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METRICALLY HOMOGENEOUS GRAPHS
OF DIAMETER THREE

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ABSTRACT. We classify countable metrically homogeneous graphs of diameter 3.

CONTENTS

1. Introduction and outline

A connected graph is *metrically homogeneous* if it is homogeneous when considered as a metric space in the graph metric—that is, any isometry between finite subspaces is induced by some isometry of the whole space onto itself.[1] This terminology applies also to disconnected graphs, if we allow the distance ∞, in which case it means that each connected component is metrically homogeneous in the strict sense, and the connected components are isomorphic. In the present paper, all graphs considered are countable (possibly finite).

We will give a full classification of the metrically homogeneous graphs of diameter 3 here. In diameter at most 2, the metrically homogeneous graphs are simply the connected homogeneous graphs, classified by Lachlan and Woodrow [LW80] by a subtle inductive argument. The argument in diameter 3 is actually much more direct than the argument in diameter 2, apart from the fact that the classification in diameter 2 is used in the treatment of the diameter 3 case.

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[1]This strong homogeneity condition was introduced by Urysohn and communicated in a letter from Alexandrov to Hausdorff in 1924 [Hu08, Ury25].

The problem of classifying *all* metrically homogeneous graphs was raised in passing by Larry Moss in [Mos92, §6] and more explicitly by Peter Cameron in [Cam98], in the following striking formulation: "Not even the countable metrically homogeneous graphs have been determined." The context of [Cam98] is much broader: *distance transitive* graphs—in geometric terminology, *2-point homogeneous graphs* (cf. [Wa52, Tit55]). For distance transitive graphs, there is a rich theory in the finite case, relying heavily on the classification of the finite simple groups. We will discuss the known metrically homogeneous graphs below (§1.1). It is reasonable to ask at this point whether all metrically homogeneous graphs are known. We prove that this is true in the case of graphs of diameter 3.

We also prove a number of results in a form not limited to the case of diameter 3, at the cost of some additional work. As to whether one can get a full classification without restricting the diameter, this remains to be seen. It is very likely that the methods used here can give further useful information about the general case without fundamental alteration, but one expects the resulting treatment would be substantially longer and noticeably more technical, without settling the general problem. So we have made such generalizations only when the cost was modest. The style of argument used appears to have considerable potential for further development, particularly in its later stages (§§5, 6) a point we return to in §7.5.

1.1. **The known metrically homogeneous graphs.** A catalog of the known metrically homogeneous graphs is given in [Che11a]. There is some decent evidence to support the view that this catalog should give a complete classification, or nearly so. Much of that evidence is reviewed in [Che18], with the present work cited as one of the relevant items. Another point that would be helpful in building up the case for completeness of the catalog to a more substantial level would be a full treatment of the "antipodal" case (see Definition 2.7, §2.1). A continuation of the analysis through diameter 4 would be a major advance, as the possibilities envisioned by the catalog of known metrically homogeneous graphs are not fully realized in diameter 3.

It is shown in [Che18] that if the proposed catalog is complete with respect to metrically homogeneous graphs of finite diameter, then it is also complete with respect to the case of infinite diameter. Of course, it is quite possible that the catalog is broadly correct but is missing some exceptional examples of small diameter, or of antipodal form, in which case that reduction to the case of finite diameter would need to be revisited. But if the catalog is complete, or nearly so, then a natural approach to the proof is to treat the case of finite diameter inductively, and reduce the general case to that one.

Some material that might reasonably have appeared here made its way into [Che18] instead, notably some of the discussion of Smith's theorem and the treatment of the antipodal case in diameter 3. That work was done independently by the two groups of authors before they exchanged notes.

1.2. **General Theory.** Our presentation makes use of the theory developed in [Che18]. This is merely a matter of convenience, as in most cases what the theory tells us is not hard to verify directly in diameter 3.

One point provided by the general theory is a useful way to draw the distinction between various exceptional cases and the *generic* case. This relies on the following terminology.

If Γ is a metrically homogeneous graph, we write Γ_i for the set of vertices at distance i from a fixed based point, which may be considered either as a metric space with the induced metric, or a graph with the induced edge relation. The isomorphism type of Γ_i is well determined, and it is homogeneous when viewed as a metric space with the induced metric. In particular the graph Γ_1 is a homogeneous graph, since nonadjacent pairs are at distance 2.

Recall also that a graph Γ is *imprimitive* if it carries a nontrivial $\mathrm{Aut}(\Gamma)$-invariant equivalence relation.

Definition 1.1. *Let Γ be a metrically homogeneous graph.*
 1. We say that Γ is of exceptional local type *if Γ_1 is*

- *imprimitive, or*
- *contains no infinite independent set.*

2. We say that Γ is of generic type *if Γ_1 is primitive, and for any vertex v in Γ_2, the neighbors of v in Γ_1 contain an infinite independent set.*

The main point of this definition is that we are able to give an explicit classification of the metrically homogeneous graphs of non-generic type, and that our analysis in the case of generic type depends on completely different methods from the non-generic case.

Since in a metrically homogeneous graph Γ of generic type the associated graph Γ_1 is primitive and contains an infinite independent subset, the classes of metrically homogeneous graphs of exceptional local type and generic type are disjoint. But this division leaves over a third, intermediate, class, as follows.

- Γ_1 is primitive and contains an infinite independent set, but for $v \in \Gamma_2$, the set of neighbors of v in Γ_1 contains no infinite independent set.

If we apply the Lachlan/Woodrow classification of homogeneous graphs, then our intermediate class may be characterized more simply as follows, as we shall explain in §2.

- Γ_1 is an infinite independent set.
- For $v \in \Gamma_2$, the set of neighbors of v in Γ_1 is finite.

This is clearly a very special case, and it turns out that in this case the graph Γ must be a regular infinitely branching tree ([Che11a, Lemma 8.6]).

We will review the general theory in §2. An important part of this theory concerns the *local structure* of Γ, by which we mean the structure of Γ_i for $i \leq \delta$.

Recall that Γ_i consists of the points at distance i from a given base point, considered in the first place as a metric space with the induced metric. Then Γ_i is an integer valued homogeneous metric space whose structure is independent of the choice of base point. If Γ_i contains no edges (i.e., the distance 1 does not occur), then this space is not so useful in our present state of knowledge. But if Γ is connected, and Γ_i contains an edge, then Γ_i with the induced graph structure turns out to be a metrically homogeneous graph. We will give more precise statements of this in §2.

1.3. The result.

The classification that results from our analysis is as follows. The statement uses the notation of the catalog of [Che18], and we leave its detailed explanation to §2. But modulo the general theory, we will need to focus on group (2) below, and specifically group (2c), for which there are essentially 5 possibilities, corresponding to the numerical parameters K_1, K_2 shown, subdivided further according to the parameter C into a total of 10 classes of examples. In addition to the numerical parameters K_1, K_2, C our main theorem involves an auxiliary parameter $\mathcal{S}$, which is a family of $(1,3)$-*spaces,* that is, spaces in which every distance is 1 or 3. In the chart below, additional constraints on the members of $\mathcal{S}$ are given; these depend on the values of the numerical parameters. The set $\mathcal{S}$ plays a secondary role in the statement of our main theorem below, and also for much of our analysis, but this parameter moves to center stage eventually in the course of §6. The roles played by these parameters are described in §2.3 in a more general setting.

The division of group (2) into three cases comes from the general theory. In general we write δ for the diameter of our metrically homogeneous graph. Here we specialize to the case where $\delta = 3$, but we write the cases $(2b, 2c)$ below in a form which remains relevant for larger values of δ. When $\delta = 3$ the case $(2b)$ is very marginal; for larger δ, that case allows a wider variety of examples than we see here.

Recall that according to our conventions, a metrically homogeneous graph of finite diameter is connected.

Theorem 1 (Classification Theorem, Diameter 3)**.** *Let Γ be a metrically homogeneous graph of diameter $\delta = 3$. Then Γ is one of the following.*

    (1) Finite:
        (a) An n-cycle for $n = 6$ or 7;
        (b) The antipodal double of C_5 or of $L[K_{3,3}]$;
        (c) The bipartite complement of a matching between finite sets.

    (2) The graphs $\Gamma^3_{K_1,K_2;C,C';\mathcal{S}}$ with admissible parameters, as follows:
        (a) If $K_1 = \infty$:
            Then $K_2 = 0$, $C = 7$, C' is 8 or 10, $\mathcal{S}$ is empty. With $C' = 8$ this is the bipartite complement of a matching between infinite sets and with $C' = 10$ it is the generic bipartite graph.

(b) *If $K_1 < \infty$ and $C \le 2\delta + K_1$ $(C \le K_1 + 6)$:*
    Then $K_1 = 1$, $K_2 = 2$, $C = 7$, $C' = 8$, and $\mathcal{S}$ is empty. This is
    the generic antipodal graph of diameter 3.
(c) *If $C > 2\delta + K_1$ $(C \ge K_1 + 7)$: Then $C' = C + 1$.*
    The various possibilities in this last case are shown in Table 1
    below.

K_1	K_2	C	$\mathcal{S}$
1	2	9 or 10	cliques and anticliques
1	3	8, 9, or 10	If $C = 8$ then $\mathcal{S}$ is empty.
2	2	9, 10	anticliques
2	3	9, 10	anything not involving a 3-clique K_3
3	3	10	empty

TABLE 1. $\delta = 3$: Case $(2c)$

Here part (1) covers exceptional local type, and part (2) covers generic type, in the sense of Definition 1.1; cf. §2.1. For the definition of antipodality in our context, see Definition 2.7.

The third column of Table 1 is devoted to the set of *Henson constraints* $\mathcal{S}$; cf. §1.6. We remark that in our classification, the set $\mathcal{S}$ must necessarily be finite (see §2.3), and in particular there are only countably many isomorphism types of metrically homogeneous graphs of diameter 3 (the classification conjecture for unrestricted diameter would imply, similarly, that there are only countably many isomorphism types of metrically homogeneous graph).

For the proof, we first quote general theory to reduce to generic type, then deal with imprimitive cases of generic type by special methods (§2). We then devote our attention exclusively to primitive metrically homogeneous graphs of generic type, beginning in §3.2. The first step is to recover the parameters $\delta, K_1, K_2, C, C', \mathcal{S}$ from the structure of the graph in a useful way (Definition 3.1). The final step is to show that our graph Γ and the "target" graph $\Gamma^\delta_{K_1,K_2,C,C',\mathcal{S}}$ have the same finite metric subspaces. Between these two steps there are two others.

- Show that the parameters satisfy various numerical conditions (admissibility) which ensure that the "target" $\Gamma^\delta_{K_1,K_2,C,C',\mathcal{S}}$ actually exists! (Proposition 3.19).
- Prove that Γ and $\Gamma^\delta_{K_1,K_2,C,C',\mathcal{S}}$ have the same isometrically embedded triangles.

This last point may seem like just one very special case of the more general problem of determining the finite subspaces of Γ, but as the point of the numerical parameters is to specify a set of forbidden triangles for $\Gamma^\delta_{K_1,K_2,C,C',\mathcal{S}}$, this really is part of the initial set-up.

In §5 we will review the preliminary steps of our approach, and lay out the plan of attack for the final analysis.

In our classification, we remark that the graphs falling under the cases $C = 8$ or $K_1 = 3$ occur in the appendix to [Che98] in a list of all primitive homogeneous structures with three or four nontrivial 2-types, all of them symmetric, not allowing "free" amalgamation, and which are determined by a set of forbidden triangles. It turns out that a number of examples in that list can be interpreted either as metrically homogeneous graphs (mostly of diameter 4) or as generic expansions of metrically homogeneous graphs of diameter 3 to a richer language. A number of examples still remain that cannot be so interpreted, and it would be interesting to find a way to account for them.

1.4. **Terminological conventions and general assumptions.** We take note of a few points of language which may require elucidation.

Some common terms make sense in both the graph theoretic and metric contexts, generally with different meanings.

In the metric setting, any three points constitute a *triangle,* with the associated metric structure. The term *clique* is used in the graph theoretic sense (the metric term would be *simplex*).

Paths in the metric sense are more general than paths in the graph theoretic sense. Given a sequence of points $(a_0, \ldots, a_n)$ in a metric space one may consider this as a labeled graph in two distinct senses, namely by labelling the pairs (a_i, a_{i+1}) with their distances, or by labelling *all* pairs (a_i, a_j) with their distances (i.e., taking the full induced metric structure). In the metric setting, we are particularly interested in *geodesic paths* (or more briefly, geodesics), in which the induced metric structure is the path metric from the induced path structure. In particular we speak of *geodesic triangles;* this means that the vertices can be arranged to form a geodesic path.

In a metrically homogeneous graph of diameter δ, all geodesics of length δ are realized.

Of course, the underlying graph will generally contain some paths in the graph theoretic sense which are not geodesics. However, any two points will be connected by some path which is a geodesic. As noted in [Cam98], this property characterizes the metric spaces associated with metrically homogeneous graphs, within the broader class of integer valued metric spaces.

We use the notation K_n for an n-clique, and, in particular, K_3 denotes a triangle in the graph theoretic sense. At the same time, we use the notations K_1, K_2 for certain numerical parameters, as discussed above. The meaning of these notations should be unambiguous, in context.

Finally, as the paper is about metrically homogeneous graphs of diameter $\delta = 3$, the condition $\delta = 3$ is in force through much of the paper, starting with §3.2. The first part of the paper prepares material based on very general considerations, and most of it applies to the case $\delta \geq 3$. In particular, we take up the discussion of *generic type* (Definition 1.1) in §2.3, which makes

sense for any diameter $\delta \geq 2$, and we explain why the full classification problem reduces to the case of generic type with $\delta \geq 3$. What we quote from the general theory holds for any diameter $\delta \geq 3$, and will for the most part be given in its general form before being specialized to the case $\delta = 3$. In particular, §3.1 provides some useful information in full generality, just before the main part of the analysis gets underway.

We also work with the assumption $\delta = 3$ in §2.5. Something can be said at a greater level of generality, but the general analysis is incomplete and the case $\delta = 3$ can be handled efficiently on its own, so we take that approach here. Thus, once we begin the substantive portion of the analysis in §3.2, we will be supposing that $\delta = 3$, and that the graph in question is primitive and of generic type.

Information about the case $\delta \leq 2$ is also very useful, and indeed is fundamental to our approach, but this case is covered completely by the classification theorem of Lachlan/Woodrow [LW80] (Fact 2.1, §2.1).

1.5. **Homogeneity, Amalgamation, and Classification.** In the study of homogeneous structures, methods of remarkable generality and power have been found for analyzing the topological and dynamical properties of the associated automorphism groups, cf. [KPT05, KeR07]. This assumes however that one knows the homogeneous structures well, in the sense that one has firm control of the associated amalgamation class of finite structures and can work with it combinatorially. This theory calls on structural Ramsey theory and descriptive set theory, notably the existence of *ample generics;* from ad hoc beginnings, the combinatorial methods developed have evolved into systematic theories, represented by [HuN18, HuN19, HeL00, Sol05].

With regard to classification problems for homogeneous structures in specified languages the situation is less clear. In the case of *finite structures* there is a highly developed theory, which makes systematic use of the ideas of stability theory as well as the classification of the finite simple groups. This theory applies in fact to the classification of stable homogeneous structures for a finite relational language. This is discussed in [Lac87, Lac96, Che00], and is extended in [ChH03].

More recently, the particular case of homogeneous structures for a language consisting of a finite set of linear orders has been completely settled by the application of ideas of the type associated with neo-stability theory (NIP) in model theory [Sim18, BrS18]. This is a very interesting case in which the application of direct amalgamation theoretic arguments is adequate in the case of at most three linear orders, but when approached in this fashion the natural analysis appears to blow up with the size the language. It is also of interest that in the imprimitive case certain *generalized metric spaces* (with values in a finite lattice) appear, and one of the points in the classification is that these themselves must be homogeneous. Thus the subject is not as distant from our present concerns as might first appear, as

metrically homogeneous graphs may also be usefully viewed as generalized metric spaces in the following way.

1.5.1. *The "magic" semigroup.* The theory of the known metrically homogeneous graphs has benefited from their relation to another type of generalized metric space.
    The appropriate general setting for this is the following.

Definition 1.2. *A* distance semigroup *is a commutative semigroup D together with a partial order $\preceq$ which extends the "natural" order*

$$a \preceq_{nat} b \iff \exists x \, b = a + x$$

If D is a distance semigroup then a D-valued metric space is a set A equipped with a symmetric D-valued distance $d(x, y)$ defined for x, y distinct, satisfying the triangle inequality

$$d(x, y) \leq d(x, z) + d(y, z)$$

for x, y, z distinct.

In the case studied by Braunfeld and Simon, D is a finite lattice and $\preceq$ is the lattice order. The corresponding class turns out to be an amalgamation class if and only if the lattice is distributive, in which case one can amalgamate using the *shortest path* construction, which is defined as follows.

Definition 1.3. *Let D be a distance semigroup and A a D-labeled graph. For a path P in A, the length of P, denoted $\|P\|$, is the sum of the labels on P.*
    The shortest path metric d on A is defined as follows.

$$d(x, y) = \inf(\|P\| \mid P \text{ a path from } x \text{ to } y)$$

For this definition to make sense, however, the indicated infima must exist. In particular if the graph A in question is not connected, and x, y are chosen to lie in distinct components, then the infimum is taken over the empty set, and thus denotes the maximum element of D—so we require D to have a unique maximum element.
    Our goal here is to use distance semigroups to understand metrically homogeneous graphs and possibly other structures for symmetric binary languages. The usual way to treat the set $[\delta] = \{1, \ldots, \delta\}$ as a semigroup is via truncated addition

$$x \oplus_\delta y = \min(x + y, \delta)$$

But what we really aim at is captured by the following definitions.

Definition 1.4. *Let $\mathcal{A}$ be a class of edge-labeled graphs and let D be a distance metric semigroup on the set of labels.*
    A partial $\mathcal{A}$-structure *is a weak substructure of an $\mathcal{A}$-structure. In the cases of interest, the underlying graph structure on structures in $\mathcal{A}$ will be complete (and every edge is labeled) and a partial $\mathcal{A}$-structure is a subgraph*

(not necessarily induced) of an $\mathcal{A}$-structure A in which each edge which is present inherits its label from A.

To complete *a partial $\mathcal{A}$-structure means to embed it as an edge-labeled subgraph into a structure in $\mathcal{A}$ (normally, on the same set of vertices).*

D is compatible *with $\mathcal{A}$ if every structure in $\mathcal{A}$ is D-metric.*

Suppose that D is compatible with $\mathcal{A}$.

- D gives an amalgamation method *for $\mathcal{A}$ if for every amalgamation diagram in $\mathcal{A}$, the shortest path completion relative to D is well defined and lies in $\mathcal{A}$.*
- D gives a completion method *for partial $\mathcal{A}$-structures if for every partial $\mathcal{A}$-structure A, the shortest path completion relative to D lies in $\mathcal{A}$.*

What interests us here are amalgamation methods for strong amalgamation classes (i.e., classes for which amalgamation can be carried out without identification of points). If the class $\mathcal{A}$ in question is in fact a strong amalgamation class, then any completion procedure includes an amalgamation method for the class.

In order for the distance semigroup structure on D to give an amalgamation method for $\mathcal{A}$, the shortest path completions required must exist, and in particular the infima taken over the set of all lengths of paths between a given pair of points must exist. This is not purely a condition on D: for example, forbidden cycles in $\mathcal{A}$ can be read in various ways as pairs of path lengths which will not occur simultaneously in any $\mathcal{A}$-structure, and whose infima are therefore not required.

We can now state a fundamental result.

Fact 1.5. *Let Γ be a known primitive metrically homogeneous graph of generic type. Suppose that Γ has diameter δ, auxiliary numerical parameters K_1, K_2, C, C', and Henson constrains $\mathcal{S}$. Let $\mathcal{A}$ be the corresponding amalgamation class of $[\delta]$-labeled graphs and let M be fixed so that Γ_M is a connected graph of diameter δ.*

Then there is a distance semigroup structure $D_{M,C}$ on $[\delta], \preceq$ for which M is the maximum element, which gives a completion method for the class $\mathcal{A}$, and, in particular, an amalgamation method for $\mathcal{A}$.

The condition imposed on M here can be given explicitly as follows.

$$\max(\delta/2, K_1) \leq M \leq \min(K_2, (C - \delta - 1)/2)$$

Namely, $K_1 \leq M \leq K_2$ to allow distance 1 to occur in Γ_M, and $\delta/2 \leq M \leq (C - \delta - 1)/2$ to allow δ to occur.

Furthermore, the partial order on $D_{M,C}$ is the natural order derived from the associated operation $+_{M,C}$ on $[\delta]$, which for $M = \delta$ is the usual truncated addition. In general $+_{M,C}$ is defined as follows: $a +_{M,C} b$ is the value in the interval

$$[|a - b|, \min(a + b, (C - 1) - (a + b))]$$

which is closest to M (observe that this is either M or one of the two end-points).

Now from what we have said so far it is not clear why $D_{M,C}$ should be a semigroup, why Γ is $D_{M,C}$-metric, why the shortest path completion is always defined when needed, or why, when it is defined, the completion is again D-metric and in $\mathcal{A}$. And the route to this at present remains roundabout. It begins with the detailed computations used to establish the amalgamation property in [Che18] and then the completion property of [ABW+17], followed by the analysis in [HKK18, HKN2x]. A similar formulation to the one we give here is found in [EHKL19], which uses the terminology of *Stationary Independence Relations,* another way of discussing canonical amalgamation procedures.

In principle, one would hope that this point of view would give a cleaner proof of the amalgamation property for admissible sets of parameters in the primitive case, but this has not yet been worked out in a self-contained way. In particular the parameter C' is not actually used in the algebraic construction, but the inequalities relating to this parameter necessarily play a role in the proof.

In the case of *imprimitive* metrically homogeneous graphs of generic type there is a very similar completion procedure but it cannot be expressed as simply in purely algebraic terms (in particular, in the antipodal case, we no longer have strong amalgamation).

In the context of the classification in the case $\delta = 3$, this algebraic structure is too limited to contribute much to our understanding.[2] But in general, this point of view contributes something to an understanding of the roles of the numerical parameters and the nature of the parameter constraints. In particular the corresponding natural order structure on $[\delta]$ induces the usual linear order on $\{1, \ldots, M\}$ and the reverse of the usual order on $\{M, \ldots, \delta\}$, and the extremal values $1, \delta$ appear as the two *minimal* elements of $[\delta]$. If the methods used in the present paper can be extended to larger diameters, then the associated semigroup may become more prominent, and may suggest useful ways of organizing the ideas, at least.

1.5.2. *Partial structures and forbidden homomorphisms.* Very recently another approach has emerged which gives a satisfying explanation for the form of the conjectured classification, though not, as yet, a proof.

The original goal of the classification in generic type, apart from the antipodal case, is to show that the constraints on Γ consist of triangles and $(1, \delta)$-spaces (from which the general classification then follows using known results, if the antipodal case can be handled separately). An alternative point of view runs as follows, if one restricts attention to metrically homogeneous graphs for which the associated amalgamation class has strong amalgamation.

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[2]See however [ABW+17, Table 1]: typically $M = 2$ in this case, and distance 2 will play a special role in our analysis.

In this case, it is useful to focus on the partial substructure (reduct) $\Gamma_{1,\delta}$, obtained from Γ by forgetting all distances other than 1 and δ. The structure Γ can be recovered definably from $\Gamma_{1,\delta}$, using the triangle inequality to bound all other distances above and below using geodesic paths involving only distance 1 and δ. The weak substructures of $\Gamma_{1,\delta}$ obtained by removing additional edges and vertices can be characterized by a family $\mathcal{F}$ of homomorphically forbidden $(1,\delta)$-spaces (the homomorphic images of these structures are forbidden substructures). In terms of $\Gamma_{1,\delta}$ the classification problem takes on the following form: *show that $\mathcal{F}$ may be taken to consist solely of cycles and complete graphs.*

Hubička, Konečný, and Nešetřil have shown in work in preparation that under reasonable hypotheses on the family $\mathcal{F}$, the desired conclusion follows, and that furthermore the precise classification in terms of numerical parameters then follows by a direct argument involving relationships among the various cycles, more naturally than via direct amalgamation arguments.

The key to their analysis is the characterization of the relational complexity of the original structure Γ in terms of the sizes of so-called *minimal separating cuts* in the forbidden structures in $\mathcal{F}$, worked out in general in [HuN16]; here the relational complexity in question is 2, and this gives strong information about the structures in $\mathcal{F}$.

As mentioned, this analysis starts with the assumption of strong amalgamation as well as an additional condition on the family $\mathcal{F}$, namely *minimality*: that is, none of the structures in $\mathcal{F}$ contains a homomorphic image of another structure in $\mathcal{F}$. Minimality can easily be achieved if $\mathcal{F}$ is finite, and for primitive metrically homogeneous graphs of known type it turns out, after the fact, that $\mathcal{F}$ is finite. On the other hand, a typical case involving an infinite set of constraints would be the generic bipartite graph of diameter δ, in which odd cycles of arbitrary length must be omitted, and there is no corresponding minimal family as the shorter cycles are homomorphic images of the longer ones.

It seems reasonable to say that the analysis by Hubička, Konečný, and Nešetřil provides some very good evidence that the classification conjecture should be (largely) correct—the antipodal case departs from this model, as it involves a different notion of Henson constraint, but one expects to handle imprimitive cases separately.

It remains very unclear how to apply this point of view to prove classification results at an appropriate level of generality. There are two questions to be resolved.

- Can strong amalgamation be proved for primitive metrically homogeneous graphs of generic type; in particular, can this be done for diameter 3 without first deriving the full classification?
- Can the reduction to cycles and Henson constraints be accomplished without a minimality condition on the family $\mathcal{F}$? (Note that the

result is true also in the bipartite case even though the minimality assumption on $\mathcal{F}$ is not achievable, as noted above.)

In particular, a test case for this method would be the classification of primitive metrically homogeneous graphs of generic type and diameter 3 under the hypothesis of strong amalgamation. If this can be done relatively efficiently, then one should consider whether the work in this paper can be replaced by some more direct approach to a proof of strong amalgamation.

To conclude, we state the general model theoretic result which lies behind the approach of Hubička, Konečný, and Nešetřil's.

Theorem 2 (Hubička, Konečný, and Nešetřil, in preparation). *Let U^* be a structure homogeneous in a binary language, L^* its canonical language and $L \subseteq L^*$ a sublanguage containing all unary relations of L^*. Denote by U the L-reduct of U^*. Assume the following.*

(1) All relations of U^* are definable in U.
(2) U is the unique existentially complete universal structure for its age, up to isomorphism.
(3) There exists a minimal family of finite connected L-structures $\mathcal{F}$ such that the age of U equals $Forb_h(\mathcal{F})$ (forbidding homomorphisms from structures in $\mathcal{F}$).

Then U^* is described by triangle constraints and L-Henson constraints (that is, L^*-structures whose L-reducts are complete). Moreover, $\mathcal{F}$ consists of irreducible structures (complete labeled graphs) and cycles.

In the application to metrically homogeneous graphs, U is the reduct to the language $\{1, \delta\}$, conditions (1,2) hold, and in the presence of strong amalgamation a form of (3) holds, but without the minimality condition.

1.5.3. *Combinatorial methods based on amalgamation.* Classification techniques based on combinatorial methods via the theory of amalgamation can also be sophisticated and conceptual. The present paper does not require the full range or sophistication of the existing techniques. In order to clarify how our analysis does proceed, and what additional resources this approach offers, we will first survey the general methodology and then discuss in more detail how our arguments go here.

The methods based more or less directly on the theory of amalgamation classes have the effect of efficiently reducing a classification problem of a suitable type to a definite finite set of checkable lemmas, each of which is in fact a special case of the result aimed at. One difficulty with this approach is that the number of cases may be very large and in principle there is no a priori bound on the length of the checking procedure, either. To be more precise, the required lemmas are checkable *if they are true;* but the associated decision problem (presented in more detail below) may be undecidable; this problem, raised by Lachlan, is one of the central theoretical questions in the subject.

Classification methods based on the theory of amalgamation are of the following four types, which are of varying degrees of sophistication and power.

- Direct amalgamation arguments;
- First reduction: Induction via auxiliary amalgamation classes [LW80, Lac84];
- Second reduction: A Ramsey-theoretic lemma [Lac84];
- Third reduction: Change of category (expansion of language) [Lac84].

.

These methods sufficed for the classification of the homogeneous directed graphs [Che98], and in the classification of the homogeneous ordered graphs [Che18], which generalize both [Lac84] and [LW80] while adhering closely to the method of [Lac84], where all four ingredients already occur. Generally speaking, these methods are most easily applied when characterizing *free* amalgamation classes, and one tends to dispose of the other cases first by more ad hoc and elementary methods.

For example, there are uncountably many homogeneous directed graphs associated with free amalgamation classes, but only countably many which are *not* free. The latter have an exceptional character; after dealing with these exceptional cases, the remaining uncountable family can be dealt with uniformly in an inductive framework, by what amounts to a reduction to finitely many cases (modulo some inductive parameters). This is lengthy, and depends on several applications of Ramsey's theorem, but it is quite manageable.

Let us be more explicit about the nature of these four types of argument.

- *Direct amalgamation*

For $\mathcal{A}$ and $\mathcal{B}$ finite sets of finite structures we write

$$\bigwedge \mathcal{A} \implies \bigvee \mathcal{B}$$

if every amalgamation class containing the structures in $\mathcal{A}$ contains some structure in $\mathcal{B}$. Frequently $\mathcal{B}$ consists of the single structure B and then we write the following.

$$\bigwedge \mathcal{A} \implies B$$

Such an assertion, when true, can always be verified in finitely many steps by performing a suitable search, naturally organized as a search in a tree of all possible results of all possible amalgamation problems. The resulting arguments are called *direct amalgamation arguments* as they involve posing appropriate amalgamation problems and considering the possible solutions. Though the relevant search tree tends to be exponentially large, the shortest path to the desired conclusion tends to be short, in practice. This is convenient, since the techniques used require several such direct amalgamation lemmas to be verified along the way, as we will see in the body of the present work, notably in §6.

Still, the complexity of such direct amalgamation problems is not known. Lachlan has conjectured that *the associated decision problem is decidable* [Lac87], and has proposed another conjecture which in the binary case is stronger: any amalgamation class is the intersection of finitely constrained amalgamation classes; in particular, when such a direct implication fails, the failure is witnessed by a finitely constrained class.

We are agnostic on this point. Conjectures in this area are not much constrained by evidence. When classification results have been obtained, while the proofs may be long, the lists have been compact and comprehensible. But it is worth bearing in mind that we may be considering just a few instances of an undecidable problem. Other model theoretic problems such as determining whether the joint embedding problem holds, or whether there is a countable universal model, become undecidable at a comparable level of generality [Bra19, Che11b].

As we have mentioned, the more sophisticated approaches available work most easily when aiming to characterize free amalgamation classes; and this is the point at which more straightforward methods tend to break down. This applies to both the first and second reductions (induction over amalgamation classes, Ramsey-theoretic arguments), and in their absence, the third reduction does not arise.

Any classification results in this subject are likely to make *some* use of direct amalgamation arguments along the way—though as we now have a body of general theory derived in this manner, valid for any diameter, some of this material belongs to the foundations of the subject and will not need to be revisited.

- *Induction over amalgamation classes*

This is a very powerful method, introduced in [LW80], and developed in new directions in [Lac84]. Subsequent work has tended to use the latter variant.

This type of induction is an elegant technique for proving that certain amalgamation classes are very large. For example, if an amalgamation class $\mathcal{A}$ of tournaments contains all tournaments of order 4, one wishes to prove that this class must contain *all* tournaments: this is the final and major step in the classification of countable homogeneous tournaments.

The idea in such cases is to define a subclass $\mathcal{A}^*$ of $\mathcal{A}$ which is also an amalgamation class, but which has stronger properties; for example, one may wish to require that every extension of a structure in $\mathcal{A}^*$ by a single vertex must lie in $\mathcal{A}$.

The original target theorem would imply that $\mathcal{A}^*$, like $\mathcal{A}$, itself contains all tournaments. However it suffices to prove that $\mathcal{A}^*$, like $\mathcal{A}$, contains all tournaments of order 4; then a trivial inductive argument proves the desired result. So the question is this: how should we define $\mathcal{A}^*$ so as to have both of the following properties?

— Any extension of a structure in $\mathcal{A}^*$ by a single vertex lies in $\mathcal{A}$.

— $\mathcal{A}^*$ is an amalgamation class.

The following definition works: $\mathcal{A}^*$ *is the class of finite tournaments A such that any expansion of A by a linearly ordered tournament belongs to the class $\mathcal{A}$.* Now the statement that every tournament of order 4 belongs to $\mathcal{A}^*$ decodes into the statement that any tournament which is an expansion of a linearly ordered tournament by four vertices belongs to $\mathcal{A}$.

The earlier argument of [LW80] in the context of graphs relies on the symmetry of the language and involves a very different choice of $\mathcal{A}^*$, but with a similar mechanism in operation to complete the proof.

● *The Ramsey-theoretic argument and a change of categories*

Induction over amalgamation classes has replaced a problem about arbitrary tournaments by a problem about tournaments obtained from a linear order by adding four vertices. These are still hard to handle.

In a second reduction, by a suitable mix of an argument based on Ramsey's theorem and amalgamation, this class can be replaced by the class of tournaments obtained by adding a single vertex to a *linear stack* of small tournaments; then after moving to a suitable larger category the stack can be reduced to a stack of height two, by an inductive argument. At this point, in the case of tournaments, all structures in view have at most nine vertices and one can arrange matters in this case to end very rapidly indeed. And one is also ready to generalize to arbitrary homogeneous directed graphs along similar lines, following the same general plan.

The difficulty that arises when one applies this set of techniques in the context of *homogeneous metric spaces* is that the triangle inequality generally eliminates the possibility of free amalgamation and thus poses obstacles to the application of the Ramsey-theoretic approach. More precisely, when the class $\mathcal{A}$ excludes certain forbidden substructures, the definition of $\mathcal{A}^*$ has to take this into account, and unless $\mathcal{A}$ is a free amalgamation class this is likely to break the Ramsey-theoretic argument, unless one works with a rather tightly controlled version of $\mathcal{A}^*$; and if one takes that route, the method proves less. However the tool remains a powerful one to have in one's repertoire even in such cases, and the information it is capable of providing may be very helpful in the general case.

In the case of metrically homogeneous graphs, there are a number of points to be settled before reaching the general case of the classification problem: these include the bipartite case, and local analysis as in [Che18], which is reviewed in 3.1. In a sense the case $\delta = 3$ treated here also belongs to this phase of preliminary analysis: if one hopes to make use of induction on diameter, this would have to be the base of the induction, as the classification in diameter at most 2 departs from the general pattern in some ways. It would not have been shocking to find additional "sporadic" structures at this stage; the fact that none appear is encouraging.

All of this—the bipartite case, the local analysis in its present state, and the analysis of the case of diameter 3—can be handled by direct amalgamation arguments. The question for the future is whether other ingredients will become more prominent at later stages.

It is curious and unexpected that while the classification of the homogeneous graphs (that is, the case $\delta = 2$) requires the more sophisticated methods we have described (either as given in [LW80] or in the alternative approach derived from [Lac84]), in the case of diameter 3 we are able to bootstrap from the diameter 2 case using considerably more direct methods. But the diameter 2 case is invoked in the reduction of this problem to the case of generic type.

How, in fact, does this analysis go?

1.6. **Structure of the Proof** ($\delta = 3$). The general classification theory, valid for any diameter δ, reduces the classification problem to *generic type,* (Definition 1.1). For generic type there is a reasonably uniform description of the expected classification in terms of five numerical parameters (one of which is the diameter δ) and a family of side constraints called *Henson constraints.* These are finite metric spaces of a specific form, typically $(1, \delta)$-spaces, that is spaces in which every distance is 1 or δ (with a variant in the case of antipodal graphs which is not relevant in our case, $\delta = 3$). In particular, if all distances equal 1 these are forbidden *cliques,* and if all distances equal δ we will call them *anticliques.*

For the case $\delta = 3$, all possibilities are shown in Table 2, page 35. From a practical point of view these should be grouped as follows.[3]

- Imprimitive: Bipartite and antipodal graphs (§2.5)
- Primitive, extremal (large K_1 or small C): $K_1 = 3$ or $C = 8$ (§5)
- Primitive, typical: $K_1 \leq 2$, $C \geq 9$ (§6)

Imprimitive cases with $\delta = 3$ are very special and can be handled easily by direct arguments. In general one has a decent grasp of the imprimitive case, which divides into the bipartite and antipodal cases. The bipartite case has already been handled in full generality, in an inductive framework. It would be very nice to have a similar analysis of the antipodal case in general, and it would be very natural to single out this case (and the natural analog of the second group as well) for special treatment before approaching the more typical cases of the general problem.

Thus all of the real work in the present paper will come in the primitive case. But between §2.5, dealing with the imprimitive cases, and §5, where the first of the primitive cases is taken up, some extensive preparations are required.

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[3]The first two cases listed have natural generalizations to $\delta \geq 3$ and should probably be treated as special cases in that context as well; the last case, "none of the above," is exceptionally well behaved for $\delta = 3$ and does not provide a clear model for the general case.

1.6.1. *Preparatory steps.* Let Γ be a metrically homogeneous graph of diameter 3 and generic type (and primitive, but this is a secondary issue at this stage).

The first stages of our analysis proceed as follows.

(1) Associate appropriate parameters $(\delta, K_1, K_2, C, C', \mathcal{S})$ to the graph Γ (Definition 3.1).
(2) Prove that these parameters are admissible, so that there is a unique known metrically homogeneous graph Γ^* of generic type with the same parameters (Proposition 3.19).
(3) Prove that Γ embeds into Γ^* (Corollary 4.3 and Remark 4.4).
(4) Prove that Γ^* embeds into Γ (Propositions 5.3 and 6.2).

In §3.2 we explain how the various numerical parameters are defined, and we show that for $\delta = 3$ these parameters take on only the values anticipated by our classification. One would very much like to carry through all of this analysis in general. Indeed, some of the work makes no assumption on the diameter (e.g., Corollary 3.18).

In the case of the known metrically homogeneous graphs, these numerical parameters determine precisely the triangles which embed into the given metrically homogeneous graph. In §4 we check that for $\delta = 3$ these parameters fulfill their intended purpose, determining which triangles are realized, in the expected manner. Again, some of the results obtained in this section are not limited to the case $\delta = 3$ (see §4.1). It will be seen that this analysis completes steps (1–3) of our plan, and also touches on the simplest case of (4), namely the embedding of triangles into Γ.

Throughout these preliminary developments one encounters a number of facts proved by explicit amalgamation arguments—and one would encounter more if we had not chosen to take advantage of a body of material where a number of useful principles are already proved quite generally. The classification results to be proved predict what facts of this type should be available at each stage, and amalgamation arguments provide a natural and easy way to check that these facts hold when needed.

But when we address the fourth point we will see less straightforward uses of the direct amalgamation method later, where the focus is on the identification of an appropriate *inductive framework* for the main amalgamation arguments (something the more sophisticated Ramsey-theoretic arguments alluded to above are intended to accomplish more forcefully, when they apply).

So at this point in the analysis, we have identified the expected target of our analysis, the known graph Γ^* which is expected to be isomorphic to Γ, and we have established the embedding in what may be expected to be the easier direction, namely the embedding of Γ into Γ^*.

This brings us to our two main results of the present work, the embedding theorem in the extremal cases (§5) and the embedding theorem in the typical cases (§6).

1.6.2. *A further reduction of the extremal cases.* There is one more reduction before we arrive at the heart of the matter, which applies only to the two extremal cases considered in §5. The two extremal cases, $C = 8$ and $K_1 = 3$, are in a sense the next cases after the imprimitive ones, since $C = 7$ gives the antipodal case, and $K_1 = 4$ (or $K_1 = \infty$, since $\delta = 3$) gives the bipartite case. These two "almost imprimitive" cases are *dual* in a way that the antipodal and bipartite cases are not. This is explained in §5.1: the two graphs we seek to characterize are isomorphic *up to a permutation of the language*—that is, up to a permutation of the set of distances $[\delta] = \{1, \ldots, \delta\}$. Thus in the treatment of the extremal case it suffices to deal with the case

$$K_1 = 3$$

Metric spaces which remain metric after permuting the distances are an oddity, but a known oddity in the context of distance regular graphs and more generally in the theory of (finite) association schemes [BaB80, Gar80]. Rebecca Coulson has shown [Cou18] that there is a similar phenomenon in the context of metrically homogeneous graphs: namely in every diameter δ there is a pair of ostensibly very different metrically homogeneous graphs which in fact become isomorphic after a permutation of the set of distances; here one of the graphs has $C = 2\delta + 2$ and the other has $K_1 = \delta$; furthermore, there are no other such pairs (though there are some other instances of metrically homogeneous graphs allowing a non-trivial permutation of the distances, which however results in an isomorphic graph).

This fortunate duality cuts the number of extremal cases requiring consideration in half, leaving just one special case rather than two. And by Coulson's result, the same holds for any value of δ: there are two extremal cases which appear to be quite different but can be treated as one.

1.6.3. *The embedding lemma: inductive framework.* There are broad similarities in the treatment of the extremal primitive case and the more typical primitive cases. We will discuss the more typical case, but much of this applies in some form also in the extremal case, with some simplifications.

In our treatment of the main embedding lemma (Proposition 5.3 for the extremal case, and Proposition 6.2 for the typical case) the key point will be to set up an appropriate *inductive framework* for the proof. Writing Γ^* for the graph of known type with the same numerical parameters as Γ, the embedding lemma states that any finite configuration embedding into Γ^* also embeds into Γ, from which the conclusion $\Gamma \cong \Gamma^*$ follows rapidly.

The main difficulty is to find *numerical measures of complexity* for finite configurations which we are able to reduce systematically by appropriate amalgamation constructions, even in cases where the factors of the amalgam have more vertices than the target structure. This provides the inductive framework which ultimately makes our analysis successful.

We will discuss this inductive framework as implemented in the case of the typical (that is, non-extremal) primitive cases of generic type and diameter

3. These cases are shown in Table 4, p. 63. Observe that no forbidden triangle involves the distance 2. Our claim is that no minimal forbidden configuration involves the distance 2, either; in other words, the minimal forbidden configurations involve only the distances 1 and 3. Phrased as an embedding lemma, this takes on the following form (Proposition 6.1): *if A is a finite [3]-valued metric space such that every triangle and every $\{1,3\}$-valued subspace embeds into our given metrically homogeneous graph Γ, then we claim that A does as well.* To conclude this outline, we will define an appropriate measure of complexity for such configurations A, and then sketch the main lines of the proof of the embedding theorem in terms of these complexity measures.

We first consider the graph G_A induced on A by taking as edge relation "$d(x,y) \in \{1,3\}$"; the trivial case is that in which G_A is a clique (A is a $(1,3)$-space). The two associated parameters of interest are the following.[4]

$$\mu = \max(|K| \,|\, K \subseteq G_A \text{ a clique})$$
$$\nu = |\{v \in G_A \,|\, \deg(v) \geq 2\}|$$

For the proof of the embedding theorem, we work inductively, and consider a putative counterexample A which minimizes first μ, and then ν. The case in which G_A has a unique clique of maximal size is handled by a direct amalgamation construction (introducing some auxiliary vertices to simplify the situation with respect to our measure of complexity) in Lemma 6.3. Then one can argue that in the minimal case one has $\nu \leq 2$ (Lemma 6.5). At this point our choice of numerical parameters has paid off substantially in terms of a reduction in complexity of the essential cases. But this still leaves much to do.

Various direct arguments take care of the cases in which G_A is a star or $\nu = 0$ (Lemmas 6.15, 6.18).[5]

After these preparations, in our inductive analysis we consider a finite metric space A whose triangles and $(1,3)$-subspaces embed into Γ, but which is supposed not to embed into Γ, chosen so as to minimize, in succession, the parameter μ, then ν, then the number of non-trivial components of G_A (which will be edges), and in last place, the cardinality of A.

At this point the configuration is tightly constrained. The graph Γ_2 inherits all of the essential properties of Γ, so induction on $|A|$ allows one to suppose that every vertex of A has a neighbor in the graph G_A,[6] while by the prior reductions there are either one or two vertices of degree at least 2 in G_A, and if there are two then they lie on an edge.

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[4]Cf. Definition 6.4 in a different notation.

[5]The treatment of the "base" case $\nu = 0$ relies very directly on the assumption $\delta = 3$.

[6]A typical reduction of the type which applies only with $\delta = 3$, as it passes through the subgraph Γ_2.

One then shows that G_A is connected, and the resulting configurations are handled by one additional induction involving direct amalgamation constructions. In this case, the auxiliary amalgamations tend to involve adding two additional points to the original configuration to fix one distance via the triangle inequality; here again one needs inductive measures of complexity which can go down while the number of points involved actually increases. Typical arguments of these kinds are seen in the final proof of Proposition 6.1.

The method used in the proof of the two forms of the embedding theorem has considerable potential for further development, and one would expect that in general the main case division would be the division between the cases $C \leq 2\delta + 2$ (or the variant $K_1 = \delta$) and the rest. This topic will be discussed in §7.5.

1.7. **Prospects.** The present article is intended as a trial run for more general results. There is still no clear strategy for a complete classification by direct amalgamation arguments (or even with the more sophisticated Lachlan/Woodrow and Lachlan approaches) but there are many ingredients ripe for generalization, and certain less well-defined elements that have appeared repeatedly.

In particular, there are useful results not tied to a bound on the diameter δ, which hold in an inductive context [Che18, Chaps. 16, 17]: namely, the reduction of the bipartite case to "prior" cases (when the diameter is finite, this simply means to cases of smaller diameter), as well as the reduction of the infinite diameter case to the finite diameter case (if the conjectured classification is true in the finite diameter case). In such cases a suitable induction can be found along lines similar to those we use here, though in each instance it has been necessary to work out a particular variation on the idea.

There is also a possibility that a variant of the classification conjecture for metrically homogeneous graphs may actually hold more generally at the level of homogeneous structures for a finite symmetric binary language with trivial algebraic closure (corresponding to strong amalgamation classes).

We return to this topic in §7.

1.8. **On Length, and Methodology.**

1.8.1. *A methodological question.* The referee of this paper raised a natural question, in the following terms.

> . . . interesting insights were given mostly at the beginning (with [§5.1] being an exception) and later on it really was mostly mechanical work. On the other hand it seems to be the spirit of the whole area and one wonders if there are really no better methods.

This is in a sense the question that lies behind work in this area, and on reflection we felt that it deserved serious discussion here.[7]

In §1.5 we have discussed the current state of the methodology in this area, notably the four elements of the combinatorial approach, as currently understood (only some of which are exploited in the present paper), together with alternative methods which have been applied to related problems. In particular, the idea of treating metrically homogeneous graphs as generalized metric spaces with values in a partially ordered semigroup may prove directly relevant.

We also took note of Lachlan's decidability conjecture in §1.5. This conjecture raises another fundamental methodological question about classification of homogeneous structures for finite relational languages in general; and our approach requires all instances of Lachlan's decision problem that we actually encounter to be resolved (sometimes in the negative, in which case one has discovered a previously unknown structure, as occasionally happens in practice).

In §1.6 we aimed to clarify the strategy followed in the technical core of the proof of our main result, and to extract the main ideas behind the sequence of technical lemmas. The main point is that at its core our approach requires a suitable measure of the complexity of a finite structure, one which can be forced to decrease in the course of a direct amalgamation argument even as the number of points involved increases. This complexity measure controls the structure of the final stages of the analysis, and if it is kept in view, then the structure of the final arguments then emerges naturally.

In §1.7 we touched on the implications for the general problem of the classification of metrically homogeneous graphs. This last point is not a settled question. We have developed an inductive scheme which allows the conjectured classification to be established by direct amalgamation arguments in the case of diameter 3. The main difficulty in general, for this direct approach, has been the absence of an analogous strategy for dealing with primitive metrically homogeneous graphs of generic type in larger diameter. Recently a re-examination of the treatment of diameter 3 has led to the development of a plausible strategy along parallel lines appropriate to the general case; see §7.5. Typically such a strategy reduces the proof of a general classification result to a number of more isolated results, each itself a consequence of the proposed classification, requiring their own direct treatment (possibly with machine assistance at that stage, and possibly leading to the identification of further "sporadic" cases).

We can summarize our views on the methodological question raised as follows.

(A) If our methods can be generalized to arbitrary diameter, then the length and the mechanical quality will no doubt be even greater,

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[7]The introduction to our first draft ended with §1.4.

but should be proportional to that seen in other combinatorial or algebraic classification problems.

(B) The mechanical quality of the numerous specific lemmas required (all of them instances of Lachlan's decision problem) suggests an opportunity for machine assisted proof.

(C) Whether or not these combinatorial methods can be brought to bear in the general case, the question of alternative methodologies remains very relevant, and the evidence on this point is mixed.

Under point (A), the other results we have in mind include the Classification of the Finite Simple Groups, the Strong Perfect Graph Theorem, the Graph Minor Theorem, the 4-Color Theorem, and Kepler's Conjecture on sphere packing, all of which have inordinately long and combinatorially complex proofs, some of them machine-assisted. These developments have produced a vigorous discussion, sometimes contentious and often enlightening, with an ample literature in its own right.

One set of responses to the rise of a class of proofs challenging the usual approaches to exposition, verification, transmission, and understanding by virtue of their length or reliance on machine computation is found in the collection of essays [Bun05]. See also [Avi17, Hal12, HAB+17, Kra11], and the talk by Hales at the Newton Institute 2017 program on "Big Proof" [Hal17].[8]

We will discuss these developments further in the remainder of this introduction. Thus we now move beyond the context of metrically homogeneous graphs to a much broader perspective. None of this is needed for the body of the paper.

1.8.2. *Intricate inductions and classification theorems.* As mentioned above, the last half century has seen the rise of very long and combinatorially complex solutions of simply posed problems in algebra, combinatorics, and geometry, sometimes machine-assisted in an essential way, notably the following.

(1) The 4-Color Theorem (announced 1976, published 1977);
(2) The Classification of the Finite Simple Groups (announced 1983, completed 2004);
(3) The Graph Minor Theorem (Robertson-Seymour Theorem) (2004);
(4) The Strong Perfect Graph Theorem (announced 2002, published 2006);
(5) Kepler's conjecture and other sphere packing problems (announced 1998, published 2005).

Here entries (2,4) are classification problems, while the proof for entry (3) is based on a qualitative structure theorem of great generality. Entries (1,5)

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[8]A program incorporating strongly positive views on automatic theorem verification was launched by Voevodsky on very different grounds: he viewed computer assistance as a solution to weaknesses of the usual approach, rather than a source of difficulties. Perhaps ultimately the burden of refereeing will be shifted to the author via appropriate mathematical software, as is often the case for mathematical typesetting.

are combinatorial results of a different character, where machine assistance plays a critical and controversial role. We will first discuss entries (2,3,4).

The *Feit-Thompson theorem,* at 255 pages, served in more innocent times as an example of a general classification result (for the empty set) with an extraordinarily long proof; the 25,000 journal pages of the proof of the *Classification of the Finite Simple Groups* sparked a spirited debate of the methodological issues between Atiyah and Gorenstein. This debate is neatly summarized in the MathSciNet review (MR0818060) of [Gor86] in terms which remain relevant today. That classification result in turn was applied to permutation group theory and a very wide variety of other branches of mathematics (including model theory), which then may be said to inherit the methodological issue, sometimes compounding it by further analyses which are quite long in their own right, or require machine assistance.

The Robertson-Seymour *graph minor theorem* is proved in over 500 pages, spread over 20 articles published over two decades. This is an extremely general finiteness theorem, giving among other things non-effective proofs of polynomial time computability for many natural problems. Lovasz has said [Lov06]

> The proof . . . is based on a very general theorem about the structure of large graphs: If a minor-closed class of graphs does not contain all graphs, then every graph in it is glued together in a tree-like fashion from graphs that can almost be embedded in a fixed surface. . . . Roughly speaking, . . . if a graph does not contain a certain minor, then it is 2-dimensional.

Thus a major element of the proof is a family of very general structure theorems of a qualitative type. We emphasize that a single, arbitrary, forbidden minor is in question here, and that it is shown that the resulting class of graphs is both severely limited and highly structured.

Another striking development with a similar character was the proof of the strong perfect graph theorem by Chudnovsky, Robertson, Seymour, and Thomas [CRST06]. The statement, the mathematical context, and the structure of the proof are all covered in [CRST03]. We cannot review this here, but we can cite the same paper by Lovász for an indication of its relevance (about which, Lovász says considerably more).

> The excluded minor characterizations and the structure theorems discussed above can serve as prototypical examples of a paradigm that leads to many difficult but important results. Perhaps most dramatic of these is the recent resolution of the Strong Perfect Graph Conjecture by Chudnovsky, Robertson, Seymour and Thomas [6]. Here again, the key to the proof is a structure theory . . .

1.8.3. *Machine-assisted proofs.* On the combinatorial side, machine-assisted proofs came into the picture very early in a substantial way, sparking a more vigorous reaction than sheer length ever did.

The first such case to receive attention was the proof of the *Four Color Theorem* by Haken and Appel. The reviewer for Math Reviews, Frank Allaire, made the following remark.[9]

> Their procedure is described by some 500 diagrams .... The proof that it discharges every U-avoiding 5-connected planar triangulation involves several enumerations of subcases. The weakest part of the proof lies in these enumerations performed by the authors' first computer program.

Robert Wilson wrote as follows in 2002 [Wil02].

> This was the first major theorem whose proof involved a substantial amount of computer calculation, and as such it was bound to cause controversy. It has stimulated a wide-ranging debate over the past quarter of a century, ...

A computer-assisted proof of *Kepler's conjecture* on sphere packing was found by Hales and Ferguson [HFL11]. This was originally published as a series of six papers carefully refereed over eight years. At the end of this long series of arguments one has arrived at the point of understanding how, in principle, a computer might verify the result by a calculation, and why that calculation should end in a reasonable amount of time. In the first chapter of [HFL11] Jeffrey Lagarias explains how the proof goes, what issues were raised by it, how they were dealt with at the time, and what some of the typical reactions were, including the policy set by the *Annals of Mathematics* for dealing with proofs of this type. Here we cannot go into any of the details, but the interested reader should certainly look into at least §§3–7 of that article (pp. 14-21).[10]

Such machine assisted methods were also viewed as promising in dimensions 8 and 24 (where E_8 and the Leech lattice reside). Very recently these two cases have been handled by showing that a so-called "magic function" required to apply the linear programming bound of Cohn and Elkies can be constructed from quasi-modular forms, providing the kind of spectacular connection and insight that many or most mathematicians live for [Via17, CKM+17]. But there is no reason to expect anything similar in dimension 3. This is an interesting example of a case in which complex combinatorial methods vie with very elegant conceptual methods, and it is not at all clear a priori which approach is more likely to be effective in a given setting.

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[9]Joint reviews for entries MR0543797, MR0543796. MR0543795, MR0543793, and MR0543792.

[10]Also noteworthy is an earlier proof proposed by W.-Y. Hsiang (cf. [Hsi95]), and the subsequent fully formalized proof by Hales et al.; these topics are also covered by Lagarias.

1.8.4. *Challenges.* Proofs of this type generally evolve over a very long period of time, and so do the details of the underlying proof strategy, with the outcome very much in doubt. And not everything of the sort which has been tried has proved successful; one should not consider only those cases in which the problematic strategy has proved successful (or, at least, has been claimed).

Lagarias makes a remark (p. 12, op. cit.) concerning the method of local inequalities used by Hales et al., which could as well be applied across the board at the point where the proof remains under development.

> There now arises a psychological difficulty, which is that the "optimality" of the local density inequality is only certified after the fact, when a proof is found. This means that one must first do a very large amount of work, with the downside risk of eventually determining that the inequality is not optimal.

Mutatis mutandis, similar remarks would apply to every proof of this type. One anticipates that a failed program with a strong conceptual motivation may very well deliver valuable insights even if its original goal is not achieved; but the same is less likely to apply to more directly combinatorial or computational approaches.

An interesting case currently under investigation, where the eventual outcome remains highly uncertain, is the *Erdős-Hajnal conjecture*, concerning the sizes of maximal cliques or anticliques (Ramsey theory) in graphs supposed to omit some particular induced subgraph (the forbidden induced subgraph being arbitrary). There are remarkably few cases in which it is known whether this conjecture is true or false; but for forbidden graphs up to order 5, the conjecture has been verified, making use along the way of the strong perfect graph theorem.

According to this conjecture, for any specified graph H, the class $\mathcal{C}$ of H-free graphs should be extremely limited, from the point of view of Ramsey theory.[11] It is natural to try to proceed as in the case of the Graph Minor Theorem to derive the conjecture from structural information, a line which has been actively explored in specific cases. We refer to [Chu13] for an extensive discussion of this problem and, in particular, the reduction to the case of prime graphs (having no non-trivial congruence) and the case in which the forbidden graph is the "bull," a particular graph on five vertices, where the Strong Perfect Graph Theorem can be brought to bear.

The classification conjecture for metrically homogeneous graphs is in a similar state: an explicit classification is conjectured and various reductions and special cases have been dealt with. The reduction of the case of infinite diameter to the case of finite diameter allows us to envision an analysis which proceeds by induction on the diameter.

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[11]Improved bounds (polynomial) in the statement of Ramsey's theorem for graphs.

A robust way of distinguishing the exceptional and generic cases a priori has been found, and the exceptional cases have been handled in full generality, while the generic case has been handled in the bipartite case, under a suitable inductive hypothesis, in all diameters. The generic case for diameter 3 will be dealt with mainly in §5 and §6; it completes the full classification in diameter 3.

1.9. **Acknowledgment.** We thank Konečný and Hubička for discussions, notably of §§1.5.1,1.5.2.

2. Preliminary results

2.1. **Exceptional local type and generic type.** We will first explain in detail the parts of the classification in diameter 3 which do not need to be addressed here, having been covered elsewhere without any limitation on δ. First we present a general framework for the classification of metrically homogeneous graphs.

The classification of homogeneous graphs by Lachlan and Woodrow may be summarized as follows. This is the case $\delta \leq 2$ from the metrically homogeneous point of view, so all open questions concern the case $\delta \geq 3$.

Fact 2.1 (Summary of [LW80]). *A homogeneous graph is either finite, imprimitive, or of one of the following forms.*

- *Complete, or an independent set of vertices;*
- *A Henson graph H_n with $n \geq 3$, or its complement;*
- *The random (homogeneous universal) graph G_∞.*

Here the Henson graph H_n is the unique homogeneous graph which is universal subject to containing no n-clique [Hen71].

If Γ is a metrically homogeneous graph then Γ_1 (as defined in the introduction) is a homogeneous graph. Certain cases clearly have a special character. For example, if Γ_1 is complete then Γ is a disjoint union of complete graphs.

We have divided the metrically homogeneous graphs into three broad classes in the introduction, mainly according to the structure of Γ_1. At the extremes we have exceptional local type, and generic type, with a residual class left in the middle.

This residual class is characterized, a priori, as follows.

- Γ_1 is primitive and contains an infinite independent set, but for $v \in \Gamma_2$, the set of neighbors of v in Γ_1 contains no infinite independent set.

By metric homogeneity, the structure of the set of common neighbors for a pair of points at distance 2 is independent of the pair considered. Therefore, if Γ_1 is a Henson graph H_n with $3 \leq n < \infty$, or a random graph, then such a set always contains an infinite independent set.

So in this class, Γ_1 cannot be a Henson graph or a random graph, yet is primitive and contains an infinite independent set. In view of the classification of homogeneous graphs, Γ_1 must be an independent set, and for $v \in \Gamma_2$

the set of neighbors of v in Γ_1 must be finite. As noted in the introduction, one can then argue that in the connected case Γ is a regular infinitely branching tree [Che11a, Lemma 8.6].

Thus we have the following.

Fact 2.2 (cf. [Che11a]). *Any metrically homogeneous graph falls into precisely one of the following categories.*

- *Exceptional local type;*
- *A regular tree with infinite branching;*
- *Generic type.*

We are interested only in the finite diameter case here, so the second entry falls away.

The classification of homogeneous graphs also gives a more useful way to think about the definition of generic type.

Remark 2.3. *Let Γ be a metrically homogeneous graph of generic type. Then Γ_1 is one of the following.*

- *An independent set;*
- *A Henson graph H_n with $3 \leq n < \infty$;*
- *A random graph G_∞.*

Lemma 2.4. *Let Γ be a metrically homogeneous graph of generic type, and u, v two vertices at distance 2 in Γ. Then the graph Γ' induced on the set of their common neighbors is isomorphic to Γ_1.*

Proof. Since Γ' is a homogeneous graph, it suffices to check that it embeds into Γ_1 and that Γ_1 embeds into Γ'. If we take u to be the basepoint for Γ then $\Gamma' \subseteq \Gamma_1$ and the first point is clear. So it suffices to show that Γ_1 embeds into Γ'.

This is evident if Γ_1 is an independent set, since Γ is of generic type. So we may suppose that Γ_1 is a random graph or a Henson graph H_n with $n \geq 3$.

Now we may take as the basepoint u_* a vertex in Γ'. Then $u, v \in \Gamma_1$ and $(\Gamma')_1 = \Gamma' \cap \Gamma_1$.

If Γ_1 is the random graph then since $u, v \in \Gamma_1$, $\Gamma' \cap \Gamma_1$ is again the random graph, and thus Γ_1 embeds into Γ'.

Suppose finally that

Γ_1 is a Henson graph H_n, the generic graph omitting an n-clique,

and $n \geq 3$.

Then

$$(\Gamma')_1 \cong H_{n-1}$$

(for $n = 3$ this is just an infinite independent set).

In particular $(\Gamma')_1$ contains a clique of order $n - 2$ and an infinite independent set. So Γ' contains a clique of order $n - 1$, an infinite independent set, as well as a path of order 3.

If Γ' is primitive, then by the classification of homogeneous graphs, this is already enough to show that H_n embeds in Γ'. So we consider the case in which Γ' is imprimitive.

In this case, as Γ' contains a path of order 3, it follows that Γ' does not contain $K_1 + K_2$ (a disjoint sum of an isolated vertex and a clique of order 2). But for $n \geq 4$ this graph is already in $(\Gamma')_1$. So we suppose

$$n = 3; \ \Gamma' \text{ is complete multipartite, with infinite parts.}$$

Since Γ' embeds in Γ_1, which is triangle-free, Γ' is complete bipartite, with infinite parts.

We aim at a contradiction.

We now adjust our notation so that $u = u_*$ is viewed as basepoint and $v \in \Gamma_2$. For $i \geq 1$ let A_i be the graph induced on

$$\{x \in \Gamma_1 \mid d(v, x) = i\}$$

Then A_i is empty for $i \geq 4$. But we will show by induction on i that A_i is complete bipartite and non-empty for all $i \geq 1$.

In the base case $i = 1$ and the statement is true by assumption. So suppose $i > 1$ and the claim holds for $j < i$. Set

$$B = \bigcup_{j < i} A_j$$

As B has finite chromatic number, $B \neq \Gamma_1$ (or more directly, if one selects representatives of the parts of the A_j for $j < i$, then any vertex of Γ_1 with no edges to the representative vertices must lie outside B). As Γ_1 is connected there are $b \in B$, $a \in \Gamma_1 \setminus B$ so that (a, b) is an edge. This forces $b \in A_i$, $a \in A_{i-1}$.

We now consider the structure (A_{i-1}, A_i), which is homogeneous as a graph with unary predicates denoting the two parts. Furthermore there are edges between the two parts, and therefore any vertex in one part has a neighbor in the other part. Since A_{i-1} is complete bipartite and Γ_1 is triangle free, the relation $E(x, y)$ on A_i defined by

$$x, y \text{ have neighbors } x', y' \text{ in } A_{i-1} \text{ in the same part}$$
$$\text{(i.e., not forming an edge)}$$

is an equivalence relation with two parts, definable without parameters. Thus A_i is either complete bipartite or the disjoint union of two cliques.

Case 1. There are b_1, b_2 at distance 2 in A_{i-1} and $a \in A_i$ a neighbor of b_1 but not b_2.

Then $d(a, b_2) = 2$ and the set of common neighbors of a and b_2 is complete bipartite, with infinite parts. Let X be the set of common neighbors of a, b_2, and the basepoint u_*. Then X lies in Γ_1 and by the triangle inequality $X \subseteq A_{i-1} \cup A_i$. On the other hand X cannot meet either part of A_{i-1} and so $X \subseteq A_i$. Thus A_i contains an infinite independent subset and must be complete bipartite. This completes the induction in this case.

Case 2. Each vertex of A_i is adjacent to all vertices in one part of A_{i-1}.

Then the parts of A_i are independent sets. Hence A_i must either be complete bipartite, or reduce to a single pair of vertices with no edge between them.

In this last case we consider the dependence of A_i on the vertex $v \in \Gamma_2$; we write $A_i = A(v)$. Each pair of vertices in Γ_1 at distance 2 arises as $A(v)$ for some $v \in \Gamma_2$. Between two disjoint pairs of points $\{a_1, b_1\}$ and $\{a_2, b_2\}$ with $d(a_\ell, b_\ell) = 2$, there can be from 0 to 4 edges occurring. Thus in Γ_2 there are at least five 2-types realized by distinct pairs of vertices, relative to the basepoint of Γ. But the diameter of Γ_2 is at most 4 and so this is a contradiction.

Thus, by induction, the A_i are complete bipartite and nonempty, giving a contradiction for $i \geq 4$. $\qquad\square$

2.2. **Exceptional Local Type.**
An explicit classification of the metrically homogeneous graphs of exceptional local type is given in [Che11a], and reviewed in [Che18]. Leaving aside those of infinite diameter, the list in diameter $\delta \geq 3$ is as follows.

Fact 2.5 ([Che11a, Theorem 10]). *A metrically homogeneous graph of exceptional local type and finite diameter $\delta \geq 3$ is finite.*

More explicitly, it is one of the following

- *An n-cycle, with $n \geq 6$;*
- *An antipodal graph of diameter 3, with Γ_1 finite:*
  - *The bipartite complement of a perfect matching between two finite sets;*
  - *The antipodal double of C_5 or $L[K_{3,3}]$.*

Specializing to diameter $\delta = 3$, the relevant n-cycles have diameters $n = 6$ or 7.

Corollary 2.6. *Let Γ be an infinite metrically homogeneous graph of finite diameter. Then Γ is of generic type.*

Proof. By Facts 2.2 and 2.5. $\qquad\square$

We now explain the antipodal case, which appears again in group $(2b)$ of our Classification Theorem.

Definition 2.7. *Let Γ be a graph of diameter $\delta \geq 2$. Then Γ is antipodal if for each vertex $v \in \Gamma$ there is a unique vertex v' with $d(v, v') = \delta$.*

This is not the standard definition from the theory of distance transitive graphs, but rather a modification more suited to the context of metrically homogeneous graphs, at least when the diameter is at least 3. This definition is meaningful for $\delta \leq 2$, but less useful in that context. For $\delta = 2$ the antipodal graphs in our sense are complete multipartite graphs which happen to have classes of size two.

The structure of antipodal graphs is quite special, as the following indicates.

Fact 2.8 ([Che11a, Theorem 11]). *Let Γ be a metrically homogeneous antipodal graph of diameter δ. Then the pairing $v \leftrightarrow v'$ defined by $d(v, v') = \delta$ defines a central involution in $\mathrm{Aut}(\Gamma)$. In particular $\Gamma_i \cong \Gamma_{\delta-i}$ for $0 \leq i \leq \delta$.*

Note that in diameter 3 we have

$$\Gamma_1 \cong \Gamma_2$$

with the pairing $v \leftrightarrow v'$ giving a canonical isomorphism.

The antipodal graphs of diameter 3 listed in Fact 2.5 are those of exceptional local type. In these graphs Γ_1 is either a finite independent set, or one of two exceptional finite primitive homogeneous graphs, of orders 5 and 9 respectively.

Below we will give the general classification of antipodal graphs of diameter 3, which is similar (Lemma 2.17).

2.3. **Generic Type.** Having disposed of the non-generic types, as discussed above, we can rephrase our main theorem more concretely as follows—where we still have to explain the main notations.

Proposition 2.9 (Main Theorem, generic type). *Let Γ be a metrically homogeneous graph of diameter 3 and generic type. Then Γ is a graph of the form $\Gamma^3_{K_1,K_2;C,C';S}$ with admissible parameters, as follows:*

- *(a) If $K_1 = \infty$ (Γ is bipartite):*
  Then $K_2 = 0$, $C = 2\delta+1 = 7$, C' is 8 or 10, S is empty. With $C' = 8$ this is the bipartite complement of a matching between infinite sets and with $C' = 10$ it is the generic bipartite graph.
- *(b) If $K_1 < \infty$ and $C \leq 2\delta + K_1$ (Γ is antipodal, not bipartite):*
  Then $K_1 = 1$, $K_2 = 2$, $C = 7$, $C' = 8$, and S is empty. This is the generic antipodal graph of diameter 3.
- *(c) $C > 2\delta + K_1$ (Γ is primitive):*
  Then $C' = C + 1$.
  The various possibilities under this heading are as listed above in Table 1.

For a full listing of all possibilities, see also Lemma 2.13 below.

In [Che11a] a class of metrically homogeneous graphs

$$\Gamma^\delta_{\bar{K},\bar{C},S}$$

defined by 5 numerical parameters $\delta, K_1, K_2, C_0, C_1$, and one more geometric parameter S was introduced. We set $\bar{K} = (K_1, K_2)$ and $\bar{C} = (C_0, C_1)$, where C_0 is even and C_1 is odd. We also work with the notation $\bar{C} = (C, C')$, where $C = \min(C_0, C_1)$, $C' = \max(C_0, C_1)$; that is, we may write these parameters in increasing order rather than according to their parity, and vary the notation to suit this point of view.

Here δ denotes the diameter, while the parameters $\bar{K}, \bar{C}$ serve to define a set of forbidden triangles $\mathcal{T}(\delta, \bar{K}, \bar{C})$. The parameter S consists of a set of finite $(1, \delta)$-metric spaces, that is, metric spaces in which all distances are

equal to 1 or δ.[12] Since $\delta > 2$, this is a union of cliques with distinct cliques at distance δ from each other.

As we remarked earlier, if the sets $\mathcal{S}$ are irredundant then they are finite. We now elaborate on the combinatorial content of this remark. The elements of $\mathcal{S}$ form an antichain by irredundancy: that is, no space in $\mathcal{S}$ embeds isometrically in any other space in $\mathcal{S}$. These isomorphism types are naturally encoded by finite multi-sets of positive integers (clique sizes), and the embedding relation corresponds to a natural relation between these multi-sets which is well known, by a lemma of Higman, to be well-quasi-ordered [Hig52]; this term simply means, in this context, that there are no infinite antichains.

The notation $\Gamma^{\delta}_{\bar{K},\bar{C},\mathcal{S}}$ stands for the so-called *Fraïssé limit* (if it exists) of the following class $\mathcal{A}^{\delta}_{\bar{K},\bar{C},\mathcal{S}}$.

> Finite metric spaces M containing no isometric copy of a forbidden triangle in $\mathcal{T}(\delta, \bar{K}, \bar{C})$, and no isometric copy of a $(1,\delta)$-space in $\mathcal{S}$.

That is $\Gamma^{\delta}_{\bar{K},\bar{C},\mathcal{S}}$ denotes a countable metrically homogeneous graph Γ with the property that the class of finite metric spaces embedding isometrically in Γ coincides with the class $\mathcal{A}^{\delta}_{\bar{K},\bar{C},\mathcal{S}}$.

Such a metrically homogeneous graph, if it exists, is determined up to isomorphism by the class $\mathcal{A}^{\delta}_{\bar{K},\bar{C},\mathcal{S}}$, and hence by the data $\mathcal{T}(\delta, \bar{K}, \bar{C})$ and $\mathcal{S}$.

Via the theory of Fraïssé, the existence of the graph $\Gamma^{\delta}_{\bar{K},\bar{C},\mathcal{S}}$ is equivalent to the *amalgamation property* for the class $\mathcal{A}^{\delta}_{\bar{K},\bar{C},\mathcal{S}}$. This involves two sets of constraints on the parameters $(\delta, \bar{K}, \bar{C}, \mathcal{S})$; a mild set of constraints which we call *acceptability*, and a more severe set of constraints we call *admissibility* [Che18].

Acceptability consists of the requirements which are imposed on the parameters by their definitions and elementary arguments; for example, we cannot have $C > 3\delta + 1$, since there are no triangles of perimeter $3\delta + 1$. Subject to the acceptability constraints, admissibility consists of the substantive constraints on the parameters for the existence of $\Gamma^{\delta}_{\bar{K},\bar{C},\mathcal{S}}$, that is, admissibility consists of a precise set of requirements on the parameters which are equivalent to the condition

> $\mathcal{A}^{\delta}_{\bar{K},\bar{C},\mathcal{S}}$ is an amalgamation class.

We will give the precise constraints on $\bar{K}$ and $\bar{C}$ in detail, below.

Now the general theory as developed in [Che18] is not of much help in proving our result in diameter 3. It is of use in predicting the result, organizing the proof in its broad outlines, and explaining the result. The general theory also provides existence proofs for the cases listed. We will also take advantage of some information about Γ_2 and Γ_3 given by the general theory, which saves us the trouble of some preliminary analysis.

The organization of the proof in general terms is as follows.

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[12]For $\delta > 3$ some further variations arise.

Let Γ be a metrically homogeneous graph of diameter 3 and generic type, and $\mathcal{A}$ the associated class of all finite metric spaces which embed isometrically in Γ.

- Determine relevant (candidate) values of the parameters $\delta, \bar{K}, \bar{C}, \mathcal{S}$ for Γ.
- Show that the triangles occurring in $\mathcal{A}$ are the triangles occurring in the class

$$\mathcal{A}^{\delta}_{\bar{K}, \bar{C}}$$

  (i.e., the class $\mathcal{A}^{\delta}_{\bar{K}, \bar{C}, \emptyset}$).
- Show that $\mathcal{A} = \mathcal{A}^{\delta}_{\bar{K}, \bar{C}, \mathcal{S}}$, and apply uniqueness (Fraïssé theory).

Now the second step is a special case of the third, but it has the merit of identifying one of the two sets of constraints defining our graph completely. In particular, at this point we know what amalgamation strategy *should be appropriate* for the class $\mathcal{A}$. This last point motivates the main lines of the analysis.

2.4. **Acceptability, Admissibility, and Forbidden triangles.** The constraints on the numerical parameters $\delta, K_1, K_2, C_0, C_1$ needed to obtain an amalgamation class are fairly complicated. In fact, they are too complicated to be fully illustrated by examples for which we have $\delta = 3$. We have given the possibilities corresponding to the case $\delta = 3$ explicitly in the statement of the Classification Theorem, and we will repeat this at the end. But to put this in context, we first consider the constraints in their general form.

Definition 2.10. *Let $(\delta, K_1, K_2, C_0, C_1)$ be a sequence of natural numbers, and let $\mathcal{S}$ be a set of finite $(1, \delta)$-spaces. Write $\bar{K} = (K_1, K_2)$ and $\bar{C} = (C_0, C_1)$ for brevity.*

1. The sequence of parameters $\delta, \bar{K}, \bar{C}, \mathcal{S}$ is acceptable if the following conditions are satisfied.

- $\delta \geq 2$;
- $1 \leq K_1 \leq K_2 \leq \delta$, *or else* $K_1 = \infty$ *and* $K_2 = 0$;
- C_0 *is even and* C_1 *is odd*;
- $2\delta + 1 \leq C_0, C_1 \leq 3\delta + 2$;
- $\mathcal{S}$ *is* irredundant *(see below).*

In particular if $\delta = \infty$ then $C_0, C_1 = \infty$ and $\mathcal{S}$ consists of a set of cliques (in fact, of just one clique).

2. An acceptable sequence of parameters is admissible *if one of the following sets of conditions is satisfied.*

  I $K_1 = \infty$:
  - $K_2 = 0$, $C_1 = 2\delta + 1$; *this is the* bipartite *case.*

II $K_1 < \infty$ and $C \le 2\delta + K_1$:
- $\delta \ge 3$;
- $C = 2K_1 + 2K_2 + 1$;
- $K_1 + K_2 \ge \delta$;
- $K_1 + 2K_2 \le 2\delta - 1$; and

  IIA $C' = C + 1$, or

  IIB $C' > C + 1$, $K_1 = K_2$, and $3K_2 = 2\delta - 1$.

III $K_1 < \infty$ and $C > 2\delta + K_1$:
- If $\delta = 2$ then $K_2 = 2$ and $\mathcal{S}$ consists of a single clique or anti-clique;
- $K_1 + 2K_2 \ge 2\delta - 1$ and $3K_2 \ge 2\delta$;
- If $K_1 + 2K_2 = 2\delta - 1$ then $C \ge 2\delta + K_1 + 2$;
- If $C' > C + 1$ then $C \ge 2\delta + K_2$;
- If $K_1 = \delta$ or $C = 2\delta + 2$, then $\mathcal{S}$ is empty.

For the notion of irredundance, we need to specify the class $\mathcal{T}(\delta, \bar{K}, \bar{C})$ of forbidden triangles associated with $\delta, \bar{K}, \bar{C}$. Then the set $\mathcal{S}$ is said to be irredundant if no space in $\mathcal{S}$ contains an isometric copy of a forbidden triangle, or of another space in $\mathcal{S}$. In other words, $\mathcal{S}$ consists of minimal forbidden $(1, \delta)$-spaces, with the proviso that any forbidden triangles will be provided by the numerical parameters.

Finally, we give the definition of $\mathcal{T}(\delta, \bar{K}, \bar{C})$ explicitly. If T is a metric triangle (a metric space with 3 points), then the type of T is the (unordered) triple of distances i, j, k involved, and the perimeter is the sum

$$p = i + j + k$$

We write the type as an ordered triple (i, j, k) but bear in mind that the order here is irrelevant.

Definition 2.11. *Let $\delta, K_1, K_2, C_0, C_1$ be an acceptable sequence of parameters (take $\mathcal{S} = \emptyset$). Then a triangle of type (i, j, k) and perimeter $p = i + j + k$ belongs to $\mathcal{T}(\delta, \bar{K}, \bar{C})$ iff one of the following holds.*

(1) p is odd and at least one of the following holds:
- *$p < 2K_1 + 1$, or*
- *$p > 2K_2 + 2\min(i, j, k)$, or*
- *$p \ge C_1$; or—*

(2) p is even and $p \ge C_0$.

Thus the parameter $\bar{K} = (K_1, K_2)$ controls the triangles of small odd perimeter, while $\bar{C} = (C_0, C_1)$ controls the triangles of large perimeter, according to their parity. The condition on K_2 is somewhat obscure, but for $i = 1$ it is complementary to the first condition. And if the graph happens to be antipodal then the condition on K_1 works out to be equivalent to the second condition with the value $K_2 = \delta - K_1$. So we may say that K_2 imposes a kind of antipodal dual constraint relative to K_1, regardless of whether the graph in question is antipodal.

The point of admissibility, as we said at the outset, is the following.

Fact 2.12 ([Che18]). *The sequence of parameters $\delta, K_1, K_2, C_0, C_1, \mathcal{S}$ is admissible if and only if $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$ is an amalgamation class.*

Now, as promised, let us specialize the conditions for admissibility to the context $\delta = 3$. We have already given the result in the statement of the Classification Theorem for $\delta = 3$.

Lemma 2.13. *Let $\delta, K_1, K_2, C_0, C_1, \mathcal{S}$ be an admissible sequence of parameters with $\delta = 3$. Then the possibilities are as shown in Table 2.*

We note that the case $C = 9$ requires the spaces in $\mathcal{S}$ to decompose into at most two cliques; this condition should be added as appropriate in the last column, but has been omitted to reduce clutter. For example, in the first primitive case listed, namely $K_1 = 1$, $K_2 = 2$, when $C = 9$ this will force $\mathcal{S}$ to consist of cliques, and when $C = 10$ then $\mathcal{S}$ may contain both cliques and anticliques.

Type	Case	K_1	K_2	C	C'	$\mathcal{S}$
Bipartite	(I)	∞	0	7	8 or 10	empty
Antipodal	(II)	1	2	7	$C+1$	empty
Primitive	(III)	1	2	9 or 10	$C+1$	cliques and anticliques
"	(III)	2	2	9 or 10	$C+1$	anticliques
"	(III)	1	3	8, 9, or 10	$C+1$	If $C = 8$ then $\mathcal{S}$ is empty.
"	(III)	2	3	9 or 10	$C+1$	anything not involving K_3
"	(III)	3	3	10	$C+1$	empty

TABLE 2. Admissible parameters for $\delta = 3$

Proof. In the rightmost column, we find the constraints on the set $\mathcal{S}$ which correspond to the values of the numerical parameters $\delta = 3$, and K_1, K_2, C, C' as shown (i.e., the triangle constraints already imposed) as well as the relevant clauses in the definition of admissibility applicable directly to $\mathcal{S}$. That is we must specify which $(1, \delta)$-spaces can occur—irredundantly—as Henson constraints, in each case.

There are three cases to consider:

(I) $K_1 = \infty$; (II) $K_1 < \infty$, $C \leq 2\delta + K_1$; (III) $K_1 < \infty$, $C > 2\delta + K_1$.

The definition of admissibility varies according to the case considered.

(I) When $K_1 = \infty$ the graph is bipartite and we immediately arrive at the first line of the table, bearing in mind that δ is odd, so there are no $(1, \delta)$-spaces compatible with the numerical constraints with more than two points.

(II) When $C \leq 2\delta + K_1$ the conditions

$$K_1 + 2K_2 \leq 5,\ K_2 \geq K_1,\ \text{and}\ K_1 + K_2 \geq 3$$

give $K_1 = 1$ and $K_2 = 2$ and then the rest follows from the definition of admissibility.

(III) This is the more typical case and it allows for several possibilities.

A *clique* is defined metrically as a set of points at mutual distance 1. An *anticlique* is defined dually as a set of points at mutual distance δ (i.e., 3).

The constraint $3K_2 \geq 2\delta$ becomes $K_2 \geq 2$. Then we see the five possibilities with $2 \leq K_2 \leq 3$ and $1 \leq K_1 \leq K_2$ listed. The corresponding lower bound on C is $2\delta + K_1 + 1$ except in the case $K_1 = 1$, $K_2 = 2$ where it is $2\delta + K_1 + 2$. And the value $C = 3\delta + 1 = 10$ imposes no constraint, since all triangles have perimeter at most 3δ.

Now we have to consider the possibility that $C' > C + 1$ in case (III). But if $C \geq 3\delta$ then $C' = C + 1$ by acceptability, so this leaves only the case $C = 8$ with $K_2 = 3$, and then as $C < 2\delta + K_2$ we again have $C' = C + 1$. $\square$

In particular, we cannot see the point of having both C_0 and C_1 when we look at the case of diameter 3: we just see the uniform bound on perimeter given by $C = \min(C_0, C_1)$.

2.5. **Imprimitive Cases.** A metrically homogeneous graph is said to be *imprimitive* if it carries a non-trivial automorphism-invariant equivalence relation. The qualitative analysis of the imprimitive case comes under the general heading of Smith's theorem in the context of distance transitive graphs, but it takes on a slightly different form here. This was presented in detail in [Che11a]. The main point is the following.

Fact 2.14 ([Che11a, §7.1]). *Let Γ be an imprimitive connected metrically homogeneous graph of diameter at least 3 and vertex degree at least 3. Then Γ is either bipartite or antipodal.*

Part of the content of this result is the particular meaning of "antipodal" in our context, which was discussed earlier.

In the case of diameter 3 we may proceed to identify the imprimitive graphs of generic type as follows.

First we consider the bipartite case.

Lemma 2.15 (Bipartite case, cf. [Che11a, Proposition 9.1]). *Let Γ be a bipartite metrically homogeneous graph of diameter 3. Then Γ is one of the following: the complement of a perfect matching, or a generic bipartite graph.*

Proof. In this case Γ is homogeneous as a bipartite graph, since the relation $d(x, y) = 3$ is simply the "non-edge" relation between the two sides. So Γ or its complement must be one of the following (cf. [GGK96]).

- complete;
- perfect matching;
- generic bipartite.

As Γ has diameter 3 and is connected and bipartite, the claim follows. $\square$

Now we turn to the antipodal case, which is richer. This involves the following construction.

Definition 2.16 (Antipodal Double). *Let G be a graph. Let G' be a second copy of G, with a fixed isomorphism $\iota : G \to G'$; we write v' for $\iota(v)$.*

The antipodal double of G is the graph $\Gamma = G \cup G' \cup \{*, *'\}$, with two additional vertices $*$ and $*'$, and with edges given by

- $(*, v)$ *($v \in G$) or $(*', v')$ ($v' \in G'$);*
- (u, v), (u', v') *($u, v \in G$, $d(u, v) = 1$);*
- (u, v'), (u', v) *($u, v \in G$, $d(u, v) = 2$).*

Thus in the antipodal double we have $\Gamma_1 = G$, $\Gamma_2 = G'$, and the antipodal pairing gives an isomorphism (namely, ι) between Γ_1 and Γ_2.

Lemma 2.17 (Antipodal case, cf. [Che99, §1 (II,III)] or [Che11a, Theorem 15]). *Let Γ be an antipodal metrically homogeneous graph of diameter 3. Then Γ is the antipodal double of one of the following graphs.*

- *an independent set I_n ($n \leq \infty$);*
- *the pentagon (5-cycle) C_5;*
- *the line graph $L(K_{3,3})$ of the complete bipartite graph $K_{3,3}$ (this is a graph of order 9);*
- *the random graph G_∞.*

Proof. Let Γ be metrically homogeneous and antipodal of diameter 3. Fix a basepoint v_* for Γ, and set $G = \Gamma_1$, $G' = \Gamma_2$. Then G is a homogeneous graph, and the antipodal pairing gives an isomorphism between G and G'. It is easy to see that the edge rule in Γ is the one we have described.

It remains only to see that homogeneity of Γ forces G to be as stated. In fact, going forward, we will require only vertex transitivity of Γ.

Note that if G is complete, then Γ is disconnected and therefore cannot be metrically homogeneous of finite diameter, so we set this case aside.

For $v \in G$, let G_v^* be the graph derived from G by switching edges and non-edges between the neighbors and non-neighbors of v. We claim

$$\Gamma_1(v) \cong G_v^*$$

Let $G_1(v)$ be the set of neighbors of v in G, and $G_2(v)$ the set of non-neighbors of v (excluding v). Consider the map from G_v^* to $\Gamma_1(v)$ which fixes $G_1(v)$, is the antipodal map on $G_2(v)$, and sends v to v_*. This gives an isomorphism of G_v^* with $\Gamma_1(v)$.

By vertex transitivity of Γ, we also have

$$\Gamma_1(v) \cong G$$

and thus

$$G_v^* \cong G$$

It remains to go through all possibilities for G, and to see which ones satisfy this condition.

The homogeneous graphs G come in complementary pairs. If G^c is the complement of G, then $(G_v^*)^c = (G^c)_v^*$, so it suffices to consider one graph G in each complementary pair.

When G is imprimitive, we may suppose it is a disjoint sum of at least two non-trivial cliques. Then in G_v^*, any neighbor of v becomes adjacent to all vertices of G_v^*, so G_v^* is not isomorphic to G.

If G is a Henson graph G_n, then $G_2(v)$ contains a clique $K \cong K_{n-1}$, and switching edges and nonedges with $G_1(v)$ will extend K to a clique of order n. So this case is eliminated, along with its complement.

The only cases remaining are those listed. $\qquad\square$

For the homogeneity of antipodal graphs of these four types, the argument is easily reversed. Alternatively, the first case is bipartite and the complement of a perfect matching, which is clearly homogeneous, all finite cases are discussed in [Cam80], and for the last case it suffices to check that there is a metrically homogeneous graph of diameter 3 and antipodal type with Γ_1 the random graph. The generic antipodal graph of diameter 3 has these properties. In our notation, this is the graph

$$\Gamma^3_{1,3,7,8,\emptyset}$$

3. ADMISSIBILITY

We aim at the classification of all metrically homogeneous graphs of diameter 3, and as we have seen we may suppose that

Γ is of generic type.

Our overall plan is to assign some useful meaning to the parameters K_1, K_2, C, C' before proving anything, and then gradually show that they control the structure of our graph in the expected fashion. Our definition actually works with the very similar parameters K_1, K_2, C, C'. Here $\{C_0, C_1\} = \{C, C'\}$ but in this notation we take

- $C < C'$;
- C_0 is even and C_1 is odd.

When studying this pair of parameters—as opposed to using them—their parity is more relevant than their size. In particular, to define these parameters for metrically homogeneous graph, not necessarily of known type, one determines C_0 and C_1, and then recovers C, C' from them.

Definition 3.1. *Let Γ be a metrically homogeneous graph of diameter at least* 3 *and of generic type.*

(1) δ is the diameter of Γ, which we generally suppose is 3 *(occasionally working more generally).*

(2) K_1 is the least k such that Γ contains a triangle of type $(1, k, k)$, and K_2 is the greatest such.

(3) We take $K_1 = \infty$, $K_2 = 0$ if no such triangles occur.
(4) C_0 is the least even number greater than 2δ such that no triangle of perimeter C_0 is in Γ; and C_1 is the least such odd number.
(5) $C = \min(C_0, C_1)$ and $C' = \max(C_0, C_1)$.
(6) $\mathcal{S}$ is the collection of minimal $(1, \delta)$-spaces S such that
  - S is forbidden, i.e. does not embed in Γ; and
  - S is not a forbidden triangle relative to the specified numerical parameters (i.e., is not in $\mathcal{T}(\delta, \bar{K}, \bar{C})$—Definition 2.11), and does not contain a smaller forbidden $(1, \delta)$-space.

While the restriction to generic type is not formally necessary, it is needed if we aim to make any meaningful use of the definition.

The very first step in our analysis is to take note of some basic information about Γ_2 and Γ_3 provided by the general theory, without restriction on the diameter.

So we take this up next.

3.1. **On the structure of Γ_i.** Recall that Γ_i denotes the *metric space structure* induced on the vertices at distance i from a given base point. We also view Γ_i as a graph with edge relation "$d(x, y) = 1$," where the path metric may not agree with the induced metric, though it will agree with the path metric on any connected component, since this is both a metrically homogeneous graph and a homogeneous metric space under the induced metric, with the metrics agreeing on distance 1 (cf. [Cam98, Proposition 5.1]). This point of view is not at all useful if the graph Γ_i has no edges, and it is most useful when the graph induced on Γ_i is connected.

Therefore, the following general result on the structure of Γ_i in the case of generic type will provide a useful point of departure.

Fact 3.2 ([Che18, Theorem 1.29]). *Let Γ be a countable metrically homogeneous graph of generic type and of diameter δ, and suppose $i \leq \delta$. Suppose that Γ_i contains an edge. Then Γ_i is a countable metrically homogeneous graph.*

Furthermore, Γ_i is primitive and of generic type, apart from the following two cases.

(1) $i = \delta$;
  $K_1 = 1$; $\{C_0, C_1\} = \{2\delta + 2, 2\delta + 3\}$;
  Γ_δ *is an infinite complete graph (hence not of generic type).*
(2) $\delta = 2i$;
  Γ *is antipodal (hence Γ_i is imprimitive, namely antipodal).*

We restate this in the form relevant to the case $\delta = 3$.

Fact 3.3 (Local Analysis). *Let Γ be a countable connected metrically homogeneous graph of generic type and of diameter 3. Suppose $i = 2$ or 3, and Γ_i contains an edge. Then Γ_i is a connected metrically homogeneous graph.*

Furthermore, Γ_i is primitive and of generic type, apart from the following case.

- $i = 3$, Γ_3 is an infinite complete graph, so $K_1 = 1$, $C_0 = 8$, $C_1 = 9$.

There is a companion result guaranteeing the existence of edges in some cases.

Fact 3.4 ([Che18, Proposition 1.30]). *Let Γ be a connected metrically homogeneous graph of generic type with $K_1 \leq 2$. Then for $2 \leq i \leq \delta - 1$, Γ_i contains an edge, unless $i = \delta - 1$, $K_1 = 2$, and Γ is antipodal.*

Since K_1 is defined as the smallest number k for which Γ_k contains an edge, Fact 3.4 has no content if $K_1 = 2$ and $\delta = 3$. So for our purposes, the statement reduces to the following.

> If $\delta = 3$ and $K_1 = 1$, with Γ of generic type, then Γ_2 contains an edge.

Fact 3.5 ([Che18, Lemma 15.4]). *Let Γ be a connected metrically homogeneous graph of generic type and diameter δ. Suppose $i \leq \delta$, and suppose also that if $i = \delta$ then $K_1 > 1$. Then the metric space Γ_i is connected with respect to the edge relation defined by*

$$d(x, y) = 2$$

Note that the metric on Γ_i is induced from Γ and may not have any connection with the graph structure on Γ_i (e.g. $i = 1$, Γ triangle free, and Γ_1 is edgeless).

Another useful point which plays a role in some of the results just quoted, and remains useful in its own right, is the following.

Fact 3.6 ([Che18, Lemma 15.5]). *Let Γ be a connected metrically homogeneous graph of generic type. Suppose $1 \leq i \leq \delta$. Then for $u \in \Gamma_{i\pm 1}$, $\Gamma_1(u) \cap \Gamma_i$ is infinite.*

For $i < \delta$ this follows easily from the definition of generic type, applied to a pair of vertices u_+, u_- from $\Gamma_{i\pm 1}$. For $i = \delta$ it requires further analysis.

3.2. **The Parameters of Γ.** Now we begin to work toward the explicit classification of the (countable) metrically homogeneous graphs satisfying

- $\delta = 3$;
- Γ is of generic type;
- Γ is neither bipartite nor antipodal.

(On occasion, we will work at a greater level of generality.)

We recall the relevant content from Table 2 of Lemma 2.13, discarding the first two lines, which concern the imprimitive case. Recall that when $C = 9$, the last column of the table should be supplemented by the condition that no space in $\mathcal{S}$ involves more than two cliques.

Given a primitive infinite metrically homogeneous graph Γ of diameter 3, we proceed as follows.

(1) Extract suitable parameters $\delta, K_1, K_2, C_0, C_1, \mathcal{S}$ as in Definition 3.1.

Type	Case	K_1	K_2	C	C'	$\mathcal{S}$
Primitive	(III)	1	2	9 or 10	$C+1$	cliques and anticliques
"	(III)	2	2	9 or 10	$C+1$	anticliques
"	(III)	1	3	8, 9, or 10	$C+1$	If $C=8$ then $\mathcal{S}$ is empty.
"	(III)	2	3	9 or 10	$C+1$	anything not involving a 3-clique
"	(III)	3	3	10	$C+1$	empty

TABLE 3. Admissible parameters for $\delta = 3$, primitive case

(2) Show that these parameters control the forbidden triangles and $(1, \delta)$-spaces in precisely the expected manner.
(3) Show that the parameters are admissible.
(4) Show that the graph Γ is isomorphic with $\Gamma^\delta_{K_1,K_2,C_0,C_1,\mathcal{S}}$.

We view admissibility not so much in terms of the general definition, but in terms of the allowable parameter sequences, as listed in our table. That is, $K_2 = 2$ or 3, $1 \leq K_1 \leq K_2$, $C' = C + 1$, and the possibilities for C have been listed in each case. We will not be much concerned with $\mathcal{S}$.

Fact 3.7 ([Che18, Lemma 13.15]). *Let $\mathcal{A}$ be an amalgamation class of diameter δ, and Γ the Fraïssé limit of $\mathcal{A}$. Assume that some triangle of odd perimeter occurs in $\mathcal{A}$, and let p be the least odd number which is the perimeter of a triangle in $\mathcal{A}$. Then the following hold.*

(1) A p-cycle embeds isometrically in Γ.
(2) $p \leq 2\delta + 1$.
(3) $p = 2K_1 + 1$.

So we have the following conditions by definition (see Definition 3.1).

- $1 \leq K_1 \leq K_2 \leq \delta$;
- $2\delta + 1 \leq C \leq 3\delta + 1$, $C < C' \leq 3\delta + 2$; and of course
- $C_\epsilon \equiv \epsilon \pmod 2$.

Therefore, referring to Table 3 above, in the primitive case admissibility of the numerical parameters (for $\delta = 3$) can be expressed by the following additional conditions.

- $K_2 \geq 2$;
- $C' = C + 1$;
- $C > 6 + K_1$;
- when $K_1 = 1$ and $K_2 = 2$, then $C > 8$.

The first point ($K_2 \geq 2$) is already contained in the general theory. Indeed, Fact 3.4 includes the following.

Lemma 3.8. *Let Γ be a metrically homogeneous graph of generic type, diameter δ, which is neither bipartite nor antipodal. If $K_1 \le 2$, then $K_2 \ge \delta - 1$.*

Summing up, we have the following.

Lemma 3.9. *Let Γ be a primitive metrically homogeneous graph of generic type and diameter 3, with associated parameters K_1, K_2, C_0, C_1. Suppose the following conditions hold.*

(a) $C \neq 7$.
(b) If $C = 8$ then $K_1 = 1$, $K_2 = 3$, and $C' = 9$.
(c) If $K_1 = 3$ then $C = 10$.
Then the parameters K_1, K_2, C, C' are as shown in Table 3.

Proof. The rows of Table 3 list the possible combinations of K_1, K_2 subject to $1 \le K_1 \le K_2 \le 3$ and Lemma 3.8.

By definition we have $C \ge 2\delta + 1 = 7$, $C \le 3\delta + 1 = 10$. The cases involving $C \le 8$ are covered by (a, b). When $C \ge 9$, then $C' = C + 1$ and the latter imposes no constraint on triangles.

Taking (c) into account we have the table, apart from the specification of $\mathcal{S}$ in the last column. $\square$

So the next step is to prove points $(a\text{–}c)$.

3.3. **Admissibility.** In this section, we will not impose the condition $\delta = 3$ until the end (Proposition 3.19). When $\delta = 3$, the parameter K_1 falls under one of the two cases $K_1 \le 2$ or $K_1 = \delta$; for large values of δ, this is a split between very small values of K_1 and a very large value of K_1, and we treat these cases accordingly.

Lemma 3.10. *Let Γ be an infinite primitive metrically homogeneous graph of diameter δ. Then $C \ge 2\delta + 2$.*

Proof. We have to eliminate the possibility $C = 2\delta + 1$.

Suppose toward a contradiction that $C = 2\delta + 1$; that is, Γ contains no triangle of perimeter $2\delta + 1$.

As Γ is primitive, it is not bipartite, so K_1 is finite. In other words, Γ contains a triangle of type $(1, i, i)$ for some i.

Take a pair of vertices u, u' at distance δ and consider a vertex $v \in \Gamma_i(u)$ at distance $\delta - i$ from u'.

By the choice of i, $\Gamma_i(u)$ contains an edge. For any neighbor v_1 of v in $\Gamma_i(u)$, we have

$$\delta - i \le d(v_1, u') \le \delta - i + 1$$

The triangle (u, u', v_1) will have perimeter $\delta + i + d(u', v_1)$, so by our hypothesis $d(u', v_1) \neq \delta - i + 1$. Thus $d(u', v_1) = \delta - i$.

So if $v \in \Gamma_i(u)$ satisfies $d(v, u') = \delta - i$, then the same applies to each neighbor of v in $\Gamma_i(u)$. By Fact 3.2, $\Gamma_i(u)$ is connected, and hence

$$\Gamma_i(u) \subseteq \Gamma_{\delta - i}(u')$$

In particular $\Gamma_{\delta-i}(u')$ also contains an edge, so we may switch the parameters i and $\delta - i$, and the points u, u', to conclude

$$\Gamma_i(u) = \Gamma_{\delta-i}(u')$$

Hence for $u_1, u_2 \in \Gamma_\delta$, we have $\Gamma_{\delta-i}(u_1) = \Gamma_{\delta-i}(u_2)$.

As Γ is primitive, it follows that Γ_δ degenerates to a single vertex, but then Γ is antipodal, and we have a contradiction.   □

Lemma 3.11. *Let Γ be an infinite primitive metrically homogeneous graph of diameter δ of generic type, containing no triangle of type $(2, \delta, \delta)$. Then Γ_δ is an infinite complete graph. In particular*

$$K_1 = 1$$
$$K_2 = \delta$$

and Γ contains no triangle of perimeter greater than $2\delta + 1$.

Proof. We show first that Γ_δ contains an edge.

If not, then Γ_δ contains no pairs at distance less than 3. In particular, each $u \in \Gamma_{\delta-1}$ has a unique neighbor u' in Γ_δ. This contradicts Fact 3.6.

So by Fact 3.2, Γ_δ is connected. But there is no pair in Γ_δ at distance 2, so Γ_δ is complete. Then again by Fact 3.2, Γ_δ is infinite.

In particular, Γ contains triangles of type $(1, 1, 1)$ and $(1, \delta, \delta)$, so $K_1 = 1$ and $K_2 = \delta$.

Now if we have a triangle (a, b, c) in Γ, with $d(a, b) = i$, $d(a, c) = j$, take a as a basepoint and take $v, v' \in \Gamma_\delta$ with $d(b, v) = \delta - i$, $d(c, v') = \delta - j$. Then

$$d(b, c) \leq (\delta - i) + 1 + (\delta - j) = 2\delta + 1 - (i + j)$$

and thus the perimeter is at most $2\delta + 1$.   □

Lemma 3.12. *Let Γ be an infinite primitive metrically homogeneous graph of diameter δ, for which Γ_δ contains an edge. Then Γ contains triangles of type (i, δ, δ) for all $i \leq K_1$. Thus*

$$C > 2\delta + K_1$$

Proof. If $K_1 = 1$, then the lemma is vacuous. So we suppose $K_1 > 1$, and thus Γ_1 is an independent set.

By Fact 3.2, the metric space Γ_δ is a metrically homogeneous graph with associated path metric, and is connected as a graph. So it suffices to show that the diameter of Γ_δ is at least K_1.

If the diameter of Γ_δ is less than K_1, then Γ_δ contains no triangles of type $(1, i, i)$ for any i, and therefore Γ_δ is bipartite, by Fact 3.7 applied to the amalgamation class corresponding to Γ_δ. By Fact 3.5, since $K_1 > 1$, the space Γ_δ is connected with respect to the relation $d(x, y) = 2$, and thus no odd distances occur in Γ_δ, which is a contradiction.   □

Lemma 3.13. *Let Γ be an infinite primitive metrically homogeneous graph of generic type. If $C = 2\delta + 2$, then any $(1, \delta)$-space which does not contain a forbidden triangle is realized in Γ.*

Proof. By Lemma 3.11, Γ contains no triangle of perimeter greater than $2\delta + 1$, so a $(1, \delta)$-space with no forbidden triangle consists of at most 2 cliques, at distance δ.

By Lemma 3.11, Γ_δ is an infinite complete graph. It follows that Γ_1 contains an infinite complete graph, and is therefore the random graph (Remark 2.3).

Claim 3.13.1. *For $u \in \Gamma_1$, the set $I_u = \Gamma_\delta(u) \cap \Gamma_\delta$ is infinite.*

First, the set I_u is nonempty as Γ contains a triangle of type $(1, \delta, \delta)$.

Suppose I_u is a singleton. Then this determines a function from Γ_1 to Γ_δ. As Γ_1 is primitive, this is either a bijection, or constant. As Γ is not antipodal, the latter possibility is ruled out. If we have a bijection, then $\mathrm{Aut}(\Gamma_1)$ must act doubly transitively on Γ_1, which is not the case.

So if I_u is finite, the number $k = |I_u|$ is at least 2. By metric homogeneity any k-set in Γ_δ is I_u for some $u \in \Gamma_1$. Therefore we have pairs u, u' in Γ_1 satisfying

$$|I_u \cap I_{u'}| = i$$

for all $i < k$. As there are only 2 nontrivial 2-types in Γ_1, we find that $k = 2$.

Let $a, b, c \in \Gamma_1$ be chosen so that I_a, I_b, I_c meet pairwise. Then (a, b, c) is a clique of order 3. However it is possible for the intersection $I_a \cap I_b \cap I_c$ to be either empty or nonempty, violating metric homogeneity.

Thus I_u is infinite, and the claim is proved.

Now take $v_\delta \in \Gamma_\delta$ and set

$$A = \Gamma_1 \cap \Gamma_\delta(v_\delta) = \Gamma_\delta(v_\delta) \setminus \{v_*\} \qquad B = \Gamma_\delta \cap \Gamma_1(v_\delta) = \Gamma_\delta \setminus \{v_\delta\}$$

with v_* the chosen basepoint. Then A and B are infinite cliques which are definable from the pair (v_*, v_δ).

The distances between vertices in A and B can be either δ or $\delta - 1$, and by our claim each vertex in A lies at distance δ from infinitely many vertices in B. Taking $d(x, y) = \delta$ to define an edge relation, we find that (A, B) is a homogeneous bipartite graph in which each vertex in A has infinitely many neighbors in B. Hence (A, B) is either complete bipartite, the complement of a perfect matching, or generic bipartite.

Thus the required configurations occur in (A, B), and the lemma follows. $\qquad \square$

Lemma 3.14. *Let Γ be an infinite primitive metrically homogeneous graph of generic type, with diameter $\delta \geq 3$. If $K_1 = \delta$, then Γ_δ is a primitive metrically homogeneous graph of diameter δ for which the corresponding parameter $K_{\delta,1}$ is also equal to δ. Furthermore the parameters of Γ satisfy*

$$K_2 = \delta \qquad\qquad C = 3\delta + 1 \qquad\qquad C' = 3\delta + 2$$

and in particular the parameters (K_1, K_2, C, C') are admissible.

Proof. That $K_2 = \delta$ follows from the definitions.

As $K_1 = \delta$, Γ_δ contains an edge. By Fact 3.2, Γ_δ is connected and metrically homogeneous.

By Fact 3.5, Γ_δ is also connected with respect to the relation "$d(x, y) = 2$," and therefore is not multipartite.

By Fact 3.7, since the graph Γ_δ is not bipartite it contains some triangle of type $(1, i, i)$, for some $i \leq \delta$. As $K_1 = \delta$, we must have $i = \delta$. Thus the diameter of Γ_δ is δ, and the parameter $K_{\delta,1}$ for Γ_δ is also δ.

As Γ_δ is connected, the metric space Γ contains triangles of type (i, δ, δ) for all $i \leq \delta$. Hence $C > 3\delta$, and then by definition $C = 3\delta + 1$, $C' = C + 1$.

By Fact 3.2 and the above, Γ_δ is primitive. $\qquad\square$

We need also to deal with the constraint set $\mathcal{S}$.

Lemma 3.15. *Let Γ be an infinite primitive metrically homogeneous graph of generic type, with diameter $\delta \geq 3$. Suppose that $K_1 = \delta$, and S is a $(1, \delta)$-space consisting of two pairs of vertices, each pair at distance 1, with the distance between them equal to δ. Then S embeds into Γ.*

Proof. We attempt the amalgamation shown below. If both factors $(a_1 c u_1 u_2)$ and $(a_2 c u_1 u_2)$ embed into Γ, then in the amalgam the only possible values for the distance $d(a_1, a_2)$ are $\delta - 1$ and δ, and the value $\delta - 1$ is excluded by the parameter c. But then $a_1 a_2 u_1 u_2$ is a copy of S.

$$d(c, u_1) = 2,$$
$$d(c, u_2) = \delta$$
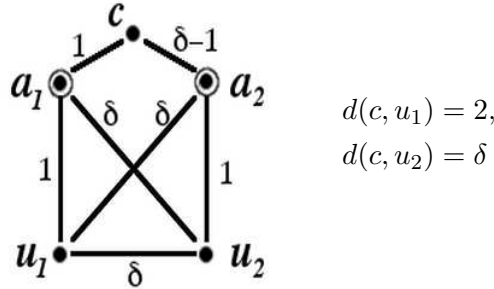
It remains to be shown that the two factors $(a_1 c u_1 u_2)$ and $(a_2 c u_1 u_2)$ both embed into Γ.

Claim 3.15.1. *The factor $(a_1 c u_1 u_2)$ embeds into Γ.*

Take a_1 as the basepoint of Γ, and $u_2 = u$ a vertex in Γ_δ. Let I_u be

$$\{v \in \Gamma_1 \,|\, d(u, v) = \delta\}$$

We need to show that $|I_u| \geq 2$.

As $K_1 = \delta$, we have $|I_u| \geq 1$. If $|I_u| = 1$, this gives a definable function from Γ_δ onto Γ_1. As Γ_δ is primitive (Lemma 3.14), this function is 1-1 or constant. As Γ_1 is nontrivial, the function is a bijection between Γ_δ and Γ_1. Hence Γ_δ realizes a unique nontrivial 2-type, a contradiction.

This proves the claim.

Claim 3.15.2. *The factor $(a_2 c u_1 u_2)$ embeds into Γ.*

Taking u_1 as the basepoint, and $c \in \Gamma_2$, we need to find a pair of vertices v, w in Γ_δ satisfying

$$d(c, v) = \delta \qquad\qquad d(c, w) = \delta - 1 \qquad\qquad d(v, w) = 1$$

Suppose this is not possible. Let $J_c = \{v \in \Gamma_\delta \,|\, d(c, v) = \delta\}$. Then J_c is nonempty as Γ_δ contains a pair at distance 2. For $v \in J_c$, and w adjacent to v, our assumption gives $w \in J_c$. As Γ_δ is connected, we find $\Gamma_\delta \subseteq J_c \subseteq \Gamma_\delta(c)$. By symmetry, $\Gamma_\delta = \Gamma_\delta(c)$. As Γ is primitive, this gives a contradiction.

Thus the second factor also embeds into Γ. $\qquad\qquad\square$

Lemma 3.16. *Let Γ be an infinite primitive metrically homogeneous graph of generic type, with diameter $\delta \geq 3$ and $K_1 = \delta$. Then*

$$\mathcal{S} = \emptyset$$

and in view of Lemma 3.14, the parameters $(K_1, K_2, C, C', \mathcal{S})$ are admissible.

Proof. The constraints in $\mathcal{S}$ must be $(1, \delta)$-spaces in which each connected component has order at most 2. We show by induction on n that a $(1, \delta)$-space S consisting of n pairs of vertices at distance 1 embeds into Γ.

Fix a pair of adjacent vertices a, b in Γ and set

$$\tilde{\Gamma} = \Gamma_\delta(a) \cap \Gamma_\delta(b)$$

Then $\tilde{\Gamma}$ is homogeneous as a metric space.

It suffices to show that $\tilde{\Gamma}$ is a primitive metrically homogeneous graph of diameter δ of generic type, with associated parameter $\tilde{K}_1 = \delta$. Then the induction hypothesis applies to $\tilde{\Gamma}$ and gives the desired configuration in Γ.

We may suppose that b is the basepoint in Γ, so that $\Gamma_\delta(a) \subseteq \Gamma_\delta \cup \Gamma_{\delta-1}$ and $\tilde{\Gamma}$ is $\Gamma_\delta(a) \cap \Gamma_\delta$.

Claim 3.16.1. $\tilde{\Gamma}$ *is connected*

Let A be a connected component of $\tilde{\Gamma}$.

We show first that the diameter of A in the induced metric is at least 2. Let u, v be an edge in A. As $\Gamma_\delta(a)$ is of generic type (Lemma 3.14), there is an isometric copy of a 4-cycle of the form (u, v, u', v') in $\Gamma_\delta(a)$. As $\Gamma_{\delta-1}$ contains no edges, one of u', v' must lie in $\Gamma_\delta(a) \cap \Gamma_\delta = \tilde{\Gamma}$, and in the connected component A.

So the diameter d of A is at least 2. Now by metric homogeneity of $\tilde{\Gamma}$, any pair of points at distance d or less lies in a single connected component of $\tilde{\Gamma}$. Supposing $\tilde{\Gamma}$ is not connected, take $v, w \in \tilde{\Gamma}$ at minimal distance $d' > d$ in $\tilde{\Gamma}$.

Take v_1, v_2 on a geodesic in $\Gamma_\delta(a)$ connecting v to w, with $d(v, v_1) = d(v_1, v_2) = 1$. If one of the vertices v_1 or v_2 lies in $\tilde{\Gamma}$, then it lies in the same connected component of $\tilde{\Gamma}$ as v, and is within distance d of w, hence lies also in the same component as w, for a contradiction. Therefore the vertices

v_1, v_2 lie in $\Gamma_\delta(a) \setminus \Gamma_\delta \subseteq \Gamma_{\delta-1}$. So $\Gamma_{\delta-1}$ contains the edge (v_1, v_2), which is a contradiction to $K_1 = \delta$.

Thus $\tilde{\Gamma}$ is connected. Since $\tilde{\Gamma}$ is homogeneous in the induced metric and connected, the induced metric agrees with the graph metric (cf. [Cam98, Proposition 5.1]), so that $\tilde{\Gamma}$ is a metrically homogeneous graph.

Claim 3.16.2. *The diameter of $\tilde{\Gamma}$ is δ.*

We seek a $(1, \delta)$-space (a, b, c, c') in which the only edge is (a, b). Taking c' as base point, we need a triangle of type $(\delta, \delta, 1)$ in Γ_δ. This is given by Lemma 3.14.

Claim 3.16.3. *$\tilde{\Gamma}$ is of generic type.*

As $\tilde{\Gamma}$ has finite diameter, it is not a regular tree, so if it is not of generic type then it is of exceptional local type, according to Fact 2.2.

As $K_1 = \delta > 1$, $(\tilde{\Gamma})_1$ is an independent set. According to the definition of exceptional local type, in this case $\tilde{\Gamma}$ must be a locally finite graph. But there are no infinite locally finite metrically homogeneous graphs of finite diameter. This is proved in [Mph82], and is included in the statement of Fact 2.5.

Claim 3.16.4. *$\tilde{\Gamma}$ contains an infinite anticlique (a set of points at mutual distance δ).*

It suffices to show that there are arbitrary large anticliques $(c_1, \ldots, c_k)$ in $\tilde{\Gamma}$. This can be proved by induction on k, viewing the configuration $(a, b, c_1, \ldots, c_k)$ as having basepoint c_k, and requiring an embedding of

$$(a, b, c_1, \ldots, c_{k-1})$$

into Γ_δ.

Claim 3.16.5. *$\tilde{\Gamma}$ contains a triangle of type $(\delta, \delta, 1)$. Thus $\tilde{K}_1 = \delta$.*

If the triangle is (c, c_1, c_2) with $d(c, c_1) = d(c, c_2) = \delta$ and $d(c_1, c_2) = 1$, the configuration $(abcc_1c_2)$ may be considered from the point of view of the basepoint c as requiring an embedding of (a, b, c_1, c_2) into Γ_δ, which is afforded by Lemma 3.15.

Claim 3.16.6. *$\tilde{\Gamma}$ is primitive.*

The previous two claims show that $\tilde{\Gamma}$ is neither antipodal nor bipartite. Thus all required properties pass to $\tilde{\Gamma}$ and we may argue inductively.

$\square$

Thus we have the following.

Lemma 3.17. *Let Γ be an infinite primitive metrically homogeneous graph of generic type with associated parameters $(\delta, K_1, K_2, C, C', \mathcal{S})$, where $\delta \geq 3$. If the parameters $(\delta, K_1, K_2, C, C')$ are admissible, then the parameters $(\delta, K_1, K_2, C, C', \mathcal{S})$ are admissible.*

Proof. The definition of admissibility imposes no further conditions on $\mathcal{S}$ (beyond irredundancy) unless

$$C > 2\delta + K_1$$

In this case, the conditions imposed are:

- If $K_1 = \delta$, then $\mathcal{S}$ is empty.
- If $C = 2\delta + 2$, then $\mathcal{S}$ is empty.

If $K_1 < \delta$, we need only concern ourselves with the case $C = 2\delta + 2$, and then Lemma 3.13 applies.

If $K_1 = \delta$, then Lemma 3.16 applies. $\square$

In particular we have the following.

Corollary 3.18. *Let* Γ *be an infinite primitive metrically homogeneous graph of generic type with associated parameters* $(\delta, K_1, K_2, C, C', \mathcal{S})$. *If*

$$K_1 \leq 2 \ or \ K_1 = \delta$$

then the associated parameters $(\delta, K_1, K_2, C, C', \mathcal{S})$ *satisfy the following conditions.*

(1) $$K_1 + 2K_2 \geq 2\delta - 1$$

(2) $$3K_2 \geq 2\delta$$

(3) $$If \ K_1 + 2K_2 = 2\delta - 1 \ then \ C \geq 2\delta + K_1 + 2$$

In particular, if $K_1 = \delta$, *or if* $K_1 \leq 2$ *and* $C' = C + 1$, *then the parameters* $(\delta, K_1, K_2, C, C', \mathcal{S})$ *are admissible.*

Proof. The definition of admissibility as a set of numerical constraints (or rather as a menu of three sets of numerical constraints) on the parameters is given in Definition 2.10. Under the hypothesis of primitivity one of the three possibilities envisioned in the definition ($K_1 = \infty$) drops away, and the main case division is between $C \leq 2\delta + K_1$ and $C > 2\delta + K_1$.

Points (1–3) repeat conditions from clause (III) of Definition 2.10.

If $K_1 = \delta$, then these points are not very significant, but they are true. Namely, Lemma 3.16 applies; in particular $K_2 = \delta$ and points (1–3) are clear. But the point of that lemma is that we have admissibility.

So we suppose

$$K_1 \leq 2$$

By Lemma 3.8, we have

$$K_2 \geq \delta - 1$$

In particular, points $(1, 2)$ are again clear.

By Lemma 3.10, we have

$$C \geq 2\delta + 2$$

For point (3), the hypothesis $K_1 + 2K_2 = 2\delta - 1$ with $K_2 \geq \delta - 1$ entails $K_2 = \delta - 1$ and $K_1 = 1$. But then $K_2 \neq \delta$ and Lemma 3.11 give $C \geq 2\delta + 3$, as required.

This disposes of points (1–3).

The final point concerns admissibility under the assumption $C' = C + 1$. By Lemma 3.17, it suffices to check admissibility of $(\delta, K_1, K_2, C, C')$,

First, we show

$$C > 2\delta + K_1.$$

Namely, as $C \geq 2\delta + 2$ and $K_1 \leq 2$, the inequality holds unless

$$C = 2\delta + 2 \qquad\qquad\qquad K_1 = 2$$

But this combination of parameters is incompatible with Lemma 3.11.

Thus, we find ourselves in case (III) of Definition 2.10. Then the definition of admissibility, given $\delta \geq 3$, reduces to points (1–3) above, and the following.

(4)   If $C' > C + 1$ then $C \geq 2\delta + K_2$.

So if $C' = C + 1$ this condition falls away.   □

Proposition 3.19. *Let Γ be an infinite primitive metrically homogeneous graph of diameter 3 and generic type, with associated parameters*

$$\delta = 3,\ K_1,\ K_2,\ C,\ C',\ \mathcal{S}$$

Then $C' = C + 1$, and the parameters $(\delta, K_1, K_2, C, C', \mathcal{S})$ are admissible.

Proof. By Corollary 3.18 it suffices to check that $C' = C + 1$.

The case $K_1 = \delta$ is handled by Lemma 3.14.

So suppose

$$K_1 \leq 2$$

If $C \geq 3\delta$ then $C' = C + 1$ by definition.

By Lemma 3.10 we have $C \geq 2\delta + 2 = 3\delta - 1$, so it suffices to treat the case $C = 2\delta + 2$. But in this case, by Lemma 3.11 we have $C' = C + 1$.   □

In particular, under the conditions of Proposition 3.19, the same parameters are associated with the graph $\Gamma^{\delta}_{K_1, K_2, C, C', \mathcal{S}}$. This result also shows that for $\delta = 3$ the parameter C' is superfluous, since it is determined by C, and we are free to drop it from our notation.

Definition 3.20. *Let Γ be a primitive metrically homogeneous graph of diameter 3 and generic type, with associated parameters $(\delta, K_1, K_2, C, C', \mathcal{S})$.*

We set $\Gamma^* = \Gamma^{\delta}_{K_1, K_2, C, C', \mathcal{S}}$.

The remaining task is to show that Γ and Γ^* are isomorphic. We will first show that Γ embeds into Γ^*.

4. Which triangles are forbidden?

In this section, we prove the following proposition.

Proposition 4.1. *Let Γ be a primitive metrically homogeneous graph of generic type with associated parameters $(\delta, K_1, K_2, C, C', \mathcal{S})$, where*

$$\delta = 3$$

Then the triangles which embed into Γ are precisely the triangles which belong to the class $\mathcal{A}^{\delta}_{K_1, K_2, C, C'}$.

We recall that the triangles in $\mathcal{A}^{\delta}_{K_1, K_2, C, C'}$ are the ones which do not belong to the set $\mathcal{T}(K_1, K_2, C, C')$ specified in Definition 2.11.

In one direction, we will not require the hypothesis $\delta = 3$.

4.1. Forbidden triangles (First embedding theorem).

The first part of Proposition 4.1 may be stated more generally as follows. Recall that we have definitions of the parameters $K_1, K_2, C, C', \mathcal{S}$ which apply to any metrically homogeneous graph of generic type (Definition 3.1). For $\delta = 3$, we know these parameters are admissible. In general, we do not know this—it is part of the main conjecture.

Proposition 4.2. *Let Γ be a primitive metrically homogeneous graph of generic type with associated parameters $(\delta, K_1, K_2, C, C', \mathcal{S})$. Suppose also the following condition is satisfied.*

$$\text{If } C' > C + 1, \text{ then } C \geq 2\delta + K_2.$$

If a triangle embeds isometrically in Γ, then that triangle belongs to the class

$$\mathcal{A}^{\delta}_{K_1, K_2, C, C'}$$

Corollary 4.3. *Let Γ be a primitive metrically homogeneous graph of generic type with associated parameters $(\delta, K_1, K_2, C, C', \mathcal{S})$. Suppose that the parameters are admissible, and also the following condition is satisfied.*

$$\text{If } C' > C + 1, \text{ then } C \geq 2\delta + K_2.$$

Then Γ embeds into $\Gamma^{\delta}_{K_1, K_2, C, C', \mathcal{S}}$.

The point here is that Γ and $\Gamma^{\delta}_{K_1, K_2, C, C', \mathcal{S}}$ also have the same Henson constraints.

Remark 4.4. *This applies in particular when $\delta = 3$, by Proposition 3.19.*

Lemma 4.5. *Let Γ be a primitive metrically homogeneous graph of generic type with associated parameters $(\delta, K_1, K_2, C, C', \mathcal{S})$. Suppose also*

$$\text{If } C' > C + 1, \text{ then } C \geq 2\delta + K_2.$$

Suppose a triangle of perimeter $p \equiv \epsilon \pmod 2$ ($\epsilon = 0$ or 1) embeds isometrically in Γ. Then

$$p < C_\epsilon$$

Proof. Let the triangle type (i, j, k) with $i \geq j \geq k$ provide a counterexample, with perimeter

$$p = i + j + k$$

and suppose k is minimal. Then by hypothesis $p \geq C_\epsilon$, and by definition we have $p \neq C_\epsilon$. Since

$$p \equiv C_\epsilon \pmod 2$$

we have $p \geq C_\epsilon + 2$.

If $k = 1$, then $p \leq 2\delta + 1 < C$ by Lemma 3.10. So $k > 1$.

Let (a, b, c) be a triangle of type(i, j, k) in Γ, with

$$d(a, b) = i, \ d(a, c) = j, \ d(b, c) = k$$

Take c' adjacent to c on a geodesic from b: $d(b, c') = k - 1$, $d(c', c) = 1$. Let $j' = d(a, c')$, and consider the triangle (a, b, c') of type
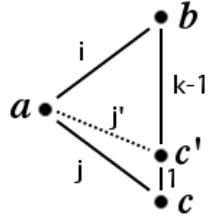
$$(i, j', k - 1)$$

The perimeter $p' = i + j' + (k - 1)$ satisfies $p' \geq p - 2 \geq C_\epsilon$ so by the minimality of k we cannot have $p' \equiv p \pmod 2$. Therefore $j' = j$, $p' = p-1$, and the triangle (a, c, c') has type $(1, j, j)$. In particular $j \leq K_2$.

Now by minimality of k we have $p' < C_{\epsilon'}$ with $\epsilon' = 1-\epsilon$, and thus $C_\epsilon + 1 \leq p' < C_{\epsilon'}$, so $C' = C_{\epsilon'} > C + 1$. Then by hypothesis $C_\epsilon = C \geq 2\delta + K_2$. But $p = i + j + k \leq 2\delta + j \leq 2\delta + K_2 \leq C_\epsilon$, a contradiction. □

Lemma 4.6. *Let Γ be a primitive metrically homogeneous graph of generic type with associated parameters $(\delta, K_1, K_2, C, C', \mathcal{S})$. Suppose a triangle of type (i, j, k) and odd perimeter p embeds isometrically into Γ. Then*

(1) $p \geq 2K_1 + 1$;
(2) $i + j \leq 2K_2 + k$.

Proof. The first claim follows by Fact 3.7. We deal with point (2).

Suppose that (a, b, c) is a triangle of type (i, j, k) with odd perimeter, and with

$$i + j > 2K_2 + k$$

chosen so that k is minimal. We may suppose that

$$d(a, b) = i, \ d(a, c) = j, \ d(b, c) = k$$

Let c' be adjacent to c on a geodesic from b to c, so $d(b, c') = k - 1$ and $d(c', c) = 1$. Consider the triangle (a, b, c') of type

$$(i, j', k - 1)$$

with $j' = d(a, c')$.
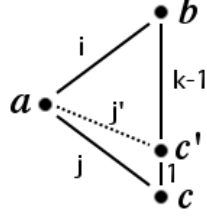
If $j' = j \pm 1$ then the perimeter $p' = i + j' + k - 1$ is again odd and hence by minimality of k we have

$$i + j' \leq 2K_2 + (k - 1)$$
$$i + j \leq i + (j' + 1) \leq 2K_2 + k$$

and we have a contradiction.

So $j' = j$ and (a, c, c') is of type $(1, j, j)$. Hence $j \leq K_2$ and we have

$$i + j \leq (j + k) + j \leq 2j + k \leq 2K_2 + k$$

again a contradiction. $\qquad \square$

Proof of Proposition 4.2. We consider a triangle of type (i, j, k) and perimeter $p = i + j + k$ embedding into Γ. The claim is that this triangle is not in the set of forbidden triangles $\mathcal{T}(\delta, (K_1, K_2), (C_0, C_1))$. According to Definition 2.11, we must check the following conditions.

- If p is odd:

$$p < C_1 \qquad\qquad p \geq 2K_1 + 1 \qquad\qquad p \leq 2K_2 + 2\min(i, j, k)$$

- If p is even:

$$p < C_0$$

Lemma 4.5 has the same hypotheses on Γ as the proposition, and yields the appropriate inequality $p < C_\epsilon$ with $p \equiv \epsilon \pmod 2$.

Lemma 4.6 has fewer conditions on Γ, and applies in the case of odd perimeter to yield the first pair of constraints, with the constraint corresponding to K_2 written in the form

$$i + j < 2K_2 + k$$

Permuting the entries i, j, k gives

$$p < 2K_2 + 2\min(i, j, k)$$

which appears to be stronger than we require, but as p is odd, equality is impossible, so this is just another way of expressing the condition required. $\qquad \square$

We remark that the side condition used here (namely, if $C' > C + 1$ then $C \geq 2\delta + K_2$) is something that one would aim eventually to prove if $C > 2\delta + K_1$, and to dispense with if $C \leq 2\delta + K_1$, but the result as stated will cover our needs in case $\delta = 3$.

4.2. **Realized Triangles.** Now we wish to prove a complementary result.

Proposition 4.7. *Let Γ be a primitive metrically homogeneous graph of generic type with associated parameters $(\delta, K_1, K_2, C, C', \mathcal{S})$, where*

$$\delta = 3$$

Then any triangle in $\mathcal{A}^{\delta}_{K_1, K_2, C, C'}$ embeds isometrically in Γ.

We resort to completely ad hoc considerations.

Lemma 4.8. *Let Γ be a primitive metrically homogeneous graph of generic type with associated parameters $(\delta, K_1, K_2, C, C', \mathcal{S})$, where*

$$\delta = 3$$

Then any triangle of even perimeter $p < C$ embeds in Γ.

Proof. As $\delta = 3$ there are few cases, particularly after we set aside the geodesics, of types $(1, 1, 2)$ and $(1, 2, 3)$.

If $p \leq 6$ then the type is $(2, 2, 2)$, and this embeds into Γ_1 since Γ_1 contains an infinite independent set, hence embeds into Γ.

The only other possibility is $p = 8$, with $C > 8$. As $C > 8$, some triangle of perimeter 8 is realized, and there is only one such type, namely $(2, 3, 3)$. $\square$

Lemma 4.9. *Let Γ be a primitive metrically homogeneous graph of generic type with associated parameters $(\delta, K_1, K_2, C, C', \mathcal{S})$, where*

$$\delta = 3$$

Then any triangle of type (i, j, k) with odd perimeter $p < C$ whose perimeter p satisfies both of the following inequalities embeds isometrically in Γ.

- *$p \geq 2K_1 + 1$;*
- *$p \leq 2K_2 + 2\min(i, j, k)$.*

Proof. We suppose $i \leq j \leq k$, and consider all cases according to the value of i.

Case 1: $i = 1$.

Then the triangle has type $(1, j, j)$ and the inequalities on p mean that $K_1 \leq j \leq K_2$.

If $j = K_1$ or K_2 then a triangle of type $(1, j, j)$ embeds into Γ by the definition of K_1, K_2 respectively.

In the remaining case $K_1 = 1$, $j = 2$, and $K_2 = 3$. By Fact 3.4, Γ_2 contains an edge, so Γ contains a triangle of type $(1, 2, 2)$.

Case 2: $i > 1$.

As the perimeter is odd, and $\delta = 3$, the triangle type is $(i, i, 3)$ with $i \geq 2$, and the perimeter is $p = 2i + 3$. The assumption $C > p$ implies that some

triangle T of this perimeter occurs in Γ. If this triangle T is not of the required type $(i, i, 3)$ then we must have $i = 2$ while T has type $(1, 3, 3)$. Thus a triangle of type $(1, 3, 3)$ embeds into Γ.

So suppose toward a contradiction that Γ_3 contains an edge, but there is no triangle of type $(2, 2, 3)$ in Γ. Fix $u \in \Gamma_2$, and $v \in \Gamma_3$ adjacent to u. For any neighbor v' of v in Γ_3, we have $d(u, v') \leq 2$ and hence $d(u, v') = 1$; otherwise, the triangle formed by u, v', and the base point has type $(2, 2, 3)$. But Γ_3 is connected (Fact 3.2) and it follows that Γ_3 is contained in $\Gamma_1(u)$.

On the other hand there is $w \in \Gamma_1$ at distance 3 from a vertex of Γ_3, and w has a neighbor w' in Γ_2, at distance at least 2 from that same vertex, which is a contradiction. $\square$

Proof of Proposition 4.7. We consider a triangle of type (i, j, k) and perimeter $p = i + j + k$ which is in

$$\mathcal{A}^{\delta}_{K_1, K_2, C, C'}$$

According to Definition 2.11, we have the following conditions.

- If p is odd:

$$p < C_1 \qquad\qquad p \geq 2K_1 + 1 \qquad\qquad p \leq 2K_2 + 2\min(i, j, k)$$

- If p is even:

$$p < C_0$$

Recall that $C' = C + 1$ in all cases, as $\delta = 3$ (Proposition 3.19 and Lemma 2.13).

Thus $p < C$, and then Lemma 4.8 applies when p is even, and Lemma 4.9 applies when p is odd. $\square$

Proof of Proposition 4.1. Propositions 4.2 and 4.7 $\square$

5. IDENTIFICATION OF Γ: $K_1 = 3$ OR $C = 8$

We proceed to the final phase of the classification of the metrically homogeneous graphs of diameter 3.

So far, we have seen that it suffices to deal with metrically homogeneous graphs Γ which are of generic type, and primitive, and that each such graph Γ has associated parameters $(\delta, K_1, K_2, C, C', \mathcal{S})$ which are also associated with a unique known metrically homogeneous graph of generic type, which we call Γ^* (Definition 3.20). The goal then is to prove the isomorphism $\Gamma \cong \Gamma^*$.

Furthermore we know, by Corollary 4.3 and Remark 4.4, that Γ embeds into Γ^*. As both Γ and Γ^* are homogeneous, to conclude it will suffice to prove that any finite metric space A embedding isometrically into Γ^* also embeds into Γ.

So far, we have proved that the required embedding holds for triangles (Proposition 4.7). This gives us a start toward an inductive proof of the general case. However, as discussed in §1.6.3, this induction will not take

place with respect to the order of the configuration A, but with respect to more subtle measures of structure, with most of the subtlety occurring in §6.

It is now time to make a further subdivision of the remaining cases according to "large" and "small" values of K_1 or C. Here (as previously in the present article) K_1 is *small* if it is at most 2, and *large* if it is δ (i.e., 3). Similarly, C is *small* if it is at most $2\delta + 2$ (8), and is *large* if it is at least 3δ (9).

The fact that this case division is exhaustive appears to be one of the more helpful features of the case $\delta = 3$.

We now divide the remainder of the analysis into two cases, as follows.

- The extremal cases: K_1 large or C is small;
- The typical cases: K_1 small and C is large.

The present section §5 is devoted to the two extremal cases, and the next section §6 is devoted to a uniform treatment of the remaining typical cases

5.1. $K_1 = 3$ **vs.** $C = 8$: **exploiting ambiguity.** Our main objective now is the following.

Proposition 5.1. *Let Γ be a primitive infinite metrically homogeneous graph of diameter 3, with $K_1 = 3$. Then*

$$\Gamma \cong \Gamma^3_{3,3,10,11}$$

is the generic graph of this type. ($\mathcal{S} = \emptyset$ and is therefore omitted from the notation here.)

We show first that Proposition 5.1 will also dispose of the case $C = 8$. This depends on an essential *ambiguity:* one cannot actually distinguish the space corresponding to $K_1 = 3$ from the space corresponding to $C = 8$, when these are presented as labeled graphs (relational structures) whose labels may be permuted. Similar phenomena occur for any $\delta \geq 3$, but only in very special cases [Cou18].

Corollary 5.2. *Let Γ be a primitive infinite metrically homogeneous graph of diameter 3, with $C = 8$. Then*

$$\Gamma \cong \Gamma^3_{1,3,8,9}$$

is the generic graph of this type.

Proof. As $C = 8$ and $\delta = 3$ we have the following assumptions on triangle types realized and omitted by Γ, bearing in mind that geodesics are realized and that the triangle inequality is satisfied.

Realized	Omitted
(1,1,2)	(1,1,3)
(1,2,3)	(2,3,3)

Recall by admissibility (and specifically, by Lemma 3.11) that Γ also contains triangles of type

$$(1, 3, 3)$$

Let $\vec{\Gamma}$ be the structure derived from Γ by cyclically permuting the relations, replacing the relations $d(x, y) = 1, 2, 3$ by $d(x, y) = 2, 3, 1$ respectively. Any permutation of the language produces another infinite primitive homogeneous structure for the same language—though not, in general, a metric space.

Then the known constraints on $\vec{\Gamma}$ are the following.

Realized	Omitted
(2,2,3)	(1,2,2)
(1,2,3)	(1,1,3)
(1,1,2)	

In particular we see from the omission of $(1, 1, 3)$ that $\vec{\Gamma}$ is in fact a metric space, and from the presence of all geodesics that it is a metrically homogeneous graph. So we have the following.

Claim 5.2.1. *The derived structure $\vec{\Gamma}$ is a primitive, infinite, metrically homogeneous graph of diameter 3.*

Writing $\vec{K}_1$ for the value of K_1 associated to $\vec{\Gamma}$, we claim

$$\vec{K}_1 = 3$$

As triangles of type $(1, 2, 2)$ do not occur, Fact 3.4 states that $\vec{K}_1 \geq 3$. On the other hand $\vec{\Gamma}$ is primitive, so $\vec{K}_1$ is finite (otherwise, $\vec{\Gamma}$ would be bipartite by Fact 3.7). Therefore $\vec{K}_1 = 3$.

So $\vec{\Gamma}$ satisfies the conditions of Proposition 5.1, and is therefore uniquely determined. It follows that Γ is also uniquely determined, and this suffices.
$\square$

5.2. **Plan of attack: The embedding property.**

We now take up the proof of Proposition 5.1. By Corollary 2.6 the graph Γ in question must be of generic type.

Notation. *For the remainder of §5, let Γ denote an infinite primitive metrically homogeneous graph of diameter 3 and generic type, with $K_1 = 3$, except where greater generality is explicitly stated.*

Γ^* is the associated known metrically homogeneous graph of generic type with the same parameters, as in Definition 3.20.

By Lemmas 3.14 and 3.16) we have

$$\Gamma^* = \Gamma^3_{3,3,10,11}$$

Thus Proposition 5.1 may be written as follows.

$$\Gamma \cong \Gamma^*$$

Furthermore we have shown that Γ embeds isometrically into Γ^* and so by Fraïssé theory it suffices to show that any finite metric space embedding into Γ^* embeds into Γ. We may state this more explicitly in the following terms.

Proposition 5.3. *With Γ as stated, let A be a finite integer-valued metric space of diameter at most 3. Suppose that A embeds into Γ^*: that is, Γ contains no triangles of type $(1,1,1)$ or $(1,2,2)$. Then A embeds into Γ.*

We will proceed in the proof by induction on the number of edges (at distance 1) in A, and then on the cardinality $|A|$. We begin with the case in which there is no such edge in A; and later we will also need to treat the case in which there is exactly one such edge, before tackling the general case by an inductive argument.

We have the option of switching our point of view whenever the condition $C = 2\delta + 2$ is more convenient, which in fact will be the case at the beginning of our analysis. Namely, if $\overleftarrow{\Gamma}$ is the structure obtained by replacing the relations $d(x, y) = 1, 2, 3$ by $d(x, y) = 3, 1, 2$, respectively, then $\overleftarrow{\Gamma}$ is again a metrically homogeneous graph, with $C = 2\delta + 2$ and $K_1 = 1$ (cf. Table 1). We take advantage of this in the proof of the next lemma.

Lemma 5.4. *With Γ as stated, Γ_2 is a random graph.*
In particular, if A is a finite metric space in which all distances are 2 or 3, then A embeds into Γ.

Here Γ_2 involves the distances $2, 3$, one of which should be viewed as the edge relation. The two conclusions given are equivalent, but some variation in phrasing will be useful when the lemma is applied.

Proof. Passing to the associated metrically homogeneous graph $\overleftarrow{\Gamma}$ with $C = 2\delta + 2$, the claim becomes that $\overleftarrow{\Gamma}_1$ is the universal homogeneous graph.

By Fact 3.3 $\overleftarrow{\Gamma}_1$ is a primitive metrically homogeneous graph of generic type.

Furthermore $\overleftarrow{\Gamma}_3$ is an infinite clique: it is a clique because in $\overleftarrow{\Gamma}$ we have $C = 2\delta + 2$ and $C' = 2\delta + 3$, and it is infinite because $\overleftarrow{\Gamma}$ is infinite, primitive, and $\aleph_0$-categorical. Since $\overleftarrow{\Gamma}$ contains an infinite clique, so does $\overleftarrow{\Gamma}_1$. By Remark 2.3, $\overleftarrow{\Gamma}_1$ is then a random graph.

The lemma follows.   □

5.3. **Minimizing edges.** For the remainder of §5, we suppose the following.

> Γ is an infinite primitive metrically homogeneous graph of diameter 3, with $K_1 = 3$.

We recall (Corollary 2.6) that Γ is then of generic type.

If A is a finite metric space, we write $e(A)$ for the number of unordered pairs u, v in A with $d(u, v) = 1$. We consider a hypothetical finite metric

space A with distances $1, 2, 3$ which omits triangles of type $(1, 1, 1)$ and $(1, 2, 2)$, but nonetheless is not isometrically embedded in our graph Γ.

We have shown in Lemma 5.4 that necessarily

$$e(A) \geq 1$$

We will deal subsequently with the case $e(A) = 1$. Now we reduce the general case to this case.

Lemma 5.5 (Reduction to $e(A) = 1$). *Let Γ be an infinite primitive metrically homogeneous graph of diameter 3 with $K_1 = 3$. Suppose that every finite metric space A with distances among $1, 2, 3$ and with the following properties embeds isometrically into Γ.*

(1) A contains no triangle of type $(1, 2, 2)$.
(2) $e(A) = 1$.

Then every finite metric space A with distances among $1, 2, 3$ which contains no triangle of type $(1, 1, 1)$ or $(1, 2, 2)$ embeds isometrically into Γ.

Proof. We consider a counterexample A with $e(A)$ minimal, and we minimize $|A|$. We view A as a graph with the usual edges given by $d(x, y) = 1$, so that $e(A)$ is the number of edges.

By Lemma 5.4 and our hypothesis, we have

$$e(A) \geq 2$$

Claim 5.5.1. *Every vertex of A lies on at most one edge.*

Suppose on the contrary that $a \in A$ is adjacent to distinct a_1, a_2. Let $A_0 = A \setminus \{a, a_1, a_2\}$. Then $d(a_1, a_2) = 2$ and this is the unique possibility (when amalgamating $A_0 \cup \{a, a_1\}$ with $A_0 \cup \{a, a_2\}$ over $A_0 \cup \{a\}$), unless it is possible to identify a_1, a_2.

To prevent the latter, adjoin c with $d(c, a_1) = 2$ and $d(c, x) = 3$ otherwise. Consider the following amalgamation.
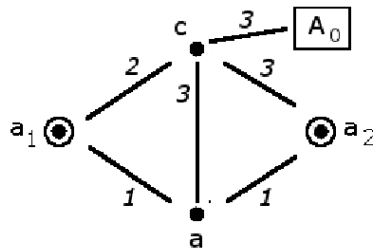
This forces the configuration A, and the factors have fewer edges, hence are in Γ by induction.

Thus the edges of A must be disjoint, as claimed.

Claim 5.5.2. *If A contains two distinct edges (a_1, a_2) and (b_1, b_2), then $d(a_i, b_j) = 3$ for $i, j = 1, 2$.*

By Claim 5.5.1 the edges are disjoint, and none of the distances $d(a_i, b_j)$ can equal 1. Our claim is that none of these distances can equal 2.

Suppose toward a contradiction that

$$d(a_2, b_1) = 2$$

Then $d(a_1, b_1) = 3$, since a_1 lies on a unique edge and A contains no triangle of type $(1, 2, 2)$.

We may view this configuration as an amalgamation problem to determine the distance $d(a_1, b_1)$ by amalgamating over a base $A_0 \cup \{a_2, b_2\}$, but this amalgamation problem has the alternate solution $d(a_1, b_1) = 1$. So we extend the diagram by an additional vertex c blocking this possibility.
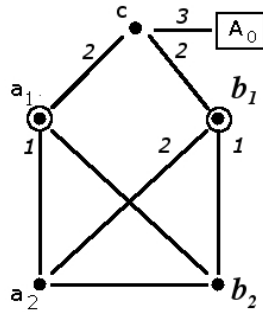
$$d(c, a_2) = d(c, b_2) = 3$$
$$A_0 = A \setminus \{a_1, a_2, b_1, b_2\}$$

As Γ does not contain a triangle of type $(1, 2, 2)$, this forces $d(a_1, b_1) = 3$. Since the factors of this amalgamation have fewer edges, they embed into Γ. It follows that A embeds into Γ, a contradiction.

This proves our second claim.

Now to prove the lemma, we fix two edges (a_1, a_2) and (b_1, b_2). These must be disjoint, with $d(a_i, b_j) = 3$ for $i, j = 1, 2$. We consider the following amalgamation, with $A_0 = A \setminus \{a_1, a_2, b_1, b_2\}$, and with two auxiliary vertices c_1, c_2, preventing the options $d(a_1, b_1) = 1$ or 2.

$$d(c_i, a) = 3 \ (a \in A_0)$$
$$d(c_i, a_2) = 3$$
$$d(c_1, b_2) = 2$$
$$d(c_2, b_2) = 3$$
$$d(c_1, c_2) = 3$$
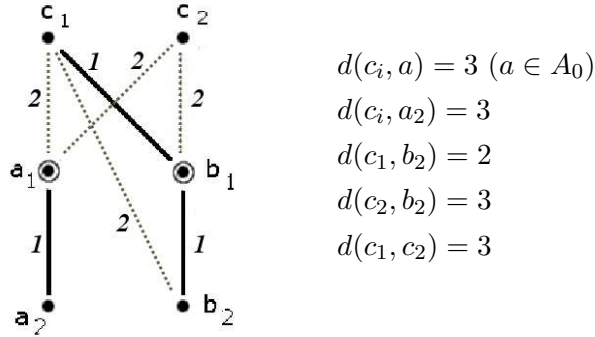
Here A_0 is present but not shown.

It suffices to check that the factors of this amalgamation embed into Γ (again, A_0 is present but not shown):
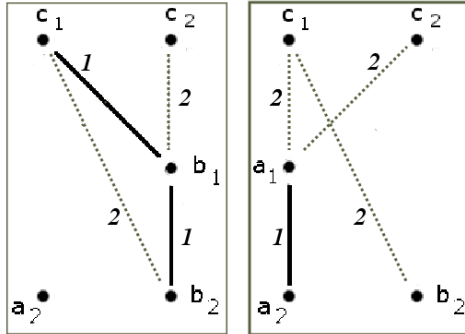
The factor omitting b_1 has fewer edges than A, so induction applies.

The factor omitting a_1 has the same number of edges as A, and the edges are not disjoint, so Claim 1 applies. □

5.4. The case $e(A) = 1$.

Lemma 5.6. *Let Γ be a metrically homogeneous graph of diameter 3 with $K_1 = 3$. Let A be a finite configuration containing a unique edge (a_1, a_2) and no triangle of type $(1, 2, 2)$.*

(1) If A does not embed into Γ then there is at least one vertex $b \in A$ with $d(a_1, b) = 2$.

(2) If in addition $|A|$ is minimized, then there is exactly one vertex $b \in A$ with $d(a_1, b) = 2$.

Proof.

Ad (1):

Suppose that there is no vertex b at distance 2 from a_1; that is, all the distances $d(a_1, b)$ with $b \in A_0 = A \setminus \{a_1, a_2\}$ are equal to 3.

Consider the following amalgamation over $a_2 c_1 c_2$.

$$d(c_1, a_2) = 2$$
$$d(c_2, a_2) = 3$$
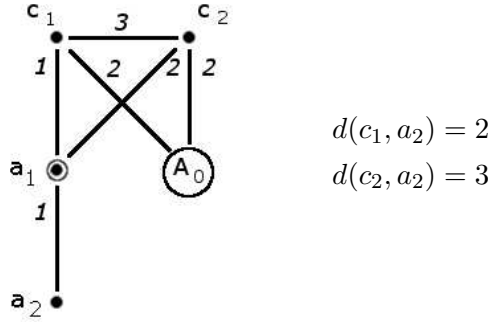
This forces the configuration A, so it suffices to check that the factors embed isometrically into Γ.

The factor omitting a_1 has no edges and thus embeds into Γ.

The factor omitting A_0 has the following form.
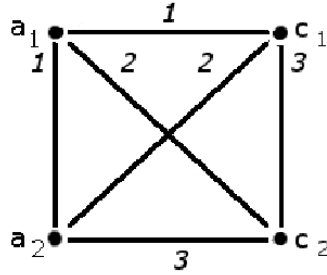
(1)

Over a_1 as base point, this means we are taking a vertex $v \in \Gamma_2$ and looking for two vertices in Γ_1 at distance 3 from v.

There is at least one such vertex, since the corresponding triangle is a geodesic.

Suppose toward a contradiction that for each vertex $v \in \Gamma_2$ there is a unique vertex v' in Γ_1 at distance 3 from v. As Γ_2 is primitive (Lemma 5.4), and Γ_1 is not a singleton, this gives a bijection between Γ_2 and Γ_1. Hence all pairs in Γ_2 should lie at the same distance. But two distances occur.

So at least one such vertex b exists.

Ad (2):

Now suppose that $|A|$ is minimal. We must show that the vertex b is unique. Suppose that $d(a_1, b_1) = d(a_1, b_2) = 2$ with b_1, b_2 distinct. Consider the following amalgamation over $\{a_1, c\} \cup A_0$, where $A_0 = A \setminus \{a_1, a_2, b_1, b_2\}$.

$$d(c, a_1) = 3$$
$$d(c, b_1) = 2$$
$$d(c, x) = 3 \ (x \in A_0)$$
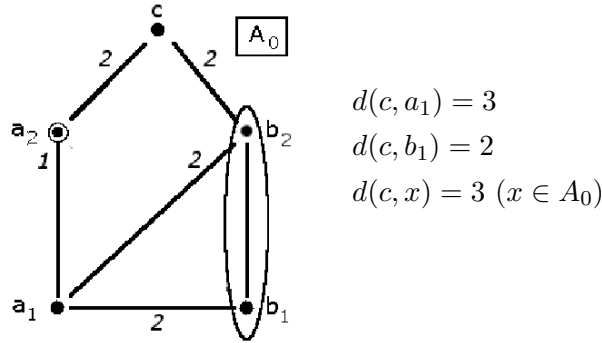
Note that $d(a_2, b_i) = 3$ for $i = 1, 2$, so this amalgamation forces a copy of A into Γ.

Now the factor omitting a_2 has no edges, and the factor omitting b_1, b_2 again has a unique edge, and is smaller, hence lies in Γ. Thus we reach a contradiction, and the vertex b is unique. $\square$

Now we take up the proof of Proposition 5.3, dealing by an amalgamation argument with the crucial remaining case.

Proof of Proposition 5.3.

By assumption Γ is primitive metrically homogeneous of generic type with $K_1 = K_2 = \delta = 3$, $C = 10$, $C' = 11$. We consider a finite metric space A with distances among $1, 2, 3$, which contains no triangles of types $(1, 1, 1)$ or $(1, 2, 2)$. Our claim is that A embeds isometrically into Γ.

By Lemma 5.5 we may suppose $e(A) = 1$. Subject to this constraint, we take $|A|$ minimal. Let (a_1, a_2) be the unique edge in A.

By Lemma 5.6, there is a unique vertex b_1 in A with $d(a_1, b_1) = 2$, and a unique vertex b_2 in A with $d(a_2, b_2) = 2$. As there is no triangle of type $(1, 2, 2)$ in A, we have $b_1 \neq b_2$.

Let $A_0 = A \setminus \{a_1, a_2, b_1, b_2\}$.

Then A has the following structure.

$$d(b_1, b_2) = 2 \text{ or } 3$$
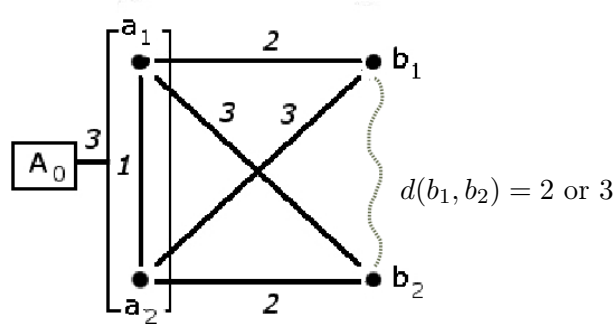
Suppose first that $|A| \geq 5$, and thus

$$A_0 \neq \emptyset$$

Adjoin elements c_1, c_2 as follows.

$$d(c_i, x) = 2 \ (x \in A_0)$$
$$d(c_i, b_j) = 3 \ (i, j = 1, 2)$$
$$d(c_1, a_1), d(c_2, a_2) = 1$$
$$d(c_1, a_2), d(c_2, a_1) = 2$$
$$d(c_1, c_2) = 3$$

Amalgamate

$A_0 \cup \{b_1, b_2, c_1, c_2\}$ with $\{a_1, a_2\} \cup \{b_1, b_2, c_1, c_2\}$ over $\{b_1, b_2, c_1, c_2\}$

For $x \in A_0$ and $i = 1, 2$ the elements c_1, c_2 prevent $d(a_i, x) = 1$ or 2, so in the amalgam necessarily $d(a_i, x) = 3$. Thus if the factors embed isometrically in Γ, then A embeds isometrically in Γ.

The factor in this amalgamation omitting (a_1, a_2) contains no edges, and hence embeds into Γ. It remains to consider the factor omitting A_0.

The factor omitting A_0 has the following form.
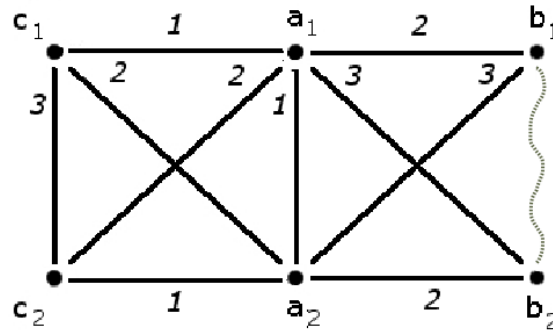
$$d(c_i, b_j) = 3 \quad (i, j = 1, 2)$$

Furthermore, the distance $d(a_1, c_2)$ is forced to be 2, so it suffices to treat the two factors obtained by omitting c_2 or a_1 respectively.

These factors are as follows.

(I)   (II)
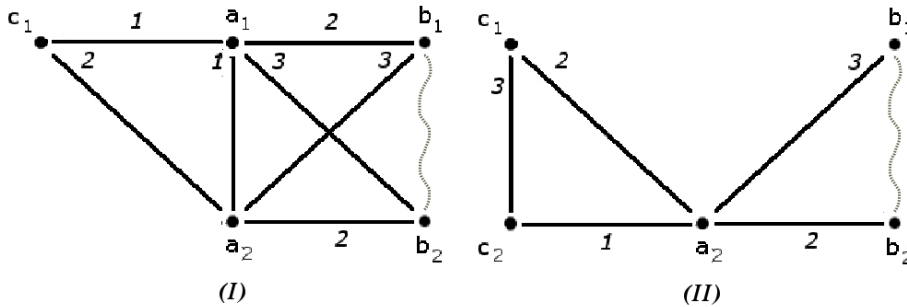
In factor (I) the distance $d(c_1, a_2)$ is forced to be 2, as otherwise (a_1, a_2, c_1) has type $(1, 1, 1)$, and thus this reduces to two factors of order 4.

The factor of (I) omitting a_2 has the unique edge (c_1, a_1), and no vertex at distance 2 from c_1. So Lemma 5.6 applies.

The factor of (I) omitting c_1 corresponds to the case $|A| = 4$ and will be dealt with below under that heading.

In factor (II) there is no vertex v with $d(c_2, v) = 2$, so Lemma 5.6 applies.

Thus we come down to the case $|A| = 4$. Then the structure of A is as follows.
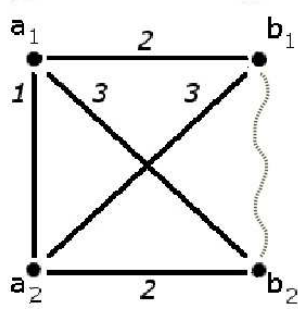
Adjoin c with $d(c, x) = 2, 3, 2$ for $x = a_1, a_2, b_2$ and with $d(c, b_1) = 2$ or 3, the last to be determined momentarily. Consider the following amalgamation over $a_2 b_1 c$.
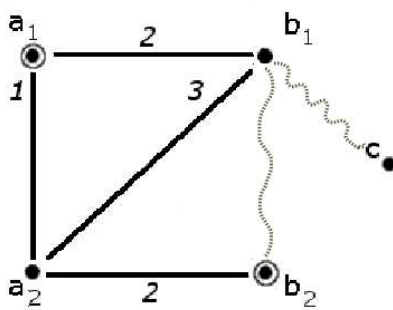
Here the vertices a_2 and c ensure that $d(a_1, b_2) = 3$.

In this amalgamation, the factor omitting a_1 has no edges. The factor omitting b_2 has the following form, with $i = 2$ or 3, to be determined.
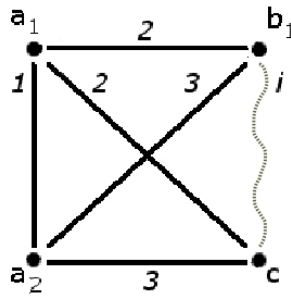
With a_2 as base point, take $a_1 \in \Gamma_1$. We require two vertices v_1, v_2 in Γ_3 with $d(a_1, v_1) = d(a_1, v_2) = 2$ (then the distance $i = d(v_1, v_2)$ will necessarily be 2 or 3).

If two such vertices v_1, v_2 cannot be found, then each $a \in \Gamma_1$ determines a unique $a' \in \Gamma_3$ by the condition $d(a, a') = 2$. As Γ_1 is primitive, this gives a bijection

$$\Gamma_1 \leftrightarrow \Gamma_3$$

But Γ_3 realizes three distinct distances, and this gives a contradiction to homogeneity.

This completes the proof.   □

Now Proposition 5.1 follows from Proposition 5.3, Corollary 4.3, and Remark 4.4.

6. Identification of Γ: $K_1 \leq 2$ and $C \geq 9$

6.1. **Statement of the problem: The embedding property.** If Γ is a metrically homogeneous graph of diameter 3, then it has been identified as a member of our catalog in the following exceptional cases.

(1) Γ is not of generic type, hence imprimitive (antipodal or bipartite) or finite; or
(2) Γ is infinite and primitive, with $K_1 = 3$ or $C = 8$.

In the remaining cases, which are the most typical ones, we have all of the following conditions.

(1) Γ is infinite, primitive, of generic type;
(2) $K_1 = 1$ or 2;
(3) $K_2 = 2$ or 3;
(4) $C = 9$ or 10.

We will again show that the graph is in our catalog. We recall the table of possibilities (Table 3), cut down to the remaining cases.

Our assumptions on the parameters can be expressed more succinctly as follows: no forbidden triangle involves distance 2.

As in §5.2, in accordance with Definition 3.20 we set

$$\Gamma^* = \Gamma^3_{K_1, K_2, C, C+1, \mathcal{S}}$$

Type	Case	K_1	K_2	C	C'	$\mathcal{S}$
Primitive	(III)	1	3	9 or 10	$C+1$	
"	(III)	1	2	9 or 10	$C+1$	cliques and anticliques
"	(III)	2	3	9 or 10	$C+1$	anything not involving a clique of order 3
"	(III)	2	2	9 or 10	$C+1$	anticliques

TABLE 4. $K_1 \leq 2$, $C \geq 9$

Our goal is the following.

Proposition 6.1. *Let Γ be an infinite primitive metrically homogeneous graph of diameter 3 with parameters*

$$K_1, K_2, C, C', \mathcal{S}$$

Suppose that $K_1 \leq 2$ and $C \geq 9$. Then

$$\Gamma \cong \Gamma^*$$

Again, by Corollary 2.6 Γ must be of generic type. By our earlier work the proposition reduces to the following.

Proposition 6.2. *Let Γ be an infinite primitive metrically homogeneous graph of diameter 3 with parameters*

$$K_1, K_2, C, C', \mathcal{S}$$

Suppose that $K_1 \leq 2$ and $C \geq 9$. Then any finite metric space which embeds isometrically in Γ^* embeds in Γ.

Recall from Proposition 4.1 that the triangles embedding isometrically in Γ are those embedding in Γ^*.

The choice of parameters K_1, K_2, C ensures that these forbidden triangles are also $(1, \delta)$-spaces, so that the minimal forbidden configurations for Γ^* are just the forbidden $(1, \delta)$-spaces.

Another way of phrasing Proposition 6.2, relevant to the proof, is the following.

> *If all the forbidden triangles of Γ are $(1, \delta)$-spaces, then the minimal $[3]$-metric spaces not embedding in Γ are all $(1, \delta)$-spaces. In other words, the amalgamation class allows free amalgamation with distance 2 used as the "default" value of the distance.*

The notion of free amalgamation suggests that this problem is one which we should expect to treat by a uniform method. The proof strategy will be inductive: we must introduce a suitable measure of complexity according to which any forbidden configuration which is not a $(1, \delta)$-space can be

reduced to some "simpler" forbidden configuration—here "simpler" does not necessarily mean smaller, but we should ensure that the reduction process terminates.

This allows us to give an argument with a reasonable degree of uniformity across the 8 cases corresponding to various specifications of the numerical parameters K_1, K_2, C in our table, as well as the infinitely many possibilities for $\mathcal{S}$.

As noted following Proposition 3.19, since C determines $C' = C + 1$, we may drop C' from our notation. In this section we will generally do so; and where appropriate, we will also use the briefer notation Γ^* introduced above.

6.2. **A Uniqueness Case.** We now treat an important case in an inductive framework. All metric configurations under consideration here have distances among $1, 2, 3$. We will not repeat this point.

Lemma 6.3. *Let Γ be an infinite primitive metrically homogeneous graph of diameter 3 and generic type with $K_1 \leq 2$ and $C \geq 9$. Let A be a $(1,3)$-space which embeds into Γ with $|A| \geq 3$, and $B \supseteq A$ finite. Suppose the following two conditions are satisfied.*

(1) Any $(1,3)$-subspace of B other than A has order less than $|A|$, and embeds isometrically in Γ.

(2) Any configuration whose $(1,3)$-spaces embed into Γ, for which all of the $(1,3)$-spaces involved have order less than $|A|$, itself embeds into Γ.

Then the configuration B embeds into Γ.

Proof. Note first that as a consequence of condition (1) and Proposition 4.7, all triangles of B embed isometrically in Γ: namely, by Proposition 4.7 any triangle in B which does not embed in Γ would lie in the set of forbidden triangles associated with the parameters of Γ and would therefore be a $(1,3)$-space, in view of the conditions on K_1 and C. But if the triangle is not A it then falls under clause (1).

We prove the lemma by induction on $|B|$.

We may suppose that $B \neq A$. Fix $b \in B \setminus A$. Our hypothesis (1) on B implies that we can choose distinct $a_1, a_2 \in A$ for which

$$d(a_1, b) = d(a_2, b) = 2$$

as otherwise we have another $(1,3)$-subspace of B of the same size as A. Let B_0 be $B \setminus \{a_1, a_2, b\}$.

Adjoin additional points c_1, c_2 satisfying the following conditions.

$$d(c_1, a_1) = d(c_2, a_1) = 1$$
$$d(c_1, b) = 1, \quad d(c_2, b) = 3$$
$$d(c_1, x) = d(c_2, x) = 2 \quad (x \in B_0)$$
$$d(c_1, c_2) = 2$$

We leave the distances $d(c_1, a_2)$ and $d(c_2, a_2)$ to be determined in the course of the construction. Observe that any triangle whose distances have been determined so far occurs in Γ, since any triangle involving at least one pair at distance 2 embeds in Γ.

Once suitable factors are constructed, we take $A_0 = B_0 \cup \{a_2, c_1, c_2\}$ and amalgamate $A_0 \cup \{a_1\}$ with $A_0 \cup \{b\}$. The points c_1, c_2 serve to ensure that $d(a_1, b) = 2$ in the result, and thus we get a copy of B. So it remains to construct the two factors appropriately.

For the factor $A_0 \cup \{a_1\}$, we determine the structure of (a_2, c_1, c_2) by taking the amalgamation of $B_0 \cup \{a_1, a_2\}$ with $B_0 \cup \{a_1, c_1, c_2\}$ over $B_0 \cup \{a_1\}$. Here, the factor omitting c_1, c_2 is $B \setminus \{b\}$, which we suppose lies in Γ by induction on $|B|$. The factor omitting a_2 contains no $(1,3)$-space of order $|A|$ (recall $|A| \geq 3$), and all of its $(1,3)$-subspaces occur in B, so by hypothesis they embed in Γ, and this factor does as well. Thus the factor $A_0 \cup \{a_1\}$ will embed in Γ if the distances $d(a_2, c_1)$ and $d(a_2, c_2)$ are chosen appropriately.

We now turn to the factor $A_0 \cup \{b\}$, whose structure is as shown.
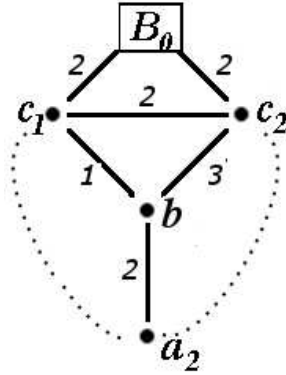
The $(1,3)$-spaces occurring here are either contained in (a_2, b, c_1, c_2), and hence of cardinality at most 2, or are contained in $B_0 \cup \{a_2, b\} = B \setminus \{a_1\}$ and hence have cardinality at most $|A| - 1$. Hence by hypothesis this factor embeds into Γ.

This completes the construction. $\square$

6.3. **Inductive Parameters.** Now we introduce some notions of complexity for finite configurations A.

Definition 6.4.

(1) If G is a graph, let G' denote the induced graph on the set of points of G of degree at least 2.

(2) Let δ be fixed and finite, and let A be an integral metric space of diameter at most δ.
   (a) Let G_A be the graph with vertex set A and edge relation given by "$d(x, y) = 1$ or δ".
   (b) A' is the metric structure induced by A on the points of $(G_A)'$.

> (c) If G_A is connected or complete, then we will say that A is $(1,\delta)$-connected or $(1,\delta)$-complete, respectively.
>
> (d) $||A|| = \max(|A_0| : A_0 \subseteq A,\ A_0\ a\ (1,\delta)\text{-subspace})$.
>
> These notions depend on δ, which we treat as fixed, as well as A, which we treat as variable.

Lemma 6.5. *Let Γ be an infinite primitive metrically homogeneous graph of diameter 3 and generic type with $K_1 \leq 2$ and $C \geq 9$. Let A be a finite configuration not embedding isometrically in Γ, but such that every triangle or $(1,3)$-space contained in A embeds isometrically into Γ. Suppose that A is chosen to minimize $||A||$, and subject to that, also minimizes $|A'|$. Then*

> *(1) A' is $(1,3)$-complete.*
> *(2) $|A'| \leq 2$.*

Proof.

Ad (1):

Suppose first that $a_1, a_2 \in A'$ are points with $d(a_1, a_2) = 2$. Consider the following amalgamation with auxiliary points c_1, c_2, where $A_0 = A \backslash \{a_1, a_2\}$.
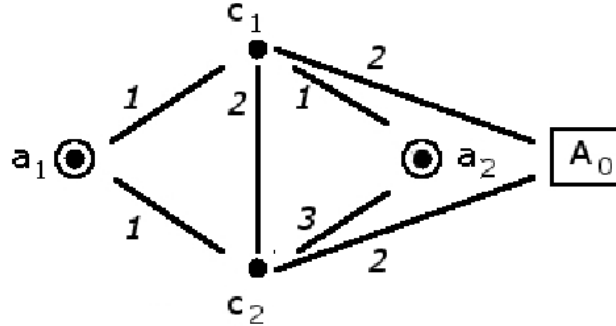
The factors F have $||F|| \leq ||A||$ and $|F'| < |A'|$, so we may conclude by induction that they embed into Γ. Then the unique amalgam has $d(a_1, a_2) = 2$ and hence contains a copy of A, for a contradiction.

So A' is $(1,3)$-complete. In particular, there is a unique maximal $(1,3)$-connected component of A containing A', and any other non-trivial $(1,3)$-connected component of A consists of a single pair of points at distance 1 or 3.

In particular, if $|A'| \geq 2$ then $||A|| = |A'|$.

Ad (2):

Suppose $|A'| \geq 3$. Then A' is the unique $(1,3)$-subspace of A of order greater than 2. We will apply Lemma 6.3 with (A, A') in place of (B, A).

Let A^* be any configuration whose $(1,3)$-subspaces and triangles embed into Γ, and whose $(1,3)$-spaces are all of order less than $|A'|$. Then $||A^*|| < ||A||$ and thus A^* embeds into Γ.

Hence Lemma 6.3 applies. Therefore A embeds into Γ. This contradiction shows that $|A'| \leq 2$. □

6.4. **Structure of Γ_2.** For the present subsection, we denote the parameters associated to Γ_2 by $\tilde{K}_1, \tilde{K}_2, \tilde{C}$. Our aim is the following.

Proposition 6.6. *Let Γ be an infinite primitive metrically homogeneous graph of diameter 3 with $K_1 \leq 2$, $K_2 \geq 2$, and $C \geq 9$. Then Γ_2 is an infinite primitive metrically homogeneous graph of diameter 3 and generic type with associated parameters $\tilde{K}_1 \leq 2$, $\tilde{K}_2 \geq 2$, and $\tilde{C} \geq 9$.*

As usual, we note that Γ is of generic type, and we include this point explicitly as a hypothesis in the following lemmas.

Lemma 6.7. *Let Γ be an infinite primitive metrically homogeneous graph of diameter 3 and generic type, with $K_1 \leq 2$, $K_2 \geq 2$, and $C \geq 9$. Then Γ_2 contains a triangle of type $(1, 2, 2)$.*

Proof. If $K_1 = 1$ then Γ_1 is a Henson or random graph, by the assumption of genericity and Fact 2.1.

In this case, set

$$\tilde{\Gamma} = \Gamma_1$$

Taking $u \in \Gamma_1$, we have $\tilde{\Gamma}_2(u) \subseteq \Gamma_2(u)$, and $\tilde{\Gamma}_2(u)$ contains the required triangle.

So now suppose

$$K_1 = 2$$

Let (a, b, c) be a triangle of type $(1, 2, 2)$ in Γ with $d(a, b) = d(a, c) = 2$. Then a, b and a, c each have infinitely many common neighbors, as Γ is of generic type.

Take u_1, u_2 distinct so that u_1 is a common neighbor of a and b, and u_2 is a common neighbor of a and c. Then

$$d(u_1, u_2) = 2$$

Take u a common neighbor of u_1, u_2, distinct from a, b, c.

Then a, b, u are neighbors of u_1, and a, c, u are neighbors of u_2.

So the triangle (a, b, c) of type $(1, 2, 2)$ lies in $\Gamma_2(u)$, and by homogeneity there is such a triangle in Γ_2. $\square$

Lemma 6.8. *Let Γ be an infinite primitive metrically homogeneous graph of diameter 3 and generic type with $K_1 \leq 2$, $K_2 \geq 2$, and $C \geq 9$. Then Γ_2 is an infinite primitive metrically homogeneous graph of diameter 3, and thus also of generic type, with associated parameters $\tilde{K}_1 \leq 2$, $\tilde{K}_2 \geq 2$, and one of the following occurs.*

- *$\tilde{C} \geq 9$;*
- *$\tilde{K}_1 = 1$, $\tilde{K}_2 = 3$, $\tilde{C} = 8$; in this case, $K_1 = 1$ and $K_2 = 3$ as well.*

Proof. As $K_1 \leq 2 \leq K_2$ (or by the previous lemma) Γ_2 contains an edge and is therefore connected, metrically homogeneous, and primitive by Fact 3.2. Γ_2 has diameter 3 since Γ contains a triangle of type $(2, 2, 3)$ (as Γ^* does).

By Lemma 6.7, Γ_2 contains a triangle of type $(1, 2, 2)$, so $\tilde{K}_1 \leq 2 \leq \tilde{K}_2$.

With these parameters, we then have either $\tilde{C} \geq 9$ or $\tilde{K}_1 = 1$, $\tilde{K}_2 = 3$, and in the latter case, $K_1 = 1$, $K_2 = 3$.   □

Thus the proof of Proposition 6.6 reduces to the elimination of the second case envisioned. We may state the required result as follows.

Lemma 6.9. *Let Γ be an infinite primitive metrically homogeneous graph of diameter 3 and generic type with*

$$K_1 = 1, \ K_2 = 3, \ and \ C \geq 9$$

Then Γ_2 contains a triangle of type $(2, 3, 3)$.

The proof is an extensive amalgamation argument, which we break up into more manageable pieces.

Lemma 6.10. *Let Γ be an infinite primitive metrically homogeneous graph of diameter 3 and generic type with*

$$K_1 = 1, \ K_2 = 3, \ and \ C \geq 9$$

Suppose that Γ_2 contains no triangle of type $(2, 3, 3)$. Then Γ contains the configuration (a, b_1, b_2, b_3) in which

$$d(a, b_i) = 1, 3, 3 \ for \ i = 1, 2, 3 \ respectively$$
$$d(b_i, b_j) = 2$$

Proof. Our target may be depicted as follows.
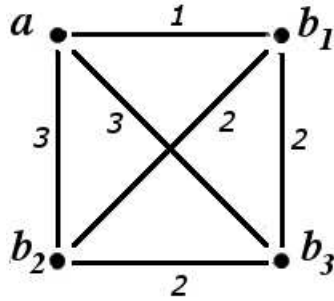
Config. (I)

For our first try, we use the following amalgamation.
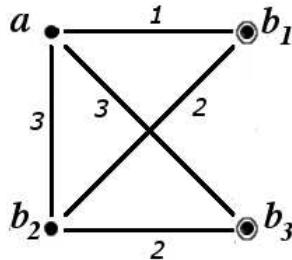
Here $d(b_1, b_3) \geq 2$, so we either get the desired configuration or the alternative with $d(b_1, b_3) = 3$.
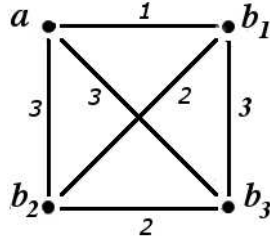
Config. (II)

In this case, we try a different approach. Adjoin a point c with

$$d(c, a) = 1$$
$$d(c, b_i) = 1, 2, 3 \text{ for } i = 1, 2, 3$$

Consider the following amalgamation.

$$d(c, b_2) = 2$$
$$d(c, b_3) = 3$$
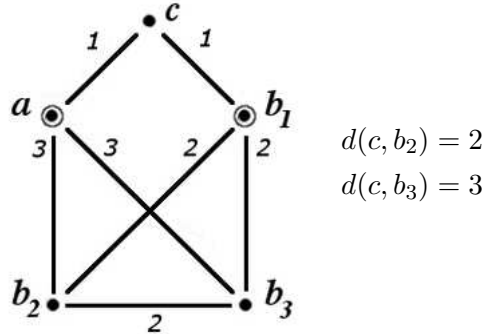
The parameter c forces $d(a, b_1) \leq 2$. In the amalgam, as a, b_2, b_3 is a triangle of type $(2, 3, 3)$ and we suppose that $\Gamma_2(b_1)$ contains no triangle of type $(2, 3, 3)$, we find

$$d(a, b_1) = 1$$

which gives the desired configuration.

Therefore it suffices to check that the factors of this last amalgamation problem embed isometrically into Γ.

The second factor $(cb_1b_2b_3)$ corresponds to a geodesic of length 3 in $\Gamma_2(b_2)$, hence embeds isometrically into Γ.

The first factor, acb_2b_3, is the Configuration (II) obtained above.   □

Now we return to Lemma 6.9.

Proof of Lemma 6.9. Our target is an embedding of a triangle of type $(2, 3, 3)$ into Γ_2, which we depict as follows.
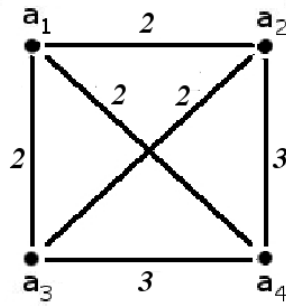
Adjoin points c_1, c_2 with $d(c_1, c_2) = 2$ and the following distances.

	a_1	a_2	a_3	a_4
c_1	2	1	1	2
c_2	1	1	3	2

We treat the resulting configuration as a 2-point amalgamation problem which determines the distance

$$d(a_2, a_3)$$

Then the auxiliary points c_1, c_2 force

$$d(a_2, a_3) = 2$$

and we get an embedding of the desired configuration into Γ.

It remains to be checked that the two factors of this amalgamation problem embed isometrically into Γ:

  (I) $a_1 a_2 a_4 c_1 c_2$ and
  (II) $a_1 a_3 a_4 c_1 c_2$

Note first that there are few triangles of perimeter greater than 7 in the configuration $(a_1 a_2 a_3 a_4 c_1 c_2)$. Such a triangle must contain a vertex lying on two edges of length 3, which must be a_3 or a_4, so the only possibilities are the triangles (a_2, a_3, a_4) and (c_2, a_3, a_4). In particular the configuration (I) omitting a_3 contains neither of these triangles, while the configuration (II) contains the second one.

Since the factor (I) omitting a_3 contains no triangles of perimeter greater than 7 we may embed it into Γ by embedding it into Γ_2. Namely, recall by Lemma 6.8 and Corollary 5.2 that under our assumptions, $\Gamma_2 \cong \Gamma^3_{1,3,8,9}$; otherwise, if $\tilde{C}$ is the parameter associated with $\tilde{\Gamma} = \Gamma_2$, we would have $\tilde{C} \geq 9$ and hence a triangle of type $(3, 3, 2)$ is already embedded into Γ_2.

Thus if the desired triangle does not occur in Γ_2, then Γ_2 contains the configuration (I) isometrically, and hence Γ does.

So we now turn our attention to the construction of the factor (II) by a suitable amalgamation.

Adjoin a point c_3 to the configuration $(a_1a_3a_4c_1c_2)$ satisfying the following.

$$d(c_3, x) = 1, 1, 2, 2, 2 \text{ for } x = a_1, a_3, a_4, c_1, c_2 \text{ respectively}$$

View the resulting configuration $(a_1a_3a_4c_1c_2c_3)$ as a 2-point amalgamation problem with the distance $d(a_1, a_3)$ to be determined.

Then the points c_2, c_3 force the distance $d(a_1, a_3)$ to be 2. So it suffices to show that the factors of this amalgamation, namely $(a_1a_4c_1c_2c_3)$ and $(a_3a_4c_1c_2c_3)$, embed isometrically into Γ.

The factor $(a_1a_4c_1c_2c_3)$ involves no pairs at distance 3 and hence no triangles of perimeter greater than 6, hence embeds isometrically in $\Gamma_2 \cong \Gamma^3_{1,3,8,9}$, and a fortiori into Γ.

So we turn to the factor $(a_3a_4c_1c_2c_3)$. This has the following form.

$$d(c_3, a_4) = d(c_3, c_2) = 2$$
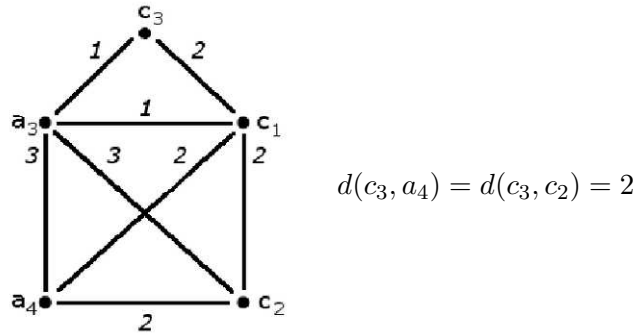
We take the following point of view here. With a_3 as base point, we want a pair in Γ_1 and a pair in Γ_3 with all distances between the pairs equal to 2. So let us now view this as a point $a = a_3$, and two pairs $A_1 = \{c_1, c_3\}$, $A_3 = \{c_2, a_4\}$, satisfying

$$d(a, x) = i \text{ for } x \in A_i$$
$$d(x, y) = 2 \text{ for } x, y \in A_1 \cup A_3$$

Now adjoin a point c adjacent to all points in $A_1 \cup A_3$ and at distance 2 from a. Then we may view aA_1A_3c as an amalgamation problem in which the distances between A_1 and A_3 all remain to be determined. But the parameter c gives the bound

$$d(x, y) \leq 2$$

for all such distances, while the parameter a gives the bound

$$d(x, y) \geq 2$$

Thus the result of this amalgamation is uniquely determined, and it suffices to show that the factors (a_3A_1c) and (a_3A_3c) occur in Γ.

(1)   (2)
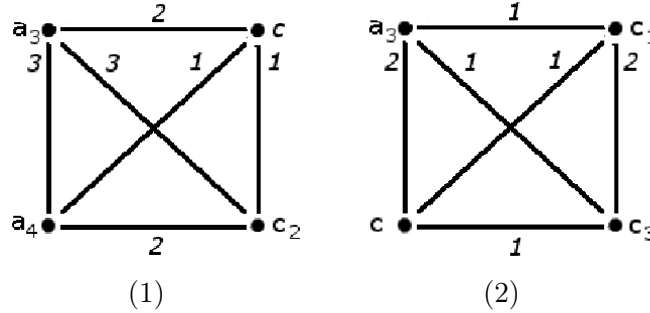
Factor (2) is available for a number of reasons—it embeds in Γ_1 and in Γ_2—so we concern ourselves with factor (1).

We adjoin a point c' with $d(c', x) = 1, 2, 2, 1$ for $x = a_3, a_4, c_2, c$ respectively. We then consider the following amalgamation.

$$d(c', a_4) = d(c', c_2) = 2$$
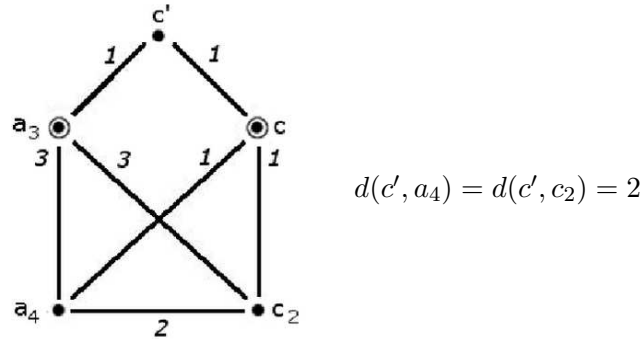
With the help of the point c', we see that $d(a_3, c)$ must be 2. So it suffices to embed the factors of this diagram isometrically into Γ.

The factors are as follows.

(3)   (4)
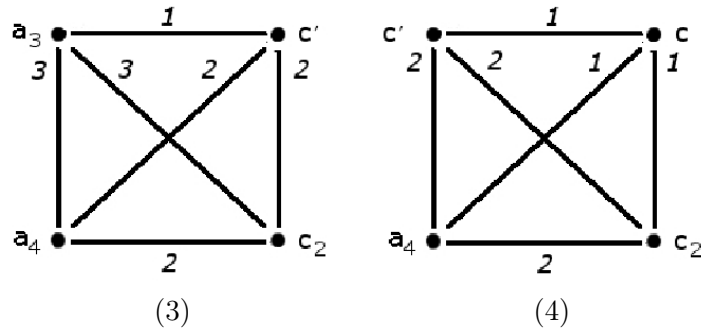
Factor (3) was treated in Lemma 6.10, while factor (4) embeds isometrically in Γ since Γ_1 contains I_3^2.

This concludes the analysis.   □

Proof of Proposition 6.6. Lemmas 6.8 and 6.9.   □

6.5. **The Parameters of Γ_2.** We continue the analysis of the previous section. Again, we denote by $\tilde{K}_1$, $\tilde{K}_2$, and $\tilde{C}$ the parameters of Γ_2.

Our aim is the following.

Proposition 6.11. *Let Γ be an infinite primitive metrically homogeneous graph of diameter 3 and generic type with $K_1 \leq 2$, $K_2 \geq 2$, and $C \geq 9$. Suppose that every infinite primitive metrically homogeneous graph $\tilde{\Gamma}$ of diameter 3 and generic type with associated parameters $\tilde{K}_1 \leq 2$, $\tilde{K}_2 \geq 2$, $\tilde{C} \geq 9$, and $\tilde{\mathcal{S}}$, and which realizes fewer triangle types than Γ, is of the form*

$$\Gamma^3_{\tilde{K}_1, \tilde{K}_2, \tilde{C}, \tilde{\mathcal{S}}}$$

Then Γ_2 is an infinite primitive metrically homogeneous graph of diameter 3 and generic type, with the same numerical parameters.

$$\tilde{K}_1 = K_1 \qquad\qquad \tilde{K}_2 = K_2 \qquad\qquad \tilde{C} = C$$

We note that the only point of the inductive hypothesis on Γ here is to identify Γ_2 in the case that Γ_2 itself realizes fewer triangle types than Γ. And the conclusion of our lemma is that this never actually occurs (under the stated hypotheses).

Lemma 6.12. *Let Γ be an infinite primitive metrically homogeneous graph of diameter 3 and generic type with $K_1 \leq 2$, $K_2 \geq 2$, and $C \geq 9$.*

Suppose Γ contains a triangle of type $(1,1,1)$ (so in fact $K_1 = 1$). Then Γ_2 also contains a triangle of type $(1,1,1)$.

Proof. If Γ contains a clique of order 4, then Γ_1 is a random graph or Henson graph containing a triangle, hence contains the desired configuration (a vertex at distance 2 from a clique of order 3). So we may suppose that

$$\Gamma \text{ contains no clique of order } 4.$$

We label the desired configuration $(ab_1b_2b_3)$ where (b_1, b_2, b_3) is a triangle of type $(1,1,1)$ and $d(a, b_i) = 2$ for $i = 1, 2, 3$.

We adjoin two auxiliary points c_1, c_2 with the following metric.

$$d(c_i, x) = \begin{cases} 1 & x \neq b_i \\ 2 & x = b_i \end{cases}$$
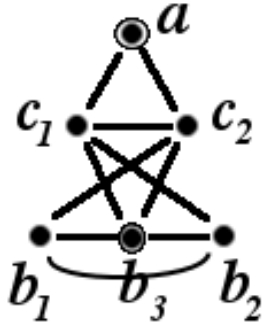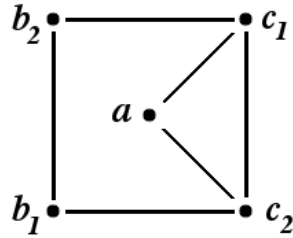

We view the resulting configuration as a 2-point amalgamation problem determining the distance $d(a, b_3)$. As all distances among a, b_3, c_1, c_2 are 1 apart from $d(a, b_3)$, and there is no clique of order 4, this forces $d(a, b_3) = 2$.

It suffices therefore to check that both of the configurations $(ab_1b_2c_1c_2)$ and $(b_1b_2b_3c_1c_2)$ embed isometrically in Γ.

Now the factor $(b_1b_2b_3c_1c_2)$ can be viewed as a 4-cycle in $\Gamma_1(b_3)$, so this embeds in Γ. Therefore it suffices to consider the factor
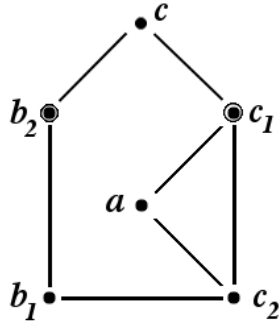
$$(ab_1b_2c_1c_2)$$

We add a point c with

$$d(c, b_2) = d(c, c_1) = 1$$
$$d(c, a) = d(c, b_1) = d(c, c_2) = 2$$

View the resulting configuration as a 2-point amalgamation problem where the distance $d(b_2, c_1)$ is to be determined.

The point c gives $d(b_2, c_1) \leq 2$.

If we have $d(b_2, c_1) = 1$, then we have the required configuration, while otherwise we have a, c_1, c_2 a triangle of type $(1, 1, 1)$ in $\Gamma_2(b_2)$, and we conclude.

So we must show that the factors $(ab_1b_2c_2c)$ and $(ab_1c_1c_2c)$ embed isometrically into Γ.

Now the factor $(ab_1b_2c_2c)$, omitting c_1, contains no clique of order 3 and therefore embeds into Γ_1 (which is a Henson graph) and hence into Γ. So we come down to the factor $(ab_1c_1c_2c)$.
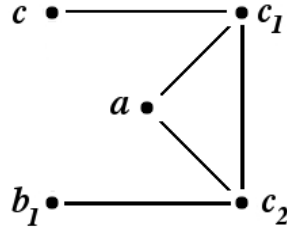
We may treat this in a similar fashion. We adjoin a vertex c' with

$$d(c', c) = d(c', c_1) = d(c', a) = 1$$
$$d(c', b_1) = d(c', c_2) = 2$$

and treat this as an amalgamation problem determining $d(c, c_1)$, which is either 1 (as desired) or 2 (which puts the triangle (a, c_1, c_2) into $\Gamma_2(c)$).

Again, the factor omitting c_1 contains no 3-clique and embeds in Γ_1. So we consider the factor $(ab_1c_1c_2c')$. This can be treated as an amalgamation problem with $d(c', a)$ to be determined, and $d(c', a) \leq 2$.
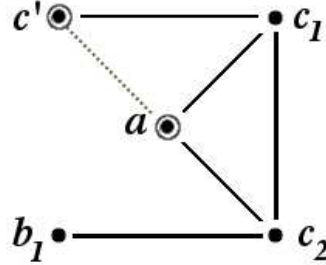
In this amalgamation, the factor omitting a has no clique of order 3 and hence embeds in Γ_1, while the factor omitting c' consists of a triangle of type $(2, 2, 1)$ in $\Gamma_1(c_2)$. Thus the factors of this diagram embed in Γ.

The completion of this diagram gives either the desired factor, where $d(a, c') = 1$, or else gives a copy of the factor $(ab_1c_1c_2c)$, and either way we may conclude.

This concludes the construction. □

Lemma 6.13. *Let Γ be an infinite primitive metrically homogeneous graph of diameter 3 and generic type with $K_1 \leq 2$, $K_2 \geq 2$, and $C \geq 9$. Suppose that every infinite primitive metrically homogeneous graph of diameter 3 and generic type which satisfies the same restrictions on its numerical parameters, and which realizes fewer triangle types than Γ, is of the form*

$$\Gamma^3_{\tilde{K}_1, \tilde{K}_2, \tilde{C}, \tilde{\mathcal{S}}}$$

for suitable parameters $\tilde{K}_1, \tilde{K}_2, \tilde{C}, \tilde{\mathcal{S}}$. If Γ contains a triangle of type $(3, 3, 1)$, then Γ_2 contains a triangle of type $(3, 3, 1)$.

Proof. We will suppose that a triangle of type $(3, 3, 1)$ does not occur in Γ_2, and then by assumption Γ_2 is of the form

$$\Gamma^3_{\tilde{K}_1, \tilde{K}_2, \tilde{C}, \tilde{\mathcal{S}}}$$

where $\tilde{K}_1$, $\tilde{K}_2$, $\tilde{C}$, $\tilde{S}$ are the parameters associated to Γ_2. By Proposition 6.6 we have $\tilde{K}_1 \leq 2 \leq \tilde{K}_2$ and $\tilde{C} \geq 9$.
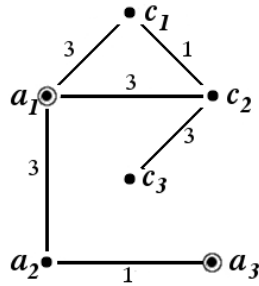
We choose notation so that the configuration in question is $(ab_1b_2b_3)$ with

$$d(b_1, b_i) = 3 \ (i = 2, 3)$$
$$d(b_2, b_3) = 1$$
$$d(a, b_i) = 2 \ (i = 1, 2, 3)$$

We take points $a_1, a_2, a_3, c_1, c_2, c_3$ with

$$d(a_1, a_2) = d(c_2, c_3) = 3$$
$$d(c_1, c_2) = d(a_2, a_3) = 1$$
$$d(a_1, c_i) = 3 \ (i = 1, 2)$$

and all other distances equal to 2, apart from $d(a_1, a_3)$, which is to be determined by completing the following amalgamation.

The point a_2 forces $d(a_1, a_3) \geq 2$. If $d(a_1, a_3) = 2$ then $(a_3a_1c_1c_2)$ is the desired configuration, and if $d(a_1, a_3) = 3$ then $(c_3a_1a_2a_3)$ is the desired configuration. So it suffices to show that the factors $(a_1a_2c_1c_2c_3)$ and $(a_2a_3c_1c_2c_3)$ of this amalgamation embed isometrically into Γ.

The factor $(a_2a_3c_1c_2c_3)$:
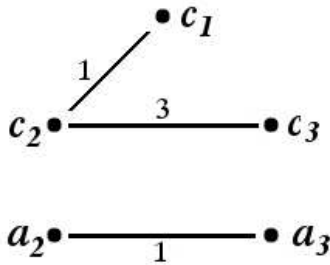
All triangles in this configuration embed isometrically into Γ_2, by inspection, in view of Proposition 6.6. So this configuration embeds into Γ_2.

*The factor $(a_1a_2c_1c_2c_3)$:*
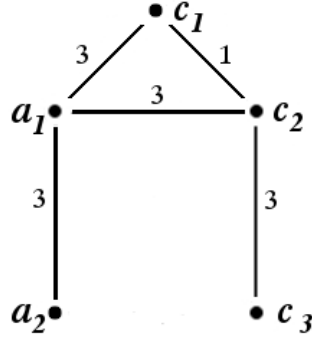
We view this as a 2-point amalgamation problem in which the distance $d(c_1, c_3)$ is to be determined. The point c_2 gives $d(c_1, c_3) \geq 2$. If $d(c_1, c_3) = 3$ then the configuration $(a_2c_1c_2c_3)$ gives a triangle of type $(3, 3, 1)$ in $\Gamma_2(a_2)$, and we are done. If $d(c_1, c_3) = 2$ then we have the correct configuration $(a_1a_2c_1c_2c_3)$.

But we must still check that the subfactors $(a_1a_2c_1c_2)$ and $(a_1a_2c_2c_3)$ embed isometrically in Γ.

The triangles occurring in the factor $(a_1a_2c_2c_3)$ all occur in Γ_2—those of type $(3, 3, 2)$ embed since $\tilde{C} \geq 9$. Therefore this configuration embeds isometrically into Γ_2, and hence into Γ.

So we must consider only the subfactor

$$(a_1a_2c_1c_2)$$
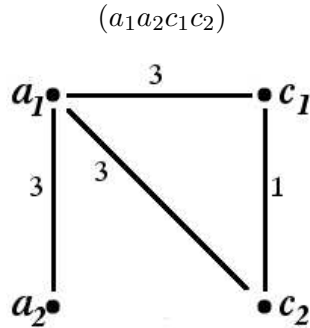
We adjoin an additional point a with

$$d(a, a_1) = 3 \qquad\qquad\qquad d(a, a_2) = 1$$
$$d(a, c_1) = \text{to be determined} \qquad\qquad d(a, c_2) = 2$$
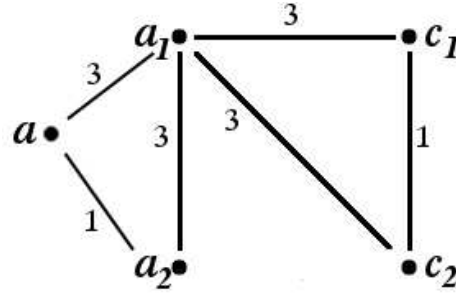
We treat the resulting configuration as a 2-point amalgamation problem with the distance $d(a_1, a_2)$ to be determined. The point a ensures that $d(a_1, a_2) \geq 2$.

If $d(a_1, a_2) = 2$, then (a_1, c_1, c_2) is a triangle of type $(3, 3, 1)$ in $\Gamma_2(a_2)$, a contradiction. If $d(a_1, a_2) = 3$ then we have the desired configuration.

It remains to be shown that the subfactors $(aa_1c_1c_2)$ and $(aa_2c_1c_2)$ of this last amalgamation embed isometrically in Γ, for some choice of the distance $d(a, c_1)$.

Consider the subfactor $(aa_2c_1c_2)$ with any value of $d(a, c_1)$ in the range 1,2,3.
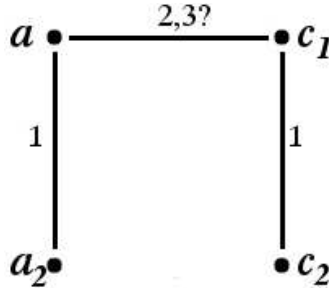
Here all triangles embed in Γ_2, so the configuration embeds isometrically in Γ_2, and hence in Γ.

So we may determine the distance $d(a, c_1)$ while constructing the subfactor $(aa_1c_1c_2)$, treating this configuration as a 2-point amalgamation problem with factors (aa_1c_2) and $(a_1c_1c_2)$, which are triangles of types $(3, 3, 2)$ and $(3, 3, 1)$, and so are realized in Γ.

This completes the construction of the factor $(a_1a_2c_1c_2c_3)$.   □

Lemma 6.14. *Let Γ be an infinite primitive metrically homogeneous graph of diameter 3 and generic type with $K_1 \leq 2$, $K_2 \geq 2$, and $C \geq 9$. Suppose that every infinite primitive metrically homogeneous graph of diameter 3 and generic type which satisfies the same restrictions on its parameters and which realizes fewer triangle types than Γ is of the form*

$$\Gamma^3_{\tilde{K}_1, \tilde{K}_2, \tilde{C}, \tilde{S}}$$

for suitable parameters $\tilde{K}_1, \tilde{K}_2, \tilde{C}, \tilde{S}$.

If Γ contains a triangle of type $(3, 3, 3)$, then Γ_2 contains a triangle of type $(3, 3, 3)$.
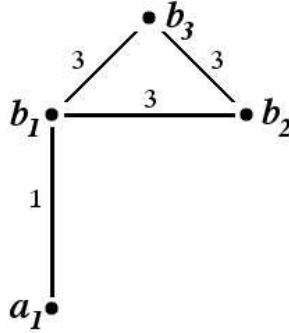
Proof.

We assume the contrary.

Then by Proposition 6.6 Γ_2 is of the form

$$\Gamma^3_{\tilde{K}_1, \tilde{K}_2, 9}$$

with $\tilde{K}_1 \leq 2 \leq \tilde{K}_2$.

Claim 6.14.1. Γ *contains a configuration $(a_1 b_1 b_2 b_3)$ in which (b_1, b_2, b_3) is a triangle of type $(3, 3, 3)$ and*
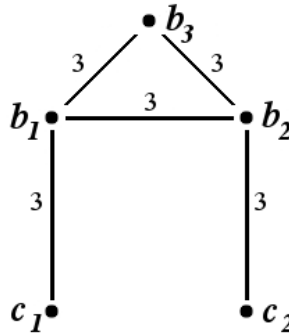
$$d(a_1, b_1) = 1$$
$$d(a_1, b_2) = d(a_1, b_3) = 2$$

If not, consider the configuration $(b_1 b_2 b_3 c_1 c_2)$ in which (b_1, b_2, b_3) is a triangle of type $(3, 3, 3)$,

$$d(c_i, b_i) = 3 \ (i = 1, 2)$$

and all other distances equal 2.

We will show first that this is realized in Γ.

We consider this configuration as a 2-point amalgamation problem in which $d(c_1, b_1)$ is to be determined. The values $d(c_1, b_1) = 1$ or 2 may be ruled out: if $d(c_1, b_1) = 1$ then we have the desired configuration already,

while if $d(c_1, b_1) = 2$ we have a triangle of type $(3, 3, 3)$ in $\Gamma_2(c_1)$, contrary to our hypothesis. So this diagram forces $d(c_1, b_1) = 3$. It suffices to check that the factors $(b_1 b_2 b_3 c_2)$ and $(b_2 b_3 c_1 c_2)$ occur in Γ.

We view $(b_1 b_2 b_3 c_2)$ as a 2-point amalgamation problem determining the distance $d(c_2, b_2)$, and again the values $d(c_2, b_2) = 1$ or 2 are ruled out, so we get $d(c_2, b_2) = 3$. As this is an amalgam of two triangles in Γ, the result is in Γ.

Now the configuration $(b_2 b_3 c_1 c_2)$ consists of the triangle (b_2, b_3, c_2) of type $(3, 3, 2)$ in $\Gamma_2(c_1)$, and as $\tilde{C} \geq 9$, this is realized in Γ.

Thus the specified configuration $(b_1 b_2 b_3 c_1 c_2)$ embeds into Γ.

Now consider the configuration $(a_1 b_2 b_3 c_1 c_2)$ in which b_2, b_3, c_1, c_2 are as in $(b_1 b_2 b_3 c_1 c_2)$, $d(a_1, c_1) = 3$, and $d(a_1, x) = 2$ otherwise.
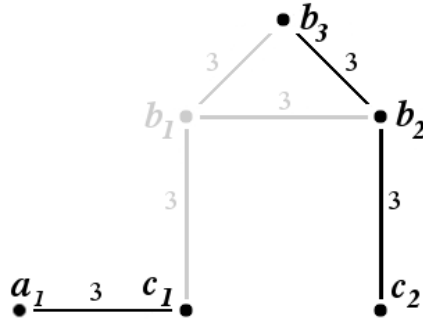
All triangles here occur in Γ_2, as $\tilde{K}_1 \leq 2 \leq \tilde{K}_2$ and $\tilde{C} = 9$, so this is present in Γ_2.

Now amalgamate $(b_1 b_2 b_3 c_1 c_2)$ with $(a_1 b_2 b_3 c_1 c_2)$ over $(b_2 b_3 c_1 c_2)$ to determine $d(a_1, b_1)$. As we assume $\Gamma_2(a_1)$ does not contain $(b_1 b_2 b_3)$, we have $d(a_1, b_1) = 1$ or 3.

If $d(a_1, b_1) = 1$ then we have the claimed configuration $(a_1 b_1 b_2 b_3)$. If $d(a_1, b_1) = 3$ then we have a triangle (a_1, b_1, c_1) of type $(3, 3, 3)$ in $\Gamma_2(c_2)$, a contradiction.
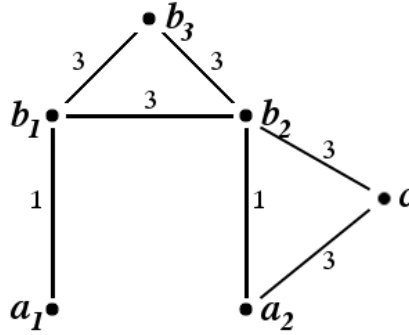
This proves the claim.

Claim 6.14.2. Γ *contains a configuration* $(a_1 a_2 b_1 b_2 b_3)$ *in which* (b_1, b_2, b_3) *is a triangle of type* $(3, 3, 3)$ *and*

$$d(a_i, b_i) = 1 \text{ for } i = 1, 2$$
$$d(a_i, b_j) = 2 \text{ for } i = 1, 2, \ j = 1, 2, 3, \ i \neq j$$
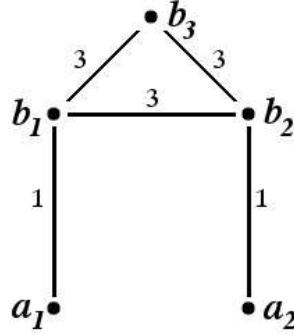
We argue similarly. We adjoin a point c with

$$d(c, x) = \begin{cases} 3 & x = a_2, b_2 \\ 2 & \text{otherwise} \end{cases}$$

Then we consider the configuration $(b_1 b_2 b_3 a_1 a_2 c)$ as a 2-point amalgamation problem determining the distance $d(a_2, b_2)$.

The value $d(a_2, b_2) = 1$ is the desired configuration.

The value $d(a_2, b_2) = 2$ would put the triangle (b_1, b_2, b_3) in $\Gamma_2(a_2)$, a contradiction.

The value $d(a_2, b_2) = 3$ would put the triangle (a_2, b_2, c) of type $(3, 3, 3)$ in $\Gamma_2(a_1)$, a contradiction.

So it suffices to check that the configurations $(a_1 b_1 b_2 b_3 c)$ and $(a_1 a_2 b_1 b_3 c)$ embed in Γ.

We can view the configuration $(a_1 b_1 b_2 b_3 c)$ as an amalgamation problem with $d(b_2, c)$ to be determined: the value $d(b_2, c) = 1$ gives the claim itself, the value $d(b_2, c) = 2$ would put the triangle (b_1, b_2, b_3) in $\Gamma_2(c)$, and the value $d(b_2, c) = 3$ is the one we aim at here.

The subfactors of this configuration are then $(a_1 b_1 b_3 c)$ and $(a_1 b_1 b_2 b_3)$. The latter is afforded by our first claim and the former is a geodesic in $\Gamma_2(c)$.
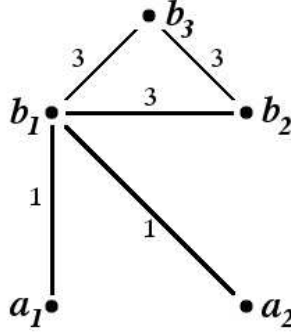
As for the second configuration $(a_1 a_2 b_1 b_3 c)$, all triangles embed in Γ_2, by inspection, and thus the configuration embeds into Γ_2, and hence into Γ.

Claim 6.14.3. *The configuration* $(b_1b_2b_3a_1a_2)$ *with* (b_1, b_2, b_3) *a triangle of type* $(3, 3, 3)$ *and*

$$d(a_i, b_1) = 1 \ i = 1, 2$$
$$d(a_i, b_j) = 2 \ i = 1, 2, \ j = 2, 3$$

embeds isometrically in Γ.

We adjoin a point c with

$$d(c, a_2) = d(c, b_2) = 1$$
$$d(c, x) = 2 \ (x \neq a_2, b_2)$$

and view the resulting configuration as a 2-point amalgamation problem determining $d(a_2, b_2)$. The points b_1 and c force $d(a_2, b_2) = 2$, giving the desired factor.

So it suffices to check that the configurations $(a_1a_2b_1b_3c)$ and $(a_1b_1b_2b_3c)$ embed isometrically into Γ.

All triangles in the factor $(a_1a_2b_1b_3c)$ embed in Γ_2, so that factor embeds isometrically in Γ_2 and hence in Γ.

The factor $(a_1b_1b_2b_3c)$ was dealt with under the previous claim.

This proves the claim.

Now we adjoin a point a_3 to the last configuration with

$$d(a_3, a_1) = 1 \qquad\qquad d(a_3, a_2) = 3$$
$$d(a_3, x) \ = 2 \text{ otherwise}$$

We view the resulting configuration as a 2-point amalgamation problem in which the distance $d(a_3, b_1)$ is to be determined. The points a_1, a_2 force $d(a_3, b_1) = 2$, but then the triangle (b_1, b_2, b_3) lies in $\Gamma_2(a_3)$, a contradiction.

So it suffices to embed the factors of this amalgamation into Γ. The factor $(a_1a_2b_1b_2b_3)$ is the configuration discussed under the last claim.

In the other factor, $(a_1a_2a_3b_2b_3)$, all triangles involved embed into Γ_2, so this factor embeds isometrically into Γ_2, and hence into Γ.

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Proof of Proposition 6.11. We consider $\tilde{\Gamma} = \Gamma_2$ and its associated parameters $\tilde{K}_1, \tilde{K}_2, \tilde{C}$. We must show these agree with the parameters of Γ.

By Proposition 6.6, we have $K_1 \leq 2 \leq K_2$ and $\tilde{C} \geq 9$.

If $K_1 = 1$ then $\tilde{K}_1 = 1$ by Lemma 6.12, and otherwise $K_1 = \tilde{K}_1 = 2$.

If $K_2 = 3$ then $\tilde{K}_2 = 3$ by Lemma 6.13, and otherwise $K_2 = \tilde{K}_2 = 2$.

We have $9 \leq \tilde{C} \leq C$ and if $C > 9$ then $\tilde{C} > 9$ by Lemma 6.14, and otherwise $C = \tilde{C} = 9$.

Thus $K_1 = \tilde{K}_1$, $K_2 = \tilde{K}_2$, and $C = \tilde{C}$, in all cases. □

6.6. **Recapitulation.** We review what is known at this point. Our standing assumption now is $\delta = 3$. We suppose that Γ is a primitive metrically homogeneous graph of generic type with associated parameters $K_1 \leq 2 \leq K_2$ and $C \geq 9$, and with associated constraint set $\mathcal{S}$ as described in Table 3. All other metrically homogeneous graphs of diameter 3 have already been classified.

Our ultimate aim is to show that Γ is isomorphic to the associated graph

$$\Gamma^* = \Gamma^3_{K_1, K_2, C, \mathcal{S}}$$

with the same parameters, including Henson constraints. (Recall that we now omit the parameter $C' = C + 1$ from our notation.)

We have reformulated what remains to be proved as an embedding problem in Proposition 6.2: any finite structure which embeds into Γ^* embeds into Γ.

We have two main results so far, and a reduction of the embedding theorem to a special case.

- Proposition 4.1: Γ contains exactly those triangles which embed into $\Gamma^3_{K_1, K_2, C, \mathcal{S}}$.
- Proposition 6.11: Suppose (inductively) that any such graph $\tilde{\Gamma}$ which realizes fewer triangles than Γ is isomorphic to the canonical metrically homogeneous graph

$$\Gamma^3_{\tilde{K}_1, \tilde{K}_2, \tilde{C}, \tilde{\mathcal{S}}}$$

  associated to its own parameters. Then taking $\tilde{\Gamma} = \Gamma_2$, this graph has the same associated numerical parameters as Γ: that is, $\tilde{K}_1 = K_1$, $\tilde{K}_2 = K_2$, $\tilde{C} = C$.
- Lemma 6.5: If there is a counterexample A to the embedding theorem then there is one such for which $|A'| \leq 2$.

After some additional preparation, Proposition 6.2 will be proved in §6.8, in an inductive framework allowing for the application of Proposition 6.11.

6.7. **An Embedding Lemma.** We now deal with isometric embedding of configurations A in which the graph with edge relation "$d(x,y) \in \{1,3\}$" is a star. This will be used to dispose of all cases of the embedding theorem in which $|A'| \leq 1$, leaving the main case $|A'| = 2$ to be addressed at the end.

Lemma 6.15 ((1,3)-Stars). *Let Γ be an infinite primitive metrically homogeneous graph of diameter 3 and generic type with $K_1 \leq 2$ and $C \geq 9$. Let*

$A = \{a\} \cup B$ with

$$\begin{cases} d(a, x) \in \{1, 3\} & (x \in B) \\ d(x, y) = 2 & (x, y \in B) \end{cases}$$

Then A embeds in Γ.

Such a configuration A will be called a $(1,3)$-star with center a.

We will use various explicit amalgamation arguments. We first deal with a special case.

Lemma 6.16. *Let Γ be an infinite primitive metrically homogeneous graph of diameter 3 and generic type with $K_1 \leq 2$ and $C \geq 9$. Then the following two configurations embed isometrically in Γ.*

(I) *The configuration $(a_1 a_2 a_3 a_4)$ in which (a_2, a_3, a_4) is a geodesic path of length 2 and a_1 satisfies*

$$d(a_1, a_i) = 3, 2, 2 \text{ for } j = 2, 3, 4$$

and

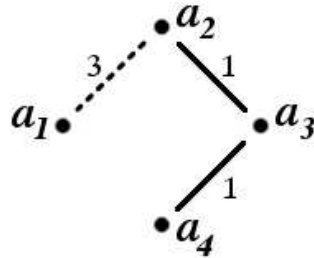(II) *the configuration $(a_1 a_2 b_1 b_3)$ which carries the following metric.*

$$d(a_i, b_j) = j$$
$$d(a_1, a_2) = d(b_1, b_3) = 2$$

Proof.

Ad (I):

In the diagrams below, we refine our practice of showing as "edges" all pairs at distance 1 or 3 by employing dashed lines to represent the distance 3.

Thus our configuration may be depicted as follows.

We introduce a point c_1 with $d(c_1, a_i) = 2, 1, 2, 3$ for $i = 1, 2, 3, 4$ respectively. We view the resulting configuration as a 2-point amalgamation determining the distance $d(a_2, a_4) = 2$.
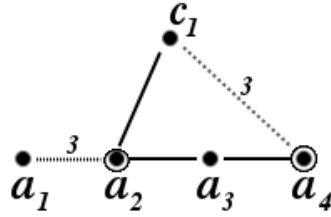
So it suffices to show that the factors $(a_1a_2a_3c_1)$ and $(a_1a_3a_4c_1)$ of this amalgamation embed isometrically into Γ.

The configuration $(a_1a_3a_4c_1)$ represents a geodesic triangle of diameter 3 (c_1, a_3, a_4) sitting in $\Gamma_2(a_1)$. The metrically homogeneous graph Γ_2 has diameter 3 by Proposition 6.6, so this configuration embeds isometrically in Γ.

Thus it suffices to consider the configuration

$$(a_1a_2a_3c_1)$$
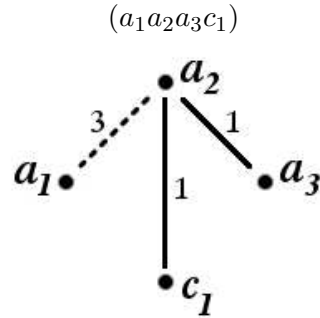
Here the points a_1, a_3, c_1 all lie at distance 2. We adjoin a point c_2 adjacent to these three points, with $d(c_2, a_2) = 2$. We consider the resulting configuration as an amalgamation problem over $\{a_2, c_2\}$ determining the distances $d(a_1, a_3) = d(a_1, c_1) = 2$.
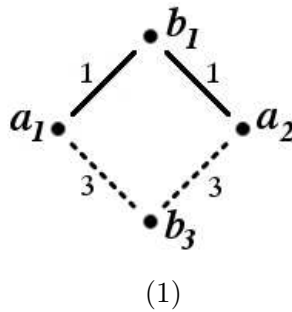
So it suffices to check that the factors $(a_1a_2c_2)$ and $(a_2a_3c_1c_2)$ of this amalgamation embed isometrically into Γ.

The factor $(a_1a_2c_2)$ is a geodesic triangle, so embeds isometrically in Γ.

The factor $(a_2a_3c_1c_2)$ is a 4-cycle, which embeds isometrically in Γ_1.

*Ad (II):*

Our configuration may be depicted as follows.

$$(1)$$

We adjoin a point c_1 with

$$d(c_1, b_1) = d(c_1, b_3) = 1$$
$$d(c_1, a_1) = d(c_1, a_2) = 2$$

We view the resulting configuration as a 2-point amalgamation problem in which $d(b_1, b_3)$ is to be determined. Then the points a_1, c_1 force $d(b_1, b_3) = 2$. So it suffices to check that the factors $(a_1 a_2 b_1 c_1)$ and $(a_1 a_2 b_3 c_1)$ embed isometrically in Γ.

The factor $(a_1 a_2 b_1 c_1)$ represents a triple (a_1, a_2, c_1) at mutual distance 2 in $\Gamma_1(b_1)$, so embeds isometrically in Γ.

We consider the configuration $(a_1 a_2 b_3 c_1)$.
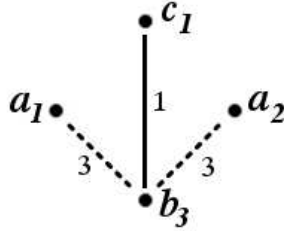
Adjoin a point c_2 with

$$d(c_2, a_2) = d(c_2, c_1) = 1$$
$$d(c_2, a_1) = d(c_2, b_3) = 2$$

View the resulting configuration as a 2-point amalgamation problem determining the distance $d(a_2, c_1) = 2$. It suffices to check that the factors $(a_1 a_2 b_3 c_2)$ and $(a_1 b_3 c_1 c_2)$ embed isometrically in Γ.

Now $(a_1 b_3 c_1 c_2)$ is the configuration (I) considered above. So it suffices to deal with

$$(a_1 a_2 b_3 c_2)$$
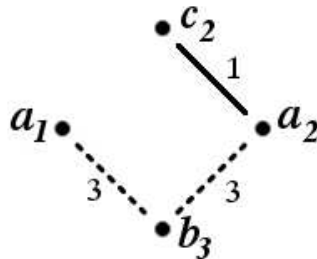
We adjoin a point c_3 with

$$d(c_3, a_1) = 3$$
$$d(c_3, a_2) = 2$$
$$d(c_3, b_3) = d(c_3, c_2) = 1$$

View the resulting configuration as a 2-point amalgamation problem determining the distance $d(b_3, c_2) = 2$. It suffices to check that the factors $(a_1 a_2 b_3 c_3)$ and $(a_1 a_2 c_2 c_3)$ embed into Γ.

The factor $(a_1 a_2 c_2 c_3)$ is isometric with the factor (I) considered above, where the geodesic path of length 2 is (c_3, c_2, a_2).

So we consider the factor

$$(a_1 a_2 b_3 c_3)$$
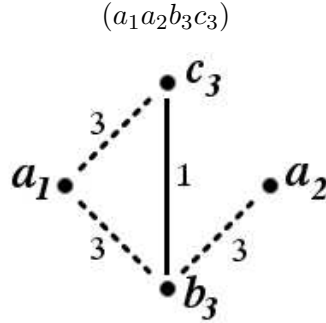
We view this as a 2-point amalgamation problem with the distance $d(a_1, c_3)$ to be determined. Here $d(a_1, c_3) \geq 2$.

If $d(a_1, c_3) = 3$ then we have the desired configuration. Otherwise, we have $d(a_1, c_3) = 2$ and this gives a configuration isomorphic to the earlier configuration $(a_1 a_2 b_3 c_1)$, and the subsequent analysis may be dispensed with. □

Lemma 6.17. *Let Γ be an infinite primitive metrically homogeneous graph of diameter 3 and generic type with $K_1 \leq 2$ and $C \geq 9$. Let $A = \{a\} \cup B$ with*

$$\begin{cases} d(a, x) \in \{1, 3\} & (x \in B) \\ d(x, y) = 2 & (x, y \in B) \end{cases}$$

Suppose that there are at most two points $x \in B$ with $d(a, x) = 3$. Then A embeds in Γ.

Proof. Adjoin a point c adjacent to all points of B, with $d(c, a) = 2$. For $i = 1$ or 3 set
$$A_i = \{x \in B \mid d(a, x) = i\}$$
View $A \cup \{c\}$ as an amalgam of $A_1 \cup \{a, c\}$ with $A_3 \cup \{a, c\}$ over the base $\{a, c\}$. Then the points a, c force
$$d(x, y) = 2$$
for $x \in A_1$, $y \in A_3$. Thus it suffices to check that the two factors embed isometrically into Γ.

Now the factor $A_1 \cup \{a, c\}$ embeds into Γ by the definition of generic type.

We consider the factor $A_3 \cup \{a, c\}$, with $|A_3| \leq 2$. We may suppose $|A_3| = 2$, and then this is the configuration (II) of Lemma 6.16. □

Now we can prove the embedding lemma for $(1, 3)$-stars.

Proof of Lemma 6.15. We suppose $A = \{a\} \cup B$, with all distances in B equal to 2 and all distances between a and B equal to 1 or 3. We may suppose $|B| \geq 3$.

Let $B' = \{b' \,|\, b \in B\}$ be a set of "duplicates" of the points in B. For $b' \in B'$, $x \in A \cup B'$, $x \neq b'$, we define

$$d(b', x) = \begin{cases} 1 & x \in B \setminus \{b\} \\ 3 & x = b \\ 2 & x \in \{a\} \cup B' \setminus \{b'\} \end{cases}$$

If we amalgamate all of the configurations $\{a\} \cup \{b\} \cup B'$ ($b \in B$) over the base $\{a\} \cup B'$, then the elements of B' ensure that all distances in B are equal to 2, and thus the desired configuration is forced. It suffices therefore to check that the configurations

$$\{a, b\} \cup B'$$

all embed isometrically into Γ.

Let $B^* = \{a\} \cup B'$. Then the configurations in question have the form $\{b\} \cup B^*$ where all distances in B^* are equal to 2 and all distances between b and B^* are 1 or 3.

Letting $B_3^* = \{x \in B^* \,|\, d(b, x) = 3\}$, we have $B^* \subseteq \{a, b'\}$. So Lemma 6.17 applies to these configurations.

This completes the construction. □

6.8. **Completion of the proof.**

Now we work in earnest toward the the proof of Proposition 6.1.

We have Γ an infinite primitive metrically homogeneous graph of diameter 3 with parameters $K_1, K_2, C, \mathcal{S}$, satisfying

$$K_1 \leq 2 \text{ and } C \geq 9$$

Our claim is that $\Gamma \cong \Gamma^*$, or equivalently that the following Embedding Principle holds.

> *Any finite Γ-constrained metric space A embeds isometrically into Γ.*

Here we use the following terminology: a finite metric space A is said to be Γ-*constrained* if all triangles in A and all $(1, 3)$-subspaces of A embed into Γ. We have shown that Γ-constraint is equivalent to Γ^*-constraint, which is equivalent to embeddability in Γ^*.

We have proved various very special cases of this principle above. Now we undertake a general inductive argument based on the parameters introduced in §6.3.

First, we minimize the number of triangle types realized in an infinite primitive metrically homogeneous graph Γ which affords some counterexample to the claim, and we keep the isomorphism type of Γ fixed throughout. This means that we may operate under the following standing assumption.

> *Any infinite primitive metrically homogeneous graph of diameter 3 realizing fewer triangle types than Γ is of the form*
>
> $$\Gamma^3_{\tilde{K}_1, \tilde{K}_2, \tilde{C}, \tilde{\mathcal{S}}}$$
>
> *for some choice of admissible parameters $\tilde{K}_1, \tilde{K}_2, \tilde{C}, \tilde{\mathcal{S}}$.*

Then by Proposition 6.11, the parameters $\tilde{K}_1, \tilde{K}_2, \tilde{C}$ associated with Γ_2 equal the parameters K_1, K_2, C associated with Γ. This is all we need to retain about our standing assumption.

Now we consider a counterexample A to our embedding principle for Γ.

Recall from Definition 6.4 that A' is the set of points $x \in A$ for which there are at least two other points $y \in A$ with $d(x, y) \neq 2$. By 6.5, if there is a counterexample to the Embedding Principle for Γ, then there is some counterexample A for which $|A'| \leq 2$. Notice that this then implies that A contains no $(1, 3)$-space of order greater than 2, and therefore the condition of Γ-constraint for A is solely a condition on the triangles in A.

We first consider the case $|A'| = 0$: the $(1, 3)$-edges are pairwise disjoint.

Lemma 6.18. *Let Γ be an infinite primitive metrically homogeneous graph of diameter 3 and generic type with parameters $K_1, K_2, C, \mathcal{S}$. Suppose the following.*

- *$K_1 \leq 2$ and $C \geq 9$.*
- *The parameters $\tilde{K}_1, \tilde{K}_2, \tilde{C}$ associated with Γ_2 agree with K_1, K_2, C.*

Let A be a finite metric space with distances among $1, 2, 3$, and suppose

$$A' = \emptyset$$

Then A embeds isometrically into Γ.

Proof. Since $A' = \emptyset$, no point of A is at distance 1 or 3 from more than one other point of A. So we may write

$$A = A_1 \cup A_3 \cup B$$

where for $i = 1$ or 3 we let A_i be the set

$$\{x \in A \mid \text{There is some } y \in A \text{ with } d(x, y) = i\}$$

Thus A_1 is a union of pairs at distance 1, A_3 is a union of pairs at distance 3, and all other distances are equal to 2.

We proceed by induction on $|A|$. If $B \neq \emptyset$, fix $b \in B$ and let $A^* = A \setminus \{b\}$. By hypothesis the metric space A^* is also Γ_2-constrained. We apply the induction hypothesis to A^* in Γ_2 to embed A^* isometrically in Γ_2. But in A, we have $A^* \subseteq \Gamma_2(b)$, so this embeds A into Γ.

So we may suppose $B = \emptyset$ and

$$A = A_1 \cup A_3$$

is a union of pairs at distance 1 or 3, which are separated by distance 2. Let us denote these pairs (a_i, b_i), with $1 \leq i \leq n$, $|A| = 2n$. We may suppose $n \geq 2$.

We adjoin additional points $\hat{a}_i$ $(1 \leq i \leq n)$ and a^* with

$$d(\hat{a}_i, a_i) = 3$$
$$d(\hat{a}_i, a_j) = 1 \; i \neq j$$
$$d(a^*, a_i) = 1$$

and with all other distances involving a^* or $\hat{a}_i$ equal to 2.

Set $\hat{A}_i = \{\hat{a}_1, \ldots, \hat{a}_n\} \cup \{b_1, \ldots, b_n\} \cup \{a_i, a^*\}$. We claim that the configurations $\hat{A}_i$ all embed into Γ.

This claim is proved inductively, where we replace the set $\{b_1, \ldots, b_n\}$ by any subset of order $k \leq n$ and proceed by induction on k.

The points b_j for $j \neq i$ lie at distance 2 from all other points of $\hat{A}_i$, so by applying the induction hypothesis to Γ_2 (invoking Proposition 6.11), we reduce to the configuration

$$\{\hat{a}_1, \ldots, \hat{a}_n\} \cup \{a_i, b_i, a^*\}$$

This configuration is a $(1,3)$-star with center a_i, so Lemma 6.15 applies.

Since the configurations $\hat{A}_i$ all embed isometrically in Γ, we consider their amalgam. The point a^* ensures $d(a_i, a_j) \leq 2$ while the points $\hat{a}_i$ ensure $d(a_i, a_j) \geq 2$. Therefore the configuration A results. $\square$

Now we may treat the general case.

Proof of Proposition 6.2. We may suppose inductively that every infinite primitive metrically homogeneous graph $\tilde{\Gamma}$ of diameter 3 and generic type with $\tilde{K}_1 \leq 2$, $\tilde{K}_2 \geq 2$, $\tilde{C} \geq 9$, and which realizes fewer triangle types than Γ is of the form

$$\Gamma^3_{\tilde{K}_1, \tilde{K}_2, \tilde{C}, \tilde{\mathcal{S}}}$$

Note that the same conclusion also holds if the parameters of $\tilde{\Gamma}$ do not satisfy the specified constraints, by our previous analysis.

We must embed an arbitrary finite Γ-constrained configuration A into Γ.

We suppose that A is a counterexample, taken so as to minimize, successively, the following.

(1) The cardinality $||A||$ of the largest $(1,3)$-space contained in A.
(2) The size of A'.
(3) The number of nontrivial $(1,3)$-connected components of A.
(4) The size of A.

Claim 6.1.1.

$$1 \leq |A'| \leq 2$$

If $|A'| = 2$ then A' is a pair of points at distance 1 or 3.

By Lemma 6.18, A' is nonempty.

The rest of the claim follows from Lemma 6.5, in view of our assumptions on A.

In particular, there is a unique $(1,3)$-connected component A_0 of A containing A'.

Claim 6.1.2. *Every point of A lies at distance 1 or 3 from at least one other point of A.*

Otherwise, take $a \in A$ at distance 2 from all points of A, and consider $A^* = A \setminus \{a\}$. By the minimality of $|A|$, the inductive assumption on Γ, and Proposition 6.11, the configuration A^* is Γ_2-constrained, hence embeds isometrically into Γ_2 by induction. Thus A embeds isometrically into Γ.

So any $(1,3)$-connected component of A other than A_0 has order 2.
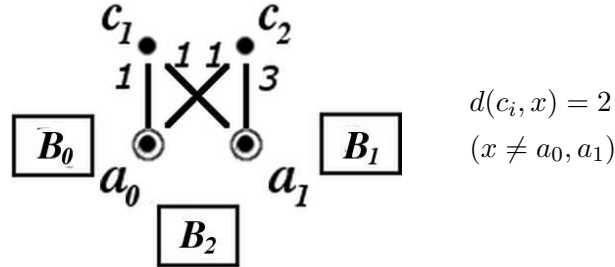
Claim 6.1.3. *A_0 is the only $(1,3)$-connected component of A.*

Suppose A_1 is another $(1,3)$-connected component (a $(1,3)$-edge disjoint from A_0). Take $a_0 \in A' \subseteq A_0$, and $a_1 \in A_1$.
Consider the following amalgamation, where

$$B_0 = A_0 \setminus \{a_0\} \qquad B_1 = A_1 \setminus \{a_1\} \qquad B_2 = A \setminus (A_0 \cup A_1)$$

(so B_1 is a singleton) and c_1, c_2 are additional points.

$$d(c_i, x) = 2$$
$$(x \neq a_0, a_1)$$

The points c_1, c_2 ensure $d(a_0, a_1) = 2$ and thus it suffices to embed the factors of this amalgamation isometrically into Γ.

The factor omitting a_1:

This has fewer non-trivial $(1,3)$-connected components, so embeds in Γ by hypothesis.

The factor F omitting a_0:

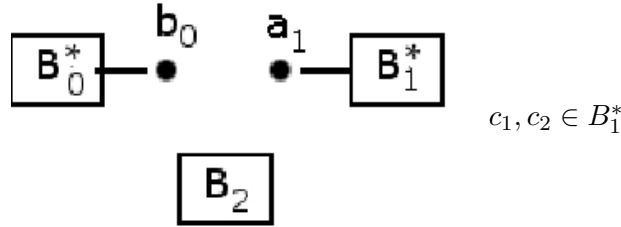Suppose first that

$$|A'| = 1$$

That is, A_0 is a $(1,3)$-star with center a_0.
Then $|F'| = 1$, with a_1 replacing a_0 in the role of center. The number of nontrivial $(1,3)$-connected components is reduced by 1, so by our induction hypothesis F embeds in Γ.
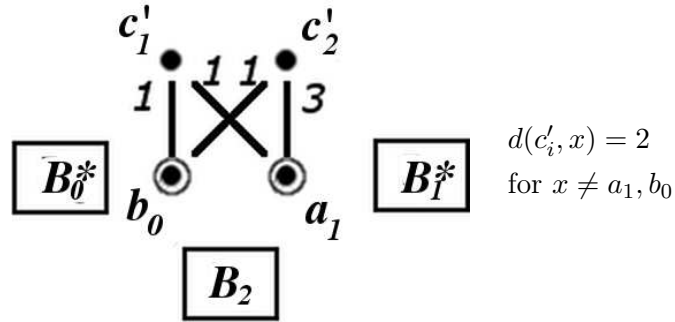
Now suppose
$$|A'| = 2$$

Then $A' = \{a_0, b_0\}$ for some point $b_0 \in B_0$. The structure of the factor F is then as follows, with $B_0^* = B_0 \setminus \{b_0\}$, $B_1^* = B_1 \cup \{c_1, c_2\}$.

$c_1, c_2 \in B_1^*$

We form an amalgamation with two additional points c_1', c_2' as follows.

$d(c_i', x) = 2$
for $x \neq a_1, b_0$

The factors here both have $|A'| = 1$, so we conclude by induction.
This proves the claim in all cases.

The final analysis:

By Claims 2 and 3, $A = A_0$ is $(1, 3)$-connected.
If $|A'| = 1$, then A is a $(1, 3)$-star, and this case is covered by Lemma 6.15. So we suppose $|A'| = 2$, and fix notation as follows.

$$A' = \{a_1, a_2\}$$

By Claim 1, this is a $(1, 3)$-edge.
Let us take A to minimize

$$\min(\deg(a_1), \deg(a_2))$$

where here and below the degree of a point is taken in the sense of the graph G_A introduced earlier, that is, the number of points at distance 1 or 3. We may suppose $\deg(a_2) \leq \deg(a_1)$.

For $i = 1$ or 2 let B_i be the set of $(1, 3)$-neighbors of a_i in $A \setminus A'$. Fix $a \in B_2$.

The configuration consists of two stars with their centers joined by an edge. We represent this schematically as follows.
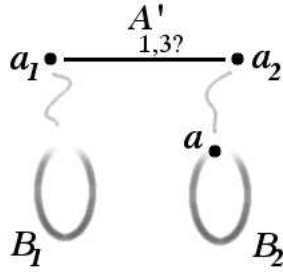
Form a configuration with two auxiliary points c_1, c_2 satisfying

$$d(c_1, a_1) = 1 \qquad\qquad d(c_2, a_1) = 1$$
$$d(c_1, a) = 1 \qquad\qquad d(c_2, a) = 3$$
$$d(c_i, x) = 2 \text{ otherwise}$$

View this configuration as an amalgamation diagram in which the distance $d(a_1, a)$ is to be determined. The auxiliary points c_1, c_2 ensure $d(a_1, a) = 2$. Thus it suffices to embed the factors of this amalgamation diagram isometrically into Γ.

The factor omitting a has smaller $\deg(a_2)$.

The factor F omitting a_1 has a_1, a_2 replaced by (at worst) a, a_2, but again $\deg(a_2)$ is reduced. So either $|F'| < |A'|$, or $F' = \{a, a_2\}$ with

$$\min(\deg(a), \deg(a_2)) < \min(\deg(a_1), \deg(a_2))$$

By induction the factor F embeds isometrically in Γ. $\square$

Now Proposition 6.1 follows from Proposition 6.2, Corollary 4.3, and Remark 4.4.

With this, the classification of the infinite primitive metrically homogeneous graphs of diameter 3 is complete, and as the finite and imprimitive cases are also classified, this proves the Classification Theorem for $\delta = 3$.

7. Conclusion

Theorem 1 supports the conjecture that all metrically homogeneous graphs are known, and are as described in [Che11a] (compare the discussion in §§1.1, 2.3).

Other supporting results are the following.

- The metrically homogeneous graphs which are not of generic type are classified [Che11a, Thm. 10, Lemma 8.6].
- The 3-constrained metrically homogeneous graphs are classified [Che18, Thm. 1.16].
- If the conjecture holds in finite diameter, then it also holds in infinite diameter [Che18, Thm. 1.23].
- Under a suitable inductive hypothesis, the classification conjecture holds for bipartite graphs [Che18, Thm. 1.27].

The methods used to treat the reduction to finite diameter and the bipartite case have some elements in common with methods used here, but the form they take on here is more robust and more promising as a guide toward further progress. Our current view of the natural road forward, in the light of the present work, is given in some detail in §7.5.

A more theoretical source of support for the classification conjecture is discussed in §1.5.2. That line of thinking actually supports a more ambitious conjecture, broader in scope though technically not strictly comparable with the classification conjecture for metrically homogeneous graphs, as we shall see.

7.1. **Broad conjectures.** The line of thought described in §1.5.2 suggested the following, which however turns out to be too strong.

Problem 1. *Let Γ be a homogeneous structure in a finite binary symmetric language and assume that algebraic closure is trivial on Γ (i.e., the associated amalgamation class has strong amalgamation). Does it follow that the age of Γ can be obtained from a 3-constrained amalgamation class $\mathcal{A}$ by imposing Henson constraints?*

To define Henson constraints at this level of generality, we suppose that we have an amalgamation class $\mathcal{A}$, and we fix a set S of 2-types such that every amalgamation diagram can be completed without introducing 2-types from S. In the case of graphs, the set S may be taken to consist either of the type of an edge, or the type of a non-edge; then correspondingly we get the constraints considered by Henson (cliques or independent sets). In the metrically homogeneous case the set S will usually be $\{1, \delta\}$.

The condition of finiteness of the language is certainly necessary. As an example, take the homogeneous structure corresponding to the class of affinely independent finite subsets of Euclidean spaces with rational distances.

The condition of symmetry is also necessary, as is shown by the example of the generic local order.

However even this formulation is refuted by the *semi-generic* structure in a symmetric binary language with one equivalence relation E and an edge relation R with the property that between two pairs of points lying in distinct E-classes, there are an even number of R-edges.

The analogous question remains open in the infinite primitive case.

Problem 2. *Let Γ be a primitive infinite homogeneous structure in a finite binary symmetric language. Does it follow that the age of Γ can be obtained from a 3-constrained amalgamation class $\mathcal{A}$ by imposing Henson constraints?*

So it remains possible that there is a complete and reasonably explicit classification of primitive homogeneous structures for finite binary symmetric languages. There is also an explicit classification conjecture for finite primitive homogeneous structures in binary languages, but of a completely different character.

In the case of metrically homogeneous graphs it is useful to divide the generic type case into primitive and imprimitive cases rather than according to the behavior of algebraic closure, but it is possible that these issues are related.

Conjecture 1. *Let Γ be a homogeneous structure for a finite symmetric binary language. Then*

- *Algebraic closure is unary; that is, it is determined by an equivalence relation with finite classes. In particular:*
- *If Γ is primitive then algebraic closure is trivial.*

7.2. **Other special cases.**

Returning to the case of metrically homogeneous graphs, the work to be done (along current lines) falls into the following areas.

- Special cases (as in the present paper);
- Local analysis (inductive tools);
- Inductive strategies.

There remains one natural special case which one would like to see treated on its own: the antipodal case.

Indeed, prior results mentioned above allow us to limit our attention to generic type graphs of finite diameter, and one would like to deal separately with the imprimitive cases. As the bipartite case has already been treated in an inductive setting, this brings us to the antipodal case.

Problem 3 (Antipodal case)**.** *Let $\delta \geq 4$ be finite. Assume the classification of metrically homogeneous graphs of diameter less than δ is as conjectured. Show that any antipodal graph of finite diameter δ has known type.*

The known antipodal graphs of diameter $\delta \geq 4$ and generic type are denoted

$$\Gamma_{a,n}^{\delta}$$

These are, in an appropriate sense, the generic antipodal graphs omitting the clique K_n; but in this setting the role of the "Henson constraints" is taken on by a set of $(1, \delta - 1)$ spaces.

Indeed, any K_n-free antipodal graph must omit every other graph on n vertices obtained from K_n by replacing some vertices by their antipodal vertices. So we consider the family of metric spaces K_{n_1,n_2} for $n_1 + n_2 = n$, defined as follows. The vertex set of K_{n_1,n_2} has the form $A_1 \cup A_2$ with $|A_1| = n_1$, $|A_2| = n_2$, and the metric d satisfies the following.

$$A_i \cong K_{n_i}, \text{ i.e, } d \equiv 1 \text{ on } A_1 \text{ or } A_2;$$

$$d(a,b) = \delta - 1 \text{ for } a \in A_1,\, b \in A_2.$$

Then $\Gamma_{a,n}^{\delta}$ is defined as the Fraïssé limit of the class of finite antipodal graphs of diameter at most δ which omit all the configurations K_{n_1,n_2} for $n_1 + n_2 = n$. The existence of $\Gamma_{a,n}$ is proved in [Che11a, Theorem 14]. We may set $n = \infty$, or drop the n, to get the generic antipodal graph of diameter δ.

The treatment of the antipodal case appears to involve significant challenges. Because of the antipodal pairing, the relevant embedding theorem involves spaces in which the distance δ does not occur. If δ is odd one may treat this as a problem in local analysis, aiming at the inductive determination of $\Gamma_{\frac{\delta-1}{2}}$.

If δ is even it is less clear how to proceed. Local analysis may be useful if $K_1 < \delta/2$; if $K_1 = \delta/2$ then the only obvious candidate for local analysis is $\Gamma_{\delta/2}$, which however should have the same numerical parameters (and, in theory, be isomorphic to) the original graph Γ. So it would appear that this problem is a large one.

We learned in the present paper that the cases $C = 2\delta + 2$ or $K_1 = \delta$ could be treated together and deserved special attention; as they are in some sense the next cases after the antipodal and bipartite cases, it may be reasonable to single them out as well for separate analysis.

7.3. **Local Analysis.** Much of the further development of the subject along the present lines would involve a considerable extension of the local analysis. In general, local analysis aims to prove that in a metrically homogeneous graph Γ of generic type, the induced metric structures Γ_i have the numerical parameters predicted by the classification (in terms of the numerical parameters associated with Γ). In general, they should still be determined by a combination of constraints on triangles and Henson constraints (most of the latter fall away when the diameter of Γ_i is less than δ).

When $K_1 \leq i \leq K_2$ and the diameter of Γ_i is less than δ, this would then imply that Γ_i is the expected metrically homogeneous graph. It may be desirable to extend the inductive framework to include all the spaces which can arise as Γ_i. The metrically homogeneous graphs are integer valued metric spaces containing geodesic paths of each length; the spaces Γ_i contain the geodesics compatible with their set of distances, which however may not contain the distance 1.

This richer form of local analysis does not yet exist.

Identification of Γ_1, starting from Lachlan/Woodrow classification, was very useful. However when $K_1 > 1$ the structure of Γ_1 is trivial and so in defining "generic type" we were forced to look beyond Γ_1. One possible target for local analysis at a relatively early stage would be a detailed analysis of the possibilities for Γ_2.

Problem 4. *Determine all possible structures for Γ_2 when $\delta \geq 4$, and show that the structure of Γ_2 is determined as expected by the parameters of Γ.*

As $\delta \geq 4$, Γ_2 has diameter 4. There are three cases to consider.

(1) $K_1 \leq 2$: *Γ_2 is a metrically homogeneous graph of diameter 4.*
These must be fully classified, and it must be shown that the numerical parameters and Henson constraints are as predicted for Γ_2 (in particular, if $\delta > 4$, the only Henson constraints should be cliques).

(2) $K_1 = 3$: Γ_2 is a homogeneous metric space with distances $2, 3, 4$ (making the triangle inequality vacuous).

This metric space should contain all triangles not excluded by a perimeter bound in Γ, and no Henson constraints unless $\delta = 4$ and an anti-clique is forbidden.

(3) $K_1 > 3$:

As the distance set in Γ_2 is $\{2, 4\}$, Γ_2 may be viewed as a connected homogeneous graph containing K_∞ (by the local theory and the assumption that Γ is of generic type) and should be generic up to the possible exclusion of an anti-clique (in the case $\delta = 4$).

This requires both some local analysis which has not yet been given, and the analog of our results for $\delta = 4$; so it is a large problem with several different aspects.

Another point of local analysis which plays a fundamental role in the semigroup point of view is the following.

Problem 5. *Let Γ be a primitive metrically homogeneous graph of generic type.*

- *Show that for some i, the induced structure Γ_i contains an edge and has diameter δ.*
- *Show that for any such i, if A, B are two finite structures embedding in Γ then the direct sum $A +_i B$ (in which $d(x, y) = i$ for $x \in A$, $y \in B$) also embeds in Γ.*

If Γ_i contains an edge and has diameter δ then the known results of local analysis show that Γ_i realizes all distances in $[\delta]$, or in other words Γ contains all direct sums $A +_i B$ with $|A| = 1$, $|B| = 2$.

7.4. **General analysis: Admissibility and triangles.** We took pains at certain points to formulate results which apply to our problem in any diameter.

The first point addressed was admissibility. Our treatment depends heavily on local analysis. We proved in Corollary 3.18 that in the case of primitive metrically homogeneous graphs of generic type with $K_1 \leq 2$ or $K_1 \geq \delta$, various inequalities associated with admissibility hold, which under the assumption $C' = C + 1$ yield admissibility of the associated parameters. As a step in the direction of a general proof of admissibility, it would be good to complete the analysis of this case.

Problem 6. *Let Γ be a primitive metrically homogeneous graph of generic type with $K_1 \leq 2$ or $K_1 = \delta$. Show that the associated parameter sequence*

$$(\delta, K_1, K_2, C_0, C_1, \mathcal{S})$$

is admissible.

Under the stated restriction on K_1, what remains to be proved under this heading is the following.

- If $C \leq 2\delta + K_1$ and $C' > C$ then $K_1 = K_2$ and $3K_2 = 2\delta - 1$.
- If $C > 2\delta + K_1$ and $C' > C + 1$ then $C \geq 2\delta + K_2$.

After completing the proof of admissibility for the case $\delta = 3$ our next step was to analyze the triangles embedding in our metrically homogeneous graph. Half of this was carried out at a high level of generality.

Namely, assuming admissibility of parameters, and also that $C' = C + 1$ or $C' \geq 2\delta + K_2$, we showed in Proposition 4.2 that any triangle embedding into Γ is consistent with the associated numerical parameters (in other words, embeds in the associated metrically homogeneous graph). The case omitted here is that in which

$$K_1 = K_2 \text{ and } C = 4K_2 + 1 = 2\delta + K_2 - 1$$

This case arises first with $\delta = 5$ and $K_1 = K_2 = 3$.

The converse to Proposition 4.2, that the triangles which are allowed by the numerical parameters actually do embed into Γ, was handled only under the assumption $\delta = 3$ by entirely ad hoc considerations, in Proposition 4.7.

On the other hand, there is some further information available in general about the realization of triangles. In [Che18] the following result is proved.

Fact 7.1 ([Che18, Lemma 15.19]). *Let Γ be a metrically homogeneous graph of generic type and diameter $\delta \geq 3$. Suppose that $K_1 \leq 2$. Then every triangle of even perimeter at most 2δ embeds into Γ.*

A weaker form of this result which holds without constraints on K_1 is found in [Cou18, Prop. 7].

The proof of Fact 7.1 passes through the following notion.

Definition 7.2. *Let $\mathcal{A}$ be a class of finite integer valued metric spaces of maximum diameter δ. Then $\mathcal{A}$ has the* Interpolation Property *if whenever $\mathcal{A}$ contains triangles of types*

$$(i - 1, j - 1, k) \qquad\qquad (i - 1, j + 1, k)$$

where

$$2 \leq i \leq \delta \qquad\qquad 2 \leq j < \delta \qquad\qquad 1 \leq k \leq \delta$$

then $\mathcal{A}$ contains a triangle of type (i, j, k).

This property holds for classes $\mathcal{A}$ associated with the known metrically homogeneous graphs of generic type, and can be shown to hold generally for metrically homogeneous graphs with $K_1 \leq 2$, and to imply the conclusion of Fact 7.1.

So there is some relevant theory in place with regard to the realization of triangles, which needs to be developed considerably farther, and appears to call for more local analysis.

7.5. **Inductive Strategies.** The most delicate part of the argument in the present paper comes in the proof of the embedding theorems, Propositions 5.3 and 6.2. These two propositions together may be reformulated as follows, using the notation of Definition 3.20, which itself depends on Proposition 3.19 (admissibility).

Proposition 7.3. *Let Γ be a primitive metrically homogeneous graph of generic type and diameter 3. Then any finite metric space which embeds isometrically in Γ^* embeds isometrically into Γ.*

If we relax the restriction on the diameter then the natural formulation of the result would be the following conjecture.

Conjecture 2. *Let Γ be a metrically homogeneous graph of generic type and finite diameter, for which the sequence of associated parameters*

$$(\delta, K_1, K_2, C, C', \mathcal{S})$$

is admissible. Then any finite metric space which embeds isometrically in Γ^* embeds isometrically into Γ.

Remark 7.4. *We do not include the assumption of primitivity in the statement of Conjecture 2. In an inductive framework, the bipartite case has already been treated. The antipodal case was quickly set aside in the present paper, as it was treated very directly and rapidly at the outset. One would anticipate that in larger diameter, the antipodal case would require much the same methods as the primitive case.*

Prior work reduces the case of infinite diameter to the case of finite diameter.

This conjecture is of course a consequence of the classification conjecture for metrically homogeneous graphs; conversely, an inductive strategy for a proof of this conjecture would be a significant contribution toward a treatment of the classification conjecture in general.

We would not actually envision a direct proof of this conjecture; it is more natural to put it in an inductive setting, so that any metrically homogeneous graph simpler than Γ in any reasonable sense (for example, of smaller diameter) would be assumed to be of known type.

While one must also treat the admissibility of the parameters, and the embeddability of Γ into Γ^*, points which were handled easily but in an ad hoc matter for the case of diameter 3, Conjecture 2 appears to be the most challenging of the three points to be dealt with, and is certainly the one which the present paper sheds the most light on. So this is what we will discuss at present.

We may stratify the problem as follows.

Definition 7.5. *Let Γ, Γ^* be metric structures and $d \geq 1$.*

Then $\mathcal{E}_d(\Gamma^*, \Gamma)$ is the following embedding property.

> Any finite metric substructure of Γ^* of diameter at most d
> embeds into Γ.

Conjecture 2 reduces to the statement that $\mathcal{E}_\delta(\Gamma^*, \Gamma)$ holds when Γ, Γ^* satisfy the stated conditions.

In our proof of the embedding property for the case $\delta = 3$ we focus almost entirely on the case $d = \delta$, but one may observe some use of the case $d = \delta - 1$ in the proof. We feel that the following more specialized conjecture receives concrete support from the work we have done here.

Conjecture 3 (Embedding Bootstrap). *Let Γ be metrically homogeneous of generic type and finite diameter δ, with admissible associated parameters*

$$\delta, K_1, K_2, C, C', \mathcal{S}$$

and let $\Gamma^* = \Gamma^\delta_{K_1, K_2, C, C', \mathcal{S}}$. Set $d = \max(K_1, \delta/2)$. If $\mathcal{E}_d(\Gamma^*, \Gamma)$ holds, then $\mathcal{E}_\delta(\Gamma^*, \Gamma)$ holds.

What we envision here is a proof by induction on d with the specified value of d as the base case, following the general approach used in the case $\delta = 3$, which can be viewed as a derivation of the embedding property for $d = \delta$ from the case $d = \delta - 1$. For $d < \delta$ such an argument would reduce the problem to the case of $(1, d)$-spaces. As these are not Henson constraints (other than in the case of antipodal graphs with $d = \delta - 1$), one further step in the argument would be needed to treat this case.

To see why the arguments we have used have this potential, one needs to understand why we choose the value $\max(K_1, \delta/2)$ as the base case; and we should also say something about how the base case is to be treated. So we take up these points next.

In our treatment of the embedding property the main point is that distances between 1 and d can be witnessed by geodesic paths in ways that do not add much complexity to the configuration, according to our way of measuring complexity, and even contribute eventually to its simplification. Our constructions were given explicitly, but in a more general setting one has to consider an amalgamation problem of a fairly general type, in which a given configuration is extended by a pair of geodesic paths, and then completed to a $[d]$-metric space, *without introducing any further pairs at distance 1 or d.* We state the relevant amalgamation lemma, a little more generally than is required.

Here we rely on the ideas of §1.5.1 and in particular on Definition 1.4.

Lemma 7.6. *Let Γ^* be a known metrically homogeneous graph of generic type and let A be a finite partial substructure of Γ^*, of diameter*

$$k > \max(\delta/2, K_1)$$

If Γ^* is antipodal, assume that $k < \delta$.

If $C > 2\delta + 2$ or $k < \delta$ then A has a completion $\hat{A}$ which embeds into Γ^*, for which all added distances $d(a, b)$ satisfy

$$1 < d(a, b) < k$$

If $C = 2\delta + 2$ and $k = \delta$ then we can impose the same condition

$$1 < d(a, b) < k$$

on the new distances, except where there is a point c at distance δ from both a and b.

We only require this in the case in which A can be interpreted as an amalgamation problem, but the most efficient proof uses the "shortest path completion" method described in §1.5.1, where the "magic" parameter is set equal to $\max(K_1, \delta/2)$.

As far as achieving the base case is concerned, this is most accessible when we have the following condition.

$$K_1 < \delta/2$$

Then we can hope to apply local analysis to Γ_i with

$$i = \lfloor (\delta - 1)/2 \rfloor$$

as Γ_i will be a metrically homogeneous graph of diameter $\delta' = \delta - 1$ or $\delta - 2$ in this case. We may suppose an inductive hypothesis applies, so that $\mathcal{E}_{\delta'}((\Gamma_i)^*, \Gamma_i)$ holds by induction. Then via local analysis one should be able to derive $\mathcal{E}_{\delta'}(\Gamma^*, \Gamma)$, though the relevant local analysis has not yet been carried through. As $\delta' \geq K_1, \delta/2$, we then have $\mathcal{E}_d(\Gamma^*, \Gamma)$ for $d = \max(K_1, \lfloor \delta/2 \rfloor)$, as desired.

When $K_1 \geq \delta/2$, the base case for our inductive approach is the property $\mathcal{E}_{K_1}(\Gamma^*, \Gamma)$. There are prospects for a separate inductive treatment of this particular case along other lines, not involving a reduction to $\mathcal{E}_{K_1 - 1}(\Gamma^*, \Gamma)$. In the diameter 3 case, this issue arises briefly in connection with the treatment of the case $K_1 = 2$, but not in a way that points toward a particular inductive approach.

A long-standing unresolved issue is the treatment of the antipodal case under an inductive hypothesis. Here one has good prospects for treating admissibility and the "easy" direction of embedding, at which point the methods described here become applicable; then most cases fall on the easier side (namely, $K_1 < \delta/2$). The exceptional case has δ even and $K_1 = \delta/2$, and so one needs a direct proof of $\mathcal{E}_{K_1}(\Gamma^*, \Gamma)$ in this particular case, which may be easier than the corresponding statement in general.

7.6. **Strong Amalgamation.** One may also head in a very different direction, namely one may look for a proof of the following.

Conjecture 4. *Suppose that Γ is a metrically homogeneous graph of generic type and finite diameter which does not have an $\emptyset$-definable equivalence relation with finite classes, and that Γ satisfies the inductive hypothesis*

- *Any metrically homogeneous graph of diameter strictly less than that of Γ is of known type.*

Then the associated amalgamation class has strong amalgamation.

Rather than aiming at more concrete cases of the classification conjecture, this aims to clarify the relationship of that conjecture with the ideas of §1.5.2.

In the analysis one may assume that Γ is of generic type. As in the case of the classification conjecture itself, one encounters the problem that induction does not apply to all structures of the type Γ_i, and hence it may be necessary to broaden the statement so as to include them in the induction hypothesis. It is not clear whether this problem is amenable to a more direct approach than the full classification problem, but it is worth investigating.

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