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A Novel Technique for Solving Fully Fuzzy Nonlinear Systems Based on Neural Networks

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Predicting the solutions of complex systems is a crucial challenge. Complexity exists because of the uncertainty as well as nonlinearity. The nonlinearity in complex systems makes uncertainty irreducible in several cases. In this paper, two new approaches based on neural networks are proposed in order to find the estimated solutions of the fully fuzzy nonlinear system (FFNS). For obtaining the estimated solutions, a gradient descent algorithm is proposed in order to train the proposed networks. An example is proposed in order to show the efficiency of the considered approaches.

Keywords: Estimated solutions; complex system; fully fuzzy nonlinear system; neural network.

1. Introduction

Artificial neural networks have many applications in different fields.[1–11] Fuzzy neural network with various properties such as learning ability, generalization as well as

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nonlinear mapping is considered as a more attractive approach.[12] The standard neural network is taken to be an approximator.[13,14] In Ref. 15, a learning algorithm based on triangular fuzzy weights in order to train the fuzzy neural network is proposed. In Ref. 16, a fuzzy delta learning rule for training the fuzzy neural network is suggested. In Ref. 17, a neural network approach is suggested in order to obtain the solution of fuzzy problems. In Ref. 18, a neural network method is used in order to find the estimated solution of fully fuzzy matrix equation. In Ref. 19, a static neural network is suggested for obtaining the solution of fuzzy polynomials. In Ref. 20, a dynamic neural network is used for extracting the approximate solution of dual fuzzy polynomials. Different techniques for modeling as well as fuzzy identification are suggested in Refs. 21–27. In Ref. 28, the state space model of a linear system is developed into the fuzzy case. In Ref. 29, the homotopy technique in order to resolve the fuzzy nonlinear system is proposed. In Ref. 30, a novel algorithm for dynamical nonsingleton fuzzy control system is suggested. In Ref. 31, the Adomian technique in order to solve fuzzy system of linear equations is proposed. In Ref. 32, the homotopy method for obtaining the solution of a system of fuzzy nonlinear equations is proposed.

This paper is an extended version of the work originally presented in Ref. 33. In this work, two approaches on the basis of neural networks are proposed for finding the Z-number solutions of FFNS. A learning algorithm based on the gradient descent technique is suggested in order to adjust the Z-number weights. The simulation outcomes demonstrate that the suggested techniques are effective in obtaining the Z-number solutions of fully fuzzy nonlinear system (FFNS). The novelty of this paper compared with Ref. 34 is introducing a new model based on neural networks to obtain the Z-number solutions of FFNS.

The remaining of the paper is organized as follows. In Sec. 2, some basic definitions related to the Z-numbers are given. The proposed approaches for obtaining the Z-number solutions of FFNS are demonstrated in Sec. 3. An example is given in Sec. 4. Section 5 concludes the work and provides discussions on further work.

2. Preliminaries

The mathematical description of FFNS is given by the following definition.

Definition 1. Suppose c is:

(1) normal, there is $\varsigma_0 \in \Re$ where $c(\varsigma_0) = 1$;
(2) convex, $c(\beta\varsigma + (1-\beta)\varsigma) \geq \min\{c(\varsigma), c(\varpi)\}$, $\forall \varsigma, \varpi \in \Re$, $\forall \beta \in [0,1]$,
(3) upper semi-continuous on $\Re$, $c(\varsigma) \leq c(\varsigma_0) + \varepsilon$, $\forall \varsigma \in N(\varsigma_0)$, $\forall \varsigma_0 \in \Re$, $\forall \varepsilon > 0$, $N(\varsigma_0)$ is a neighborhood;
(4) $c^+ = \{\varsigma \in \Re, c(\varsigma) > 0\}$ is compact, so c is a fuzzy variable, $c \in E : \Re \to [0,1]$.

The fuzzy variable c is demonstrated as

$$c = (\underline{c}, \bar{c}), \tag{1}$$

in which $\underline{c}$ is the lower-bound variable and $\bar{c}$ is the upper-bound variable.

Definition 2. The Z-number is made up of two components $Z = [c(\varsigma), p]$. The first component $c(\varsigma)$ is the restriction on a real-valued uncertain variable ς. The second component p is a measure of reliability of c. p can be reliability, strength of belief, probability or possibility. The Z-number can be stated as Z^+-number in the case where $c(\varsigma)$ is a fuzzy number and p is the probability distribution of ς. If $c(\varsigma)$ as well as p are fuzzy numbers, then the Z-number can be stated as Z^--number.

The Z^+-number contains more information when compared with the Z^--number. In this paper, the definition of Z^+-number is utilized, i.e. $Z = [c, p]$, where c is a fuzzy number and p is a probability distribution.

The most common membership functions which define the fuzzy numbers are the triangular function

$$\mu_c = H(s,u,v) = \begin{cases} \dfrac{\varsigma - s}{u - s}, & s \le \varsigma \le u; \\ \dfrac{v - \varsigma}{v - u}, & u \le \varsigma \le v; \end{cases} \quad \text{otherwise } \mu_c = 0; \tag{2}$$

and the trapezoidal function

$$\mu_c = H(s,u,v,w) = \begin{cases} \dfrac{\varsigma - s}{u - s}, & s \le \varsigma \le u; \\ \dfrac{w - \varsigma}{w - v}, & v \le \varsigma \le w; \\ 1, & u \le \varsigma \le v; \end{cases} \quad \text{otherwise } \mu_c = 0. \tag{3}$$

The probability measure can be stated as

$$P = \int_{\Re} \mu_c(\varsigma) p(\varsigma) d\varsigma, \tag{4}$$

where p is the probability density of ς and $\Re$ is the restriction on p. For discrete Z-numbers we have

$$P(c) = \sum_{\iota=1}^{\gamma} \mu_c(\varsigma_\iota) p(\varsigma_\iota). \tag{5}$$

Definition 3. The α-level for fuzzy number c is stated as

$$[c]^\alpha = \{\varsigma \in \Re : c(\varsigma) \ge \alpha\}, \tag{6}$$

where $0 < \alpha \le 1$, $c \in E$.

Therefore $[c]^0 = c^+ = \{\varsigma \in \Re, c(\varsigma) > 0\}$. As $\alpha \in [0,1]$, $[c]^\alpha$ is bounded, $\underline{c}^\alpha \le [c]^\alpha \le \bar{c}^\alpha$. The α-level of c between $\underline{c}^\alpha$ and $\bar{c}^\alpha$ can be defined as

$$[c]^\alpha = (\underline{c}^\alpha, \bar{c}^\alpha), \tag{7}$$

where $\underline{c}^\alpha$ as well as $\bar{c}^\alpha$ are functions of α. We define $\underline{c}^\alpha = d_A(\alpha)$, $\bar{c}^\alpha = d_B(\alpha)$, $\alpha \in [0,1]$.

Definition 4. The α-level of the Z-number $Z = (c, p)$ is defined as follows:

$$[Z]^\alpha = ([c]^\alpha, [p]^\alpha), \tag{8}$$

where $0 < \alpha \le 1$. $[p]^\alpha$ is computed by Nguyen's theorem,

$$[p]^\alpha = p([c]^\alpha) = p([\underline{c}^\alpha, \bar{c}^\alpha]) = [\underline{P}^\alpha, \overline{P}^\alpha], \tag{9}$$

where $p([c]^\alpha) = \{p(\varsigma) | \varsigma \in [c]^\alpha\}$. Therefore, $[Z]^\alpha$ is stated as

$$[Z]^\alpha = (\underline{Z}^\alpha, \overline{Z}^\alpha) = ((\underline{c}^\alpha, \underline{P}^\alpha), (\bar{c}^\alpha, \overline{P}^\alpha)), \tag{10}$$

where $\underline{P}^\alpha = \underline{c}^\alpha p(\underline{\varsigma}_\iota^\alpha)$, $\overline{P}^\alpha = \bar{c}^\alpha p(\bar{\varsigma}_\iota^\alpha)$, $[\varsigma_\iota]^\alpha = (\underline{\varsigma}_\iota^\alpha, \bar{\varsigma}_\iota^\alpha)$.

Similar to the fuzzy numbers,[1] three main operations are defined for the Z-numbers: $\oplus, \ominus$ and $\odot$, which indicate the sum, subtract and multiply, respectively. In this paper, the proposed operations are different from the ones in Ref. 33.

Suppose $Z_1 = (c_1, p_1)$ as well as $Z_2 = (c_2, p_2)$ are two discrete Z-numbers expressing the uncertain variables ς_1 and ς_2, so $\sum_{k=1}^{\gamma} p_1(\varsigma_{1k}) = 1, \sum_{k=1}^{\gamma} p_2(\varsigma_{2k}) = 1$. The following operation is defined:

$$Z_{12} = Z_1 * Z_2 = (c_1 * c_2, p_1 * p_2), \tag{11}$$

where $*\{\oplus, \ominus, \odot\}$.

The operations used for the fuzzy numbers are stated as[1]

$$\begin{aligned}
&[c_1 \oplus c_2]^\alpha = [\underline{c}_1^\alpha + \underline{c}_2^\alpha, \bar{c}_1^\alpha + \bar{c}_2^\alpha],\\
&[c_1 \ominus c_2]^\alpha = [\underline{c}_1^\alpha - \underline{c}_2^\alpha, \bar{c}_1^\alpha - \bar{c}_2^\alpha],\\
&[c_1 \odot c_2]^\alpha = \begin{pmatrix} \min\{\underline{c}_1^\alpha \underline{c}_2^\alpha, \underline{c}_1^\alpha \bar{c}_2^\alpha, \bar{c}_1^\alpha \underline{c}_2^\alpha, \bar{c}_1^\alpha \bar{c}_2^\alpha\} \\ \max\{\underline{c}_1^\alpha \underline{c}_2^\alpha, \underline{c}_1^\alpha \bar{c}_2^\alpha, \bar{c}_1^\alpha \underline{c}_2^\alpha, \bar{c}_1^\alpha \bar{c}_2^\alpha\} \end{pmatrix}.
\end{aligned} \tag{12}$$

For the discrete probability distributions, the following relation is defined for all $p_1 * p_2$ operations:

$$p_1 * p_2 = \sum_{\iota} p_1(\varsigma_{1,\iota}) p_2(\varsigma_{2,(\gamma-\iota)}) = p_{12}(\varsigma). \tag{13}$$

The Hukuhara difference is defined as[35]

$$\begin{aligned} &Z_1 \ominus_H Z_2 = Z_{12},\\ &Z_1 = Z_2 \oplus Z_{12}. \end{aligned} \tag{14}$$

In case that $Z_1 \ominus_H Z_2$ exists, the α-level can be defined as

$$[Z_1 \ominus_H Z_2]^\alpha = [\underline{Z}_1^\alpha - \underline{Z}_2^\alpha, \overline{Z}_1^\alpha - \overline{Z}_2^\alpha]. \tag{15}$$

Clearly, $Z_1 \ominus_H Z_1 = 0$, $Z_1 \ominus Z_1 \neq 0$.

Moreover, the generalized Hukuhara difference is defined as[36]

$$Z_1 \ominus_{gH} Z_2 = Z_{12} \Longleftrightarrow \begin{cases} Z_1 = Z_2 \oplus Z_{12}, \\ Z_2 = Z_1 \oplus (-1) Z_{12}. \end{cases} \tag{16}$$

By taking into consideration the α-level, we have $[Z_1 \ominus_{gH} Z_2]^\alpha = [\min\{\underline{Z}_1^\alpha - \underline{Z}_2^\alpha, \overline{Z}_1^\alpha - \overline{Z}_2^\alpha\}, \max\{\underline{Z}_1^\alpha - \underline{Z}_2^\alpha, \overline{Z}_1^\alpha - \overline{Z}_2^\alpha\}]$ and if $Z_1 \ominus_{gH} Z_2$ and $Z_1 \ominus_H Z_2$ exist, then $Z_1 \ominus_H Z_2 = Z_1 \ominus_{gH} Z_2$. The conditions for the existence of $Z_{12} = Z_1 \ominus_{gH} Z_2 \in E$ are

$$\begin{cases} \underline{Z}_{12}^\alpha = \underline{Z}_1^\alpha - \underline{Z}_2^\alpha \quad \text{and} \quad \overline{Z}_{12}^\alpha = \overline{Z}_1^\alpha - \overline{Z}_2^\alpha \\ \text{with } \underline{Z}_{12}^\alpha \text{ increasing, } \overline{Z}_{12}^\alpha \text{ decreasing, } \underline{Z}_{12}^\alpha \le \overline{Z}_{12}^\alpha; \end{cases}$$
$$\begin{cases} \underline{Z}_{12}^\alpha = \overline{Z}_1^\alpha - \overline{Z}_2^\alpha \quad \text{and} \quad \overline{Z}_{12}^\alpha = \underline{Z}_1^\alpha - \underline{Z}_2^\alpha \\ \text{with } \underline{\mathbf{Z}}_{12}^\alpha \text{ increasing, } \overline{Z}_{12}^\alpha \text{ decreasing, } \underline{Z}_{12}^\alpha \le \overline{Z}_{12}^\alpha, \end{cases} \tag{17}$$

where $\forall \alpha \in [0,\, 1]$.

Suppose c is a triangular function, the absolute value of the Z-number $Z = (c, p)$ is defined as

$$|Z(\varsigma)| = (|s_1| + |u_1| + |v_1|, p(|s_2| + |u_2| + |v_2|)). \tag{18}$$

Let c_1 as well as c_2 be triangular functions, the supremum metric for the Z-numbers $Z_1 = (c_1, p_1)$ and $Z_2 = (c_2, p_2)$ is expressed as

$$D(Z_1, Z_2) = d(c_1, c_2) + d(p_1, p_2), \tag{19}$$

where $d(\cdot, \cdot)$ is the supremum metric for fuzzy sets.[1] $D(Z_1, Z_2)$ has the properties mentioned in the following:

$$\begin{aligned} &D(Z_1 + Z, Z_2 + Z) = D(Z_1, Z_2), \\ &D(Z_2, Z_1) = D(Z_1, Z_2), \\ &D(bZ_1, kZ_2) = |b| D(Z_1, Z_2), \\ &D(Z_1, Z_2) \le D(Z_1, Z) + D(Z, Z_2), \end{aligned} \tag{20}$$

where $b \in \Re$, $Z = (c,\, p)$ is the Z-number and c is a triangle function.

Definition 5. Suppose $\widehat{Z}$ is the space of Z-numbers. The α-level of the Z-number-valued function $H : [0, s] \to \widehat{Z}$ is defined as

$$H(c, \alpha) = [\underline{H}(c, \alpha), \overline{H}(c, \alpha)], \tag{21}$$

where $c \in \widehat{Z}$, for each $\alpha \in [0, 1]$.

Using the definition of generalized Hukuhara difference, the gH-derivative of H at c_0 is defined as

$$\frac{d}{dt} H(c_0) = \lim_{\zeta \to 0} \frac{1}{\zeta} [H(c_0 + \zeta) \ominus_{gH} H(c_0)]. \tag{22}$$

In (22), $H(c_0 + \zeta)$ as well as $H(c_0)$ represent symmetric pattern with Z_1 and Z_2, respectively, given in (16).

3. Neural Network Approach for *Z*-Number Solution Approximation

Here two novel techniques based on neural networks are suggested for obtaining the numerical solutions of FFNS.

3.1. *Fully fuzzy nonlinear systems*

Let us take into consideration the following system:

$$\begin{cases} S_{11} \odot \varphi \odot \psi \oplus \cdots \oplus S_{1\mathrm{n}} \odot \varphi^n \odot \psi^n = G_1, \\ S_{21} \odot \varphi \odot \psi \oplus \cdots \oplus S_{2n} \odot \varphi^n \odot \psi^n = G_2, \end{cases} \tag{23}$$

where $S_{1j}, S_{2j}, \phi, \psi, G_1, G_2$ belong to Z-number set (for $j = 1, \ldots, n$). For obtaining the approximated solutions, a feedback neural network is developed. The suggested neural network is demonstrated in Fig. 1.

3.2. *Calculation of Z-number output*

Here a feedback neural network is suggested such that the α-level sets of the Z-number parameters S_{qj} are considered to be nonnegative, i.e. $0 \leq \underline{S}_{qj}^{\alpha} \leq \overline{S}_{qj}^{\alpha}$ where $j = 1, \ldots, n$ and $q = 1, 2$. The following relations are generated:

- input units:

$$[\phi]^\alpha = (\underline{\phi}^\alpha, \overline{\phi}^\alpha), \\ [\psi]^\alpha = (\underline{\psi}^\alpha, \overline{\psi}^\alpha); \tag{24}$$

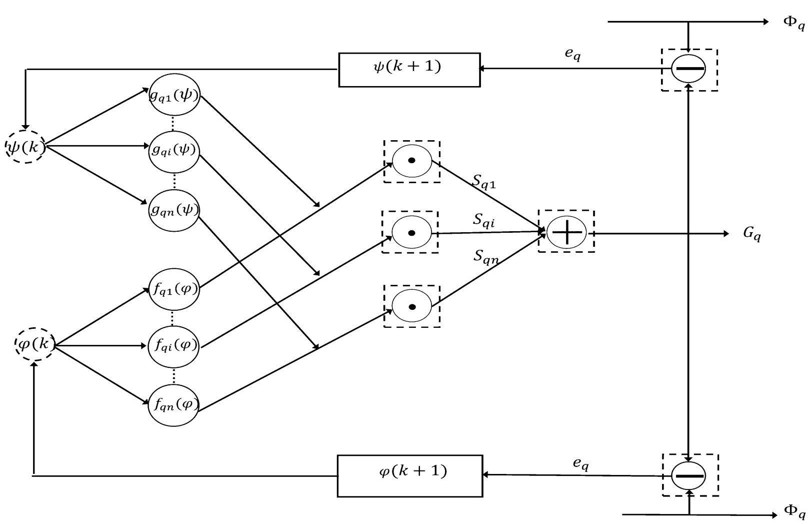

Fig. 1. FFNS in the form of feedback neural network.

- hidden units in the first layer:

$$\begin{aligned}[u_j]^\alpha &= (\underline{u}_j^\alpha, \overline{u}_j^\alpha), \quad j = 1, \ldots, n, \\ [v_j]^\alpha &= (\underline{v}_j^\alpha, \overline{v}_j^\alpha), \quad j = 1, \ldots, n,\end{aligned} \tag{25}$$

where

$$\underline{u}_j^\alpha = \begin{cases} \underline{\phi}^\alpha(\underline{\phi}^\alpha)^{j-1}, & \underline{\phi}^\alpha \geq 0, \\ \overline{\phi}^\alpha(\overline{\phi}^\alpha)^{j-1}, & \underline{\phi}^\alpha < 0, \quad j \text{ is even}, \\ \underline{\phi}^\alpha(\underline{\phi}^\alpha)^{j-1}, & \underline{\phi}^\alpha < 0, \quad j \text{ is odd}, \end{cases} \tag{26}$$

$$\overline{u}_j^\alpha = \begin{cases} \overline{\phi}^\alpha(\overline{\phi}^\alpha)^{j-1}, & \overline{\phi}^\alpha \geq 0, \\ \underline{\phi}^\alpha(\underline{\phi}^\alpha)^{j-1}, & \overline{\phi}^\alpha < 0, \quad j \text{ is even}, \\ \overline{\phi}^\alpha(\overline{\phi}^\alpha)^{j-1}, & \overline{\phi}^\alpha < 0, \quad j \text{ is odd}, \end{cases} \tag{27}$$

$$\underline{u}_j^\alpha = \begin{cases} \underline{\psi}^\alpha(\underline{\psi}^\alpha)^{j-1}, & \underline{\psi}^\alpha \geq 0, \\ \overline{\psi}^\alpha(\overline{\psi}^\alpha)^{j-1}, & \underline{\psi}^\alpha < 0, \quad j \text{ is even}, \\ \underline{\psi}^\alpha(\underline{\psi}^\alpha)^{j-1}, & \underline{\psi}^\alpha < 0, \quad j \text{ is odd}, \end{cases} \tag{28}$$

$$\overline{u}_j^\alpha = \begin{cases} \overline{\psi}^\alpha(\overline{\psi}^\alpha)^{j-1}, & \overline{\psi}^\alpha \geq 0, \\ \underline{\psi}^\alpha(\underline{\psi}^\alpha)^{j-1}, & \overline{\psi}^\alpha < 0, \quad j \text{ is even}, \\ \overline{\psi}^\alpha(\overline{\psi}^\alpha)^{j-1}, & \overline{\psi}^\alpha < 0, \quad j \text{ is odd}, \end{cases} \tag{29}$$

- hidden units in the second layer:

$$[o_j]^\alpha = (\underline{o}_j^\alpha, \overline{o}_j^\alpha), \quad j = 1, \ldots, n, \tag{30}$$

where

$$\underline{o}_j^\alpha = \begin{cases} \underline{u}_j^\alpha \, \underline{v}_j^\alpha, & \underline{u}_j^\alpha \geq 0, \quad \underline{v}_j^\alpha \geq 0, \\ \underline{u}_j^\alpha \, \overline{v}_j^\alpha, & \underline{u}_j^\alpha < 0, \quad \underline{v}_j^\alpha \geq 0, \\ \overline{u}_j^\alpha \, \overline{v}_j^\alpha, & \underline{u}_j^\alpha < 0, \quad \underline{v}_j^\alpha < 0, \\ \overline{u}_j^\alpha \, \underline{v}_j^\alpha, & \underline{u}_j^\alpha \geq 0, \quad \underline{v}_j^\alpha < 0, \end{cases} \tag{31}$$

and

$$\overline{o}_j^\alpha = \begin{cases} \overline{u}_j^\alpha \, \overline{v}_j^\alpha, & \overline{u}_j^\alpha \geq 0, \quad \overline{v}_j^\alpha \geq 0, \\ \overline{u}_j^\alpha \, \underline{v}_j^\alpha, & \overline{u}_j^\alpha < 0, \quad \overline{v}_j^\alpha \geq 0, \\ \underline{u}_j^\alpha \, \overline{v}_j^\alpha, & \overline{u}_j^\alpha \geq 0, \quad \overline{v}_j^\alpha < 0, \\ \underline{u}_j^\alpha \, \underline{v}_j^\alpha, & \overline{u}_j^\alpha < 0, \quad \overline{v}_j^\alpha < 0; \end{cases} \tag{32}$$

- output unit:

$$[\Phi_q]^\alpha = (\underline{\Phi}_q^\alpha, \overline{\Phi}_q^\alpha), \quad q = 1, 2, \tag{33}$$

where

$$[\Phi_q]^\alpha = \left(\begin{array}{c} \sum_{j\varepsilon M} \underline{S}_{qj}^{\alpha} \underline{o}_j^{\alpha} + \sum_{j\varepsilon N} \overline{S}_{qj}^{\alpha} \underline{o}_j^{\alpha}, \\ \sum_{j\varepsilon C} \overline{S}_{qj}^{\alpha} \bar{o}_j^{\alpha} + \sum_{j\varepsilon D} \underline{S}_{qj}^{\alpha} \bar{o}_j^{\alpha} \end{array} \right), \tag{34}$$

such that $M = \{j|\underline{o}_j^\alpha \geq 0\}$, $N = \{j|\underline{o}_j^\alpha < 0\}$, $C = \{j|\bar{o}_j^\alpha \geq 0\}$ and $D = \{j|\bar{o}_j^\alpha < 0\}$.

A cost function for α-level sets of the Z-number output Φ_q and the corresponding target output G_q is stated as follows:

$$\begin{aligned} e_q &= \underline{e}_q^\alpha + \bar{e}_q^\alpha, \\ \underline{e}_q^\alpha &= \frac{1}{2}(\underline{G}_q^\alpha - \underline{\Phi}_q^\alpha)^2, \\ \bar{e}_q^\alpha &= \frac{1}{2}(\overline{G}_q^\alpha - \overline{\Phi}_q^\alpha)^2. \end{aligned} \tag{35}$$

3.3. *Learning algorithm*

Let $\phi = ((\phi^1, \phi^2, \phi^3, \phi^4), p)$ and $\psi = ((\psi^1, \psi^2, \psi^3, \psi^4), p)$ be initialized at random Z-numbers. We have the following relation for Z-number variable ϕ (Ref. 14):

$$\begin{aligned} \phi^r(k+1) &= \phi^r(k) \oplus \Delta\phi^r(k), \quad r = 1,2,3,4, \\ \Delta\phi^r(k) &= -\eta \frac{\partial e_q}{\partial \phi^r} \oplus \gamma \Delta\phi^r(k-1), \end{aligned} \tag{36}$$

where η is considered to be the learning rate and γ is considered to be the momentum term constant. $\frac{\partial e_q}{\partial \phi^r}$ is calculated as follows:

$$\frac{\partial e_q}{\partial \phi^r} = \frac{\partial \underline{e}_q^\alpha}{\partial \phi^r} + \frac{\partial \bar{e}_q^\alpha}{\partial \phi^r}. \tag{37}$$

Hence,

$$\frac{\partial \underline{e}_q^\alpha}{\partial \phi^r} = \frac{\partial \underline{e}_q^\alpha}{\partial \underline{\Phi}_q^\alpha} \frac{\partial \underline{\Phi}_q^\alpha}{\partial \phi^r} = -(\underline{G}_q^\alpha - \underline{\Phi}_q^\alpha) \frac{\partial \underline{\Phi}_q^\alpha}{\partial \phi^r}, \tag{38}$$

where

$$\frac{\partial \underline{\Phi}_q^\alpha}{\partial \phi^r} = \sum_{j\varepsilon M} \frac{\partial \underline{\Phi}_q^\alpha}{\partial \underline{o}_j^\alpha} \frac{\partial \underline{o}_j^\alpha}{\partial \underline{u}_j^\alpha} \frac{\partial \underline{u}_j^\alpha}{\partial (\underline{\phi}^\alpha)^j} \frac{\partial (\underline{\phi}\alpha)^j}{\partial \phi^r} + \sum_{j\varepsilon M} \frac{\partial \underline{\Phi}_q^\alpha}{\partial \underline{o}_j^\alpha} \frac{\partial \underline{o}_j^\alpha}{\partial \underline{u}_j^\alpha} \frac{\partial \underline{u}_j^\alpha}{\partial (\underline{\phi}^\alpha)^j} \frac{\partial (\underline{\phi}\alpha)^j}{\partial \phi^r} \tag{39}$$

and

$$\frac{\partial \bar{e}_q^\alpha}{\partial \phi^r} = \frac{\partial \bar{e}_q^\alpha}{\partial \overline{\Phi}_q^\alpha} \frac{\partial \overline{\Phi}_q^\alpha}{\partial \phi^r} = -(\overline{G}_q^\alpha - \overline{\Phi}_q^\alpha) \frac{\partial \overline{\Phi}_q^\alpha}{\partial \phi^r}, \tag{40}$$

where

$$\frac{\partial \overline{\Phi}_q^\alpha}{\partial \phi^r} = \sum_{j\varepsilon C} \frac{\partial \overline{\Phi}_q^\alpha}{\partial \bar{o}_j^\alpha} \frac{\partial \bar{o}_j^\alpha}{\partial \bar{u}_j^\alpha} \frac{\partial \bar{u}_j^\alpha}{\partial (\overline{\phi}^\alpha)^j} \frac{\partial (\overline{\phi}\alpha)^j}{\partial \phi^r} + \sum_{j\varepsilon C} \frac{\partial \overline{\Phi}_q^\alpha}{\partial \bar{o}_j^\alpha} \frac{\partial \bar{o}_j^\alpha}{\partial \bar{u}_j^\alpha} \frac{\partial \bar{u}_j^\alpha}{\partial (\overline{\phi}^\alpha)^j} \frac{\partial (\overline{\phi}\alpha)^j}{\partial \phi^r}, \tag{41}$$

$\frac{\partial(\underline{\phi}^\alpha)^j}{\partial\phi^r}$ as well as $\frac{\partial(\overline{\phi}^\alpha)^j}{\partial\phi^r}$ are calculated as follows:

$$\frac{\partial(\underline{\phi}^\alpha)^j}{\partial\phi^r} = \frac{\partial(\underline{\phi}^\alpha)^j}{\partial\underline{\phi}^\alpha}\,\frac{\partial\underline{\phi}^\alpha}{\partial\phi^r} = \begin{cases} j(\underline{\phi}^\alpha)^{j-1}\begin{cases}1-\alpha, & r=1,\\ \alpha, & r=2,\\ 0, & r=3,\\ 0, & r=4,\end{cases} & \underline{\phi}^\alpha \geq 0,\\ j(\overline{\phi}^\alpha)^{j-1}\begin{cases}0, & r=1,\\ \alpha, & r=2,\\ 1-\alpha, & r=3,\\ 0, & r=4,\end{cases} & \underline{\phi}^\alpha < 0,\ j \text{ is even},\\ j(\underline{\phi}^\alpha)^{j-1}\begin{cases}0, & r=1,\\ \alpha, & r=2,\\ 1-\alpha, & r=3,\\ 0, & r=4,\end{cases} & \underline{\phi}^\alpha < 0,\ j \text{ is odd},\end{cases} \tag{42}$$

and

$$\frac{\partial(\overline{\phi}^\alpha)^j}{\partial\phi^r} = \frac{\partial(\overline{\phi}^\alpha)^j}{\partial\overline{\phi}^\alpha}\,\frac{\partial\overline{\phi}^\alpha}{\partial\phi^r} = \begin{cases} j(\overline{\phi}^\alpha)^{j-1}\begin{cases}1-\alpha, & r=1,\\ \alpha, & r=2,\\ 0, & r=3,\\ 0, & r=4,\end{cases} & \overline{\phi}^\alpha \geq 0,\\ j(\underline{\phi}^\alpha)^{j-1}\begin{cases}0, & r=1,\\ \alpha, & r=2,\\ 1-\alpha, & r=3,\\ 0, & r=4,\end{cases} & \overline{\phi}^\alpha < 0,\ j \text{ is even},\\ j(\overline{\phi}^\alpha)^{j-1}\begin{cases}0, & r=1,\\ \alpha, & r=2,\\ 1-\alpha, & r=3,\\ 0, & r=4,\end{cases} & \overline{\phi}^\alpha < 0,\ j \text{ is odd}.\end{cases} \tag{43}$$

The connection weights ϕ_j are updated as follows:

$$\phi_j(k+1) = (\phi(k+1))^j, \quad j = 2, \ldots, n. \tag{44}$$

We can adjust the Z-number parameter ψ like ϕ.

The solution of (23) can also be obtained with another type of neural network, see Fig. 2. In this neural network, the α-level sets of the fuzzy input S_{qj} are nonnegative, i.e. $0 \leq \underline{S}^{\,\alpha}_{qj} \leq \overline{S}^{\,\alpha}_{qj}$, where $j = 1, \ldots, n$ and $q = 1, 2$. The following relations are generated:

- input units:

$$[S_{qj}]^\alpha = (\underline{S}^{\,\alpha}_{qj}, \overline{S}^{\,\alpha}_{qj}), \quad q = 1, 2,\ \ j = 1, \ldots, n; \tag{45}$$

- hidden units:

$$[o_{qj}]^\alpha = (\underline{o}^{\,\alpha}_{qj}, \bar{o}^{\,\alpha}_{qj}), \quad q = 1, 2,\ \ j = 1, \ldots, n; \tag{46}$$

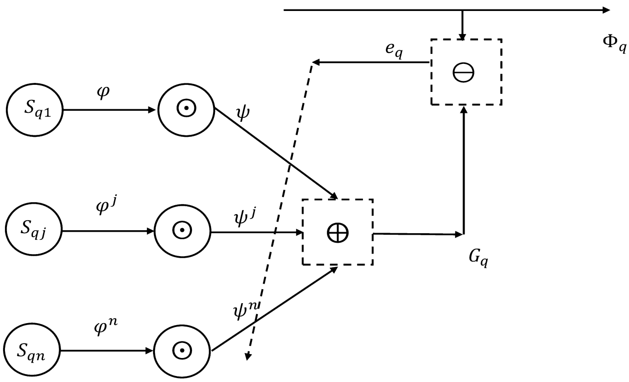

Fig. 2. FFNS in the form of feed-forward neural network.

where

$$\underline{o}^{\alpha}_{qj} = \begin{cases} \sum_{j\in C} \underline{S}^{\alpha}_{qj}(\underline{\phi}^{\alpha})^j, & \underline{\phi}^{\alpha} \geq 0, \\ \sum_{j\in D} \underline{S}^{\alpha}_{qj}(\overline{\phi}^{\alpha})^j, & \underline{\phi}^{\alpha} < 0, \quad j \text{ is even}, \\ \sum_{j\in F} \overline{S}^{\alpha}_{qj}(\underline{\phi}^{\alpha})^j, & \underline{\phi}^{\alpha} < 0, \quad j \text{ is odd}, \end{cases} \tag{47}$$

where $C = \{j|\underline{\phi}^{\alpha} \geq 0\}$, $D = \{j|\underline{\phi}^{\alpha} < 0,\ j \text{ is even}\}$ and $F = \{j|\underline{\phi}^{\alpha} < 0, j \text{ is odd}\}$, and

$$\bar{o}^{\alpha}_{qj} = \begin{cases} \sum_{j\in C'} \overline{S}^{\alpha}_{qj}(\overline{\phi}^{\alpha})^j, & \overline{\phi}^{\alpha} \geq 0, \\ \sum_{j\in D'} \overline{S}^{\alpha}_{qj}(\underline{\phi}^{\alpha})^j, & \overline{\phi}^{\alpha} < 0, \quad j \text{ is even}, \\ \sum_{j\in F'} \underline{S}^{\alpha}_{qj}(\overline{\phi}^{\alpha})^j, & \overline{\phi}^{\alpha} < 0, \quad j \text{ is odd}, \end{cases} \tag{48}$$

where $C' = \{j|\overline{\phi}^{\alpha} \geq 0\}$, $D' = \{j|\overline{\phi}^{\alpha} < 0, j \text{ is even}\}$ and $F' = \{j|\overline{\phi}^{\alpha} < 0,\ j \text{ is odd}\}$;

- output unit:

$$[\Phi_q]^{\alpha} = (\underline{\Phi}^{\alpha}_q, \overline{\Phi}^{\alpha}_q), \quad q = 1, 2, \tag{49}$$

where

$$\underline{\Phi}^{\alpha}_q = \begin{cases} \sum_{j\in A} \underline{o}^{\alpha}_{qj}(\underline{\psi}^{\alpha})^j, & \underline{\psi}^{\alpha} \geq 0, \\ \sum_{j\in B} \underline{o}^{\alpha}_{qj}(\overline{\psi}^{\alpha})^j, & \underline{\psi}^{\alpha} < 0, \quad j \text{ is even} \\ \sum_{j\in H} \bar{o}^{\alpha}_{qj}(\underline{\psi}^{\alpha})^j, & \underline{\psi}^{\alpha} < 0, \quad j \text{ is odd} \end{cases} + \begin{cases} \sum_{j\in K} \underline{o}^{\alpha}_{qj}(\overline{\psi}^{\alpha})^j, & \underline{\psi}^{\alpha} \geq 0, \\ \sum_{j\in L} \underline{o}^{\alpha}_{qj}(\underline{\psi}^{\alpha})^j, & \underline{\psi}^{\alpha} < 0, \quad j \text{ is even}, \\ \sum_{j\in T} \bar{o}^{\alpha}_{qj}(\overline{\psi}^{\alpha})^j, & \underline{\psi}^{\alpha} < 0, \quad j \text{ is odd}, \end{cases} \tag{50}$$

where $A = \{j|\underline{o}_{qj}^{\alpha} \geq 0, \underline{\psi}^{\alpha} \geq 0\}$, $B = \{j|\underline{o}_{qj}^{\alpha} \geq 0, \underline{\psi}^{\alpha} < 0, j \text{ is even}\}$, $H = \{j|\underline{o}_{qj}^{\alpha} \geq 0, \underline{\psi}^{\alpha} < 0, j \text{ is odd}\}$, $K = \{j|\underline{o}_{qj}^{\alpha} < 0, \underline{\psi}^{\alpha} \geq 0\}$, $L = \{j|\underline{o}_{qj}^{\alpha} < 0, \underline{\psi}^{\alpha} < 0, j \text{ is even}\}$ and $T = \{j|\underline{o}_{qj}^{\alpha} < 0, \underline{\psi}^{\alpha} < 0, j \text{ is odd}\}$, and

$$\overline{\Phi}_q^{\alpha} = \begin{cases} \sum_{j\in A'} \bar{o}_{qj}^{\alpha}(\overline{\psi}^{\alpha})^j, & \overline{\psi}^{\alpha} \geq 0, \\ \sum_{j\in B'} \bar{o}_{qj}^{\alpha}(\underline{\psi}^{\alpha})^j, & \overline{\psi}^{\alpha} < 0, \quad j \text{ is even} \\ \sum_{j\in H'} \underline{o}_{qj}^{\alpha}(\overline{\psi}^{\alpha})^j, & \overline{\psi}^{\alpha} < 0, \quad j \text{ is odd} \end{cases} + \begin{cases} \sum_{j\in K} \bar{o}_{qj}^{\alpha}(\underline{\psi}^{\alpha})^j, & \overline{\psi}^{\alpha} \geq 0, \\ \sum_{j\in L} \bar{o}_{qj}^{\alpha}(\overline{\psi}^{\alpha})^j, & \overline{\psi}^{\alpha} < 0, \quad j \text{ is even}, \\ \sum_{j\in T} \underline{o}_{qj}^{\alpha}(\underline{\psi}^{\alpha})^j, & \overline{\psi}^{\alpha} < 0, \quad j \text{ is odd}, \end{cases} \tag{51}$$

where $A' = \{j|\bar{o}_{qj}^{\alpha} \geq 0, \overline{\psi}^{\alpha} \geq 0\}$, $B' = \{j|\bar{o}_{qj}^{\alpha} \geq 0, \overline{\psi}^{\alpha} < 0, j \text{ is even}\}$, $H' = \{j|\bar{o}_{qj}^{\alpha} \geq 0, \overline{\psi}^{\alpha} < 0, j \text{ is odd}\}$, $K' = \{j|\bar{o}_{qj}^{\alpha} < 0, \overline{\psi}^{\alpha} \geq 0\}$, $L' = \{j|\bar{o}_{qj}^{\alpha} < 0, \overline{\psi}^{\alpha} < 0, j \text{ is even}\}$ and $T' = \{j|\bar{o}_{qj}^{\alpha} < 0, \overline{\psi}^{\alpha} < 0, j \text{ is odd}\}$.

For this neural network the training algorithm is similar to (36).

4. Numerical Example

In this section, a numerical example is used to demonstrate how to apply feedback neural network and feed-forward neural network in order to find the solutions of FFNS.

Example. Consider the following FFNS:

$$\begin{cases} ((7,9,11), p(0.7,0.81,0.9)) \odot \varphi \odot \psi \oplus ((2,5,7), p(0.7,0.8,0.9)) \odot \varphi^2 \odot \psi^2 \\ \qquad = ((29,1201,6011, p(0.7,0.8,0.91)), \\ ((9,10,12), p(0.7,0.8,0.9)) \odot \varphi \odot \psi \oplus ((4,6,9), p(0.7,0.85,0.9)) \odot \varphi^2 \odot \psi^2 \\ \qquad = ((55,1501,8001), p(0.75,0.8,0.9)), \end{cases} \tag{52}$$

where $\varphi = ((-5, -3, -2), p(0.8, 0.9, 1))$ and $\psi = ((-4, -3, -1), p(0.8, 0.9, 1))$ are the exact solutions. The neural networks shown in Figs. 1 and 2 are used for estimating the solutions φ and ψ. The maximum learning rate of neural networks is $\eta = 0.001$.

The neural networks arbitrarily start from $\varphi(0) = ((-8, -6, -5), p(0.7, 0.8, 0.9))$ and $\psi(0) = ((-7, -6, -4), p(0.75, 0.8, 0.9))$. The approximation results are demonstrated in Tables 1 and 2 for neural networks shown in Fig. 1 (feedback neural network) and Fig. 2 (feed-forward neural network), respectively. It can be seen that feed-forward neural network method and feedback neural network method can approximate the Z-number solutions. Feedback neural network method is more suitable than the feed-forward neural network method. The reason behind it is that the approximation error of feedback neural network can be lower than that of feed-forward neural network, while the former needs lower number of training iterations. In Tables 1 and 2, $\varphi(k)$ and $\psi(k)$ are approximate solutions of FFNS (52), k is the

Table 1. Results of approximating the Z-number solutions $\varphi(k)$ and $\psi(k)$ with an example feedback neural network obtained during the experiments.

k	$\varphi(k)$	$\psi(k)$
1	$((-7.9145, -5.9551, -4.9346), p(0.6, 0.8, 0.85))$	$((-6.9123, -5.9453, -3.9233), p(0.7, 0.8, 0.87))$
2	$((-7.4657, -5.5512, -4.4139), p(0.75, 0.8, 0.9))$	$((-6.6112, -5.6451, -3.6231), p(0.6, 0.8, 0.9))$
3	$((-7.1678, -5.2195, -4.1897), p(0.7, 0.8, 0.9))$	$((-6.1812, -5.2651, -3.1933), p(0.7, 0.85, 0.9))$
⋮	⋮	⋮
69	$((-5.0089, -3.0096, -2.0102), p(0.8, 0.9, 1))$	$((-4.0082, -3.0071, -1.0093), p(0.8, 0.96, 1))$
70	$((-5.0073, -3.0051, -2.0057), p(0.8, 0.94, 1))$	$((-4.0062, -3.0051, -1.0071), p(0.8, 0.9, 1))$
71	$((-5.0021, -3.0032, -2.0041), p(0.8, 0.9, 1))$	$((-4.0033, -3.0021, -1.0042), p(0.8, 0.95, 1))$

Table 2. Results of approximating the Z-number solutions $\varphi(k)$ and $\psi(k)$ with an example feed-forward neural network obtained during the experiments.

k	$\varphi(k)$	$\psi(k)$
1	$((-7.8113, -5.8451, -4.8611), p(0.6, 0.8, 0.85))$	$((-6.7981, -5.8321, -3.8112), p(0.7, 0.8, 0.87))$
2	$((-7.5983, -5.6678, -4.6451), p(0.75, 0.8, 0.9))$	$((-6.5811, -5.5431, -3.4981), p(0.6, 0.8, 0.9))$
3	$((-7.3184, -5.3433, -4.3891), p(0.7, 0.8, 0.9))$	$((-6.3129, -5.2679, -3.2351), p(0.7, 0.85, 0.9))$
⋮	⋮	⋮
109	$((-5.0112, -3.0123, -2.0118), p(0.8, 0.9, 1))$	$((-4.0223, -3.0119, -1.0114), p(0.8, 0.96, 1))$
110	$((-5.0082, -3.0094, -2.0086), p(0.8, 0.94, 1))$	$((-4.0102, -3.0092, -1.0081), p(0.8, 0.9, 1))$
111	$((-5.0055, -3.0078, -2.0071), p(0.8, 0.9, 1))$	$((-4.0065, -3.0059, -1.0052), p(0.8, 0.95, 1))$

number of iterations and p is the measure of probability. The error between the approximate solution and the exact solution for both approaches is demonstrated in Fig. 3. In Fig. 3 the approximation error of the feedback neural network method is smaller than the feed-forward neural network method.

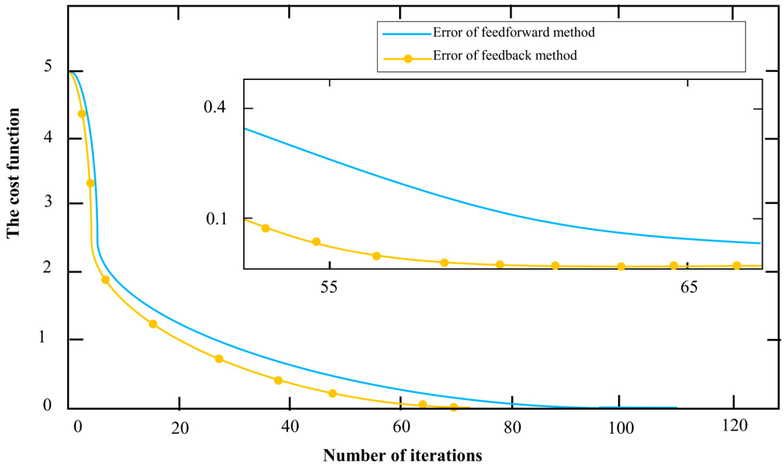

Fig. 3. The errors between the approximate solution and the exact solution with feedback and feed-forward neural networks.

5. Conclusion

In this paper, two approaches on the basis of neural networks are suggested for obtaining the estimated Z-number solutions of FFNS. A learning algorithm based on the gradient descent technique is used in order to generate the estimated solutions of FFNS. An example is proposed in order to show the efficiency of the proposed approaches. The simulation results show that these new models are effective to estimate the Z-number solutions of FFNS. The comparison of the feedback neural network method with the feed-forward neural network method shows that the feedback neural network method is better or at least more suitable than the feed-forward neural network method. The reason behind it is that the approximation error of feedback neural network can be lower than that of the feed-forward neural network, while the former needs lower number of training iterations. Further work is to study the stability of training algorithm.

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