbs measures. We discuss solution concepts and available results for systems of point vortices with deterministic and random circulations, and further generalizations of the point vortex model.

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1. Introduction

Turbulent flows play a prominent role in fluid dynamics, meteorology and engineering (e.g. in combustion phenomena). A very prominent property of turbulent flows is self-organization and formation of large scale coherent jets and vortices, also on large scales, e.g. Jupiter's Great Red Spot [19]. Such turbulences might change some features, e.g. size and brightness, but have kept the same patterns over a long time. Atmospheric vortices, and likewise ocean eddies, are examples of strongly stratified, rapidly rotating flows, where the motion is layer-wise, and 2D turbulence models can be applied to explain the macroscopicly arising vortex structures.

The description of the dynamics of vortices in two space dimensions goes back to Helmholtz, who in 1858 established vortices as the "particles" of fluid mechanics [3,57], paving the way to the application of methods from statistical mechanics to turbulence theory. As in two space dimensions the vorticity of each fluid element is conserved in time, it is possible to consider a model in which only a finite number of particles carry vorticity. We may consider a two dimensional flow involving isolated blobs of vorticity in an otherwise irrotational fluid, or point vortices, if the vorticity is concentrated in blobs of infinitesimal radius. In Helmholtz words: "each vortex line remains continually composed of the same elements of fluid, and swims forward with them in the fluid." Kirchhoff [1] already showed that such vortex blobs obey approximately a Hamiltonian particle dynamics

$$\begin{aligned}\gamma_i \frac{dx_i^1}{dt} &= \frac{\partial \mathcal{H}}{\partial x_i^2} \\ \gamma_i \frac{dx_i^2}{dt} &= -\frac{\partial \mathcal{H}}{\partial x_i^1}.\end{aligned} \tag{1.1}$$

Lin [67, 68] proved in 1941 that the motion of N vortices in a bounded domain is a Hamiltonian system conserving total kinetic energy, see Section 2. We refer the reader interested in the historical developments to the excellent survey [41], which contains also a study of Onsager's unpublished work on turbulence.

Several strains of research have emerged in the study of turbulence, ranging from incorporating dynamical systems approaches by setting the vortex dynamics in a complex plane, see e.g. [4], to methods from partial differential equations and statistical physics. In this work we review the equilibrium statistical mechanics approach, where the long time behavior of a fluid has been modelled by a space of fluid configurations endowed with a Gibbs measure, and the configurations were chosen on a phenomenological ground to mimic coherent vortex structures. The investigation of turbulence in two-dimensional fluids via point vortex models is deeply influenced by the work of Onsager [85] in 1949, who postulated that the generation of large-scale vortices was a consequence of the inviscid Euler equations.

Though the distributions of vorticity in the actual flow of normal fluids are continuous, in many cases a set of discrete vortices provides a reasonable approximation. Therefore, point vortex models, i.e. systems of N equations of the form

$$\begin{cases} \dot{X}_j = \sum_{k \neq j} \gamma_k \nabla^{\perp} G(X_j, X_k), \\ X_j(0) = x_j, \end{cases} \qquad j = 1, 2, \dots, N, \tag{1.2}$$

where G is the Green function of the Laplacian, have been studied, in different settings with different assumptions, over the last decades. In Section 2, we summarize the derivation, existence theory and approximation property of the point vortex model and discuss its assumptions. In Section 3 we review some results obtained in the passage to the limit as the number of vortices goes to infinity. One expects that the vortex positions become independent of one another, so when the field created by the vortices converges to a mean field, the k-point correlation functions factorize and behave like a product of k copies of 1-point correlation functions. This decorrelation in the limit is called *propagation of chaos*, and holds for deterministic circulations and positive temperature. In the negative temperature case, correlations may persist and we get a mixture of correlation functions with a certain mixing measure characterized by the following *variational principle*: The weak cluster points of the Gibbs measures (or, more precisely, the correlation functions) are minimizers of a certain functional $\mathcal{F}$, which, in certain circumstances, can be identified with the free energy functional, see (3.8). The uniqueness of solutions to the mean-field equation is related to the propagation of chaos: Uniqueness of the mean field equation means uniqueness of the minimizer of the functional $\mathcal{F}$ and then the mixing measure is a Dirac measure, hence propagation of chaos holds.

Viewing the vortex method as a way to approximate stationary solutions to the Euler Equation, it is natural to investigate the speed of convergence, leading to Large Deviation Principles, see Section 3.5, and the behavior of the fluctuations of the empirical measure around the limiting law, see Section 3.4.

In the 1990s, a generalization of Onsagers ideas to the 2D Euler equations with a continuous vorticity field, the *Miller-Robert-Sommeria theory* (MRS theory), has been proposed [78, 93–96]. The MRS theory includes the previous Onsager theory and determines within which limits the theory will give relevant predictions and results. In particular it predicts that most microscopic states concentrate into a single equilibrium macrostate, which is characterized by the maximization of an entropy with some constraints related to dynamics invariants. We will not review the Miller-Robert-Sommeria theory here and refer to the above references and the lecture notes [11] for more information, also on applications of MRS theory, for both the two-dimensional Euler and quasi-geostrophic equations: As outlined in [11], MRS theory predicts phase transitions in different contexts and was applied as a model for the Great Red Spot, ocean vortices and jets [12].

Moreover, there is a vast literature on the study of the mean-field equations, which we do not review in detail here. For example, Bartolucci [6] examines in detail the mean field supercritical thermodynamics of the vorticity distribution and Bartolucci, Jevnikar, Lee and Yang [7] analyze bubbling solutions of the mean field equation. In order to take into account variable vortex intensities, certain non-local elliptic equation which contain an exponential type nonlinearity are studied, see e.g. Chavanis [20], Ricciardi and Takahashi [90, 92] or the recent work [91].

In the case of three-dimensional fluids, ensembles of vortex filaments have been introduced and deeply analysed, starting with works of Chorin [22, 23, 25], who proposed an interesting comparison with the classical Kolmogorov-Obukhov theory. Other ansatzes include almost parallel vortex filaments, which is relatively close to the 2D setting [70]. The models of vortex filaments proposed by Chorin are mainly based on probabilistic structures on a 3D lattice, like paths of self-avoiding walk or percolation clusters, or use a relation with the intersection local time [42]. We review some of the results in Section 4.

In the last part of this work, we briefly discuss some variations on the theory, namely vortex dynamics with noise in Section 5 and generalized models, i.e. the system of equations

$$\begin{cases} \dot{X}_j = \sum_{k \neq j} \gamma_k \nabla^{\perp} G_m(X_j, X_k), \\ X_j(0) = x_j, \end{cases} \qquad j = 1, 2, \ldots, N, \tag{1.3}$$

where G_m is the Green function of the fractional Laplacian $(-\Delta)^{\frac{m}{2}}$, in Section 6.

2. The point vortex model

In this work we review classical and recent results when methods of statistical mechanics were used to study Euler and surface-quasigeostrophic flows whenever the vorticity field is a linear combination of delta functions concentrated in points of the physical space. We start with a sketch of the derivation of the point vortex equation and discuss its solution and relationship to Euler's equation.

2.1. Euler's equations and the point vortex model

Call $\theta(x, t)$ the vorticity of a fluid and $u = (u_1, u_2)$ the velocity. Recall that

$$\theta(x, t) = \operatorname{curl} u(x, t) = \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right)(x, t). \tag{2.1}$$

The Euler equations for the vorticity of a fluid in a bounded flow domain $\Lambda \subset \mathbb{R}^2$ read

$$\begin{aligned} \partial_t \theta + (u \cdot \nabla)\theta &= 0 \\ \operatorname{curl} u &= \theta \\ \nabla \cdot u &= 0 \\ \theta(x,0) &= \theta_0(x) \\ u \cdot n|_{\partial\Lambda} &= 0. \end{aligned} \tag{2.2}$$

If u decays at infinity, the incompressibility condition allows us to reconstruct the velocity field by means of the vorticity. In fact, there exists a function ψ, called the stream function, such that $u = \nabla^\perp \psi$, hence

$$\begin{aligned} -\Delta\psi &= \theta \\ \psi &= 0 \text{ on } \partial\Lambda. \end{aligned} \tag{2.3}$$

If $-G_\Lambda(x,y)$ is the Green's function of the Laplacian with zero Dirichlet boundary conditions, then

$$u(x,t) = \int k(x-y)\theta(y)dy \tag{2.4}$$

with $k = \nabla^\perp G_\Lambda$, which is often called the Biot-Savart law.

The existence theory for the Euler equation has been developed over decades, among the vast literature we mention only a few results which are of use in the discussion of the point vortex system. In two dimensions, for bounded domains and when the initial vorticity is bounded, existence, uniqueness and global regularity of solutions were shown by Wolibner [107] and Yudovich [58]; these results were extended, in the framework of weak solutions, to the case where the initial vorticity belongs to L^p with $p > 1$, see [16], and even for $p = 1$ when the vorticity is some finite measure [37]. We also point out the exciting developments on non-uniqueness for the Euler equations initiated by De Lellis and Székelyhidi in [35]. See also Section 5, where solution concepts are discussed in the framework of vortex dynamics with noise.

2.1.1. The point vortex model

Consider the situation in which the vorticity θ is initially concentrated in N infinitesimal regions of the physical space $\Lambda \subset \mathbb{R}^2$. To fix notation, let each vortex θ_i have a support concentrated in a point $x_i = (x_i^1, x_i^2)$. For positions $x_i = x_i(0)$, denote the initial distribution of the vorticity by

$$\theta_0(dx) = \sum_{i=1}^{N} \gamma_i \delta_{x_i}(dx) \tag{2.5}$$

and we may call the individual component $\gamma_i \delta_{x_i}(dx)$ of the measure (2.5) a *point vortex*. The real number γ_i is called the *intensity* of the point vortex or the *circulation* of the vortex localized in x_i. In the case that the supports of the vortices are not points, but disjoint sets, one calls $\int \theta_i(x)dx = \gamma_i$ the *net circulation* which the i-th vortex carries. By Kelvin's theorem about the conservation of circulation, the intensities γ_i remain constant in time. As a generalization, several authors consider the intensities to be independent and identically distributed random variables with respect to some probability law, see Sections 3.3, 4.1, 6.2 and 6.3. Combining (2.5) and (2.4), we get that the velocity field for N point vortices at time zero reads, in the case of $\Lambda = \mathbb{R}^2$

$$u(x,0) = \sum_{i=1}^{N} \gamma_i \nabla^{\perp} G_{\Lambda}(x, x_i). \tag{2.6}$$

We may therefore write the *point vortex system* as

$$\begin{aligned} \frac{d}{dt} x_i(t) &= -\nabla_i^{\perp} \sum_{j=1, j\neq i}^{N} \gamma_j \, G_{\Lambda}(x_i(t) - x_j(t)) \\ x_i(0) &= x_i, \end{aligned} \tag{2.7}$$

or, defining the vector field $K : \mathbb{R}^2 \to \mathbb{R}^2$ as $K = \frac{x^{\perp}}{\|x\|}$

$$\dot{x}_i(t) = \sum_{j=1, j\neq i}^{N} \gamma_j \, K(x_i(t) - x_j(t)). \tag{2.8}$$

Given N initially pairwise disjoint point vortex positions with non-zero circulations, the vortex dynamics (2.8) is locally well-posed as a Cauchy problem. It has a global solution if the vortices do not collapse, this will be discussed in Theorem 2.2 below. We remark, first of all, that (2.7) makes sense only if we assume that a vortex does not move under the action of its own field. We will discuss this assumption in Section 2.2.1. The fundamental problem, however, is that the function u in (2.6) becomes singular whenever x tends to x_i, and the point vortex model considers precisely this situation when the mass of the measure is concentrated just on the points x_i. So, due to the singularity of G_{Λ} at the origin, (2.4) makes sense a priori only for absolutely continuous signed measures $\theta(dx) = \theta(x)dx$ with density $\theta(x) \in L^p(\mathbb{R}^2)$ for some suitable p. A sufficient condition would be $L^1 \cap L^{\infty}$, see [74].

To allow measures with atoms such as (2.5), Delort [36] used a symmetrization trick to tame the singularity of the Biot-Savart kernel: For a measure-valued initial condition $\theta_0 \in H^{-1}(\mathbb{T})$ such that the velocity $u_0 \in L^2$, test the equation against a test function $\phi \in C^{\infty}(\mathbb{T})$, only in terms of θ, to get

$$\langle \theta_t, \phi \rangle \ = \ \langle \theta_0, \phi \rangle \ + \ \int_0^t \int_{\mathbb{T}} \int_{\mathbb{T}} K(x-y)(\nabla\phi(x) - \nabla\phi(y))\theta_s(dx)\theta_s(dy)\, ds$$

where $K = \nabla^{\perp} G$, which plays the role of the Biot-Savart kernel. The new kernel $K(x-y)(\nabla\phi(x)-\nabla\phi(y))$ is bounded and smooth outside the diagonal, but discontinuous along the diagonal.

The existence of a global smooth solution to the point vortex system (2.7) for fixed N and almost every initial condition is then proved by a regularization of the kernel to deal with the singularity. Such a regularization of the kernel is equivalent to an approximation by vortex blobs [21], by which one intends finite blobs of vorticity of diameter approximately ϵ. We refer to Theorem 2.2 for details.

2.1.2. The point vortex equation as an approximation of the Euler equation

A core question is in which sense the dynamics of a finite number of point vortices can be seen as an approximation of Euler vortex dynamics. In other words, under which assumptions do empirical measures of the form

$$\theta_t^N(dx) = \sum_{i=1}^{N} \gamma_i \delta(x_i(t) - x)(dx), \tag{2.9}$$

which are obtained by the vortex motion, satisfy a weak form of the Euler equation?

In some special cases, namely for smooth flows [52] and for bounded initial vorticity [58], when uniqueness of the solution to the Euler vortex equation is known, positive results are known. In general, we want to show that if we approximate an initial condition by point vortices (2.5), then the point vortex measure (2.9) approximates solutions to the Euler equation in a suitable measure sense. As above, to make sense of the kernel appearing in a weak formulation of the Euler vortex equation, the measures should be absolutely continuous, which is not the case for (2.9). As a remedy, one can perform a regularization of the kernel, and prove that if we start at time zero with N vortex blobs located in a ball of radius R, then these blobs cannot leave a ball of radius R_* within time T. To obtain compactness of the regularizing sequence, uniform L^p-bounds of the vorticity are needed, which result in technical condition on ϵ. The below theorem illustrates this approach under the assumption that the vortex intensities are positive and normalized:

Theorem 2.1. *[[76], Theorem 5.3.1.] Let $\theta_0 dx$ be a probability measure on $\mathbb{R}^2$ and assume $\theta_0(x) \in L^1 \cap L^\infty$. Let θ_t be the (weak) solution of the Euler equation with initial data θ_0. Let θ^N as in* (2.9) *solve*

$$\frac{d}{dt}\theta_t^N(f) = \theta_t^N(u \cdot \nabla f), \tag{2.10}$$

where f is a smooth function, $\theta^N(f) = \int f(x)\theta^N(dx)$ and u is given by the right hand side of (2.8). *Let $\theta_0(x)$ be the initial condition of* (2.10) *and assume that $\sum_{i=1}^N \gamma_j = 1$. Suppose that at time zero $\theta_0^N \longrightarrow \theta_0 \in L^\infty \cap L^1(\mathbb{R}^2)$ weakly as $N \to \infty$ for bounded and continuous f. Assume the Euler equation has the property that its solutions are weakly continuous w.r.t. the initial conditions, and choose appropriately δ such that $\epsilon \approx N^{-\delta}$.*

Then at time t, the distribution of the vortices evolved by the regular Euler equations converges to the empirical vorticity at time t, in formula

$$\theta_t^N(f) \longrightarrow \theta_t \qquad \text{weakly as } N \to \infty. \tag{2.11}$$

in the sense that for the Wasserstein distance W_1 holds

$$\lim_{N\to\infty} \sup_{t\in[0,T]} W_1(\theta_\epsilon^N(t), \theta(t)) = 0 \tag{2.12}$$

The proof of Theorem 2.1 is done by replacing K by K_ϵ in (2.8), i.e. to smooth the logarithmic divergence when $x \approx x_i$ and to estimate the Wasserstein distance between $\theta_\epsilon^N(t)$ and $\theta(t)$. This means one considers finite blobs of vorticity of diameter approximately ϵ instead of point vortices and takes the simultaneous limit with scaling $\epsilon \approx N^{-\delta}$. However, due to a possible hyperbolicity of the motion, the error estimates at time t are quite unsatisfactory, namely not better than $N^{-p}C(p)e^{Ct}$, with p arbitrarily large and $C(p)$ diverging with p, see [9]. We end with some literature remarks: While we cited the book [76] as a classical reference, the original result was proven by the same authors in [73] and an analogous result has been recently proved for $m \in (2, 3)$ in [54]. An result similar to Theorem 2.1 holds also for general point vortex systems (1.3) as recently shown in [51], see Section 6.2, Theorem 6.2 for details.

2.1.3. Motion of the center of vorticity

Property (2.11) of the vortex flow says that the point vortex model provides a finite-dimensional approximation for the solutions of the Euler equation in the plane. One may also ask whether we recover the point vortex system in the limit as $\epsilon \to 0$, if we initialize the continuum Euler model with an initial condition that contains N distinct blobs of vorticity with radius ϵ. The answer is only partially positive: Marchioro and Pulvirenti [76], Theorem 4.4.1. show that the motion of the center of vorticity of a single vortex blob converges to the motion of a single point vortex. However, the motion of the center of vorticity is much more regular than the motion of a given fluid particle: in general, it cannot be inferred that the motion of the fluid particles supporting the vorticity θ_ϵ converge to a vortex flow, as indeed the singularity of the kernel K may cause such an irregular flow that the vortex blobs do not converge at all. Recently, similar results have been obtained for certain generalized models in [51], see Section 6.2.

2.1.4. Collapse of vortices

The existence theory of solutions to the Hamiltonian system (1.1), which is a system of ordinary differential equations, is not trivial because of the logarithmic divergence of the Hamiltonian: If two vortices are at the same point in space, the second equation does not make sense anymore. Whether such a collapse can happen depends on the sign of the (constant) circulation of the vortices: In the full plane $\mathbb{R}^2$ the case of three vortices is integrable due to conservations of center of vorticity, moment of inertia and Hamiltonian. If all three vortices have the same sign of circulation, then conservation of Hamiltonian and rotational invariance exclude the possibility for a collapse. If, however, the circulations have different signs, a self-similar solution can be constructed which rotates and contracts at the same time, until all three vortices collapse at the same point, see [76], Chapter 4. Recently, a similar result was obtained for generalized models in [5].

One may hope that cases of collapse are exceptional, in the sense that the Lebesgue measure of the set of initial conditions for which collapse can happen is zero. A non-collapse result for almost all initial conditions can be established under certain conditions on the circulations, see [76] for the Euler case and [51] for the case of generalized models.

Theorem 2.2. *[[76], Corollary 4.2.2.] Suppose that all N vortices are contained in a circular bounded domain. Under the assumption that*

$$\sum_{i\in J} \gamma_i \neq 0, \qquad \text{for all } J \subset \{1, 2, \dots, N\}, \tag{2.13}$$

and outside a bounded, measurable set of initial conditions of Lebesgue measure zero, the initial value problem associated to the point vortex dynamics (1.2) *has a global smooth solution.*

The result is proven via approximation by vortex blobs. The idea of the proof is to ensure that the vortices stay at a minimal distance, so there is no collapse of vortices, and to infer the existence of a smooth solution from it. Recently, this result was extended to generalized models (1.3) in [51]:

Theorem 2.3. *[[51], Theorem 3.1.] Suppose x_i move according to the generalized point vortex dynamics* (1.3) *with* $1 < m < 2$ *and that* (2.13) *holds. Then, outside a set of initial conditions of Lebesgue measure zero, the initial value problem associated to the vortex equation has a global smooth solution.*

Similar as in 2.2, the main part is to prove that there exists a constant $c > 0$ independent of ϵ and on the initial condition, such that

$$\max_{1\le j\le N}\sup_{t\in[0,T]} |x_i^\epsilon(t) - x_i^\epsilon(0)| \le c \tag{2.14}$$

The proof is based on the conservation of the center of pseudo-vorticity $\sum_{j=1}^N \gamma_j x_j$. The assumption (2.13) is essential, while it is only required that $|\nabla G_\alpha|$ goes to zero at infinity. Moreover, Flandoli and Saal [45] prove that there is a global dynamics if the initial condition is sampled according to an invariant distribution of vortex positions (this is the case $\beta = 0$ in the Gibbs language (3.1)).

2.2. *Onsager's theory of turbulence*

In the above section we saw that the point vortex model (1.2) may provide an approximation of the Euler equation in vorticity form, which describes the Euler flows whenever the vorticity field can be described as a linear combination of delta functions concentrated in points of the physical space. In this section we focus our viewpoint and discuss the point-vortex model in the framework of Gibbsian equilibrium statistical mechanics.

The general assumption is that the elements of the system are in thermodynamical equilibrium among themselves. This is important, as otherwise it is not possible to assign a temperature to the system. Moreover, it is assumed that the vortex system is energetically isolated and that the large-time statistics is a microcanonical equilibrium (by ergodicity of the point vortex dynamics). Moreover, it is assumed that no point vortices collide, which means that the number of vortices does not change, and the circulation γ_j remains a fixed attribute of the j-th point vortex.

We now present the general setting. Consider a system of N identical point vortices in a smooth, bounded, connected open domain $\Lambda \subset \mathbb{R}^2$ and let the positions of the vortices be denoted by $(x_1, \dots x_N) \in \Lambda^N$. Let $\beta \in \mathbb{R}$ denote the inverse temperature of the system. The fundamental object in the point vortex model is a finite-dimensional Hamiltonian $\mathcal{H}$, sometimes called "Kirchhoff Hamiltonian" [1], representing the fluid kinetic energy. It reads

$$\mathcal{H}_\Lambda(x_1, \dots, x_N) = \frac{1}{2}\sum\sum_{1\le i\le j\le N} \gamma_i\gamma_j G_\Lambda(x_i, x_j) \;+\; \sum_{i=1}^N \gamma_i^2 g_\Lambda(x_i). \tag{2.15}$$

with $G_\Lambda(x, y)$ the Green's function of the Poisson equation in Λ with Dirichlet boundary conditions, of which we know that it look as

$$G_\Lambda(x, y) = -\frac{1}{2\pi}\log|x - y| + g_\Lambda(x, y) \qquad \text{on } \Lambda \times \Lambda \tag{2.16}$$

and $g_\Lambda : \Lambda \times \Lambda \to \mathbb{R}$ is symmetric and harmonic in each variable. The $g_\Lambda(x)$ in (2.15) is defined as

$$g_\Lambda(x) := \frac{1}{2} g_\Lambda(x, x) \quad \text{on } \Lambda, \tag{2.17}$$

see Section 2.2.1 for details. In the case $\Lambda = \mathbb{R}^2$ or $\Lambda = \mathbb{T} \subset \mathbb{R}^2$ the flat torus, the term $g_\Lambda(x_i)$ is absent and the Hamiltonian simplifies to the *interaction energy*

$$\mathcal{H}_\Lambda(x_1, \dots, x_N) = \frac{1}{2}\sum\sum_{1\le i\le j\le N} \gamma_i\gamma_j G_\Lambda(x_i, x_j)\,. \tag{2.18}$$

A word on the assumptions. The energetic isolation assumption is, in fact, never really strictly satisfied, as the system looses energy due to viscosity. However, it was observed (see e.g. [77]) that large vortex structures are only weakly dissipated by viscosity. This tendency for energy to flow and reside in large scales, as observed in two-dimensional turbulence, is often called the *inverse energy cascade* and was predicted by Kraichnan [64].

The vortex model fails to to describe the evolution of vortices in fluids when effects of kinematic vorticity, such as the diffusion of vorticity, play a role: Indeed, in fluids governed by the Navier-Stokes equations, the effects of vorticity manifest on time scales proportional to the inverse of the viscosity, as stated in [40], page 836.

It is generally assumed that a vortex on $\mathbb{R}^2$ does not move under the action of its own field. To justify this, we approximate the point vortices by so-called "vortex blobs":

Lemma 2.4. *Let $\Lambda = \mathbb{R}^2$. Consider a sequence of smooth functions $\theta_\epsilon \in C^\infty$ such that $\theta_\epsilon \to \delta$ in the weak sense as $\epsilon \to 0$. Assume that θ_ϵ are spherically symmetric and denote by $u_\epsilon = K * \theta_\epsilon$ the velocity field generated by θ_ϵ. Let θ_ϵ satisfy* (2.7). *Then, $u_\epsilon(0) = 0$.*

The proof is straightforward and uses the symmetry of the distribution, see [76], Chapter 4, page 135.

The approximation by vortex blobs is one way around the problem that u becomes singular whenever x tends to x_i. However, also the smoother vortex blob model breaks down as an approximation to the Euler dynamics once the vortex blobs come too close to each other: In such a situation, a stronger vortex may stretch a nearby weaker vortex into a long ribbon of vorticity under the shear of its velocity field, as pointed out by Overman and Zabusky [86].

The point vortex model may avoid such a situation as, recalling Theorem 2.2, as the radius of the blob goes to zero, the initial conditions for which this stretching may happen have vanishing Liouville measure.

2.2.1. The self-energy

In Section 2, we discussed the point vortex model on the whole plane $\mathbb{R}^2$ or on the torus $\mathbb{T} \subset \mathbb{R}^2$, to get around the fact that the presence of a boundary creates an effect of self-interaction on point vortices. In fact, the additional term with $\sum_{i=1}^N \gamma_i^2 g_\Lambda(x_i)$ in the Hamiltonian (2.15) represents the individual energies of each vortex with the image charges necessary to maintain the boundary conditions and with an (infinite) self-energy substracted:

$$g_\Lambda(x) = \lim_{y \to x} \frac{1}{2} \left(G_\Lambda(x, y) - G_\infty(x, y) \right) \tag{2.19}$$

This is a consequence of a structure theorem for the Green function, which says that the Green function G_Λ on a bounded domain can be written as

$$G_\Lambda(x, y) \; = \; G(x, y) + g_\Lambda(x, y), \tag{2.20}$$

with $g_\Lambda : \Lambda \times \Lambda \to \mathbb{R}$ as in (2.15) above. Moreover, using (2.17), we get that g_Λ is bounded from above on Λ and $g_\Lambda(x) \to -\infty$ as $\mathrm{dist}(x, \partial\Lambda) \to 0$.

The question on self-interaction and self-energy depends on the domain Λ: On the whole plane $\mathbb{R}^2$, Lemma 2.4 argues that a single vortex blob stays almost at rest, implying that the (in principle infinite) self-interaction of the vortex is negligible.

This is fundamentally different in the the case of bounded domains: with the same strategy as in Lemma 2.4, one can argue that $\lim_{\epsilon\to 0}\theta_\epsilon(0) = \frac{1}{2}\nabla^\perp g_\Lambda(0)$, so a single vortex moves just because of the presence of a boundary.

A heuristic motivation for the self-interaction term can be seen as follows. Consider a single vortex blob of intensity γ centred at $x_0 \in \Lambda$, for example we may denote $\theta_\epsilon(x) = \gamma\epsilon^{-2}\eta(x/\epsilon)$. Assume moreover that η is radial. The velocity field corresponding to θ_ϵ is

$$u_\epsilon(x) = \int_\Lambda \nabla_x^\perp G_\Lambda(x,y)\theta_\epsilon(y)\,dy.$$

By (2.20), we can write

$$u_\epsilon(x_0) = \int_\Lambda \nabla_x^\perp G(x_0,y)\theta_\epsilon(y)\,dy + \int_\Lambda \nabla^\perp g_\Lambda(x_0,y)\theta_\epsilon(y)\,dy$$

The first integral is zero by symmetry, and since $\theta_\epsilon \rightharpoonup \gamma\delta_0$, the second integral converges,

$$\int_\Lambda \nabla^\perp g_\Lambda(x_0,y)\theta_\epsilon(y)\,dy \longrightarrow \gamma\nabla^\perp g_\Lambda(x_0,x_0).$$

In conclusion $u_\epsilon(x_0) \to \gamma\nabla^\perp g_\Lambda(x_0,x_0)$, and $\gamma\nabla^\perp g_\Lambda(x_0,x_0)$ can be considered the velocity field generated by the vortex itself.

This heuristics can also explain how the self-interaction term disappears in the point vortex system on the torus: By translation invariance, we have that $g_{\mathbb{T}}(x,y) = g_{\mathbb{T}}(x-y)$ and $g_{\mathbb{T}}$ is bounded, and so $\gamma\nabla^\perp g_{\mathbb{T}}(x_0,x_0)$ is zero on the torus.

Notice that this heuristic argument strongly depends on the symmetry of the vortex blob. If the blob shape is not symmetric, then the integrals above may diverge.

The presence or non-presence $g_\Lambda(x,y)$ makes a fundamental difference also in the techniques for proving limit theorems: For a bounded domain with Dirichlet boundary conditions, the maximum principle tells us that $G(x,y) \geq 0$ and $g_\Lambda(x,y) \geq \frac{1}{2\pi}\log\operatorname{dist}(y,\partial\Lambda)$. More precisely, for every fixed $y \in \Lambda$ we have

$$\begin{aligned} -\Delta_x g_\Lambda(x,y) &= 0 \qquad \text{in } \Lambda \\ g_\Lambda(\cdot,y) &= \frac{1}{2\pi}\log|\cdot - y| \geq \frac{1}{2\pi}\log\operatorname{dist}(x,\partial\Lambda) \quad \text{on } \partial\Lambda, \end{aligned} \tag{2.21}$$

hence

$$g_\Lambda(x) \geq \frac{1}{2\pi}\log\operatorname{dist}(x,\partial\Lambda) \qquad \text{on } \Lambda. \tag{2.22}$$

The lower bound (2.22) is very useful when proving bounds on the partition function (3.2): In fact, for Dirichlet boundary conditions on Λ, the partition function is finite for $\beta \in (-8\pi,\infty)$, as we can apply (2.22) to get an upper bound on the partition function for all positive inverse temperatures. On the other hand, in the torus case $g_\Lambda(x,y) = 0$, and this tool is not available. We refer to Section 3 for details.

From a modelling point of view, the appearance of a self-energy is one of the issues in the study of turbulence which seem still not to be completely understood. For example, computing the energy spectrum via the canonical Gibbs measure for a point vortex system, one finds a self-energy of order k^{-1} of each point vortex, which is quite unphysical, as noted in [88]. Sufficient regularity of initial data can solve this dilemma: For an initial vorticity distribution with density in $L^1 \cap L^\infty$, Theorem 2.1 states that the solutions to the point vortex system converge weakly to solutions to the Euler equations, which says that the self-energy is negligible, at least for short times t.

2.2.2. Temperature regimes and negative temperatures

In his paper [85], Onsager noticed that the gas of vortices exhibits three different temperature regimes. First, for positive and large inverse temperature, the vortices are mostly close to the boundary. Second, for positive but small inverse temperature they will be more or less uniformly distributed. Third, Onsager argued that negative temperature states exist when the energy of the system is increased.

Negative absolute temperature is loosely defined as a decrease of the entropy as a function of the mean energy. The model predicts this to happen once the mean energy exceeds a critical value. A negative temperature canonical distribution describes the phenomenon that at high energy, the vortices of the same sign are forced to be close to each other, and indeed the creation of local clusters of vortices of the same sign has been observed in numerical experiments by Joyce and Montgomery [79]. In addition, Montgomery, Matthaeus, Stribling and Martinez [80] conducted simulations showing that two-dimensional incompressible flows with high Reynolds number are well described by solutions to the vortex mean field equation (see e.g. (3.6)) for negative β.

From the mathematical side, it was argued in [13] that the mean field scaling is relevant for the study of this negative temperature phase, and indeed Eyink and Spohn [40] proved that the microcanonical ensemble yields negative temperatures for the regularized vortex Hamiltonian in the mean field limit. The correct scaling limit procedure is crucial, as Fröhlich and Ruelle [47] argued that the negative temperature regime does not exist in the standard thermodynamical limit.

One may summarize that Onsager's theory can explain the spontaneous appearance of large-scale vortices in 2D flows if one accepts that point vortices might yield states of negative absolute temperature at sufficiently high energy.

3. Gibbs measures

The basic idea of statistical mechanics is that the system's state should be described by a probability measure on the state space, which is called Gibbs measure, and it is defined via the Hamiltonian $\mathcal{H}$, (2.15), of the system. It reads formally

$$\mu_{\beta,N}(dX^N) = \frac{1}{Z_{\beta,N}} e^{-\beta\mathcal{H}(X^N)} dX^N \tag{3.1}$$

where $X^N := (x_1, \ldots, x_N) \in \Lambda$, are the positions of the point vortices, Z_N is the partition function defined as

$$Z_{\beta,N} = \int_{\Lambda^N} e^{-\beta\mathcal{H}(X^N)} dX^N \tag{3.2}$$

and dX refers to a suitable a priori measure on the state space. The precise form of the Gibbs measure depends on the Hamiltonian, and especially on the shape and boundary conditions on Λ and the circulations. Therefore we will state the Gibbs measure and partition function for each result we discuss.

The Gibbs measure (3.1) gives the probability of the system being in a specific state, and their marginals, the k-point correlation functions or k-particle distribution functions, correspond to the probability density of finding the first k particles in the positions $x_1, \ldots, x_k$

$$\rho_k^N(x_1, \ldots, x_k) = \int \mu_{\beta,N}(x_1, \ldots, x_N)\, dx_{k+1} \ldots dx_N. \tag{3.3}$$

The k-point correlation functions will be important tools in the analysis of typical configurations of the point vortex system.

In this section we will present some important cases where results have been obtained, we will state the precise Gibbs measure (3.1) in every case. Note that the Hamiltonian (2.15), which it assigns to each configuration a potential energy, is finite dimensional in case of the point vortex system. While Hamiltonian and Gibbs measure are also defined for infinite systems it turned out that Gibbs measures were not a good description for the continuous case, in fact the free fields computed by them were far from those observed experimentally, see [13, 14] and references therein.

Last, a word on the notation: In the case of constant circulation, the configuration space consists ony of the position of the vortices, while in case of random circulations, each configuration is a position - circulation pair (x_i, γ_i), which is often abbreviated by $\tilde{x}_i$ for the sake of notational simplicity, see e.g. (3.18).

3.1. Limits of Gibbs measures: the deterministic setting in 2D

Kiessling [59] and, independently, Caglioti, Lions, Marchioro and Pulvirenti [13] investigated the mean field limit of the point vortex model with constant, positive circulations $\gamma > 0$ on a smooth, bounded, connected open domain $\Lambda \subset \mathbb{R}^2$. In this case, the state space consists of the positions of the vortices. The intensities of the vortices are chosen as $\gamma = 1/N$, constant and the same for all vortices, which means that the Hamiltonian (2.15) depends only on the positions $X^N := (x_1, \ldots, x_N) \in \Lambda^N$ of the vortices and therefore reads $\mathcal{H}_\Lambda(X^N) = \frac{1}{2} \sum\sum_{1 \le i \le j \le N} G_\Lambda(x_i, x_j) \; + \; \sum_{i=1}^N g_\Lambda(x_i)$. The Gibbs measure in this setting reads

$$\mu_{\gamma,\beta,N}(dX^N) = \frac{1}{Z_{\gamma,\beta,N}} e^{-\beta\gamma^2 \mathcal{H}(X^N)} dX^N \tag{3.4}$$

where $Z_{\gamma,\beta,N}$ denotes the partition function

$$Z_{\gamma,\beta,N} = \int_{\Lambda^N} e^{-\beta\gamma^2 \mathcal{H}(X^N)} dX^N. \tag{3.5}$$

The range of β where the partition function is finite defines the range of (values proportional to the inverse) temperatures for which the Gibbs measure (3.4) makes sense. In the case of Dirichlet boundary conditions on Λ, this is true for $\beta \in \left(-\frac{8\pi}{\gamma^2 N}, \infty\right)$. Recalling $\gamma = 1/N$ and rescaling the inverse temperature by $1/N$ yields $\beta \in (-8\pi, \infty)$. Dirichlet boundary conditions are essential as they ensure that the interaction G is positive, $g(x)$ is bounded from below in a ball around the origin and diverges logarithmically when x approaches the boundary of the domain Λ, see (2.22) and the discussion in section 2.2.1.

We are interested in the asymptotic behavior of the Gibbs measures (3.4) in the limit $N \to \infty$. For fixed inverse temperature β and constant intensities, μ_N is a sequence (only) in N. As the elements of the Gibbs sequence are points in different function spaces, i.e. they are functions defined on different domains, it is of advantage to define another quantity, the correlation functions, which give the probability density of finding the first j particles (actually, any particles by symmetry) in the positions $x_1, \ldots, x_j$. Taking a mean field type limit, the scope is to prove that the one point correlation functions converge in some sense to solutions ρ of a mean field equation

$$\begin{aligned} -\rho_\beta(x) &= \frac{e^{-\beta\psi}}{\int_\Lambda e^{-\beta\psi} dx} \\ -\Delta\psi &= \rho_\beta \quad \text{in } \Lambda \\ \psi &= 0 \quad \text{on } \partial\Lambda \end{aligned} \tag{3.6}$$

and to study the variational principles associated to (3.6). The reality is more involved, and we will summarize it in the following. At first, it was shown that, if the empirical distributions of the vortex system $\frac{1}{N}\sum_{j=1}^N \delta_{x_j}(dx)$ are converging, with large probability, weakly to a (smooth) vorticity profile ρ, any weak limit point of the Gibbs measure (3.4) is an average over infinite product measures:

Proposition 3.1 (Caglioti, Lions, Marchioro and Pulvirenti [13])**.** *Let ρ_k^N be the k-particle distribution function* (3.3) *associated to the Gibbs measure $\mu_{\beta,N}$. Then, there exists a limit $\rho_k^N \to \rho_k$ as $N \to \infty$ and we have*

$$\rho_k(x_1, \ldots, x_k) = \int \rho^{\otimes k} \pi(d\rho) \tag{3.7}$$

where π is a Borel probability measure on the space $\mathcal{M}_1^+$ of all probability measures on Λ.

The proof of Proposition 3.1 uses the De Finetti theorem and therefore needs exchangeability of the vortices, which fails in the case of random interactions, see Section 3.3. We observe that factorization of ρ_k, i.e. $\rho_k^N \to \rho^{\otimes k}$ weakly is true only if there is a unique solution to the mean field limit.

A-priori, the one-particle distribution function $\rho^N(x)$ converges to a *superposition* of solutions ρ of the mean field equation, characterized by the measure π, which is a measure on the space of probability measures. Further information can be drawn from the support of π:

Proposition 3.2. *The mixing measure π is concentrated on those solutions $\rho \in L^\infty(\Lambda)$ of the mean field equation* (3.6) *which minimize (for $\beta > 0$) or maximize (for $\beta < 0$) the free energy*

$$\mathcal{F}(\rho) = \frac{1}{2}\int_{\Lambda\times\Lambda} \rho(x) G_\Lambda(x-y)\rho(y) dxdy + \frac{1}{\beta}\int_\Lambda \rho(x)\log[|\Lambda|\rho(x)]\, dx \tag{3.8}$$

with the constraints $\rho \geq 0$ and $\int \rho dx = 1$.

If there exists a unique solution ρ to (3.6), the full sequence ρ_k^N converges to $\rho^{\otimes k}$ and the mixing measure π is a Dirac measure; in this case, we say that the weak limit of the Gibbs measures as $N \to \infty$ obeys a propagation of chaos property. Uniqueness holds in the case $\beta > 0$, and in some special cases when $\beta < 0$. Before we discuss these, we relate the solution to the mean field equation to solutions of the Euler equation. This is done via the stream function ψ, defined as (2.3) and we get finally that one-point distribution functions ρ^N defined as (3.3) converge to a superposition of solutions of the mean field equation (3.6). In particular, the one-particle distribution function converges to the unique solution of the Euler-Lagrange equation for the free energy $\mathcal{F}$, as described in Proposition 3.2, which means that the limit measures concentrate on very particular stationary solutions of the 2D Euler equation with velocity field $u = (-\partial_2\psi, \partial_1\psi)$.

3.1.1. Negative temperatures

For $\beta < 0$, uniqueness of solutions to (3.6) is unknown in general. From the above results, we can conclude a sufficient condition, summarizing what we know about (3.8):

Corollary 3.3 ([13], Corollary 4.1.). *If $\beta < 0$ and we assume that there exists a unique $\rho \in L^\infty(\Lambda)$ which maximizes* (3.8) *over all $\rho \in \mathcal{M}_1^+$ with the constraints $\rho \geq 0$ and $\int \rho dx = 1$, then ρ_k^N and $\rho_k^N \log \rho_k^N$ converge almost everywhere and in $L^1(\Lambda^k)$.*

We refer to [13] for the proof and note here just that (3.8) is defined and finite for absolutely continuous ρ with density in $L^p(\Lambda)$ such that $\rho \log \rho \in L^1(\Lambda)$.

The general case of negative temperature $\beta < 0$ corresponds to the so-called "one species version" of the mean field equation (3.6), as predicted by [79]. Their mathematical analysis is more delicate: Roughly, the solution should maximize the energy-entropy functional

$$\mathcal{G}(\psi) = -\frac{1}{2}\int_\Lambda |\nabla\psi|^2 - \frac{1}{\beta}\log\int_\Lambda e^{-\beta\psi} \tag{3.9}$$

with $\psi = -\Delta^{-1}\rho$, see e.g. [13], Proposition 7.2. This works, however, only for $\beta > -8\pi$, due to the shape of the partition function. Further results for $\beta \to -8\pi$ are reviewed in [13], they depend on the exact properties of the domain Λ. For example, in case of the unit disk, the vorticity distribution ρ takes the explicit form

$$\rho = -\Delta\psi = \frac{1 - c_\beta}{\pi}\frac{1}{(1 - c_\beta r^2)^2} \tag{3.10}$$

with $c_\beta = \frac{\beta}{8\pi+\beta}$. This solution concentrates at zero when $\beta \to -8\pi$ and $\rho \to \delta_0$ weakly in the sense of measures. In an annulus, there is not a particular point at which radial solutions can concentrate as $\beta \to -8\pi$, so the authors suggest that a unique radial solution can be obtained for all $\beta \in \mathbb{R}$, see [13], Section 7. Similar results are possible for radial solutions in connected domains with rotational symmetry. Moreover, a non-existence result for star-shaped domains was proven, with $\beta < 0$ and its exact value depends on the boundary of the domain, see [13] Proposition 7.1.

Furthermore, Kiessling [59], Theorem 5, showed that for Λ a ball and $\beta < -4\pi$, the Dirac measure is a weak solution to the Euler Lagrange equation for the free energy functional. He proposes to use this result to extend the definition of an equilibrium state to all temperatures, with the following physical pictures in mind (see [59], page 52): When the inverse temperature approaches the critical value -8π from above, the thermal motion of the particles cannot prevent the singular attractive interactions from forcing the system to condense to a point located at the center of the ball (Kraichnan supercondensation phenomenon, [59] page 52). If the temperature drops below the critical value, the system will stay in the collapsed state. As the Dirac measure is admissible as an equilibrium measure of a collapsed system, this could be a possible extension of the definition of equilibrium state. As the free energy functional is ∞ at the Dirac measure for $-4\pi \geq \beta \geq -8\pi$ and $-\infty$ for $\beta < -8\pi$), he classifies this as an extreme case of a second-order phase transition, i.e. a critical behavior of the canonical free energy at $\beta = -8\pi$. However, it remains open whether the Dirac measure is also the equilibrium measure for $N < \infty$.

3.2. Gibbs measures for vortices of positive and negative intensities on the torus

We have seen above that a negative value for β complicates significantly the situation. While in the case of constant positive circulations results can be obtained by assuming $\beta > 0$, this advantage disappears when the intensity of the vortices itself has a sign. Now we discuss a result in this direction from [9]. Consider the point vortex equation on a two-dimensional flat torus $\mathbb{T} \subset \mathbb{R}^2$. In this case, by periodicity of u and the circulation theorem, the following neutrality property of the vorticity holds:

$$\int_{\mathbb{T}} \theta(x)dx = 0. \tag{3.11}$$

The torus case offers several advantages, in particular the fundamental solution to Poisson's equation on the torus can be written in Fourier expansion as $G_{\mathbb{T}}(x,y) = \frac{1}{(2\pi)^2}\sum_{k\in\mathbb{Z}^2,k\neq 0} k^{-2}e^{ik\cdot(x-y)}$ which gives also a more explicit expression for the Hamiltonian, which, in the torus case, contains only the interaction term (2.18). The existence of the Gibbs measure for this system was proven by Fröhlich and Seiler in [48]. Also in this case, one is interested in proving statements saying, morally, that the solutions to the point vortex dynamics can be interpreted as generalized solutions of the Euler vortex equations in the sense of a generalization of Theorem 2.1 for positive and negative vortices [36, 42, 99]. It turns out that the limit measure is Gaussian and can be characterized by methods of Quantum Field Theory. In fact, in this special case, it is known that for $\beta, \eta > 0$ the sum

$$\frac{\beta}{2}\int_{\mathbb{T}} u^2(x)dx \;+\; \frac{\eta}{2}\int_{\mathbb{T}} \theta^2(x)dx \;=\; \frac{1}{2}\langle \theta, (\eta\mathbf{1} - \beta\Delta^{-1})\theta\rangle \tag{3.12}$$

is a quadratic form generating a Gaussian measure

$$\mu_{\beta,\eta}(d\theta) \;=\; \prod_{k\in\mathbb{Z}^2,k\neq 0} \frac{\exp(-\frac{1}{2}|\hat{\theta}_k|^2(\eta+\beta/k^2))}{2\pi(\eta+\beta/k^2)^{-1}} d\hat{\theta}_k \tag{3.13}$$

These formally invariant measures are, however, not a good tool as they are concentrated on a set of distribution for which the initial value problem does not easily make sense, in fact the set of all vorticities $\theta \in L^\infty(\mathbb{T})$ has $\mu_{\beta,\eta}$-measure zero.

To prove that also in the case of positive and negative intensities on the torus, the Gibbs measures associated with an interacting system of vortices converges to an equilibrium measure, a regularization of $G_{\mathbb{T}}(x,y)$ was introduced, which smoothes the logarithmic divergence when $x \approx y$:

$$G_{\epsilon,\mathbb{T}}(x,y) = \frac{1}{(2\pi)^2}\sum_{k\in\mathbb{Z}^2,k\neq 0} \frac{\exp(ik\cdot(x-y))\exp(-\epsilon k^2)}{k^2}. \tag{3.14}$$

Consider a gas of vortices interacting via the regularized Green's function, let the vortices interact with strength $\gamma_i = \pm\sqrt{\sigma}$, $\sigma > 0$, and assume the neutrality condition

$$\sum_{i=1}^{N} \gamma_i = 0 \tag{3.15}$$

Using the rescaling $\sigma = \frac{2\pi^2}{\eta N}$ so that $\sqrt{\sigma} \to 0$ and $N\sigma \to$ const, one obtains the following result:

Theorem 3.4 (Benfatto, Picco and Pulvirenti [9]). *Let the vortices θ_i have each a small support of diameter ϵ centered at $x_i = (x_i^1, x_i^2)$, let intensity of the vortices be $\gamma_i = \pm\sqrt{\sigma}$ and let the neutrality condition* (3.15) *hold. Let Then the sequence of canonical Gibbs measures $\mu_{N,\beta,\sigma}$ associated with a gas of vortex blobs interacting via a two-body interaction* (3.14) *converges weakly in the simultaneous limit $N \to \infty$ and $\epsilon \to 0$, with $\epsilon \approx N^{-\delta}$ for some $0 < \delta < \frac{\eta}{2\pi\beta}$ to the Gaussian measure* (3.13)*, which is invariant for the two-dimensional Euler Equation.*

Therefore, at least in statistical terms as in the situation of Theorem 3.4, a long-time control of the Euler flow by means of the vortex dynamics can be obtained. A version of this result without cutoff, read in terms of fluctuations of a mean field limit, has been proved on the disc by [10] and on the torus by [53].

3.3. Gibbs measures for random intensities

A more general setting is to consider point vortices with **random intensities**, as studied by Neri [82, 83], Bodineau and Guionnet [10] and, more recently, by Sawada and Suzuki [98] in 2D, and by Kiessling and Wang [60] on the 3D sphere. The study of random intensities is a way to provide a mathematical explanation to certain results in statistical physics. An example is the work of Joyce and Montgomery on "neutral systems" [79], which are defined by an equal number of vortices of intensity ± 1 with probability $1/2$.

Limit distributions for a system of point vortices with random intensities were studied by Bodineau and Guionnet [10], Neri [82] and Sawada and Suzuki [98]. We first sketch the results of [82], which generalize the approach taken in [13], and return to [10], where also a Central Limit Theorem and a Large Deviation Principle was shown, in the next sections. Note that the setting in [82] differs from [98], in which each vortex has a fixed circulation value and the ratio of the number of the vortices with a certain circulation to the number of whole point vortices are given by a probability measure.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space and define by $\mathcal{M}_1^+(D)$ the space of probability measures on a set D. There are now two possibilities to define the Gibbs measure: In the averaged setting, we define the Gibbs measures on the product space $\mathcal{M}_1^+((\Lambda\times[-1, 1])^N)$, in the quenched setting, we assume that the circulations observe a certain ratio of positive and negative signs, and keep γ as a parameter, defining the Gibbs measure only over Λ^N.

A classical setup is to work in the averaged setting , i.e. with the configuration space being the product space of pairs of positions and intensities $(\Lambda \times [-1, 1])^N$. We use again the short-hand notation $X^N = (x_1, \dots, x_N)$ for the positions and $\gamma^N = (\gamma_1, \dots, \gamma_N)$ for the random circulations. The Hamiltonian, which now depends both positions and intensities, reads

$$\mathcal{H}_\Lambda(X^N, \gamma^N) = \frac{1}{2}\sum\sum_{1\le i\le j\le N}\gamma_i\gamma_j G_\Lambda(x_i, x_j) \; + \; \sum_{i=1}^{N}\gamma_i^2 g_\Lambda(x_i). \tag{3.16}$$

with $G_\Lambda(x, y)$ the Green's function of the Poisson equation in Λ with Dirichlet boundary conditions.

The averaged Gibbs measure on $\mathcal{M}_1^+((\Lambda \times [1, 1])^N)$ reads

$$d\mu_{\beta,N}(X^N, \gamma^N) = \frac{1}{Z_N^\gamma(\beta)} e^{-\frac{\beta}{N}\mathcal{H}(X^N,\gamma^N)} d\mathcal{L}^N \otimes dP^N \tag{3.17}$$

with a Borelian probability measure P on $[-1, 1]$.

Proposition 3.5 ([82], Corr. 4). *Consider an N point vortex system on a bounded domain $\Lambda \subset \mathbb{R}^2$ and assume that the vortices are distributed according to the Lebesgue measure $\mathcal{L}$. Let the vortex intensities γ_i be random variables identically distributed w.r.t a Borelian probability measure P on $[-1, 1]$ and consider a rescaled temperature $\beta/N \in (-8\pi, 8\pi)$.*

Then we have that the k-point correlation functions ρ_k^N are bounded in $L^p((\Lambda\times[-1, 1])^k)$ for all k and N large enough and hence, if $p > 1$, there exists a subsequence $\rho_k^{N_j} \rightharpoonup \rho_k$ weakly in $L^p((\Lambda \times [-1, 1])^k)$.

The limit ρ_k is an average of product measures w.r.t. some mixing measure π, which is a Borelian probability measure supported inside a ball of finite radius of $L^\infty(\Lambda \times [-1, 1])$ centered at the origin.

The crucial property here is the exchangeability of the position-circulation pairs, which allows to apply a Hewitt-Savage theorem and to conclude the existence of a mixing measure π as in the deterministic case, see Proposition 3.1. Note that in order to obtain the symmetry of ρ_k, which allows to define the limit energy in terms of the two-point distribution function, infinite exchangeability is needed. In this case, one can deduce the following variational principle, where we used the abbreviation $\tilde{x}_i := (x_i, \gamma_i)$ for the position - circulation pair.

Proposition 3.6 ([82], Theorem 11). *The weak cluster points $\mu_* = (\rho_k)_{k\in\mathbb{N}}$ are minimizers of the functional*

$$F^*(\rho) := \frac{\beta}{2} \int_{(\Lambda\times[1,1])^2} \mathcal{H}(\tilde{x}_1, \tilde{x}_2)\rho_2(\tilde{x}_1, \tilde{x}_2)\, d\tilde{x}_1 d\tilde{x}_2 \ + \ \lim_{k\to\infty} \frac{1}{k} \int_{(\Lambda\times[1,1])^k} \rho_k \log \rho_k \, d(\tilde{x}_1, \ldots, \tilde{x}_k) \qquad (3.18)$$

and the appearing limit is finite for symmetric ρ_j which are bounded in $L^\infty((\Lambda \times [1, 1])^k)$.

Without the symmetry of ρ_k, which is a consequence of the infinite exchangeability, a representation of the energy part in (3.18) as a two-marginal is not possible, This makes the analysis of the quenched case much harder. In [10], where a functional similar to the energy-entropy functional (3.18) with a general energy term

$$\mathcal{E}(\nu) = \int\int \gamma\gamma' G_\Lambda(x, y) d\nu(x, \gamma) d\nu(y, \gamma') \qquad (3.19)$$

appears as a good rate function, new techniques are used: for example, entropic inequalities are used to control the logarithmic singularity of the energy by the relative entropy, in order to achieve lower semicontinuity of the energy-entropy functional. Furthermore, it is used that for any probability measure ν with finite entropy, the energy (3.19) coincides with a modified energy $\hat{\mathcal{E}}$ where Hamiltonian is not integrated across the diagonal $x = y$, and that this nicer modified energy is quasi-continuous, see [10], Section 3.2. for details.

To return to the discussion of the mean field limit of Gibbs measures with random intensities, Neri showed that for each weak cluster point μ_* of the Gibbs measure, there exists a mixing measure $\pi_* \in \mathcal{M}_1^+(\Lambda \times [-1, 1])$. Moreover, the 1-point distribution function on Λ can be associated to some solutions of a non-linear Poisson equation, more precisely:

Theorem 3.7 ([82], Th. 16). *For π-almost every $\mu \in supp\,\pi$, its potential u is in C^∞ and satisfies the following mean field equation*

$$\begin{aligned} -\Delta u(x) &= \left(\int_\Lambda \int_{-1}^1 e^{-\beta\gamma' u(y)} dy P(d\gamma') \right)^{-1} \int_{-1}^1 \gamma e^{-\beta\gamma u(x)} P(d\gamma) \quad \forall x \in \Lambda \\ u(x) &= 0 \qquad \forall x \in \partial\Lambda \end{aligned} \qquad (3.20)$$

This result is consistent with the analysis for constant circulations of intensity one, in fact setting $P = \delta_1$, we recover the mean field equation of [13] and [59].

Note that the mean fields are essentially Newton potentials associated with minima of an energy-entropy functional $\mathcal{G}$, as the deterministic case (3.9), and the mean-field equation is the Euler-Lagrange equation associated to this functional. The functional acts on the potentials and reads

$$\mathcal{G}(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{1}{\beta}\log\left(e^{-\beta\rho u(x_1)}d\tilde{x}_1\right) \tag{3.21}$$

Note that $\mathcal{G}$ is not equal to the energy-entropy functional (3.18), but related: Neri shows the existence of minimizers for $\mathcal{G}$ using the sharp form of Moser-Trudinger inequality, and that $\mathcal{G}$ preserves the minimizers of F in the sense that the potential u of a minimizer μ of (3.18) is a minimizer of $\mathcal{G}$. For positive inverse temperature $\beta > 0$, the functional $\mathcal{G}$ is convex, implying the uniqueness of solutions to the mean field equation (3.20). For $-8\pi < \beta < 0$, every minimizing sequence of $\mathcal{G}$ is bounded. Last, note that zero is a solution of (3.20) in the case that the vortices are uniformly distributed on Λ, i.e. $\int_{-1}^{1}\gamma_1 P(d\gamma_1) = 0$. In the case that $\Lambda = [0,1]^2$, the trivial zero solution is not a physical one, in the sense that zero is not a minimizer of $\mathcal{G}$, see [82], Chapter 10.

3.4. Law of Large Numbers and Central Limit Theorem

We have seen in Sections 3.1, 3.2 and 3.3 that if there exists a unique solution ρ_* to the variational problem, the propagation of chaos property holds. Consider now a situation when the propagation of chaos property holds and additionally $\gamma_i \equiv 1$. In this case there is a Law of Large Numbers for the empirical vorticity distribution (2.9) in the sense that $\theta_N \rightharpoonup \rho_*$ weakly for any $f \in C_b(\Lambda)$ and any $\epsilon > 0$, we have

$$\left|\int_\Lambda f(x)\theta_N(x)dx - \int_\Lambda f(x)\rho_*(x)dx\right| < \epsilon \tag{3.22}$$

with probability going to one as $N \to \infty$.

The connection between the propagation of chaos property and the Law of Large Numbers becomes even more plain in the study of the microcanonical ensemble, see [69], p. 51:

Lemma 3.8 ([69], Lemma 5.1.). *Let $\Lambda \subset \mathbb{R}^2$ bounded, smooth, open and connected and* meas$(\Lambda) = 1$. *Let $\beta > -8\pi$ and $\rho \in L^1(\Lambda)$ such that $\rho \geq 0$ and $\int_\Lambda \rho dx = 1$. Then the following assertions are equivalent:*

1. *For all $\phi \in C(\bar{\Lambda})$*

$$\int_\Lambda \left|\frac{1}{N}\sum_{i=1}^{N}\phi(x_i) - \int_\Lambda \phi\rho dx\right|^2 d\mu_{\beta,N} \longrightarrow 0 \qquad \text{as } N \to \infty. \tag{3.23}$$

2. *For all $\phi \in C(\bar{\Lambda})$*

$$\int_\Lambda \sup_{\|\phi\|_{W^{1,\infty}(\Lambda)}\leq 1}\left|\frac{1}{N}\sum_{i=1}^{N}\phi(x_i) - \int_\Lambda \phi\rho dx\right|^2 d\mu_{\beta,N} \longrightarrow 0 \qquad \text{as } N \to \infty. \tag{3.24}$$

3. *$\rho_k^N \longrightarrow \prod_{i=1}^{k}\rho(x_i)$ weakly in the sense of measures for all $k \geq 1$.*

4. $\rho_k^N \longrightarrow \prod_{i=1}^k \rho(x_i)$ *weakly in the sense of measures for* $k = 1$ *and* $k = 2$.

We will investigate further the microcanonical case in this work, but just mention that in this framework, the microcanonical variational principle corresponds to the minimization of the entropy at fixed energy, while in the discussion above it was based on the (free energy) functional, which is the sum of (minus the) entropy and the energy. The free energy functional was defined for all ρ satisfying the conditions mentioned in the above Lemma, plus the condition that $\rho \log \rho \in L^1(\Lambda)$.

A unique minimizer and therefore propagation of chaos are fundamental prerequisites for a Law of Large Numbers, and are sufficient, in case of bounded interaction, as e.g. when working with a regularization as in [9], also for negative inverse temperature β (at least when β small enough in absolute value).

Indeed, Neri [82] proves that in the neutral case, $\mathcal{L}\times\nu$ is not a minimizer when β is negative and large enough in absolute value. In conclusion, there seems to be no way to study the negative temperature case $\beta < 0$, for β sufficiently large and negative, in view of a Law of Large Numbers or a Central Limit Theorem.

For positive temperature, it is possible to derive a quenched Law of Large Numbers and Central Limit Theorem in case that the total intensity satisfies the neutrality condition (3.15): For this one considers the empirical measure $\theta_N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i,\gamma_i}$ and concludes, thanks to the propagation of chaos property, that the rescaled empirical measure converges, as $N \to \infty$, to a limiting law $\rho_*(dx, d\gamma)$.

The Central Limit Theorem then asks the question: what is the behavior of the fluctuations of the empirical measure θ_N around the limiting law $\rho_*(dx, d\gamma)$?

To prove such a Central Limit Theorem is extremely difficult due to the strong interactions between the particles. Bodineau-Guionnet [10] investigate fluctuations of the empirical measure for positive inverse temperature $\beta > 0$ in the case of Λ a disk, and neutral gas. In this case, the empirical measure converges towards $\rho_*(dx, d\gamma) = \mathcal{L}(dx) \otimes P(d\gamma)$.

To fix notation, we define the operator $\Xi \in L^2(\rho_*)$ with kernel $G_\Lambda(x, y)\gamma\gamma'$ and denote by I the identity on $L^2(\rho_*)$. Note that Ξ is a non-negative operator, and consequently $(I+\beta\Xi)^{-1}$ is non-degenerate at positive temperature. In [10], the following result is obtained:

Theorem 3.9 (Central Limit Theorem for neutral gases)**.** *Let* $\beta > 0$ *and* Λ *a circular disk.*

1) quenched case*: Assume* $|\mathrm{card}\{i : \gamma_i = 1\} - \mathrm{card}\{i : \gamma_i = -1\}| = o(N^{3/4})$*. Then, for any function* $f \in L \subset L^2(\rho_*)$*, we have that*

$$\frac{1}{\sqrt{N}}\sum_{i=1}^N\left(f(x_i\gamma_i) - \int f(x,\gamma)d\rho_*(x,\gamma)\right) \tag{3.25}$$

converges in law under the quenched Gibbs measure $\mu_{\beta,N}^\gamma$ *towards a centered Gaussian random variable with covariance*

$$\int \left(f - \bar{f}\right)(I + \beta\Xi)^{-1}\left(f - \bar{f}\right)\, d\rho_* \tag{3.26}$$

where we denoted $\bar{f} := \int f d\rho_*$.

2) averaged case*: Assume the intensities are symmetrically Bernoulli distributed, i.e.* $P = \frac{1}{2}(\delta_1 + \delta_{-1})$*. Then* (3.25) *converges in law under the averaged Gibbs measure* $\mu_{\beta,N}$ *towards a centered Gaussian random variable with covariance* (3.26).

The investigation of the Law of Large Numbers and Central Limit Theorem is particularly difficult due to the logarithmic singularity of the interaction and the missing control of the boundary term of G_Λ. To deal with the logarithmic singularity of the interaction, the paper [10] employs techniques developed in [8] for strongly interacting random variables, which allow to study fluctuations as soon as the empirical measure converges. To use this approach, it is necessary to restrict the test functions f to a subset $\mathcal{U} \subset L^2(\rho_*)$, more precisely

$$\mathcal{U} = \left\{ f \in L^2(\rho_*) \mid f(x,\gamma) = (I + \beta\Xi\left(\lambda_\gamma^{-1}\nabla\cdot(\lambda_\gamma g)\right),\ g \in C^1(\Lambda), g|_{\partial\Lambda} = 0 \right\} \tag{3.27}$$

where $\lambda_\gamma(x) = \frac{\rho_*(dx,d\gamma)}{d\mathcal{L}(x)dP(\gamma)}$.

Moreover, good controls of the partition function are necessary: The authors of [10] can show for neutral gases at positive inverse temperature $\beta > 0$ and Λ a disk that

$$\limsup N^{-1/2} \log Z_{N,\beta} = 0. \tag{3.28}$$

The upper bound is implied by an estimate that the partition function grows at most polynomially in N. For the lower bound, in the quenched setting we get

$$\log Z^\gamma_{N,\beta} \ \geq\ -\beta \sum_{i<j} \gamma_i\gamma_j \int G_\Lambda(x,y)d\mathcal{L}(x)d\mathcal{L}(y) \tag{3.29}$$

and the right hand side of (3.29) is of order $o(\sqrt{N})$ if it can be assured that $\sum_{i=1}^N \gamma_i = o(N^{3/4})$, which explains the assumption on γ_i in Theorem 3.9. The problem of the Central Limit Theorem has been recently reconsidered in [53].

In the negative temperature regime it is not clear which is the range of validity of the Law of Large Numbers and of the Central Limit Theorem. To fix ideas, consider a dilute gas of point vortices on the torus. For positive β the limit of measures (3.17), for $N \uparrow \infty$, is $\ell \otimes P$, where ℓ is the normalized measure on the torus and P is the prior distribution on vortex intensities. The same holds for small negative values of β,see [69], up to a critical value $\beta_c \in (-8\pi, 0)$. For values of $\beta \leq \beta_c$ the measure $\ell \otimes P$ is not anymore a minimizer of the free energy, although is still a critical point and thus a solution of the mean field equation. It is an open problem to prove any kind of fluctuations result for the negative regime. We do not know if the Gaussian CLT holds up to β_c. We notice that if we look at the candidate limit covariance, the same given in 3.9, we immediately realise that the candidate covariance is positive definite up to the value β_c. Based on this, we conjecture that the CLT holds up to β_c.

3.5. Large Deviations

Viewing the vortex method as a way to approximate stationary solutions to the Euler equation, it is natural to investigate the speed of convergence. This leads to the question of Large Deviation Principles. Large Deviation Principles state that for any $\epsilon > 0$ and $f \in C_b(\Lambda)$

$$\mu_N\left(\left\{\left| \int_\Lambda f(x)\theta_N(x)dx\ -\ \int_\Lambda f(x)\rho_*(x)dx \right| \geq \epsilon\right\}\right) \leq e^{-N\cdot\Delta} \tag{3.30}$$

where Δ is the difference between the maximum entropy $S(\rho_*)$ and the maximum of the entropy $S(\rho)$ under the constraint that $|\rho(f) - \rho_*(f)| \geq \epsilon$. A Large Deviation upper bound in case of bounded potentials (including the boundary term) was proved in [40].

In case of an a bounded domain $\Lambda \subset \mathbb{R}^2$ and the logarithmic potential G_Λ (2.16) with Dirichlet boundary conditions, a Large Deviation principle was proven for neutral gases by Bodineau-Guionnet [10] and, more recently, by Leble, Serfaty and Zeitouni [65]. We describe their results in the following.

Let, as in Section 3.3, $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space and $\mathcal{M}_1^+(\Lambda \times \{-1,1\})$ the space of probability measures on the product space $\Lambda \times \{-1,1\}$ of position and intensity. Let the vortices be distributed according to the Lebesgue measure $\mathcal{L}$ and the circulations be identically distributed with Bernoulli law P. Denote $\mathcal{M}_P := \left\{\nu \in \mathcal{M}_1^+(\Lambda \times \{-1,1\}) : \pi_2 \circ \nu = P\right\}$ the set of probability measures with intensity marginal P.

We consider the energy functional

$$\mathcal{E}(\nu) = \int\int \gamma\gamma' G_\Lambda(x,y) d\nu(x,\gamma) d\nu(y,\gamma') \qquad \forall \nu \in \mathcal{M}_1^+(\Lambda \times \{-1,1\}) \tag{3.31}$$

and the lower-semicontinuous mixed energy-entropy functional

$$\mathcal{F}_\beta(\nu) = H(\nu|\mathcal{L} \otimes P) + \frac{\beta}{2}\mathcal{E}(\nu)\mathcal{M}_1^+(\Lambda \times \{-1,1\}) \tag{3.32}$$

where $H(\nu|\mathcal{L} \otimes P)$ is the relative entropy of ν w.r.t. the product measure $\mathcal{L} \otimes P$.

The quenched setting refers to a fixed ratio of ± 1-valued intensities γ_i, which do not satisfy the neutrality condition (3.15). Abbreviate, as always, $X^N = (x_1, \ldots, x_N)$. The quenched Gibbs measures reads

$$d\mu^\gamma_{\beta,N}(X^N) = \frac{1}{Z_N^\gamma(\beta)} e^{-\frac{\beta}{N}\mathcal{H}(X^N,\gamma)} d\mathcal{L}^N(X^N) \tag{3.33}$$

Theorem 3.10 (Quenched Large Deviation Principle)**.** *For any $\beta \in (-8\pi, \infty)$, if for some measure P holds $\frac{1}{N}\sum_{i=1}^N \delta_{\gamma_i} \longrightarrow P$ then the law of the empirical measure $\theta_N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i,\gamma_i}$ under the quenched Gibbs measure $\mu^\gamma_{\beta,N}$ satisfies a Large Deviation Principle with rate function*

$$\mathcal{G}_q(\mu) := \mathcal{F}_\beta(\mu) - \inf_{\mu \in \mathcal{M}_P} \mathcal{F}_\beta \qquad \text{if } \mu \in \mathcal{M}_P \tag{3.34}$$

and $\mathcal{G}_q(\mu) = \infty$ otherwise.

Note that, thanks to the property that there are roughly as many positive vortices as negative vortices, the limit of the empirical measure $\rho_*(dx, d\gamma) = \mathcal{L}(dx) \otimes P(d\gamma)$ is a minimum of $\mathcal{G}_q$. Therefore, for not too negative temperatures, the positive and negative vortices are both uniformly distributed over Λ.

Note that the convexity of the rate function is not clear at all in the negative temperature case.

The averaged setting. As already mentioned in 3.3, the averaged setting is preferable due to the possibility to obtain a mixing measure by applying a Hewitt-Savage type theorem. The averaged setting deals with i.i.d. Bernoulli distributed random intensities, and we denote the occurring Bernoulli law by $P^N = P^{\otimes N}$. Define furthermore by $\mathcal{M}_P$ the set of probability measures with marginal P. We recall the averaged Gibbs measure from (3.17), this time on $\mathcal{M}_1^+((\Lambda \times \{-1,1\})^N)$.

Due to the singularity of the interaction, there is not enough control on the partition function to ensure exponential tightness for large values of β. Therefore, we either need to restrict to only positive circulations $\gamma_i = 1$ or, in case of positive and negative circulations, to a temperature range which is symmetric around zero.

Theorem 3.11 (Averaged Large Deviation Principle)**.** *For any $\beta \in (-8\pi, 8\pi)$, if for some measure P holds $\frac{1}{N}\sum_{i=1}^{N}\delta_{\gamma_i} \longrightarrow P$, then the law of the empirical measure $\theta_N = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i,\gamma_i}$ under the averaged Gibbs measure $\mu_{\beta,N}$ satisfies a Large Deviation Principle with rate function*

$$\mathcal{G}_a(\mu) := \mathcal{F}_\beta(\mu) - \inf_{\mu \in \mathcal{M}_1^+(\Lambda\times\{-1,1\})} \mathcal{F}_\beta \tag{3.35}$$

The proof of the Large Deviation Principle is done by controlling the singularities of the Hamiltonian (3.16). For this, it is shown that the energy functional (3.31) is quasi-continuous for probability measures with finite entropy. The main difficulty comes from the logarithmic singularity of the interaction and the missing control of the boundary term of G_Λ. To control the partition function, it is necessary to restrict to the case of a disk, yet the analysis is extremely delicate.

3.5.1. Large Deviation Principle for two-component plasma in a box

Leble, Serfaty and Zeitouni [65] derive a Large Deviations Principle for the case of a two-dimensional two-component plasma in a box. This case is different from the model considered above due to the different scaling of β: While in [10] β scales as $1/N$, [65] consider the thermodynamical limit with constant β in a box of finite size, with particles living in Λ. Also the techniques needed to handle this situation are completely different, and we will sketch them briefly here.

By two-component plasma we call an ensemble with $2N$ particles, of which N particles of positive charge and N particles of negative charge, which interact logarithmically. Call $X^N = (x_1, \dots, x_N)$ the points in the unit cube $\Lambda = [0, 1]^2$ carrying a positive charge $+1$, and $Y^N = (y_1, \dots, y_N)$ the points in Λ carrying a negative charge -1 . The Gibbs measures associated to the two-component plasma in the plane reads

$$d\mu_{\beta,N}(X^N, Y^N) = \frac{1}{Z_{N,\beta}} e^{-\beta/2} w_N(X^N, Y^N) dX^N dY^N \tag{3.36}$$

with $dX^N dY^N$ the Lebesgue measure on Λ^{2N}, the partition function

$$Z_{N,\beta} := \int_{\Lambda^{2N}} e^{-\beta/2} w_N(X^N, Y^N) dX^N dY^N \tag{3.37}$$

and the logarithmic interaction

$$w_N(X^N, Y^N) := \sum_{1\le i\ne j\le N} - \log|x_i - x_j| - \log|y_i - y_j| + 2 \sum_{1\le i,j\le N} \log|x_i - y_i|\,. \tag{3.38}$$

Due to a different choice of constants in the interaction, the critical case is now not $|\beta| = 8\pi$ but $\beta = 2$.

The neutral case is more difficult than the case of only positive circulations, as the interaction energy is no longer bounded from below. In [65] the authors worked with "dipoles" of pairs of particles with opposite sign, matched by nearest neighbour pairing, and express the interaction energy via a electric potential generated by the system of charges. We will summarize this setup in the following:

Let $\mathcal{X}$ be the set of locally finite signed point configurations with the topology of local convergence and let $\mathcal{P}(\mathcal{X})$ denote the set of probability measures on $\mathcal{X}$. Let τ_λ denotes the action of translation by a vector $\lambda \in \mathbb{R}^2$. We may identify the pair of N-tuples of points (X^N, Y^N) in the square Λ as an element of the space $\mathcal{X}$ by associating to X^N and resp. Y^N the point configuration $\nu_N^+ = \sum_{i=1}^N \delta_{x_i}$ and $\nu_N^- = \sum_{i=1}^N \delta_{y_i}$. Then we rescale the finite signed configurations by a factor of $\sqrt{N}$ to get $\hat{\nu}_N^+ = \sum_{i=1}^N \delta_{\sqrt{N}x_i}$ and $\hat{\nu}_N^- = \sum_{i=1}^N \delta_{\sqrt{N}y_i}$. We define the map

$$
\begin{aligned}
i_N : (\mathbb{R}^2)^N \times (\mathbb{R}^2)^N &\longrightarrow \mathcal{P}(\Lambda \times \mathcal{X}) \\
(X^N, Y^N) &\mapsto P_{(X^N,Y^N)} := \int \delta_{(x, \tau_{\sqrt{N}x} \cdot (\hat{\nu}_N^+ - \hat{\nu}_N^-))} dx
\end{aligned}
\tag{3.39}
$$

The variable x is a "tag" that keeps track of the point $x \in \Lambda$ around which the configuration was blown up, and in this way we build from any signed point configuration the law $P_{(X^N,Y^N)}$ of a tagged signed point process. We denote by $\mathcal{P}_{inv,1}(\Lambda \times \mathcal{X})$ the set of stationary laws P of signed point processes such that P has total intensity 1, i.e. there is, on average, one point of each sign per unit volume. On $\mathcal{P}_{inv,1}(\Lambda \times \mathcal{X})$ we define the sum of the interaction energy $\mathbb{W}$ and the specific relative entropy

$$
\mathcal{F}_\beta(P) = \frac{\beta}{2}\mathbb{W}(P) + \mathrm{Ent}(P) \qquad \text{if } \mathrm{Ent}(P) < \infty \tag{3.40}
$$

and infinity otherwise. The interaction energy is defined as the infinite volume limit of the logarithmic interaction in the system of charges described by the signed configurations. For tagged particles, it is simply evaluated at the corresponding distintegration measure P^x and integrated over P. The specific relative entropy for tagged particles is defined in the same manner:

$$
\mathrm{Ent}(P) := \int_\Lambda \mathrm{Ent}(P^x)dx = \int_\Lambda \lim_{R\to\infty} \frac{1}{R^2}\mathrm{Ent}(P_R|(\Pi^1 \otimes \Pi^1)_R) \tag{3.41}
$$

where the subscript R denotes the restriction of a measure to the cube $C_R = [-R/2, R/2]$, $\Pi^1 \otimes \Pi^1$ is the law of two Poisson point processes of intensity 1 and $\mathrm{Ent}(\mu|\nu) = \int \log \frac{d\mu}{d\nu} d\mu$ is the usual relative entropy.

The specific relative entropy of the law of a signed point process is therefore the infinite-volume limit of the usual relative entropy with respect to a reference measure. It favors disorder and thus tends to separate the dipole points, so it can be used to control the partition function for small β, when this term dominates. The interaction term competes with the entropy term, it gets stronger as β gets larger and favors signed configurations which minimize the logarithmic interaction. The interesting point is now that these two terms are bounded from below for $\beta < 2$, enabling us to say that, at the microscopic level, the point process induced by the Gibbs measure has a typical behavior:

Theorem 3.12 ([65], Theorem 1 and 2)**.** *Consider inverse temperatures* $0 < \beta < 2$ *and* $\Lambda = [0, 1]^2$ *the unit cube in* $\mathbb{R}^2$*. Let* $x_1, \dots, x_N$ *be the point charges carrying a positive charge of intensity* 1 *and* $y_1, \dots, y_N$ *be the point charges carrying a negative charge of intensity* -1*. Let* $\mathcal{X}$ *be the set of locally finite signed point configurations with the topology of local convergence. Denote the measure* $\widetilde{\mu}_N^\beta$ *as the push-forward of the Gibbs measure* μ_N^β *by the map* i_N *defined in* (3.39)*.*

Then sequence of probability measures $\widetilde{\mu}_N^\beta$ *satisfies a Large Deviation Principle at speed N with good rate function given by*

$$
\mathcal{F}_\beta^{sc} - \inf_{\mathcal{P}_{inv,1}(\Lambda\times\mathcal{X})} \mathcal{F}_\beta \tag{3.42}
$$

where $\mathcal{F}_\beta^{sc}$ *is the lower semicontinuous envelope of* $\mathcal{F}_\beta$*.*

Moreover, the sequence of empirical measures associated to the positive and negative charges

$$(\theta_N^+, \theta_N^-)_N := \left(\frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}, \frac{1}{N} \sum_{i=1}^{N} \delta_{y_i} \right)_N \tag{3.43}$$

converges μ_N^β- almost surely to $\mathcal{L}_\Lambda \otimes \mathcal{L}_\Lambda$, where $\mathcal{L}_\Lambda$ is the uniform probability measure on Λ.

The result is derived using that the law of a special tagged signed point process is tight, and any accumulation point as $N \to \infty$ is a stationary probability measure. Note that it is proven that both empirical measures θ_N^+ and θ_N^- converge a.s. to the uniform measure on Λ, meaning that they have the same limit. But, in contrast to the one-component case, the optimal macroscopic distribution of the points cannot be deduced so easily.

4. The 3D case

Onsager's theory aimed in explaining two-dimensional flows. Two-dimensional flows or quasi-two-dimensional flows are an important subject of study, as they appear in nature as atmospheric or geostrophic turbulence, they are useful as a theoretical toy model and more easily allow for numerical simulations. Despite this, of a major aim in turbulence theory is the analysis of the 3D case. The 3D case is much more complex than the 2D case, starting from the simple observation that the vorticity $\theta = \operatorname{curl} u$ is no longer a scalar-valued object, but a three-dimensional vector. Moreover, in 3D, there exist scale-invariant, turbulent cascade states.

We discuss here two approaches to the 3D case, namely the generalization of Onsager's theory to two-dimensional smooth manifolds, and the study of vortex filaments as equilibrium models of these 3-dimensional cascade states. We can note first that the Euler equations for inviscid, incompressible fluid flow on the sphere form an infinite-dimensional Hamilton system, just as in the 2D case. In analogy to the 2D case, the maximum entropy state is constrained by the conserved energy, angular momentum, and the Casimir integrals. The conservation of the Casimirs is equivalent to the preservation of the vorticity value distribution of the (piecewise) continuous initial vorticity.

It is a challenging problem to classify such generally turbulent Euler flows. One possible scenario, suggested by numerical integrations of the incompressible Euler equations for two-dimensional fluid flows with more general, piecewise continuous and sparsely distributed initial vorticity, which show a tendency of the vorticity field to become more and more filamentary, is that the solutions might in the long run converge weakly to some stationary vorticity field.

4.1. Random circulations on the sphere

Kiessling and Wang [60] investigated how Onsager's theory generalizes to two-dimensional smooth manifolds, in particular the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, which is of potential relevance to meteorology and planetary science. Note first that, as the Euler equations on $\mathbb{S}^2$ form a Hamiltonian system, stationary solutions of these Euler equations and solutions that are stationary in a rigidly corotating frame are exceptional, and so Euler flows on $\mathbb{S}^2$ that are launched by regular initial data will typically remain genuinely dynamical forever, never approaching a stationary state in the $C^{1,\alpha}$- topology. In the case of the sphere $\mathbb{S}$ we have the mean-zero property

$$\int_{\mathbb{S}^2} G(x-y)dy = 0. \tag{4.1}$$

As the vorticity distribution is preserved, see above, the question is whether the weak limit of the vorticity field as t becomes large can be anticipated based on its initial data.

The authors in [60] showed in parts analogous results as in the 2D case, i.e. that the normalized empirical vorticity converges in the limit as $N \to \infty$, and the limit, the "typical point vortex distributions", are special solutions of the Euler equations of incompressible, inviscid fluid flow on the sphere $\mathbb{S}^2$.

In particular, the typical states maximize a "negative 2-entropy" under some constraints, in analogy to the characterization in 2D as minimizer of an energy-entropy functional. The authors note that on the sphere, it is in general not true that the typical vorticities characterized by the limit of the point vortex system are stationary solutions of Euler's equation, but their level sets coincide, see [60], page 904.

4.2. Gibbs measures and vortex filaments

To the authors' best knowledge, the study of 3D turbulence based on equilibrium statistical mechanics started in the 1980s. Chorin [24] proposed several novel heuristic models [22, 23, 25] for collections of three-dimensional vortex filaments. The basis of these models are experiments which seem to suggest that the vorticity is concentrated in filament-like structures that dominate the small scale evolution, see e.g. the books [24, 46]. Mathematically rigorous treaties appeared since 2000, notably initiated by P-L Lions and Majda [70] and Flandoli [42], on the other hand. They differ in the precise modelling of the vortex filaments:

Lions and Majda [70] use an asymptotic theory to simplify the interaction of the vortices and to describe the filaments as functions on $\mathbb{R}^2$, after an appropriate parametrization. The parametrization is performed over the center curve of the filament, which are asymptotically close to lines parallel to the x_3-axis, explaining the name "nearly parallel vortex filaments". The work is based on the restriction that the filaments cannot fold, while folding is a major feature of general vortex filaments, necessary to prevent energy increase as a consequence of vortex stretching (see [24], Ch. 5). Flandoli [42] aimed in a model which allows for vortex folding. He used a generalization of the approach of Chorin, who modeled vortex filaments by trajectories of stochastic processes.

The ansatzes differ in several aspects, for example in the restrictions on the inverse temperature β. While in the approach of [70] only positive temperatures are admissible, both positive and negative β, up to a lower bound, similar to the Onsager 2D theory, have been considered. Moreover, in the approximation of [70] the definition of energy and Gibbs measures is not difficult, the aim is to reach a mean field result and several effective characterizations of the mean field distribution. Considering a model of vortex filaments based on 3D Brownian paths as [42] poses the challenge that the naive definition of the energy H of a Brownian filaments leads to an infinite quantity, and one has first to find the correct definition to show exponential integrability of the energy with respect to the Wiener measure, in order to show the existence of a Gibbs measure, as done in [44]. In the following subsections, we will briefly sketch both approaches.

4.2.1. Nearly parallel vortex filaments

Nearly parallel vortex filaments are filaments concentrated near a curve that is nearly parallel to the x_3-axis. They can be described by a function $x_i(\sigma, t) \in \mathbb{R}^2$ where $\sigma \in \mathbb{R}$ parametrizes the asymptotic center curve of the filament. The family $x_i(\sigma, t) \in \mathbb{R}^2$ evolves according to the $2N$ coupled system of equations

$$\gamma_j \frac{\partial x_j}{\partial t} = J\left[\alpha_j \gamma_j^2 \frac{\partial^2 x_j}{\partial \sigma^2} + \frac{1}{2}\sum_{i \neq j}^{N} \gamma_i \gamma_j \frac{x_i - x_j}{|x_i - x_j|^2}\right] \tag{4.2}$$

where $\alpha_j \in \mathbb{R}$ denotes the vortex core structure and J is the 2x2 matrix $J = (0, -1; 1, 0)$. In the limit when the core structures α_j go to zero, $x_i(\sigma, t)$ can be treated as point vortices in two dimensions for each value of the parameter σ: Roughly speaking, for each horizontal slice σ, we get N point vortices moving in the plane independently from those in the other slices. Similarly, when the core structures α_j go to $+\infty$, each filament becomes a straight line, i.e. $x_j(\sigma)$ becomes a constant point in $\mathbb{R}^2$, and all the horizontal slices of these lines yield the same system of N point vortices in two dimensions.

The simplified asymptotic equations (4.2) have been derived by Klein, Majda and Damodaran [63] and are valid as long as the Reynolds number is very large and the separation distance of the filaments is much larger than the core thickness of each filament. Moreover, the wavelength of the nearly parallel filament perturbations has to be much longer than the separation distance between the filaments.

The term involving $\frac{\partial^2 x_j}{\partial \sigma^2}$ arises from the linearized self-induction of the individual filaments, which is a purely three-dimensional effect not present in the two-dimensional point vortex dynamics. In fact, special solutions without σ-dependence coincide with solutions to the 2D point vortex equations.

The equilibrium statistical mechanics of nearly parallel vortex filaments was studied in [70] in the case of positive inverse temperature $\beta > 0$, constant positive circulations $\gamma_j = 1$, filaments which are periodic in σ with period L and equal cores $\alpha_j = \alpha$.

Thanks to these assumptions, the Hamiltonian can be written as

$$\mathcal{H}^{3D} = \sum_{j=1}^{N} \frac{\alpha}{2} \int_0^L \left|\frac{\partial x_j}{\partial \sigma}\right|^2 d\sigma - \frac{1}{2}\sum_{i \neq j}^{N} \int_0^L \log|x_j(\sigma) - x_i(\sigma)| d\sigma. \tag{4.3}$$

In contrast to the 2D case, the Gibbs measure in the 3D case is much more involved and is defined via additional conserved quantities, it involves function space intergrals w.r.t. to a discounted conditional Wiener measure. To define the Gibbs measure more precisely, some notation is needed: Let $\Omega^N = (\omega_1, \ldots, \omega_N)$ denote periodic continuous paths with $\omega_j \in C([0, 1], \mathbb{R}^2)$ and $\omega_j(0) = \omega_j(1)$ for all $1 \leq j \leq N$. Let ν^β be the Wiener measure on $(\mathbb{R}^2)^N$ with diffusion constant $1/\beta$ conditioned on periodic paths. We may write ν^β as $\nu^\beta_{x,x} dx$ where $\nu^\beta_{x,x}$ is the conditional Wiener measure conditioned on paths such that $\omega(0) = \omega(1) = x \in \mathbb{R}^2$. Note that $\int d\nu^\beta = \infty$, so ν^β is not a bounded measure on the Banach space $\left(\Omega^N, \max_{i,t\in[0,1]} |\omega_i(t)|\right)$. Moreover, $\nu^\beta_{x,x}$ is not a probability measure as $\int d\nu^\beta_{x,x} = \frac{1}{2\pi\beta t}$, and so the rigorous construction of $\nu^\beta_{x,x}$ goes via the explicit statement of its marginals through its action on bounded continuous function, which we will skip here due to the heavy notation involved. The Gibbs measure reads

$$\mu^{3D}_\beta = \frac{1}{Z_{N,\beta}} \exp\left(-\int_0^1 d\sigma \left[\frac{\beta L^2}{\alpha A^2 2}\right] + \sum_{j=1}^{N} \lambda_1 \cdot \omega_j(\sigma) + \lambda_2 \sum_{j=1}^{N} |\omega_j(\sigma)|^2\right) d\nu^\beta_{x,x}(\Omega) dx \tag{4.4}$$

with $\frac{1}{Z_{N,\beta}}$ the partition function associated to μ_β^{3D}, A the amplitude of the curves, $\lambda_1 \in \mathbb{R}^2$ a dilation factor, λ_2 constants. For special cases, see [70], page 85, the Gibbs measure is the law of a "quantum oscillator process", and under its law, the process $\Omega(t)$ is Gaussian and $\omega_1(t), \ldots, \omega_N(t)$ are independent.

In the *broken path models*, the continuous paths $\omega_j(\sigma)$ are approximated by discrete curves, the so-called *broken path discretization* in σ. This is done by discretizing the Hamiltonian and the conserved quantities, i.e. we consider broken chains

$$x_j^\sigma, \quad 1 \le j \le N,\ 0 \le \sigma \le M\ ,\ M\delta = 1\ , \tag{4.5}$$

with the periodicity condition $x_j^M = x_j^0$ and replace in the definition of the Hamiltonian the term $\log|x_j(\sigma) - x_i(\sigma)|$ by $\sum_{\sigma=0}^{M-1} \delta \log|x_j^\sigma - x_i^\sigma|$.

The broken path discretization is conceptually good as one recovers in the case of the extremely coarse broken path with one single element the Gibbs measure for the 2D point vortex theory, and also the 2D moment of inertia instead of the respective discrete conserved quantity in 3D. and the Gibbs measure for the continuum 3D problem in the infinitely fine discretization case.

The analogue to the empirical measure in the three-dimensional vortex filament case is the empirical distribution of the filament curves, for which Lions and Majda can characterize the mean field limit:

First, the empirical distribution of the filament curves converges to a probability density $\rho(x)$ on $\mathbb{R}^2$ which is independent of σ. In formula (set the period $L = 1$, i.e. the vortices are 1-periodic)

$$\mathbb{P}_{\mu_{3D}}\left(\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i(\sigma)} \;=\; \rho(x) \quad \text{for each } \sigma\right) \;=\; 1. \tag{4.6}$$

Moreover, the probability density $\rho(x)$ is given by

$$\rho(x) = \frac{p(x,x,1)}{\int_{\mathbb{R}^2} p(x,x,1)} \tag{4.7}$$

where p satisfies the following mean field PDE:

$$\begin{aligned} \frac{\partial p}{\partial t} \;-\; \frac{1}{2\beta}\Delta p + a\beta\left[\left(-\frac{1}{2\pi}\log|x|\right) * \rho\right] p \;+\; \mu|x|^2 p \;=\; 0 \quad \text{in } \mathbb{R}^2 \times (0,1) \\ p|_{t=0} = \delta_y(x) \end{aligned} \tag{4.8}$$

with inverse temperature β, μ derived from a conserved quantity, and a a constant. It is worth noting that the mean field limit for the broken path approximations, which is derived along the lines of [13] and [69], converges to the continuum mean field equation (4.7) and (4.8).

4.2.2. Vorticity fields based on stochastic processes

Under the assumption that the vorticity field is concentrated along a curve $\Theta(t)$, $t \in [0,T]$, the vorticity field $\theta(x) = \operatorname{curl} u(x)$ is formally defined as

$$\theta(x) = \gamma \int_0^T \delta(x - \Theta(t))\dot{\Theta}(t)\,dt \tag{4.9}$$

where γ is the circulation, used here as a parameter.

Now, recall that we can write, under suitable regularity and decay assumptions, the kinetic energy $H(u) = \frac{1}{2}\int_{\mathbb{R}^3} |u(x)|^2 dx$ in terms of the vorticity field as

$$H = \frac{1}{8\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\theta(x)\cdot\theta(y)}{|x-y|}dxdy. \tag{4.10}$$

Using (4.9), the kinetic energy (4.10) for vorticity fields on curves $\Theta(t)$ can be reformulated as

$$H = \frac{\gamma^2}{8\pi}\int_0^T\int_0^T \frac{\dot\Theta(t)\cdot\dot\Theta(s)}{|\Theta(t)-\Theta(s)|}dtds. \tag{4.11}$$

The problem is, however, that the energy of a smooth curve is infinite, due to the non-integrable divergence along the diagonal. The first hope was that for fractal curves like a Brownian trajectory, the foldings of the curve would have reduced its energy sufficiently, in the sense that $|\Theta(t) - \Theta(s)|$ could be less divergent and very fast changes in direction may produce cancellations in $\dot\Theta(t)\cdot\dot\Theta(s)$. That hope was not rigorously confirmed, in part due to problems arising with the very frequent self-intersections of non-smooth curves.

The cut-off necessary to get a finite energy can be obtained via lattice approximation. Indeed, inspired by the lattice vortex filaments introduced in [23], Flandoli [42] introduced a filament-like vortex structure based on a 3D Brownian path: Let the vorticity fields be concentrated over sets of the form

$$C_{\mathcal{A}} = \{x + W_t, x \in \mathcal{A}, t \in [0,T]\} \tag{4.12}$$

for W_t Brownian Motion in $\mathbb{R}^3$ and $\mathcal{A} \subset \mathbb{R}^3$ a compact set supporting a probability measure ρ such that

$$\int_{\mathcal{A}}\int_{\mathcal{A}} \frac{1}{|x-y|}\rho(dx)\rho(dy) < \infty. \tag{4.13}$$

Flandoli [42] prove that under the assumption (4.13), the total energy of the vortex structure is indeed finite, though the interaction energy H_{xy}, defined formally as

$$H_{xy} = \frac{\gamma^2}{8\pi}\int_0^T\int_0^T \frac{1}{|x+W_t-(y+W_s)|}\circ dW_s\circ dW_t \tag{4.14}$$

gives an infinite contribution to the energy on the diagonal. This is done by showing that H_{xy} behaves like $\frac{1}{|x-y|}$ up to lower order terms, which are, in fact, just logarithmic corrections. Thanks to these scaling properties of the interaction energy when $|x-y| \to 0$, it can be shown that the kinetic energy of the vortex structure $(C_{\mathcal{A}}, \rho)$ is a well defined real valued random variable with finite moments of every order.

5. Vortex dynamics with random initial condition

Another viewpoint on the point vortex model is coming from the study of the weak vorticity formulation of the 2D Euler equations with white noise initial condition. To introduce the weak vorticity formulation, recall from Section 2 the classical results on the torus $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$: Wolibner [107] and Yudovich [58] showed that for initial data $\theta_0 \in L^\infty(\mathbb{T})$, there exists a unique weak solution of class $L^\infty([0,T]\times\mathbb{T}) \cap C([0,T];L^p(\mathbb{T}))$ for every $p \in [1,\infty)$, satisfying

$$\langle\theta_t,\phi\rangle = \langle\theta_0,\phi\rangle + \int_0^t \langle\theta_s, u_s\cdot\nabla\phi\rangle\, ds \tag{5.1}$$

for every $\phi \in C^\infty(\mathbb{T})$. If instead the initial data $\theta_0 \in L^p(\mathbb{T})$ for some $p \in [1, \infty)$, there exists a global weak solution of class $C([0, T]; L^p(\mathbb{T}))$, but uniqueness is an open problem.

In a search for weaker and weaker concepts of solutions, Delort [36], see also [38, 99], could prove that for a measure-valued initial condition $\theta_0 \in H^{-1}(\mathbb{T})$ such that the velocity $u_0 \in L^2$, there is a global solution measure-valued solution of class $L^\infty([0, T]; \mathcal{M}(\mathbb{T}) \cap H^{-1}(\mathbb{T}))$, which satisfies for every $\phi \in C^\infty(\mathbb{T})$

$$\langle \theta_t, \phi \rangle = \langle \theta_0, \phi \rangle + \int_0^t \int_{\mathbb{T}} \int_{\mathbb{T}} K_\phi(x, y) \theta_s(dx) \theta_s(dy)\, ds \tag{5.2}$$

where

$$K_\phi(x, y) = K(x - y)(\nabla\phi(x) - \nabla\phi(y)) \tag{5.3}$$

with K the Biot-Savart Kernel on the torus. Here $\mathcal{M}(\mathbb{T})$ denotes the space of finite signed measures. Note that in $K_\phi(x, y)$ the singularity of the Biot-Savart kernel has been removed, $K_\phi(x, y)$ is bounded and smooth outside the diagonal, but discontinuous along the diagonal. For this result, with a much weaker solution concept, it was crucial to analyze precisely the concentration of vorticity along the diagonal of the kernel $K_\phi(x, y)$.

Interpreting the Euler equations in weak vorticity form (5.2), Albeverio and Cruzeiro [2] showed that Euler equations in weak vorticity form have a stochastic solution which is a stationary process with time marginal given by white noise on the two-dimensional torus $\mathbb{T}$.

While Albeverio and Cruzeiro used a Fourier formulation, Flandoli [43] recently proved the existence of a solution as a limit of random point vortices, in the sense of a random version of the classical existence result of a measure-valued solution, as outlined in the above chapters and e.g. in [76]: Given an initial condition of point vortex form $\theta_0(dx) = \sum_i \gamma_i \delta_{x_0^i}$, with real-valued intensities $\gamma_1, \ldots, \gamma_N$ and the vector of initial positions $(x_0^1, \ldots, x_0^N)$ belonging to a set of full Lebesgue measures in $\mathbb{T}^N$, there exists a unique measure-valued solution of the form $\theta_t(dx) = \sum_i \gamma_i \delta_{x_t^i}$ fulfilling (5.2).

The point vortex approximation therefore interprets the white noise solution of Albeverio and Cruzeiro as a limit of randomly distributed vortices with positive and negative random vorticities. Moreover, the point vortex approximation gives an intuition why the white noise distribution solutions do not concentrate on the diagonal, as it was discussed in the deterministic case, e.g. by Delort [36], Schochet [99], Poupaud [87] and Di Perna and Majda [38]: at every instance in time, the point vortices are distributed at random uniformly in space, independently from one another.

We outline the setting and results in the following. Consider N point vortices with positions $X_t^{i,N}$ on the torus with random intensities $\gamma_1, \ldots, \gamma_N$. Denote the rescaled point vortex system

$$\dot{x}_t^{i,N} = \sum_{j=1}^N \frac{1}{\sqrt{N}} \gamma_j K\left(x_t^{i,N} - x_t^{j,N}\right) \qquad i = 1, \ldots, N \tag{5.4}$$

with initial positions away from the generalized diagonal on the torus

$$\Delta_N = \left\{(x^1, \ldots x^N) \in (\mathbb{T}^2)^N \ : \ x^i = x^j \quad \text{for some } \ i \neq j, \ i, j = 1 \ldots N\right\} \tag{5.5}$$

This corresponds to the time evolution of the measure-valued vorticity field

$$\theta_t^N = \frac{1}{\sqrt{N}} \sum_{n=1}^N \gamma_n \delta_{X_t^n}. \tag{5.6}$$

By the results of Section 2, independently of the sign of $\gamma_1, \ldots, \gamma_N$, for $\otimes_N Leb_{\mathbb{T}}$ - almost all initial condition with values on the tous, the positions $\left(x_t^{1,N}, \ldots, x_t^{N,N}\right)$ remain different for all times.

Moreover, the measure $\otimes_N Leb_{\mathbb{T}}$ is invariant, in the sense that for an initial condition which is a random variable with distribution $\otimes_N Leb_{\mathbb{T}}$, the stochastic process $\left(x_t^{1,N}, \ldots, x_t^{N,N}\right)$ is stationary, with invariant marginal law $\otimes_N Leb_{\mathbb{T}}$.

Let now $(\Xi, \mathcal{F}, \mathbb{P})$ be a probability space, the intensities (γ_n) be an i.i.d. sequence of $N(0,1)$ random variables and let the initial positions (X_0^n) be an i.i.d. sequence of torus-valued random variables, independent of the intensity sequence and uniformly distributed. Note that under these conditions, the random vector $\left((\gamma_1, X_0^1), \ldots (\gamma_N, X_0^N)\right)$ has the law

$$\lambda_N^0 := \otimes_N (N(0,1) \otimes Leb_{\mathbb{T}}) \tag{5.7}$$

The initial random distribution θ_0^N is centered, as γ_n and $\langle \delta_{X_0^n}, \varphi \rangle$ are independent and γ_n is centered, and so $\mathbb{E}[\gamma_n \langle \delta_{X_0^n}, \varphi \rangle] = 0$. Moreover, θ_0^N has the same covariance as white noise, but it is not Gaussian. By a Hilbert space-values version of the Central Limit Theorem, we have

Proposition 5.1 ([43], Proposition 21). *Let ω_{WN} denote white noise. Then the initial random distribution converges in law*

$$\theta_0^N \longrightarrow \omega_{WN} \tag{5.8}$$

and the convergence is in $H^{-1-\delta}$ for every $\delta > 0$.

Denote now by μ the law of White Noise. Then the following reformulation of Albeverio-Cruzeiro holds:

Theorem 5.2 ([43], Theorem 24). *There exists a probability space $(\Xi, \mathcal{F}, \mathbb{P})$ with the following properties:*

1. *There exists a measurable map $\theta_\cdot : \Xi \times [0,T] \to C^\infty(\mathbb{T})'$ such that $\theta_\cdot$ is a time-stationary white noise solution of Euler equations.*
2. *On $(\Xi, \mathcal{F}, \mathbb{P})$ one can define the random point vortex system* (5.4)*; it has a subsequence which converges $\mathbb{P}$-a.s. to the solution of point (1) in $C\left([0,T]; H^{-1-}(\mathbb{T})\right)$*

Note that the solution of part (1) may not be unique. In fact, as remarked in [43], Section 4.1., a statement of uniqueness in law is an open problem, as straties to prove it are usually based on the uniqueness of the 1-dimensional marginals, which is again not known.

6. Generalized models

To study more turbulence phenomena, a more general set of models were introduced. These models deal with generalized Euler equations, where the velocity is again given by (2.4) but with G the Greens function of the fractional Laplacian $(-\Delta)^{m/2}$, namely

$$\begin{cases} \partial_t \theta + \nabla \cdot (u\theta) = 0, \\ u = k_m \star \theta, \end{cases} \tag{6.1}$$

where $k_m = \nabla^\perp G_m$ and G_m is the Green function of the fractional Laplacian $(-\Delta)^{m/2}$. The case of the Euler equations corresponds to $m = 2$, and the case $m = 1$ is the so-called inviscid surface quasi-geostrophic equation (SQG). The inviscid SQG has been derived in meteorology to model frontogenesis, namely the production of fronts due to tightening of temperature gradients [28, 55, 56] (see also [32, 97] for the first mathematical and geophysical studies about strong fronts).

The equations (6.1) are often called generalized surface quasi-geostrophic equations, as

they bridge the cases of Euler and SQG and share a series of common physical features, such as the emergence of inverse cascades [100, 104–106], and universal invariance properties.

In the geophysical literature, equations of the form (6.1) are often called α-models, as they correspond to the Biot-Savart law

$$u := \nabla^{\perp}(-\Delta)^{-1+\alpha}\theta, \tag{6.2}$$

with $\alpha = \frac{1}{2}$ the SQG case and $\alpha = 0$ the 2D Euler case.

6.1. Existence of solutions for generalized SQG equations

Generalized models share the same difficulties as the SQG. Existence of weak solutions to the SQG equation in L^2 is known since [89], see also [72] for solutions in L^p with $p > 4/3$. Weak solutions in L^2 on the torus for $m < 1$ were obtained in [17].

The SQG equation was first rigorously studied in [28] and shows, mathematically, interesting analogies with the 3D Euler equation, nurturing the hope that the study of the regularity of the SQG model could provide hints for the formation of singularities in the 3D Euler equation. Considerable effort has been taken in this direction, for example in [28] a closing saddle scenario for a finite time singularity has been suggested, but ruled out by Cordoba [30] and Cordoba and Fefferman [31]. Moreover, there are numerical results that if the SQG equations are singular, the formation of singularities must be self-similar, e.g. [101, 102].

For initial data with sufficient smoothness, a local existence result is known [17], giving unique solutions with the same regularity of the initial condition. Unlike the Euler equation, it is not known if the inviscid SQG (as well as its generalized version) has a global solution. Actually, there is numerical evidence, see [33], of emergence of singularities in the generalized SQG, for $m \in [1, 2)$. Furthermore, a numerical study by Ohkitani [84] shows that there is a value $m \in [0, 2]$ for which the solutions behave in the most singular manner.

Further results have been obtained for the evolution of vortex patches, namely solutions of SQG that take only two values, and where the main interest is about the evolution of the interface. For example, Cordoba, Fontelos, Mancho and Rodrigo [33] studied the patch problem for the SQG equation and suggested a finite time singularity, and recently [62] proved finite-time blow up for for any sufficiently small $\alpha > 0$ for the modified SQG patch equation. See also [17, 34] for relevant results.

The stability of SQG vortices was investigated, among others, by Carton [26], Dritschel [39] and many others, see e.g. [29] for references.

Generalized models, interpolating between the 2D Euler and the SQG case, have been studied widely both in the mathematical (e.g. [27, 71]) and in the geophysical literature, see the above references. The question of global regularity for these models with smooth initial data has been open for all models except the 2D Euler case. However, interestingly, for the generalized models there are no examples of solutions with unbounded growth of derivatives in time known, despite that these models are more singular than the Euler case. In fact, [61] showed that high Sobolev norms are arbitrary bounded on finite time intervals. Moreover, [34] find classes of global solutions and [18] presents a regularity criterion for classical solutions.

For existence of weak solution for the generalized SQG model one can see [17]. In [51] the following result is given.

Theorem 6.1. *[[51], Th. 2.6.] Let $\theta_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, then there is a solution θ in distributions of* (6.1) *on* $[0, \infty)$ *with initial condition* θ_0. *Moreover*

$$\|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p},$$

for every $p \in [1, \infty]$ and all $t > 0$.

Global flows of weak solution with a (formal) invariant measure, corresponding to the measure

$$\mu_{\beta,\alpha}(d\theta) = \frac{1}{Z_{\beta,\alpha}} \varepsilon^{-\beta\|(-\Delta)^{-\frac{m}{4}}\theta\|^2 - \alpha\|\theta(t)\|^2_{L^2}}\, d\theta, \tag{6.3}$$

with $\beta = 0$, as initial condition has been provided in [81].

For collapse and non-collapse results by [5] and [51], which are similar in spirit to the Euler case, we refer to the results already described in Section 2.

6.2. The point vortex evolution

In analogy to (1.2), generalized point vortex models were recently defined and studied [45, 50, 51]. They describe the evolution of vortex positions according to the system of equations

$$\begin{cases} \dot{X}_j = \sum_{k \neq j} \gamma_k \nabla^\perp G_m(X_j, X_k), \\ X_j(0) = x_j, \end{cases} \qquad j = 1, 2, \ldots, N, \tag{6.4}$$

where G_m is the Green function of the operator $(-\Delta)^{\frac{m}{2}}$. The equations (6.4) form a Hamiltonian system with Hamiltonian $H_N(\gamma^N, X^N) = \frac{1}{2}\sum_{j \neq k} \gamma_j \gamma_k G_m(X_j, X_k)$, where $X^N = (X_1, X_2, \ldots, X_N)$ and $\gamma^N = (\gamma_1, \gamma_2, \ldots, \gamma_N)$. A natural invariant distribution for the Hamiltonian dynamics (6.4) should be the measure

$$\mu^N_\beta(dX^N) = \frac{1}{Z^N_\beta} e^{-\beta H_N(X^N, \gamma^N)}\, d\ell^{\otimes N}, \tag{6.5}$$

where we denoted by ℓ the normalized Lebesgue measure. In general, (6.5) does not make sense anymore in the case $m < 2$ as the singularity of the Green function of the fractional Laplacian is too strong. Nevertheless, the program developed in [10, 13, 14, 69] for point vortices for the Euler equations can be performed, at the level of a regularized problem on the torus, i.e. for a regularization of the Green function, also in the case $\beta > 0$ and $m < 2$, see Section 6.3.

Generalized point vortex models interpolate the Euler and the SQG point vortex models. SQG point vortices have been studied e.g. by Lim and Majda [66], Taylor and Llewellyn Smith [103] and Badin and Barry [5].

In [51] the validity of the point vortex system for the model (6.1) on the whole plane $\mathbb{R}^2$ was investigated. It turns out that for the generalized models (6.1) the point vortex system provides a good approximation, as already known for the case $m = 2$, see Section 2, Theorem 2.1. For technical reasons one needs to work with a regularized vortex dynamics, i.e. the evolution

$$\dot{X}^N_{\epsilon,j} = \sum_{k=1}^{N} \gamma^N_k k^\epsilon_m(X^N_{\epsilon,j} - X^N_{\epsilon,k}), \tag{6.6}$$

with initial conditions $x_1^N, x_2^N, \dots, x_N^N$. This is a vortex system analogous to (6.4), but with the kernel $k_m = \nabla^\perp G_m$ replaced by a regularized version k_m^ϵ. We can then define the empirical measure

$$\theta_\epsilon^N(t) = \sum_{j=1}^N \gamma_j^N \delta_{X_{\epsilon,j}^N(t)}, \qquad t \geq 0.$$

and state the result:

Theorem 6.2. *[[51], Theorem 3.2.] Let $m \in (1,2)$, $\theta_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ be non-negative and of mass one, and let θ be a weak solution of* (6.1) *with initial condition θ_0. Assume that θ_0^N weakly converges (in the sense of measures) to θ_0 as $N \uparrow \infty$, where*

$$\theta_0^N = \sum_{j=1}^N \gamma_j^N \delta_{x_j^N},$$

for suitable $\gamma_1^N, \dots, \gamma_N^N \in \mathbb{R}$ and $x_1^N, \dots, X_N^N \in \mathbb{R}^2$. Then there is a sequence $(\epsilon_N)_{N\geq 1}$ such that $\theta^{N,\epsilon_N}(t)$ weakly converges to $\theta(t)$, uniformly in t in bounded sets.

Here we have simplified the statement of Theorem 6.2 for the sake of clarity. A part from the need of a regularized dynamics, another issues made the proof of Theorem 6.2 more involved and technical than in the Euler case: As uniqueness for initial conditions in $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ is not known for $m < 2$, a limit along a sequence of regularizations had to be taken. We refer to [51] for details.

Conversely it could be shown in can prove that solutions of (6.1) approximating point vortices at time $t = 0$ converge to the point vortex evolution, at least for values of the parameter m not too different from 2. A specific regularized setting is necessary, which we illustrate here for the simple case of radially symmetric vortex blobs, and refer the interested reader to [51] for details.

Fix $N \geq 1$, $\gamma_1, \gamma_2, \dots, \gamma_N \in \mathbb{R}$, and N points $x_1^0, x_2^0, \dots, x_N^0 \in \mathbb{R}^2$. Given a function $\eta \in C_c^\infty(\mathbb{R}^2)$ such that $\eta \geq 0$, η has support in $B_1(0)$ and $\int_{\mathbb{R}^2} \eta(x)\,dx = 1$, for every $\epsilon > 0$ define the following *vortex blobs* $\theta_{0,1}^\epsilon = \epsilon^{-2}\eta((x - x_1^0)/\epsilon), \dots, \theta_{0,N}^\epsilon = \epsilon^{-2}\eta((x - x_N^0)/\epsilon)$. Define

$$\theta_\epsilon(0, x) = \sum_{j=1}^N \gamma_j \theta_{0,j}^\epsilon(x), \tag{6.7}$$

where $\gamma_1, \dots, \gamma_N$ are the intensities of each vortex blob, $x_1^0, \dots, x_N^0$ are the centers, and ϵ is small enough that the balls $(B_\epsilon(x_j^0))_{j=1,\dots,N}$ are disjoint.

Theorem 6.3 ([51], Th. 3.6.)**.** *Assume $\sqrt{3} < m < 2$ and denote by θ_ϵ a solution of* (6.1)*, according to Theorem 6.1, with initial condition $\theta_\epsilon(0)$ given by* (6.7)*. Then for all $T > 0$,*

$$\lim_{\epsilon \to 0} \langle \theta_\epsilon(t), \phi \rangle = \sum_{j=1}^N \phi(X_i(t)), \qquad t \in [0, T],$$

where $(X_i)_{i=1,\dots,N}$ is the solution of the vortex evolution (6.4) *with vortex intensities $\gamma_1, \gamma_2 \dots, \gamma_N$ and with initial conditions $(x_1^0, x_2^0, \dots, x_N^0)$.*

Note that it has been assumed that the vortex evolution (6.4) with initial condition $(x_1^0, x_2^0, \ldots, x_N^0)$ has a global solution. According to Theorem 2.3, if the intensities fulfill (2.13), global existence is true for a.e. choice of $(x_1^0, x_2^0, \ldots, x_N^0)$.

A corollary of the proof of Theorem 6.3 is the so-called *localization property* , which holds also in the Euler case [75]: The evolution of (6.4) started on a vortex blob stays around the center of pseudovorticity. We refer to [51], Proposition 3.8., for the precise statement. See also [15,49].

6.3. Limit theorems for the invariant distribution

As mentioned above, the results on limit distributions developed in [10,13,14,69] for point vortices for the Euler equations cannot be easily transferred to generalized point vortices, as in general, (6.5) does not make sense for $m < 2$.

Nevertheless, some results in this direction were obtained in [50], using a regularization to deal with the singularity of the Green function of the fractional Laplacian, which is in general too strong. The original problem is recovered in the limit of infinite vortices, choosing the regularization parameter ϵ so that it goes at the same time to 0 as N goes to infinity. However, the speed of convergence of $\epsilon = \epsilon(N)$ must be at least logarithmically slow in terms of N.

In this section, we describe the setting in [50] and the obtained results.

Let us consider (6.1) on the torus with periodic boundary conditions and zero spatial average. Recall that the measure

$$\mu_\beta^N(dX^N, d\gamma^N) = \frac{1}{Z_\beta^N} \varepsilon^{-\beta H^N(X^N,\gamma^N)} d\ell^{\otimes N}\, d\nu^{\otimes N}.$$

is well defined when the intensities are all positive. When intensities are allowed to be negative, the singularity of the Green function is too strong and the exponential is not integrable. Writing the Green function in terms of the eigenvectors of the fractional Laplacian, a regularization of the Green function can be defined as

$$G_{m,\epsilon}(x) = \sum_{k=1}^{\infty} \lambda_k^{-\frac{m}{2}} \varepsilon^{-\epsilon\lambda_k} e_k(x) e_k(y), \tag{6.8}$$

so the fractional operator was regularized to reads $(-\Delta)^{m/2} e^{-\epsilon\Delta}$ and has eigenvalues $\lambda^{m/2} e^{\epsilon|k|^2}$.

Likewise we define a regularized Hamiltonian H_ϵ^N by replacing the Greens function with G with the regularized Greens function (6.8). The regularized motion is still Hamiltonian with the Hamiltionian H_ϵ^N.

Let ν be a probability measure on the real line with support on a compact set $K_\nu \subset \mathbb{R}$. In other words, assume that intensities are bounded in size by a deterministic constant. We denote the *prior* distribution on vortex intensities by the measure ν. A natural invariant distribution for the regularized Hamiltonian dynamics with random intensities then reads

$$\mu_{\beta,\epsilon}^N(d\gamma^N, dX^N) = \frac{1}{Z_{\beta,\epsilon}^N} \varepsilon^{-\frac{\beta}{N} H_N^\epsilon(\gamma^N, X^N)}\, d\ell^{\otimes N}\, d\nu^{\otimes N}, \tag{6.9}$$

where ℓ is the normalized Lebesgue measure on the torus $\mathbb{T} \subset \mathbb{R}^2$ and $Z_{\beta,\epsilon}^N$ is the normalization factor.

Under the conditions $\beta > 0$ and $\alpha < 2$, and when $\epsilon(N) \downarrow 0$, propagation of chaos can be shown, namely vortices decorrelate and in the limit are independent.

Theorem 6.4 (Convergence of finite dimensional distributions)**.** *Assume $m < 2$ and $\beta > 0$, and fix a sequence $\epsilon = \epsilon(N) \downarrow 0$ so that*

$$\epsilon(N) \downarrow 0 \quad as \quad N \uparrow \infty, \qquad \epsilon(N) \geq C(\log N)^{-\frac{2}{2-m}} \tag{6.10}$$

with C large enough (depending on ν and β). Then, as $N \to \infty$, the k-finite dimensional marginals of $\mu^N_{\beta,\epsilon}$ converge to $(\nu \otimes \ell)^{\otimes k}$. In particular, propagation of chaos holds.

Note that the above theorem as well as the other results are asymptotic both in the number of vortices and the regularization parameter ϵ, and thus they capture the behaviour of the original system. However, the results hold, only if ϵ is allowed to go to zero with a speed which is at least logarithmically slow with respect to the number of vortices.

Next, also a Law of Large Numbers for the joint empirical distribution holds:

Theorem 6.5 (Law of Large Numbers, [50] Theorem 3.2.)**.** *Consider a system of N point vortices at equilibrium, with equilibrium measure* (6.9)*, described by the N pairs $(\gamma^N_1, X^N_1), \dots, (\gamma^N_N, X^N_N)$ of intensity and position. Assume $m < 2$ and $\beta > 0$, and define the joint empirical distribution*

$$\eta_N = \frac{1}{N}\sum_{j=1}^{N} \delta_{(\gamma^N_j, X^N_j)}$$

of intensity and position of point vortices. Choose $\epsilon = \epsilon(N)$ as in the previous theorem. Then

$$\eta_N \rightharpoonup \nu \otimes \ell, \qquad \text{in probability}$$

as $N \uparrow \infty$.

In terms of θ, the limit is a stationary solution of the original equation. The Law of Large Numbers for η_N implies a Law of Large Numbers for the empirical pseudo-vorticity

$$\theta_N = \frac{1}{N}\sum_{j=1}^{N} \gamma^N_j \delta_{X^N_j}. \tag{6.11}$$

Indeed, convergence of η_N to $\ell \otimes \nu$ implies immediately the convergence of θ_N to $\nu(\gamma)\ell$, with $\nu(\gamma) = \int \gamma\, \nu(d\gamma)$.

Likewise, a Central Limit Theorem holds. The limit Gaussian distribution for the θ variable turns out to be a statistically stationary solution of the equations.

Furthermore, one can analyze fluctuations with respect to the limit stated in the previous theorem, namely the limit of the measures

$$\zeta_N = \sqrt{N}(\eta_N - \nu \otimes \ell)$$

to a Gaussian distribution.

To state properly the covariance, we define the operators $\mathscr{E}, \mathscr{G}$ as

$$\mathscr{G}\phi(x) := \int_{\mathbb{T}} G_m(x,y)\phi(y)\, \ell(dy),$$
$$\mathscr{E}\phi(\gamma, x) := \gamma \int_{K_\nu}\int_{\mathbb{T}} \gamma' G_m(x,y)\phi(\gamma', y)\, \nu(d\gamma')\ell(dy).$$

The operator $\mathscr{G}$ provides the solution to the problem $(-\Delta)^{\frac{m}{2}}\Phi = \phi$ with periodic boundary conditions and zero spatial average, and extends naturally to functions depending on both variables γ, x by acting on the spatial variable only.

Theorem 6.6 (Central limit Theorem, [50] Theorem 3.4.)**.** *Assume $\beta > 0$ and choose $\epsilon = \epsilon(N)$ as in* (6.10)*. Then $(\zeta_N)_{N\geq 1}$ converges, as $N \uparrow \infty$, to a Gaussian distribution with covariance $I - \beta(I + \beta\Gamma_\infty\mathscr{G})^{-1}\mathscr{E}$, in the sense that for every test function $\psi \in L^2(\nu\otimes\ell)$, $\langle\psi, \zeta_N\rangle$ converges in law to a real centred Gaussian random variable with variance*

$$\sigma_\infty(\psi)^2 := \langle I - \beta(I + \beta\Gamma_\infty\mathscr{G})^{-1}\mathscr{E}(\psi - \bar{\psi}), (\psi - \bar{\psi})\rangle,$$

where $\bar{\psi} = (\nu\otimes\ell)(\psi)$ and $\Gamma_\infty = \nu(\gamma^2)$.

Note furthermore that $\sqrt{N}(\theta_N - \nu(\gamma)\ell)$ converges to a Gaussian distribution with covariance $\Gamma_\infty(I + \beta\Gamma_\infty\mathscr{G})^{-1}$, in the sense that for every test function $\psi \in L^2(\ell)$, $\langle\sqrt{N}(\theta_N - \nu(\gamma)\ell), \psi\rangle$ converges in law to a real centred Gaussian random variable with variance

$$\tilde{\sigma}_\infty(\psi)^2 = \Gamma_\infty\langle(I + \beta\Gamma_\infty\mathscr{G})^{-1}(\psi - \bar{\psi}), (\psi - \bar{\psi})\rangle.$$

The Gaussian measure obtained corresponds to the invariant measure (6.3) of the original system (6.1), when one takes $\alpha = 1/\Gamma_\infty$. This yields a central limit theorem for the empirical pseudo-vorticity θ_N (6.11).

Note that the above results hold also in a quenched version, namely if intensities are non-random but given at every N. For the central limit theorem, some concentration condition on the convergence of the intensities needs to be assumed.

The above results were performed on a two-dimensional torus, where, because of the periodic boundary conditions, the Green function contains no boundary term and satisfies a zero mean property (4.1), as already mentioned in the case of $m = 2$. Limit theorems on a general bounded domain, as in [13] or [82], where Dirichlet boundary conditions are used, are much more involved and are subject of an on-going research.

6.4. Velocity statistics for turbulent flows

Another difference between the Euler case $m = 2$ and the case $m \neq 2$ is on the level of a thermodynamical limit, i.e. for $N \to \infty$, together with an infinite volume limit such that the density of vortices is constant. In the the Euler case $m = 2$ the thermodynamical limit does not exist due to the logarithmic divergence with the number of vortices.

In the case $m \neq 2$, however, it is possible to consider a proper thermodynamical limit and consequently resuls are independent of the number of point vortices in the domain considered, but depend on the value of m.

In [29] an analytical form of the probability density distribution of the velocity fluctuations for different degrees of locality is shown in the case of a neutral system with randomly distributed, uncorrelated vortices with uniform probability on a disk of (in the thermodynamical limit diverging) radius. It is shown moreover that the central region of the distribution is not Gaussian, in contrast to the case of 2D turbulence, but can be approximated with a Gaussian function in the small velocity limit. The tails of the distribution exhibit a power law behavior, withe the exponent depending on m, and self similarity in terms of the density variable.

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Conflict of interest

The authors declare that there is no conflicts of interest in this paper.

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