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Boundary delayed observer-controller design for reaction-diffusion systems

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Abstract—We consider delayed boundary stabilization of a 1D reaction-diffusion equation under boundary delayed measurements. We design an observer-based control law via the modal decomposition approach. The observer is governed by a PDE which leads to separation of the observer and the controller design. We suggest a network-based implementation of the controller in the presence of two networks: from sensor to controller, and from the controller to actuator. To reduce the workload of the second network, we suggest a novel switching-based dynamic event-triggering mechanism. We extend the results to the vector case and illustrate their efficiency by a numerical example.

I. INTRODUCTION

Sampled-data and delayed control of PDEs is becoming an active research area. General results on sampled-data control of PDEs were presented in [1]. Constructive conditions in terms of linear matrix inequalities (LMIs) for sampled-data and delayed control of PDEs that are applicable to the performance (e.g. exponential decay rate) analysis have been initiated in [2]–[5]. However, these results are confined to distributed control of parabolic PDEs.

Boundary controllers for PDEs may be designed by modal decomposition technique [6] or by backstepping approach [7]. Sampled-data and delayed implementation of such controllers is a challenging direction. For linear systems of conservation laws, event-triggered boundary control was suggested in [8]. State-feedback sampled-data boundary controllers for 1-D linear transport and heat equations were introduced recently in [9] and [10]. In [11], modal decomposition technique was combined with a predictor to compensate a constant delay in the boundary state-feedback controller for heat equation. However, the existing results on boundary sampled-data or delayed control are confined to state-feedback case.

In the present paper we introduce an observer-based boundary control of 1D reaction-diffusion equation under boundary measurements in the presence of input and output time-varying delay (that may include data sampling). We develop modal decomposition approach. Note that finite-dimensional boundary observer for such systems that is applicable under boundary Dirichlet actuation was recently constructed in [12]. Differently from [12] we consider PDE observer that allows for separation of the observer and controller design, and we study a general Robin actuation.

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We study a network-based implementation of the controller in the presence of two networks: from sensor to controller, and from the controller to actuator. To reduce the workload of the second network, we suggest a novel event-triggering (ET) mechanism. There exist two main approaches to ET design: static ET (see e.g. [13]) and dynamic ET (see e.g. [8], [14]). Both approaches may lead to Zeno behaviour. To avoid this, a static switching-based ET approach was introduced in [15]. In this paper, we propose a novel dynamic switching-based ET mechanism that avoids Zeno behaviour. Furthermore, we present an extension of the results to the vector case.

A conference version of the present paper (see [16]) was focused on the network-based controller design for scalar systems. Here we consider a more general case of time-varying delays, where we have only weak solutions, and we need more involved proofs. Moreover, we present a vector extension of the results with a vector example from [17].

II. NOTATIONS AND PRELIMINARIES

Throughout the paper $\mathbb{R}^n$ denotes the n dimensional Euclidean space with the vector norm $|v|$ for $v \in \mathbb{R}^n$, $\mathbb{R}^{n \times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathbb{R}^{n \times n}$ means that P is symmetric and positive definite, whereas $|v|_P^2 := v^T P v$.

We denote by $H^k(0,1)$ the Sobolev space of index k with the norm $\|f\|_{H^k(0,1)} := \sqrt{\sum_{j=0}^{k} \left\| f^{(j)} \right\|_{L^2(0,1)}^2}$.

Lemma 1: [18] (Wirtinger's inequality) If $f \in H^1(0,1)$ satisfies $f(0) = 0$ or $f(1) = 0$, then

$$\|f\|_{L^2(0,1)} \leq \frac{2}{\pi} \|f'\|_{L^2(0,1)}. \tag{1}$$

Lemma 2: [18] (Sobolev's inequality) If $f \in H^1(0,1)$, then for some constant $M > 0$

$$\max_{x \in [0,1]} |f(x)| \leq M \|f\|_{H^1(0,1)}. \tag{2}$$

Recall that for a regular Stürm-Liouville eigenvalue problem

$$\phi'' + \lambda \phi = 0, \quad 0 < x < 1, \tag{3}$$
$$\phi'(0) = 0, \quad \gamma_1 \phi(1) + \gamma_2 \phi'(1) = 0,$$

with

$$|\gamma_1| + |\gamma_2| > 0 \tag{4}$$

there exists a sequence of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ with corresponding eigenfunctions $\{\phi_n\}_{n=1}^{\infty}$. Moreover, the eigenvalues form an unbounded, monotone increasing sequence and the eigenfunctions are a complete orthonormal system in $L^2(0,1)$ (see [19]). Throughout this paper we assume that (4) is valid.

III. Delayed Observer-Controller Design

A. Problem formulation and well-posedness.

Consider the reaction-diffusion system

$$z_t = z_{xx} + qz, \quad t \geq 0, \tag{5}$$
$$z_x(0,t) = 0, \quad \gamma_1 z(1,t) + \gamma_2 z_x(1,t) = u(t - \tau_u(t)),$$

where $x \in (0,1)$, $\gamma_1^2 + \gamma_2^2 \neq 0$. Here $z(x,t) \in \mathbb{R}$ is the state, $u(t) \in \mathbb{R}$ is the control input, $\tau_u(t)$ is the input delay and $q \in \mathbb{R}$ is the reaction coefficient. We assume $u \equiv 0$ for $t - \tau_u(t) < 0$. We consider the boundary measurements

$$y(t) = z(0, t - \tau_y(t)), \quad t \geq 0 \tag{6}$$

with the measurement delay $\tau_y(t)$, and assume $y \equiv 0$ for $t - \tau_y(t) < 0$.

In this paper we treat two classes of input and output delays: absolutely continuous delays and sawtooth delays that correspond to network-based control. We assume that the delays τ_y and τ_u are known and are upper-bounded by τ_M:

$$\tau_u(t) \leq \tau_M, \quad \tau_y(t) \leq \tau_M \quad \forall t \geq 0.$$

For the case of absolutely continuous delays, we assume that $\tau_u(t)$ and $\tau_y(t)$ are lower bounded by $\tau_m > 0$. This assumption is employed for well-posedness only. As in [20], we assume that there exists a unique $t_* \in [\tau_m, \tau_M]$ such that

$$t - \tau(t) < 0, t < t_*, \quad t - \tau(t) \geq 0, t \geq t_*,$$

for any $\tau(t) \in \{\tau_u(t), \tau_y(t)\}$.

In the case of sawtooth delays, we consider a network-based control in the presence of two networks: from sensor to controller and from controller to actuator (see Figure 1). Denote the observer updating instances by s_k, where $0 = s_0 < s_1 < s_2 < s_3 < \cdots$, $\lim_{k \to \infty} s_k = \infty$. Let $\rho_k, k \geq 0$ be the transmission delays between the sensor and the controller. Similarly, denote the controller updating instances by $t_k, \; k = 0, 1, \ldots$, where $0 = t_0 < t_1 < t_2 < t_3 < \cdots$, $\lim_{k \to \infty} t_k = \infty$. Let $\mu_k, \; k = 0, 1, \ldots$ be the transmission delays between the controller and the actuator. We allow the transmission delays to be larger than the corresponding sampling intervals provided that the updating sequences $\{s_k + \rho_k\}$ and $\{t_k + \mu_k\}$ remain increasing. Moreover, we assume that $s_{k+1} - s_k, t_{k+1} - t_k \leq \text{MATI}, \; k = 0, 1, \ldots$, where MATI is maximum allowable transmission interval. Similarly, $\rho_k, \mu_k \leq \text{MAD}, \; k = 0, 1, \ldots$, where MAD is maximum allowable delay. The resulting NCS has a piecewise-constant control input implemented by zero-order hold device. By using the time-delay approach to NCSs [21], [22], the resulting control input can be modeled as a delayed one with $\tau_u(t) = t - t_k, \; t \in [t_k + \mu_k, t_{k+1} + \mu_{k+1})$ and $u(t - \tau_u(t)) = 0$ for $t \in [0, t_0 + \mu_0)$. Similarly, the measurements on the controller side are modeled using $\tau_y(t) = t - s_k, \; t \in [s_k + \rho_k, s_{k+1} + \rho_{k+1})$ with $y(t) = 0$ for $t \in [0, s_0 + \rho_0]$. The resulting delays τ_y and τ_u are upper-bounded by $\tau_M := MATI + MAD$.

We assume that the sampling instances as well as transmission delays are known, and there are no packet dropouts. The assumption about known ρ_k is valid e.g. when the measurements are sent together with the time-stamps. The assumption about known μ_k is less realistic, but it may be
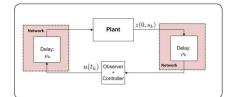

Fig. 1. Network-based control.

reasonable e.g. in the case of constant $\mu_k \equiv \mu$. Note that unknown delays do not allow separation of the controller and observer design, which is crucial in our approach.

For well-posedness, we start with sawtooth delays. Assume $z_0 \in L^2(0,1)$. By [23, Theorem 1], there exists a unique weak solution $z \in C([0, t_0 + \mu_0], L^2(0,1))$ to (5) with $u(t - \tau_u(t)) = 0$ for $t \in [0, t_0 + \mu_0]$. For $t \in [t_0 + \mu_0, t_1 + \mu_1]$, let

$$w(x,t) = z(x,t) - r(x)u(t_0), \tag{7}$$

where $r(x)$ is a quadratic polynomial satisfying $r'(0) = 0$ and $\gamma_1 r(1) + \gamma_2 r'(1) = 1$. Then

$$w_t = w_{xx} + qw + f(x,t),$$
$$w_x(0,t) = 0, \quad \gamma_1 w(1,t) + \gamma_2 w_x(1,t) = 0, \tag{8}$$
$$f(x,t) = r''(x)u(t_0) + qr(x)u(t_0).$$

Since $f \in L^1([t_0 + \mu_0, t_1 + \mu_1], L^2(0,1))$, [23, Theorem 1] implies that there exists a unique weak solution $w \in C([t_0 + \mu_0, t_1 + \mu_1], L^2(0,1))$ of (8) such that $w(\cdot, t_0 + \mu_0) = z(\cdot, t_0 + \mu_0) - ru(t_0) \in L^2(0,1)$ with $z(\cdot, t_0 + \mu_0)$ obtained on $[0, t_0 + \mu_0]$. Then $z = w + ru(t_0) \in C([t_0 + \mu_0, t_1 + \mu_1], L^2(0,1))$ is the weak solution of (5) on $[t_0 + \mu_0, t_1 + \mu_1]$. Using the same argument step by step on $[t_k + \mu_k, t_{k+1} + \mu_{k+1}]$ $(k = 1, 2, \ldots)$ with the initial condition $z(\cdot, t_k + \mu_k) \in L^2(0,1)$ obtained on the previous interval, we establish the existence of a unique weak solution $z \in C([0, \infty), L^2(0,1))$ for $z_0 \in L^2(0,1)$. Below, we take $z_0 \in H^2(0,1)$, which implies $z(\cdot, t) \in H^2(0,1)$ for $t \geq 0$ (see, e.g [24]).

For the case of absolutely continuous delays, the existence of a unique weak solution $z \in C([0, \infty), L^2(0,1))$ can be obtained using similar arguments provided $u(t)$ is an absolutely continuous function. The control law proposed hereafter will result in absolutely continuous $u(t)$.

Our objective is stabilization of (5) by using observer-based controller. In this paper we will develop a modal decomposition approach for observer-controller design (for the state-feedback sampled-data control, this approach was introduced in [24]).

B. Boundary observer design.

We construct an observer in the form of a PDE

$$\hat{z}_t = \hat{z}_{xx} + q\hat{z} - \mathcal{L}(x)[y(t) - \hat{z}(0, t - \tau_y(t))], \quad t \geq 0,$$
$$\hat{z}_x(0,t) = 0, \quad \gamma_1 \hat{z}(1,t) + \gamma_2 \hat{z}_x(1,t) = u(t - \tau_u(t)), \tag{9}$$

where $\hat{z}(x,t) = 0$ for $t \leq 0$ and $\mathcal{L}(x)$ is given by

$$\mathcal{L}(x) := \sum_{n=1}^{N} l_n \phi_n(x), \tag{10}$$

with ϕ_n satisfying (3) and some scalar gains $\{l_n\}_{n=1}^N$. For the estimation error $e(x,t) := z(x,t) - \hat{z}(x,t)$, we obtain the PDE

$$e_t(x,t) = e_{xx}(x,t) + qe(x,t) + \mathcal{L}(x)e(0, t - \tau_y(t)), \quad t \geq 0,$$
$$e_x(0,t) = 0, \quad \gamma_1 e(1,t) + \gamma_2 e_x(1,t) = 0 \qquad (11)$$

with initial condition $e(x,0) = z_0(x)$. The PDE (11) doesn't depend on the control law. This leads to separation of the observer and controller design. Note that the finite-dimensional observer suggested in [12] for the case of homogeneous boundary conditions does not allow such separation in the case of boundary control. The existence of a unique weak solution of (11) is established by the same arguments used for well-posedness of (5).

The solution to (11) can be presented as

$$e(\cdot,t) = \sum_{n=1}^\infty e_n(t)\phi_n(\cdot), \qquad (12)$$

where $e_n(t) = \langle e(\cdot,t), \phi_n \rangle$. By differentiating under the integral sign and substituting (11) we obtain

$$\dot{e}_n(t) = \int_0^1 e_t(x,t)\phi_n(x)dx \qquad (13)$$
$$= [e_x(1,t)\phi_n(1) - e(1,t)\phi_n'(1)] - (\lambda_n - q)e_n(t)$$
$$+ e(0, t - \tau_y(t)) \sum_{k=1}^N l_k \int_0^1 \phi_k(x)\phi_n(x)dx.$$

Since both $\phi_n(x)$ and $e(x,t)$ satisfy homogeneous boundary conditions for $x = 1$, the following holds:

$$e_x(1,t)\phi_n(1) - e(1,t)\phi_n'(1) = \det \begin{bmatrix} \phi_n(1) & \phi_n'(1) \\ e(1,t) & e_x(1,t) \end{bmatrix} = 0. \qquad (14)$$

Moreover, by orthonormality of $\{\phi_n\}_{n=1}^\infty$,

$$\sum_{k=1}^N l_k \int_0^1 \phi_k(x)\phi_n(x)dx = l_n \cdot \begin{cases} 1, & n \leq N, \\ 0, & \text{otherwise.} \end{cases} \qquad (15)$$

Combining (15) and (14) we obtain the following ODEs

$$\dot{e}_n(t) = -(\lambda_n - q)e_n(t) + l_n e(0, t - \tau_y(t)) , \ n \leq N$$
$$\dot{e}_n(t) = -(\lambda_n - q)e_n(t) , \ n > N. \qquad (16)$$

Let $\delta > 0$ be a desired decay rate. Since $\lim_{n\to\infty} \lambda_n = +\infty$, there exists $N \in \mathbb{N}$ such that

$$-\lambda_n + q < -\delta , \ \forall n > N. \qquad (17)$$

The required value of N can be estimated using equations (18) and (19) in [25]. For the case of Neumann or Dirichlet boundary condition, the eigenvalues and the corresponding N can be obtained explicitly.

Define

$$\zeta(x,t) := \sum_{n=N+1}^\infty e_n(t)\phi_n(x), \quad t \geq 0. \qquad (18)$$

By Parseval's equality, (17) and (11),

$$\|\zeta(\cdot,t)\|_{L^2(0,1)}^2 = \sum_{n=N+1}^\infty e^{2(-\lambda_n+q)t} |e_n(0)|^2$$
$$\leq e^{-2\delta t} \|z_0\|_{H^2(0,1)}^2, \quad t \geq 0. \qquad (19)$$

For $n \leq N$, we rewrite the ODEs in (16) as a system

$$\dot{e}^N(t) = Ae^N(t) + LCe^N(t - \tau_y(t)) + L\zeta_0(t - \tau_y(t)),$$
$$e^N(t) = \begin{bmatrix} e_1(t) \dots e_N(t) \end{bmatrix}^T, \qquad (20)$$

where

$$A = \text{diag}\left(-\lambda_1 + q, ..., -\lambda_N + q\right) \in \mathbb{R}^{N \times N},$$
$$L = \begin{bmatrix} l_1 \dots l_N \end{bmatrix}^T \in \mathbb{R}^{N \times 1}, \quad C = \begin{bmatrix} \phi_1(0) \dots \phi_N(0) \end{bmatrix}, \quad (21)$$
$$\zeta_0(t) = e(0,t) - \sum_{n=1}^N e_n(t)\phi_n(0).$$

It can be verified that $\phi_j(0) \neq 0$ for all $j \in \mathbb{N}$ and all values of $\gamma_1, \gamma_2 \in \mathbb{R}$. Therefore, since A is diagonal with non-zero diagonal entries, the pair (A, C) is observable by the Hautus lemma. Thus, for any $\delta > 0$, a vector L can be chosen such that there exists $P \in \mathbb{R}^{N \times N}$, $P > 0$ which satisfies the Lyapunov inequality

$$P(A + LC) + (A + LC)^T P < -2\delta P. \qquad (22)$$

In order to prove exponential convergence of the error equation (11), we will show that $\zeta_0(t - \tau_y(t))$ decays exponentially with a decay rate δ. The proof suggested in [12] can be applied only to the case of Dirichlet boundary condition on $x = 1$, where $z_0 \in H^1(0,1)$ with $z_0(1) = 0$. For the case of Robin boundary condition on $x = 1$, the proof in [12] cannot be applied since the eigenfunctions of (3) do not vanish on the boundaries.

Lemma 3: Assume that $z(\cdot,0) := z_0(\cdot) \in H^2(0,1)$ satisfies

$$\partial_x z_0(0) = \gamma_1 z_0(1) + \gamma_2 \partial_x z_0(1) = 0. \qquad (23)$$

Then there exists a constant $M_\zeta > 0$ such that

$$|\zeta_0(t - \tau_y(t))| \leq M_\zeta \exp(-\delta t) \|z_0\|_{H^2(0,1)}, t > 0. \qquad (24)$$

Proof: We first show that $\zeta(\cdot,t) \in H^2(0,1)$, $t > 0$, where ζ is given by (18). Let

$$g(x,t) := - \sum_{n=N+1}^\infty \lambda_n e_n(t)\phi_n(x), t > 0. \qquad (25)$$

Since $z_0 \in H^2(0,1)$ satisfies (23) we obtain

$$\|g(\cdot,t)\|_{L^2(0,1)}^2 = \sum_{n=N+1}^\infty \lambda_n^2 \exp(2(-\lambda_n + q)t) |e_n(0)|^2$$
$$\leq \exp(-2\delta t) \|z_0''\|_{L^2(0,1)}^2 \leq \exp(-2\delta t) \|z_0\|_{H^2(0,1)}^2. \qquad (26)$$

Therefore, $g(\cdot,t) \in L^2(0,1)$ for all $t > 0$. Let $v \in C_0^\infty(0,1)$ be a smooth, compactly supported function. After integrating by parts we find for any $m > N + 1$

$$\int_0^1 \left(\sum_{n=N+1}^m e_n(t)\phi_n(x) \right) v''(x)dx \qquad (27)$$
$$= \int_0^1 \left(- \sum_{n=N+1}^m \lambda_n e_n(t)\phi_n(x) \right) v(x)dx.$$

Taking $N \to \infty$, using the dominated convergence theorem, (18), (25) and (27) we find that $\zeta(\cdot,t) \in H^2(0,1)$ and $\zeta_{xx}(\cdot,t) = g(\cdot,t)$, $t > 0$.

Since $\phi'_n(0) = 0$, we have $|\phi'_n(x)| \leq \lambda_n^{\frac{1}{2}}$. Moreover, $\pi^2(n-1)^2 \leq \lambda_n \leq \pi^2 n^2$ [25]. Then for some $M > 0$

$$|\sum_{n=N+1}^{\infty} e_n(t)\phi'_n(x)| \leq M \sum_{n=N+1}^{\infty} \exp(-\lambda_n t)\lambda_n^{\frac{1}{2}} < \infty,$$

where the last inequality follows from the root test. Thus, (18) can be differentiated term by term in x (for each $t > 0$), leading to $\zeta_x(0, t) = 0$. By Wirtinger's inequality and (26)

$$||\zeta_x(\cdot, t)||^2_{L^2(0,1)} \leq \frac{4}{\pi^2} e^{-2\delta t} ||z_0||^2_{H^2(0,1)}, \quad t > 0. \quad (28)$$

By the Sobolev inequality (2),

$$|\zeta_0(t)|^2 = |\zeta(0, t)|^2 \leq M^2 ||\zeta(\cdot, t)||^2_{H^1(0,1)}. \quad (29)$$

By combining (29), (28) and (19) we obtain the result in the case $\tau_y(t) \equiv 0$. For $\tau_y(t) \geq 0$, we have

$$|\zeta_0(t - \tau_y(t))| \leq M_1 \exp(\delta\tau_M) \exp(-\delta t) ||z_0||_{H^2(0,1)}, \ t > 0,$$

which implies (24) with $M_\zeta = M_1 \exp(\delta\tau_M)$. ∎

Remark 1: For the case of $u \equiv 0$, the finite-dimensional observer of [12] can be used for Robin boundary conditions provided $z_0 \in H^2(0, 1)$.

Proposition 1: Consider the error system (11). Given $\delta > 0$, let N satisfy (17), $L = \begin{bmatrix} l_1 \dots l_N \end{bmatrix}^T \in \mathbb{R}^{N \times 1}$ satisfy (22), $z_0(\cdot) \in H^2(0, 1)$ satisfy (23) and $\mathcal{L}(x)$ be defined by (10). Let there exist matrices $P_2, P_3, G \in \mathbb{R}^{N \times N}$ and positive-definite matrices $P, S, R \in \mathbb{R}^{N \times N}$ such that

$$\Phi < 0 \text{ and } \begin{bmatrix} R & G \\ * & R \end{bmatrix} \geq 0,$$

where $\Phi = \{\Phi_{ij}\}$ is the symmetric matrix composed from

$$\Phi_{11} = A^T P_2 + P_2^T A + 2\delta P + S - e^{-2\delta\tau_M} R,$$
$$\Phi_{12} = P - P_2^T + A^T P_3, \ \Phi_{13} = e^{-2\delta\tau_M}(R - G) + P_2^T LC$$
$$\Phi_{14} = e^{-2\delta\tau_M} G, \ \Phi_{22} = -P_3 - P_3^T + \tau_M^2 R$$
$$\Phi_{23} = P_3^T LC, \ \Phi_{24} = 0, \ \Phi_{33} = -e^{-2\delta\tau_M}(2R - G - G^T)$$
$$\Phi_{34} = e^{-2\delta\tau_M}(R - G), \ \Phi_{44} = -e^{-2\delta\tau_M}(S + R) \quad (30)$$

with A, C from (21). Then there exists $M_e > 0$ such that

$$||e(\cdot, t)||_{L^2(0,1)} \leq M_e \exp(-\delta t) ||z_0||_{H^2(0,1)}, \quad t \geq 0,$$
$$|e^N(t)| \leq M_e \exp(-\delta t) ||z_0||_{H^2(0,1)}, \quad (31)$$

where e^N is defined in (20).

Proof: The proof is similar to proof of Theorem 1 in [12]. For $n \leq N$, the LMIs (30) guarantee ISS stability of (20) with respect to input ζ_0 (cf. Proposition 4.3 of [21]). Together with (24) this implies the second inequality of (31).

For $n \geq N + 1$ (16) gives

$$e_n(t) = \exp(-(\lambda_n - q) t) e_n(0).$$

By using Parseval's equality and (17), we find

$$\sum_{n=N+1}^{\infty} |e_n(t)|^2 \leq \tilde{M}_e \exp(-2\delta t) ||z_0||^2_{H^2(0,1)}.$$

By combining both bounds, we obtain the result. ∎

C. Control law design.

For the exponential stabilization of (5) by using observer (9), it is sufficient to achieve ISS of the observer PDE

$$\hat{z}_t = \hat{z}_{xx} + q\hat{z} - \mathcal{L}(x)e(0, t - \tau_y(t)), \quad t \geq 0,$$
$$\hat{z}_x(0, t) = 0, \gamma_1 \hat{z}(1, t) + \gamma_2 \hat{z}_x(1, t) = u(t - \tau_u(t)), \quad (32)$$

where $\hat{z}(x, 0) = 0$, $x \in [0, 1]$ and the input $e(0, t - \tau_y(t))$ is exponentially decaying:

$$|e(0, t - \tau_y(t))| = |Ce^N(t - \tau_y(t)) + \zeta_0(t - \tau_y(t))|$$
$$\leq M \exp(-\tilde{\delta}t) ||z_0||^2_{H^2(0,1)} \quad \forall t \geq 0, \quad \tilde{\delta} > \delta. \quad (33)$$

Here M and $\tilde{\delta}$ are some constants. Inequality (33) follows from Lemma 3 and Proposition 1, where (due to strict inequalities in LMIs) (33) holds with some $\tilde{\delta} > \delta$.

We represent the solution of (32) as

$$\hat{z}(x, t) = \sum_{n=1}^{\infty} \hat{z}_n(t)\phi_n(x). \quad (34)$$

By Parseval's equality

$$||\hat{z}(\cdot, t)||^2_{L^2(0,1)} = \sum_{n=1}^{N} |\hat{z}_n(t)|^2 + \sum_{n=N+1}^{\infty} |\hat{z}_n(t)|^2. \quad (35)$$

By performing calculations similar to (13) we obtain the following ODEs for $\hat{z}_n(t)$

$$\dot{\hat{z}}^N(t) = A\hat{z}^N(t) + Bu(t - \tau_u(t)) - Le(0, t - \tau_y(t)),$$
$$\hat{z}^N(t) = \begin{bmatrix} \hat{z}_1(t) \dots \hat{z}_N(t) \end{bmatrix}^T \in \mathbb{R}^N \quad (36)$$

and

$$\dot{\hat{z}}_n(t) = -(\lambda_n - q)\hat{z}_n(t) + b_n u(t - \tau_u(t))$$
$$\hat{z}_n(0) = 0, \quad n > N.$$

Here A is defined in (21), $B = \begin{bmatrix} b_1 \dots b_N \end{bmatrix}^T \in \mathbb{R}^N$ and

$$b_n = (\gamma_1^2 + \gamma_2^2)^{-1}[\gamma_2\phi_n(1) - \gamma_1\phi'_n(1)], \quad 1 \leq n \leq N$$

are obtained after integration by parts. Note that $b_n \neq 0$ for $1 \leq n \leq N$, by the uniqueness of solutions to (3).

By the Hautus lemma, the pair (A, B) is controllable. Therefore, for any $\delta > 0$, we can choose $K \in \mathbb{R}^{1 \times N}$ such that there exists $P^c > 0$ which satisfies

$$P^c(A + BK) + (A + BK)^T P^c < -2\delta P^c. \quad (37)$$

We propose the following controller

$$u(t - \tau_u(t)) = K\hat{z}^N(t - \tau_u(t)),$$
$$\hat{z}^N(t) = \begin{bmatrix} \langle \hat{z}(\cdot, t), \phi_1 \rangle \dots \langle \hat{z}(\cdot, t), \phi_N \rangle \end{bmatrix}^T \in \mathbb{R}^N. \quad (38)$$

The closed-loop system (36) and (38) has the form

$$\dot{\hat{z}}^N(t) = A\hat{z}^N(t) + BK\hat{z}^N(t - \tau_u(t)) - Le(0, t - \tau_y(t)) \quad (39)$$

Theorem 1: Assume that conditions of Proposition 1 hold. Given $\delta > 0$ and $K \in \mathbb{R}^{1 \times N}$ subject to (37), let there exist matrices $P_2^c, P_3^c, G^c \in \mathbb{R}^{N \times N}$ and positive-definite matrices $P^c, S^c, R^c \in \mathbb{R}^{N \times N}$ that satisfy the LMIs (30), where LC is substituted by BK. Let $z(\cdot, 0) = z_0 \in H^2(0, 1)$. Then the closed-loop system (9), (11), (38) is exponentially stable with

a decay rate $\delta > 0$, meaning that (31) holds and there exists $M_{\hat{z}} > 0$ such that

$$\|\hat{z}(\cdot, t)\|_{L^2(0,1)} \le M_{\hat{z}} \exp(-\delta t) \|z_0\|_{H^2(0,1)}, \quad t \ge 0. \quad (40)$$

Under the Dirichlet actuation, the result holds for $z_0 \in H^1(0,1)$, $z_0(1) = 0$ with $\|z_0\|_{H^2(0,1)}$ changed by $\|z_0\|_{H^1(0,1)}$ in (40).

Proof: The LMIs (30) guarantee ISS of the closed-loop system (36), (38) with respect to input $e(0, t - \tau_y(t))$ (cf. Proposition 4.3 of [21]), implying due to Lemma 3 and the second inequality (31) the following bound:

$$|u(t)|^2 = \left|K\hat{z}^N(t)\right|^2 \le M_u \exp(-2\delta t) \|z_0\|_{H^2(0,1)}^2 \quad (41)$$

for some $M_u > 0$. By variation of constants formula

$$\hat{z}_n(t) = b_n \int_0^t \exp((\lambda_n - q)(s - t)) u(s - \tau_u(s)) ds, \ n > N.$$

Then (41) yields

$$|\hat{z}_n(t)| \le M_\sigma |b_n| \frac{\exp(-\delta t)}{\lambda_n - q - \delta} \|z_0\|_{H^2(0,1)}, \quad n > N$$

with some $M_\sigma > 0$. Hence,

$$\sum_{n=N+1}^{\infty} |\hat{z}_n(t)|^2 \le \sum_{n=N+1}^{\infty} (\lambda_n - q - \delta)^{-2} |b_n|^2 \times M_\sigma^2 \exp(-2\delta t) \|z_0\|_{H^2(0,1)}^2. \quad (42)$$

To continue bounding in (42), we use arguments of Theorem 3.2 in [24], which imply

$$\sum_{n=1}^{\infty} (\lambda_n - q - \delta)^{-2} |b_n|^2 < \infty. \quad (43)$$

By combining the last estimate with (35), (41) and (42) we obtain (40).

Under the Dirichlet actuation, (24) holds with $\|z_0\|_{H^2(0,1)}$ changed by $\|z_0\|_{H^1(0,1)}$, which implies the result. ∎

Remark 2: The LMIs of Proposition 1 allow to find appropriate injection gain L. Following [21, Section 5.2], one can take $P_3 = \upsilon P_2$, where υ is a tuning parameter, and use $Y = P_2^T L$ as a new decision variable. After solving the resulting LMIs, the injection gain can be found as $L = P_2^{-T} Y$. A similar design can be applied to find appropriate controller gain $K \in \mathbb{R}^{1 \times N}$.

Summarizing, Theorem 1 implies the following observer-controller design: given (5) and a desired decay rate δ,

1) find the truncation order N according to (17),
2) find the gains L and K (see Remark 2),
3) solve the observer PDE (9) to obtain $\hat{z}(x, t)$,
4) compute the first N modes of $\hat{z}(x, t)$ to find $\hat{z}^N(t)$,
5) calculate $u(t) = K\hat{z}^N(t)$.

Moreover, for the exponential stabilization of (5) by using observer (9), it is sufficient to achieve "state-feedback" input-to-state stabilization of the ODE (36). The implementation of the proposed control law (44) requires numerical approximation of ODEs (36) and the error PDE (11). If N is very large, the computational cost can be significant.
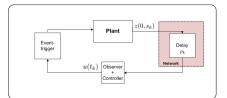

Fig. 2. Network-based control with ET from controller to actuator.

IV. SWITCHING BASED DYNAMIC EVENT-TRIGGERED CONTROL

In this section we consider a network-based implementation of the boundary observer-controller design suggested in the previous section. We consider two networks: one from the sensor to the controller and another from the controller to the actuator (see Figure 2). For the first network, we assume that the sampling intervals on the sensor side are bounded by maximum allowable transmission interval (MATI) and the transmission delay is bounded by maximum allowable delay (MAD). Then, the observer construction in (9) can be applied with $\tau_M = MATI + MAD$.

Similarly to (38), we consider a control law of the form

$$u(t - \tau_u(t)) = K\hat{z}^N(s_k), \quad t \in [s_k, s_{k+1}), \quad (44)$$

where the sequence of sampling instants s_k that preserves ISS of (36) will be found on the basis of ET mechanism that aims to minimize the number of the sent signals. We also denote

$$\beta^N(t) := \hat{z}^N(t) - \hat{z}^N(s_k). \quad (45)$$

We will introduce a novel dynamic switching-based ET mechanism that avoids Zeno behaviour.

Let the sampling be given by

$$s_{k+1} = \min\left\{ t \ge s_k + h \mid \left|\beta^N(t)\right|_\Omega^2 > \epsilon^2 \left|\hat{z}^N(t)\right|_\Omega^2 + \theta\eta(t)\right\} \quad (46)$$

where $h > 0$ is a waiting time before proceeding to verify the ET condition. Here $\eta(t)$ satisfies

$$\dot{\eta}(t) = -\alpha\eta(t) + \epsilon^2 \left|\hat{z}^N(t)\right|_\Omega^2 - \left|\beta^N(t)\right|_\Omega^2, \quad t \in [s_k + h, s_{k+1}) \quad (47)$$

and $\alpha > 0$, $\theta > 0$ are scalar parameters. Note that $\theta = 0$ correponds to static ET. Due to (46)

$$|\beta^N(t)|_\Omega^2 \le \epsilon^2 \left|\hat{z}^N(t)\right|_\Omega^2 + \theta\eta(t), \quad t \in [s_k + h, s_{k+1}). \quad (48)$$

During the waiting time, we assume that $\eta(t)$ satisfies

$$\dot{\eta}(t) = -2\delta_1\eta(t), \quad t \in [s_k, s_k + h], \\ \eta(0) = \eta_0 \ge 0 \quad (49)$$

with $\delta_1 > \delta$. The variations of constants formula and the positivity of the non-homogeneous term in (47) imply that $\eta(t) \ge 0$ for $t \ge 0$. It can be easily verified that for a given state $x(s_k)$ the next execution time under the dynamic ET mechanism (46) is not smaller than under static ET (see Remark 2.4 in [14]).

Clearly the Zeno behavior is avoided since $s_{k+1} - s_k \ge h$. ET mechanism (46) can be viewed as a switching between

the periodic sampling $s_k = kh$ for $t \in [s_k, s_k + h)$ and the continuous ET (46) for $t \in [s_k + h, s_{k+1})$. The resulting closed-loop system (39) under the ET mechanism (46) subject to (47) and (49) can be presented as a switching between the two systems. The first one (for $t \in [s_k, s_k + h)$) is governed by

$$\dot{\hat{z}}^N(t) = (A + BK)\hat{z}^N(t) - BK\beta^N(t) - Le(0, t - \tau_y(t)) \tag{50}$$

and (49). The second one (for $t \in [s_k + h, s_{k+1})$) is governed by (50) and (47) subject to (48).

For the ISS stability analysis of the resulting closed-loop system, we define the switching Lyapunov functional

$$V = |\hat{z}^N(t)|_P^2 + \eta(t) \tag{51}$$
$$+ \chi_k(t)(h - \tau_u(t)) \int_{t-\tau_u(t)}^{t} \exp(2\delta(s-t))|\dot{\hat{z}}^N(s)|_U^2 ds,$$

where $t \in [s_k, s_{k+1})$, $\tau_u(t) = t - s_k$, $P^c, U \in \mathbb{R}^{N \times N}$ are positive-definite and

$$\chi_k(t) = \begin{cases} 1, & t \in [s_k, s_k + h), \\ 0, & t \in [s_k + h, s_{k+1}). \end{cases}$$

Note that $\chi_k(t) = 1$ corresponds to periodic control (44), whereas $\chi_k(t) = 0$ corresponds to event-triggered control.

Theorem 2: Given $\delta > 0$ such that $\alpha > 2\delta + \theta$, let $\delta_1 > \delta$, N satisfy (17) and $K \in \mathbb{R}^{1 \times N}$ satisfy (37). Given a tuning parameter $\epsilon > 0$ let there exist matrices $P_2^c, P_3^c, \tilde{P}_2^c, \tilde{P}_3^c \in \mathbb{R}^{N \times N}$, positive-definite matrices $P^c, U, \Omega \in \mathbb{R}^{N \times N}$ such that

$$\Phi_1 < 0, \; \Phi_2 < 0, \; \Phi_3 < 0, \tag{52}$$

where

$$\Phi_1 = \begin{bmatrix} \phi^s & P^c - (P_2^c)^T + (A + BK)^T P_3^c \\ * & -P_3^c - (P_3^c)^T + hU \end{bmatrix}, \tag{53}$$

$$\Phi_2 = \begin{bmatrix} \phi^s & P^c - (P_2^c)^T + (A + BK)^T P_3^c & -(P_2^c)^T BKh \\ * & -P_3^c - (P_3^c)^T & -(P_3^c)^T BKh \\ * & * & -he^{-2\delta h}U \end{bmatrix}, \tag{54}$$

$$\Phi_3 = \begin{bmatrix} \psi^s & P^c - \tilde{P}_2^{cT} + (A + BK)^T \tilde{P}_3^c & -\tilde{P}_2^{cT} BK \\ * & -\tilde{P}_3^{cT} - \tilde{P}_3^c & -\tilde{P}_3^{cT} BK \\ * & * & -2\Omega \end{bmatrix}, \tag{55}$$

$$\phi^s = (A + BK)^T P_2^c + (P_2^c)^T (A + BK) + 2\delta P^c,$$
$$\psi^s = (A + BK)^T \tilde{P}_2^c + \tilde{P}_2^{cT}(A + BK) + 2\delta P^c + 2\epsilon^2 \Omega.$$

Then the closed-loop system (36), (44) under the ET mechanism (46) subject to (47) and (49) with the input that satisfies (33) is exponentially stable with a decay rate $\delta > 0$.

Proof: Note that by definition (51), V does not increase at the switching instances. This implies that it is sufficient to verify that

$$W := \dot{V} + 2\delta V - \xi |e(0, t - \tau_y(t))|^2 \leq 0 \tag{56}$$

along each subsystem of the switched system. Differentiating V along (50), (49) and following the steps in Section 3.2 in [26] we obtain

$$W \leq \frac{h - t + s_k}{h}|\tilde{\kappa}|_{\Psi_1}^2 + \frac{t - s_k}{h}|\kappa|_{\Psi_2}^2, \tag{57}$$

where $\tilde{\kappa} = \left[\hat{z}^N(t), \dot{\hat{z}}^N(t), -e(0, t - \tau_y(t))\right]^T$ and $\kappa = \left[\hat{z}^N(t), \dot{\hat{z}}^N(t), \beta^N(t), -e(0, t - \tau_y(t))\right]^T$,

$$\Psi_2 = \begin{bmatrix} & | & (P_2^c)^T L \\ \Phi_1 & | & (P_3^c)^T L \\ - & | & -\xi \\ * & | & -\xi \end{bmatrix}, \Psi_2 = \begin{bmatrix} & | & (P_2^c)^T L \\ \Phi_2 & | & (P_3^c)^T L \\ & | & 0 \\ - & | & - \\ * & | & -\xi \end{bmatrix}.$$

Differentiating V along (50) and (47) subject to (48) we obtain

$$\dot{V} = 2(\hat{z}^N)^T P^c \dot{\hat{z}}^N - \alpha\eta + \epsilon^2|\hat{z}^N|_\Omega^2 - |\beta^N|_\Omega^2.$$

By using the descriptor approach

$$0 = 2\left[(\hat{z}^N)^T (P_2^c)^T + (\dot{\hat{z}}^N)^T (P_3^c)^T\right]$$
$$\times [-\dot{\hat{z}}^N(t) + (A + BK)\hat{z}^N(t) - BK\beta^N(t)$$
$$- Le(0, t - \tau_y(t))],$$

we find

$$W \leq |\hat{\kappa}|_{\Psi_3}^2 + (-\alpha + 2\delta + \theta)\eta(t), \tag{58}$$

where $\hat{\kappa} = col\left\{\hat{z}^N(t), \dot{\hat{z}}^N(t), \beta^N(t), -e(0, t - \tau_y(t))\right\}$ and

$$\Psi_3 = \begin{bmatrix} & | & (\tilde{P}_2^c)^T L \\ \Phi_3 & | & (\tilde{P}_3^c)^T L \\ & | & 0 \\ - & | & - \\ * & | & -\xi \end{bmatrix}.$$

Since $\alpha > 2\delta + \theta$, by using (52) and applying Schur complement we arrive at $\Psi_i < 0(i = 1, 2, 3)$ for sufficiently large $\xi > 0$. The latter due to (57) and (58) implies the result. ∎

V. AN EXTENSION TO THE VECTOR CASE

Consider the vector PDE

$$z_t = Dz_{xx} + Qz, \quad t \geq 0, \tag{59}$$
$$z_x(0, t) = 0, \quad \gamma_1 z(1, t) + \gamma_2 z_x(1, t) = u(t - \tau_u(t)),$$

where $x \in (0, 1)$, $z = col\{z^1, ..., z^p\} \in \mathbb{R}^p$ is the state, γ_1 and γ_2 are scalars such that $\gamma_1^2 + \gamma_2^2 \neq 0$, $D = \text{diag}\{d_1, ..., d_p\} \in \mathbb{R}^{p \times p}$ is the diffusion matrix, $Q \in \mathbb{R}^{p \times p}$ is the matrix of reaction term and $u(t - \tau_u(t)) \in \mathbb{R}^p$ is the control input. Furthermore, we assume boundary measurements given by (6).

For z^j $(j = 1, ..., p)$ consider the Stürm-Liouville problem

$$d_j\phi'' + \lambda\phi = 0, \quad 0 < x < 1, \; j = 1, ..., p,$$
$$\phi'(0) = 0, \quad \gamma_1\phi(1) + \gamma_2\phi'(1) = 0.$$

It is easy to see that this problem admits the same eigenfunctions as (3) with the corresponding eigenvalues $\{d_j\lambda_n\}_{n=1}^{\infty}$.

The observer is constructed in the form of the vector PDE

$$\hat{z}_t = D\hat{z}_{xx} + Q\hat{z} - \mathcal{L}(x)[y(t) - \hat{z}(0, t - \tau_y(t))],$$
$$\hat{z}_x(0, t) = 0, \; \gamma_1\hat{z}(1, t) + \gamma_2\hat{z}_x(1, t) = u(t - \tau_u(t)), \tag{60}$$

where

$$\mathcal{L}(x) = \sum_{n=1}^{N} l_n\phi_n(x), \quad l_n \in \mathbb{R}^{p \times p} \tag{61}$$

For the error $e = z - \hat{z} = [e^1, ..., e^p]^T$ we have

$$e_t = De_{xx} + Qe + \mathcal{L}(x)e(0, t - \tau_y(t)), \tag{62}$$
$$e_x(0,t) = 0, \quad \gamma_1 e(1,t) + \gamma_2 e_x(1,t) = 0.$$

Given a decay rate $\delta > 0$ there exists $N \in \mathbb{N}$ such that

$$-(\min_{1 \leq i \leq p} d_i)\lambda_n < -(\delta + \max_{1 \leq i \leq p} \sum_{j=1}^{p} |q_{ij}|) \quad \forall n > N. \tag{63}$$

By presenting the error as in (12) and performing calculations similar to (13)-(15), we obtain the ODEs (20) with

$$e^N(t) = [e_1(t), \ldots, e_N(t)]^T \in \mathbb{R}^{pN},$$
$$e_j(t) = \int_0^1 e(x,t)\phi_j(x)dx \in \mathbb{R}^p, \tag{64}$$
$$A = \operatorname{diag}(-\lambda_1 D + Q, \cdots, -\lambda_N D + Q) \in \mathbb{R}^{pN \times pN},$$
$$L = [l_1, \ldots, l_N]^T \in \mathbb{R}^{pN \times p},$$
$$C = [I_p\phi_1(0), \cdots, I_p\phi_N(0)] \in \mathbb{R}^{p \times pN},$$
$$\zeta_0(t) = Ce^N(t) - e(0,t) \in \mathbb{R}^p,$$

whereas

$$\dot{e}_n(t) = (-\lambda_n D + Q)e_n(t), \quad n > N. \tag{65}$$

Applying Gershgorin's circle theorem and (63) we find that

$$Re(\lambda) < -\delta \quad \forall \lambda \in \sigma(-\lambda_n D + Q), \ \forall n > N,$$

where $\sigma(-\lambda_n D + Q)$ is the spectrum of $-\lambda_n D + Q$. This and (65) yield that for some $M_e > 0$

$$|e_n(t)|^2 \leq M_e \exp(-2\delta t) |e_n(0)|^2, \quad n > N. \tag{66}$$

We assume that the pair (A, C) is observable. This happens, for instance, if the matrices $\{-\lambda_n D + Q\}_{n=1}^N$ have no common eigenvalues. Then, for any $\lambda \in \sigma(A)$ we have $\operatorname{rank}(A - \lambda I_{pN}) \geq p(N-1)$. Since $\phi_n(j) \neq 0$ for $n \geq 1$, by combining this with the definition of C in (64) and applying the Hautus lemma, we find that the pair (A, C) is observable. This implies that (22) holds for some gain $L \in \mathbb{R}^{pN \times p}$. By employing (66) and the arguments of Lemma 3 we arrive at the estimate (3).

The controller has a form

$$u(t - \tau_u(t)) = K\hat{z}^N(t - \tau_u(t)), \ K \in \mathbb{R}^{p \times pN},$$
$$\hat{z}^N(t) = [\hat{z}_1(t), \ldots, \hat{z}_N(t)]^T \in \mathbb{R}^{pN}, \tag{67}$$
$$\hat{z}_j(t) = \int_0^1 \hat{z}(x,t)\phi_j(x)dx \in \mathbb{R}^p.$$

By using (67) and calculations similar to (13) we arrive at ODE (36) with

$$B = [b_1 D \cdots, b_N D]^T \in \mathbb{R}^{pN \times p},$$
$$b_n = (\gamma_1^2 + \gamma_2^2)^{-1}[\gamma_2\phi_n(1) - \gamma_1\phi_n'(1)], \quad n = 1, ..., N.$$

By the Hautus lemma, the pair (A, B) is controllable under the assumption that the matrices $\{-\lambda_n D + Q\}_{n=1}^N$ have no common eigenvalues. Then Theorems 1 and 2 hold with the matrix decision variables from $\mathbb{R}^{pN \times pN}$. Similarly, the reasoning in Section IV is independent of the dimension of the ODE (50). Therefore, the proposed switching-based dynamic ET mechanism can be straightforwardly generalized to the case of the vector PDE (59).
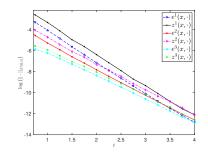

Fig. 3. $L^2(0,1)$-norms of z and e.

VI. EXAMPLE

Consider network-based control of (59) with $z \in \mathbb{R}^3$ and

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 5 & 1 \end{bmatrix} \tag{68}$$

under the Neumann actuation $z_x(1,t) = u(t_k)$, $k = 0, 1, ...$ and the measurements $z(0, s_k)$. The non-delayed state-feedback control of this system via the backstepping was suggested in [17]. Let $\delta = 2.5$ be the desired decay rate. In this case, condition (63) is satisfied for $N = 1$. The following observer and controller gains L and K were found by tuning the parameter υ and increasing τ_M while verifying that the LMIs in Remark 2 remain feasible:

$$L \approx \begin{bmatrix} -3.85 & -2.59 & -3.17 \\ -3.77 & -8.26 & -4.62 \\ 1.95 & -5.7 & -4.33 \end{bmatrix}, K \approx \begin{bmatrix} -0.87 & -0.49 & -0.73 \\ -0.84 & -1.59 & -0.67 \\ -0.38 & 0.88 & -0.58 \end{bmatrix}, \tag{69}$$
$$\tau_M = 0.0651, \quad \upsilon = 0.512.$$

The small value of τ_M is due to choosing a large δ. We note that when verifying the LMIs we observed that smaller values of δ lead to larger τ_M. Thus, for $\delta = 1$ and $\delta = 0.5$ we obtained feasibility for $\tau_M = 0.0822$ and $\tau_M = 0.1017$, respectively.

For the simulation we choose

$$z_0(x) = (\cos(\pi x) + 1) \left[-x^2 + 1, -2x^2 + 1, -3x^2 + 1 \right]^T,$$

$x \in (0, 1)$, $s_{k+1} - s_k = 0.5\tau_M$, whereas $\rho_k \in [0.2\tau_M, 0.4\tau_M]$ are selected at random. For the network between controller and actuator we choose $\mu_k = 0$ for simplicity and $t_{k+1} - t_k = 0.9\tau_M$.

The simulation is carried out for the corresponding error PDE (62), the PDE (59) and the ODE (39) (without ET). The $L^2(0,1)$ norms of the components of $e(x,t), z(x,t)$ (on a linear-logarithmic scale) are given in Figure 3 for simulation time $t_{fin} = 4$. The computed linear fits are given b $l_{z^1}(t) \approx -2.825t - 0.3736$, $l_{z^2}(t) \approx -2.833t - 1.11$, $l_{z^3}(t) \approx -2.62t - 2.54$, with similar fits for $e(x,t)$ that confirms the theoretical results.

Next, we consider event-triggered control of (59) with L and K given by (69), $\delta = 2.5$, $\alpha = 4$, $\theta = 0.6$, $\eta_0 = 1.4$ and $\delta = 2.501$. We compare the number of controller-actuator transmissions for periodic (no ET), static switching-based ET and dynamic switching-based ET. Let $t_{fin} = 10$ be the simulation time. For each $\epsilon = i \times 10^{-3}$ ($i \in \{0, ..., 10^3\}$), we find the maximum h which satisfies LMIs of Theorem 2.

	Periodic	Static	Dynamic
τ_M	0.079	0.059	0.0653
ϵ	-	0.497	0.512
avg. sent msg.	126.58	82.26	57.34

TABLE I

COMPARISON BETWEEN PERIODIC, STATIC AND DYNAMIC ET CONTROL.

For each pair (ϵ, h) we perform numerical simulations with several initial conditions

$$z_0(x) = [\cos(\pi x) + 1] \left[-mx^2 + 1, -\frac{m}{2}x^2 + 1, -m(x^2 - 1) \right]^T,$$
$$m \in \{1, ...20\}.$$

For each pair (ϵ, h) and each initial condition, we solve the observer PDE (60) and the ODEs (39), (47) and (49) while checking the ET condition (48) according to switching-based ET mechanism described in Section IV. We choose the pair (ϵ, h) which results in the minimum average number of sent updates (see Table I). It appears that the static switching-based ET reduces the average number of overall transmissions by $\approx 35\%$, the dynamic switching-based ET improves this result by another $\approx 20\%$ with no significant additional computational cost.

VII. CONCLUSION

A boundary delayed observer-controller design was introduced for 1D heat equation The design was based on the modal decomposition approach. For the network-based implementation, a novel switching-based dynamic event-triggered control was proposed.

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