Article:

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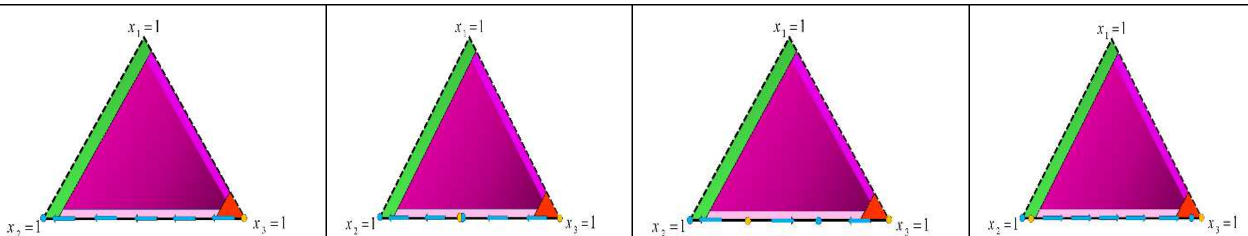
In this supplementary detailed calculations are provided for equations that have been presented in table 5.

First, our analysis in ‘’first step’’ for $x_1 = 0$ and $x_2 = 0$ boundaries are carried out and then the steps 2-4 are applied to the remaining 15 cases in the game. Related proposed conditions are discussed for each case.

Step 1

Sub-Case A2: $p_{22} > p_{32} > p_{23} > p_{33}$

Location of equilibrium points in this case is depicted in figure (1).



$\frac{r_{22} r_{33}}{r_{23}^2} > k_1$	$\frac{r_{22} r_{33}}{r_{23}^2} = k_1$	$\frac{r_{22} r_{33}}{r_{23}^2} < k_1$	$\frac{r_{22} r_{33}}{r_{23}^2} \ll k_1$

Figure 1. Location of the system equilibrium points on the $x_1 = 0$ boundary for different values of bifurcation parameter in case A2

Therefore, the figure 1(a) would be the desired situation. Our proposal in this case is:

$$\text{Our suggestion}: ((r_{22} r_{33}) / r_{23}^2) > k_1 \qquad (1)$$

Sub-Case A3: $p_{22} > p_{32}, p_{33} > p_{23}$

In this case bifurcation never happens because for any value of bifurcation parameter $s \neq 0$ in (8). Consequently, one of the solutions in equation 7 is acceptable mathematically since it is located in the range of $x_2 = 0$ to $x_2 = 1$ while the other answer is not. . The location of equilibrium points on the $x_1 = 0$ border in this case is shown in figure (5).

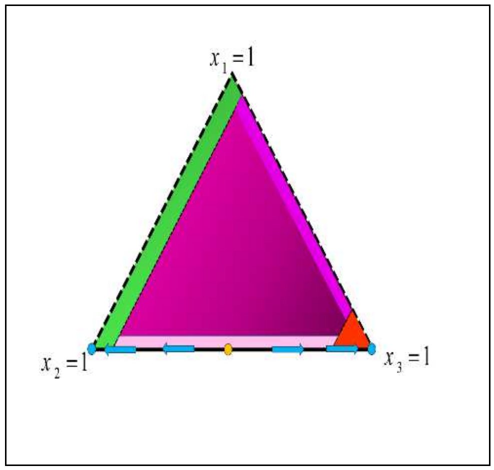
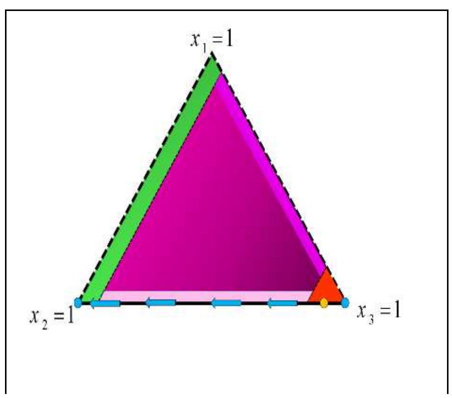


Figure 1. Location of the system equilibrium points on the $x_1 = 0$ border before applying our proposal to case A3

By changing the interaction rate parameters the position of red equilibrium point in figure (5) will change. When $r_{22} \to \infty$, the location of red equilibrium point moves as indicated in figure (6) which is the objective..



$r_{22} \rightarrow \infty$

Figure 2. Location of the system equilibrium points on the $x_1 = 0$ border after applying our proposal to case A3

Thus, our proposal in this case is :

(2)

Sub-Case A4: $p_{32} > p_{22}, p_{23} > p_{33}$

Bifurcation does not happen in this case as well. The Location of equilibrium points on the $x_1 = 0$ boundary is shown in figure (7).

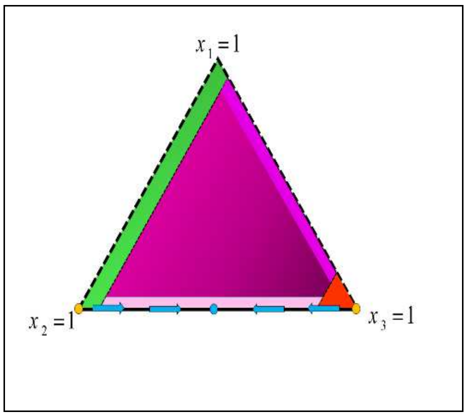


Figure 3. Location of the system equilibrium points on the $x_1 = 0$ boundary before applying our proposal to case A4

By altering the interaction rate parameters the red equilibrium point in figure (7) will move to a new position. When, $r_{33} \rightarrow \infty$, the location of red equilibrium point will move to a desired point as illustrated in figure 8..

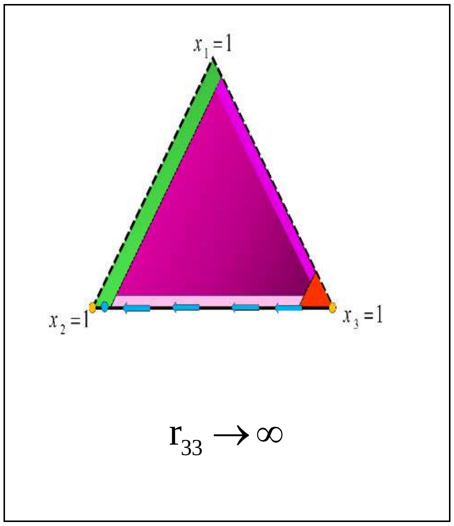


Figure 4. Location of the system equilibrium points on the $x_1 = 0$ border after applying our proposal to case A4

Thus, the proposition for this case is:

$$\text{Our} - \text{Proposal} : r_{33} \to \infty \qquad (3)$$

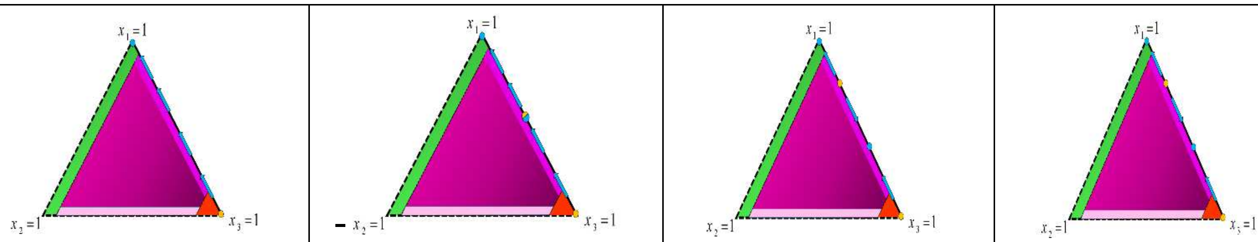
Boundary $x_2 = 0$

On the $x_2 = 0$ border, the general idea of finding different equilibrium points of the system and stability analysis is similar to the previous section but in this case we define the bifurcation parameter as $r_{11}r_{33} / r_{13}^2$ and k_2 as follows:

$$k_2 = \frac{1}{(p_{11} - p_{33})^2}\left(p_{11}(p_{13} + p_{31} - 2p_{33}) + p_{13}(p_{33} - 2p_{31}) + p_{31}p_{33}\right) + \frac{1}{(p_{11} - p_{33})^2}\left((p_{11} - p_{13})(p_{11} - p_{31})(p_{13} - p_{33})(p_{31} - p_{33})\right)^{\frac{1}{2}}$$

Sub-Case B2: $p_{11} > p_{31} > p_{13} > p_{33}$

Similar to the A2 case, the location of equilibrium points on $x_2 = 0$ boundary is shown in figure (10):



$\frac{r_{11}r_{33}}{r_{13}^2} > k_2$	$\frac{r_{11}r_{33}}{r_{13}^2} = k_2$	$\frac{r_{11}r_{33}}{r_{13}^2} < k_2$	$\frac{r_{11}r_{33}}{r_{13}^2} \ll k_2$

Figure 5. Location of the system equilibrium points on the $x_2 = 0$ border for different values of the bifurcation parameter in case B2

Figure 10(a) indicates the desired situation. Hence, our proposal for this case would be:

$$\text{Our}-\text{Proposal}:((r_{11}r_{33})/r_{13}^2) > k_2 \quad (4)$$

Sub-Case B3: $p_{11} > p_{31}, p_{33} > p_{13}$

Similar to case A3, bifurcation never happens in this case as well. The position of equilibrium points on the $x_2 = 0$ boundary in this case is depicted in figure (11).

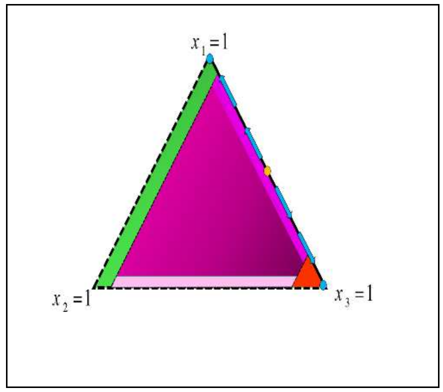


Figure 6. Location of the system equilibrium points on the $x_2 = 0$ border before applying our proposal to case B3

By changing the interaction rate parameters, the location of red equilibrium point in figure 11 change. When $r_{11} \to \infty$, the red equilibrium point moves as figure 12 which is .

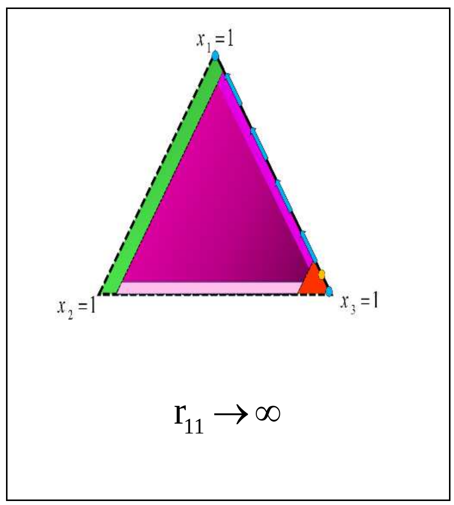


Figure 7. Distribution of the system equilibrium points on the $x_2 = 0$ border after applying our proposal to case B3

Thus, our suggestion in this case is:

$$\text{Our} - \text{Proposal} : r_{11} \to \infty \qquad (5)$$

Sub-Case B4: $p_{31} > p_{11}, p_{13} > p_{33}$

Bifurcation won’t happen in this case, too. The position of equilibrium points on the $x_2 = 0$ border is illustrated in figure (13).

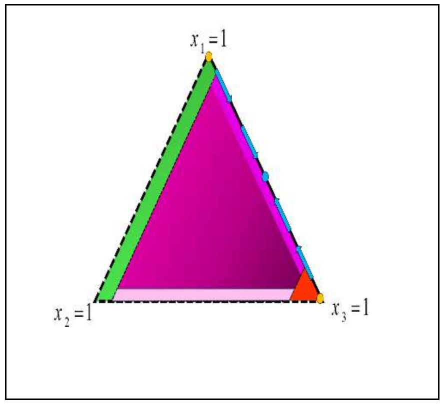


Figure 8. Position of the system equilibrium points on the $x_2 = 0$ border before applying our proposal in case B4

Altering the interaction rate parameters makes the location of red equilibrium point in figure 13 transferred and if $r_{33} \to \infty$, the red equilibrium point moves to the desired point as shown in figure.

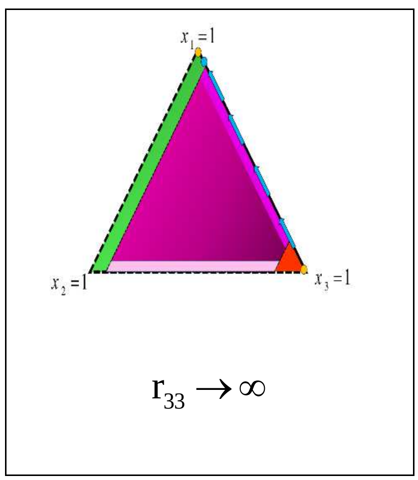


Figure 9. Location of the system equilibrium points on the $x_2 = 0$ boundary after applying our proposal to case B4

Thus, suggestion for this case is:

$$\text{Our} - \text{Proposal} : r_{33} \to \infty \qquad (6)$$

2. **Case A1B2**

According to previous parts, our proposition for this case would be:

$$\begin{cases} r_{22}r_{33} \to \infty \\ r_{23} \to 0 \end{cases} \quad \text{and} \quad \frac{r_{11}r_{33}}{r_{13}^{2}} > k_2$$

Step 2:

In this step, suggestions to exclude equilibrium points from region 2 are provided. As it has been asserted, region 2 is divided into 3 different sub-regions.

Sub-region 2.1:

In this sub-region, based on our proposition in first step, we may write:

$$\begin{cases} r_{12}x_2 \to 0 \quad , \quad r_{22}x_2 \to 0 \\ r_{23}x_3 \to 0 \quad , \quad r_{32}x_2 \to 0 \end{cases} \Rightarrow \begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{13}x_3p_{13}}{r_{11}x_1 + r_{13}x_3} \\ f_2 \to p_{21} \\ f_3 \to \dfrac{r_{31}x_1p_{31} + r_{33}x_3p_{33}}{r_{31}x_1 + r_{33}x_3} \end{cases}$$

Taking into account these equations, there might be equilibrium points in this sub-region. Thus, to eliminate any possible equilibrium points in this sub-region, extra conditions are proposed.

Our extra condition: $r_{13} \to 0 \quad (7)$

Consequently, equations reform to:

$$\begin{cases} r_{12}x_2 \to 0 & , \quad r_{13}x_3 \to 0 \\ r_{22}x_2 \to 0 & \quad r_{23}x_3 \to 0 \\ r_{31}x_1 \to 0 & , \quad r_{32}x_2 \to 0 \end{cases} \Rightarrow \begin{cases} f_1 \to p_{11} \\ f_2 \to p_{21} \\ f_3 \to p_{33} \end{cases}$$

Since $f_1 > f_3$, there would be no equilibrium points in this sub-region for this case.

Sub-region 2.2:

In this sub-region we propose the following trend:

$$\begin{cases} r_{11}x_1 \to 0 & , \quad r_{13}x_3 \to 0 \\ r_{22}x_2 \to 0 & \quad r_{23}x_3 \to 0 \\ r_{31}x_1 \to 0 & , \quad r_{32}x_2 \to 0 \end{cases} \Rightarrow \begin{cases} f_1 \to p_{12} \\ f_2 \to p_{21} \\ f_3 \to p_{33} \end{cases}$$

Since $f_2 > f_3$, there is no equilibrium point.

Sub-region 2.3:

The equation for case 1 may be written as:

$$\begin{cases} r_{13}x_3 \to 0 & , \quad r_{23}x_3 \to 0 \\ r_{31}x_1 \to 0 & , \quad r_{32}x_2 \to 0 \end{cases} \Rightarrow \begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{12}x_2p_{12}}{r_{11}x_1 + r_{12}x_2} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{22}x_2p_{22}}{r_{21}x_1 + r_{22}x_2} \\ f_3 \to p_{33} \end{cases}$$

Since, $f_2 \in (p_{22}, p_{21})$, $f_2 > f_3$ in this region and there is no equilibrium point here.

Step 3:

In this step recommendations are proposed to obtain $\dot{x}_3 < 0$, where $x_3 = 1 - \varepsilon$.

$$\begin{cases} x_3 = 1-\varepsilon \\ x_1 + x_2 = \varepsilon \end{cases} \quad \text{if } : \dot{x}_3 < 0 \quad \Rightarrow \quad f_3 < x_1 f_1 + x_2 f_2 + x_3 f_3 \quad \Rightarrow \quad \varepsilon f_3 < x_1 f_1 + x_2 f_2 \quad (8)$$

So, equations are active in step 3.

Sub-region 2.1:

In this sub-region:

$$\begin{cases} x_1 \to \varepsilon & , \quad f_1 \to p_{11} \\ x_2 \to 0 & , \quad f_2 \to p_{21} \\ x_3 = 1-\varepsilon & , \quad f_3 \to p_{33} \end{cases} \Rightarrow \begin{cases} x_1 f_1 + x_2 f_2 \to \varepsilon p_{11} \\ \varepsilon f_3 \to \varepsilon p_{33} \end{cases} \Rightarrow \varepsilon f_3 < x_1 f_1 + x_2 f_2$$

Therefore, equation 8 is always active and we don’t need any proposition.

Sub-region 2.2:

In this sub-region:

$$\begin{cases} x_1 \to 0 & , \quad f_1 \to p_{12} \\ x_2 \to \varepsilon & , \quad f_2 \to p_{22} \\ x_3 = 1-\varepsilon & , \quad f_3 \to p_{33} \end{cases} \Rightarrow \begin{cases} x_1 f_1 + x_2 f_2 \to \varepsilon p_{22} \\ \varepsilon f_3 \to \varepsilon p_{33} \end{cases} \Rightarrow \varepsilon f_3 < x_1 f_1 + x_2 f_2$$

Identically, equation 8 is always active and we don’t need any proposition.

Sub-region 2.3:

$$\begin{cases} f_1 \to \dfrac{r_{11} x_1 p_{11} + r_{12}(\varepsilon - x_1) p_{12}}{r_{11} x_1 + r_{12}(\varepsilon - x_1)} \\ f_2 \to \dfrac{r_{21} x_1 p_{21} + r_{22} x_2 p_{22}}{r_{21} x_1 + r_{22}(\varepsilon - x_1)} \\ f_3 \to p_{33} \end{cases}$$

If: $x_1 f_1 + x_2 f_2 > \varepsilon f_3 \Rightarrow r_{11} x_1 (x_1 p_{11} - M_2) > r_{12}(\varepsilon - x_1)(M_2 - x_1 p_{12})$

Where: $M_2 = \varepsilon f_3 - x_2 f_2$

If $M < x_1 p_{11}$, then the following condition is our proposition to make equation 8 active:

$$\text{Our proposition: } r_{11} > \frac{r_{12}(\varepsilon - x_1)(M_2 - x_1 p_{12})}{x_1(x_1 p_{11} - M_2)} \qquad (9)$$

Moreover, the activeness of the following equation is verified:

$$M_2 < x_1 p_{11} \qquad (10)$$

$$M_2 - x_1 p_{11} = \varepsilon \mathrm{i} - (\varepsilon - x_1)\left(\frac{r_{21} x_1 p_{21} + r_{22}(\varepsilon - x_1) p_{22}}{r_{12} x_1 + r_{22}(\varepsilon - x_1)}\right) - x_1 p_{11}$$

$$= \frac{r_{12} x_1 (\varepsilon p_{33} - (\varepsilon - x_1) p_{21} - x_1 p_{11}) - r_{22}(\varepsilon - x_1)(x_1 p_{11} + (\varepsilon - x_1) p_{22} - \varepsilon p_{33})}{r_{12} x_1 + r_{22}(\varepsilon - x_1)}$$

And: $\begin{cases} (\varepsilon p_{33} - (\varepsilon - x_1) p_{21} - x_1 p_{11}) < 0 \\ (x_1 p_1 + (\varepsilon - x_1) p_2 - \varepsilon p_{33}) > 0 \end{cases} \Rightarrow M_2 - x_1 p_{11} < 0 \Rightarrow M_2 < x_1 p_{11}$

Therefore, the equation 10 is always active and there is no need to extra proposition.

Step 4:

In this step, the proposed condition are intended to have $\dot{x}_3 < 0$, where $x_3 = \sigma$. Therefore, it is completely similar to step 3, and we just replace $1-\varepsilon$ with σ.

$$\begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{12}(\varepsilon - x_1)p_{12}}{r_{11}x_1 + r_{12}(1-\sigma - x_1)} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{22}x_2p_{22}}{r_{21}x_1 + r_{22}(1-\sigma - x_1)} \\ f_3 \to p_{33} \end{cases}$$

Our proposition: $r_{11} > \dfrac{r_{12}(1-\sigma - x_1)(N_2 - x_1p_{12})}{x_1(x_1 p_{11} - N_2)}$ (11)

Where: $N_2 = (1-\sigma)f_3 - x_2f_2$

3. **Case A1B3**

According to previous parts, our proposition for this case is:

$$\begin{cases} r_{22}r_{33} \to \infty \\ r_{23} \to 0 \end{cases} \quad \text{and} \quad r_{11} \to \infty$$

Step 2:

In this step, propositions to avoid equilibrium points in region 2 is presented. As it has been mentioned, region 2 is divided into 3 different sub-regions and.

Sub-region 2.1:

In this sub-region, $x_2 \to 0$ and the procedure continues based on following suggestions:

$$\begin{cases} r_{11}x_1 \to \infty \quad , \quad r_{22}x_2 \to 0 \\ r_{23}x_3 \to 0 \quad , \quad r_{32}x_2 \to 0 \end{cases} \Rightarrow \begin{cases} f_1 \to p_{11} \\ f_2 \to p_{21} \\ f_3 \to \dfrac{r_{31}x_1p_{31} + r_{33}x_3p_{33}}{r_{31}x_1 + r_{33}x_3} \end{cases}$$

Since $f_3 \in (p_{33}, p_{31})$, $f_1 > f_3$ and no equilibrium point is located in this sub-region.

Sub-region 2.2:

similar to sub-region 1, $x_1 \to 0$ and it can be easily noticed that:

$$\begin{cases} r_{12}x_1 \to 0 \quad , \quad r_{23}x_3 \to 0 \\ r_{31}x_1 \to 0 \quad , \quad r_{32}x_2 \to 0 \end{cases} \Rightarrow \begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{12}x_1p_{12} + r_{11}x_3p_{13}}{r_{11}x_1 + r_{12}x_2 + r_{13}x_3} \\ f_2 \to p_{22} \\ f_3 \to p_{33} \end{cases}$$

Regarding the fact that $f_2 > f_3$, there would be no equilibrium point.

Sub-region 2.3:

For case 1, equations are:

$$\begin{cases} r_{11}x_1 \to \infty \\ r_{23}x_3 \to 0 \\ r_{32}x_2 \to 0 \end{cases} \Rightarrow \begin{cases} f_1 \to p_{11} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{22}x_2p_{22}}{r_{21}x_1 + r_{22}x_2} \\ f_3 \to \dfrac{r_{31}x_1p_{31} + r_{33}x_3p_{33}}{r_{31}x_1 + r_{33}x_3} \end{cases}$$

Since, $f_3 \in (p_{33}, p_{31})$, always $f_1 > f_3$ in this region and there is no equilibrium point here.

Step 3:

In this step the following suggestions are provided to have $\dot{x}_3 < 0$, where $x_3 = 1 - \varepsilon$.

$$\begin{cases} x_3 = 1 - \varepsilon \\ x_1 + x_2 = \varepsilon \end{cases} \quad \text{if } : \dot{x}_3 < 0 \quad \Rightarrow \quad f_3 < x_1f_1 + x_2f_2 + x_3f_3 \quad \Rightarrow \quad \varepsilon f_3 < x_1f_1 + x_2f_2 \quad (12)$$

Therefore, in step 3 we will recommend propositions so that the above equation becomes active.

Sub-region 2.1:

In this region:

$$\begin{cases} x_1 \to \varepsilon & , \quad f_1 \to p_{11} \\ x_2 \to 0 & , \quad f_2 \to p_{21} \\ x_3 = 1-\varepsilon & , \quad f_3 \to p_{33} \end{cases} \Rightarrow \begin{cases} x_1 f_1 + x_2 f_2 \to \varepsilon p_{11} \\ \varepsilon f_3 \to \varepsilon p_{33} \end{cases} \Rightarrow \varepsilon f_3 < x_1 f_1 + x_2 f_2$$

It is clear that the equation is always active and there is no necessity for proposition.

Sub-region 2.2:

$$\begin{cases} x_1 \to 0 & , \quad f_1 \to \dfrac{r_{11} x_1 p_{11} + r_{12} x_1 p_{12} + r_{11}(1-\varepsilon) p_{13}}{r_{11} x_1 + r_{12} x_2 + r_{13}(1-\varepsilon)} \\ x_2 \to \varepsilon & , \quad f_2 \to p_{22} \\ x_3 = 1-\varepsilon & , \quad f_3 \to p_{33} \end{cases} \Rightarrow \begin{cases} x_1 f_1 + x_2 f_2 \to \varepsilon p_{22} \\ \varepsilon f_3 \to \varepsilon p_{33} \end{cases} \Rightarrow \varepsilon f_3 < x_1 f_1 + x_2 f_2$$

Therefore, equation (12) is active and no proposed procedure is required, as well.

Sub-region 2.3

In this sub-region:

$$\begin{cases} f_1 \to p_{11} \\ f_2 \to \dfrac{r_{21} x_1 p_{21} + r_{22}(\varepsilon - x_1) p_{22}}{r_{21} x_1 + r_{22}(\varepsilon - x_1)} \\ f_3 \to \dfrac{r_{31} x_1 p_{31} + r_{33}(1-\varepsilon) p_{33}}{r_{31} x_1 + r_{33}(1-\varepsilon)} \end{cases}$$

If: $x_1 f_1 + x_2 f_2 > \varepsilon f_3 \Rightarrow r_{22}(\varepsilon - x_1)((\varepsilon - x_1) p_{22} - M_3) > r_{12} x_1 (M_3 - (\varepsilon - x_1) p_{21})$

Where: $M = \varepsilon f_3 - x_1 f_1$

If $M < (\varepsilon - x_1)p_{22}$, then the appropriate solution to make equation (12) active is proposed as follows:

$$\text{Our proposition: } r_{22} > \frac{r_{12}x_1(M_3 - (\varepsilon - x_1)p_{21})}{(\varepsilon - x_1)((\varepsilon - x_1)p_{22} - M_3)} \quad (13)$$

Thus, we check if the following equation is active:

$$M < (\varepsilon - x_1)p_{22} \qquad (14)$$

$$M - (\varepsilon - x_1)p_{21} = \varepsilon\left(\frac{r_{31}x_1p_{31} + r_{33}\varepsilon p_{33}}{r_{31}x_1 + r_{33}\varepsilon}\right) - x_1p_{11} - (\varepsilon - x_1)p_{21}$$

$$= \frac{r_{31}x_1(\varepsilon p_{31} - x_1\, p_{11} - (\varepsilon - x_1)p_{21}) - r_{33}\varepsilon((\varepsilon - x_1)p_{21} + x_1\, p_{11} - \varepsilon p_{33})}{r_{31}x_1 + r_{33}\varepsilon}$$

And: $((\varepsilon - x_1)p_{21} + x_1\, p_{11} - \varepsilon p_{33}) > 0$

As a result, the equation (12) is always active if the following condition is applied:

$$\text{Our extra proposition: } r_{33} > \frac{r_{31}x_1(\varepsilon p_{31} - x_1\, p_{11} - (\varepsilon - x_1)p_{21})}{\varepsilon((\varepsilon - x_1)p_{21} + x_1\, p_{11} - \varepsilon p_{33})} \quad (15)$$

Step 4:

In this step, the intention is to maintain $\dot{x}_3 < 0$ (where $x_3 = \sigma$). Therefore, we just replace $1 - \varepsilon$ with σ as we did in step 3.

$$
\begin{cases}
f_1 \to p_{11} \\
f_2 \to \dfrac{r_{21}x_1 p_{21} + r_{22}(1-\sigma-x_1)p_{22}}{r_{21}x_1 + r_{22}(1-\sigma-x_1)} \\
f_3 \to \dfrac{r_{31}x_1 p_{31} + r_{33}(1-\varepsilon)p_{33}}{r_{31}x_1 + r_{33}\sigma}
\end{cases}
$$

Our proposition: $r_{22} > \dfrac{r_{12}x_1(N_3 - (1-\sigma-x_1)p_{21})}{(1-\sigma-x_1)((1-\sigma-x_1)p_{22} - N_3)}$ (16)

Our extra proposition: $r_{33} > \dfrac{r_{31}x_1((1-\sigma)p_{31} - x_1 p_{11} - (1-\sigma-x_1)p_{21})}{(1-\sigma)((1-\sigma-x_1)p_{21} + x_1 p_{11} - (1-\sigma)p_{33})}$ (17)

Where: $N_3 = (1-\sigma)f_3 - x_2 f_2$

4. **Case A1B4**

According to preceding discussions, our recommendation for this case is:

$$
\begin{cases}
r_{22}r_{33} \to \infty \\
r_{23} \to 0
\end{cases}
\quad \text{and} \quad r_{33} \to \infty
$$

Step 2:

Similar to previous parts, the desired condition is to have no equilibrium point in region 2 and to achieve that, region 2 is divided in 3 sub-regions.

Sub-region 2.1:

$x_2 \to 0$ and the proposed approach is:

$$\begin{cases} r_{12}x_2 \to 0 \\ r_{22}x_2 \to 0 \\ r_{33}x_3 \to \infty \end{cases} \Rightarrow \begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{13}x_3p_{13}}{r_{11}x_1 + r_{13}x_3} \\ f_2 \to p_{21} \\ f_3 \to p_{33} \end{cases}$$

Since in equations above $f_2 > f_3$, there are no equilibrium point in this sub-region in this case.

Sub-region 2.2:

In this sub-region $x_1 \to 0$ and:

$$\begin{cases} r_{11}x_1 \to 0 \quad , \quad r_{21}x_1 \to 0 \\ r_{23}x_3 \to 0 \quad , \quad r_{33}x_3 \to \infty \end{cases} \Rightarrow \begin{cases} f_1 \to \dfrac{r_{12}x_2p_{12} + r_{13}x_3p_{13}}{r_{12}x_2 + r_{13}x_3} \\ f_2 \to p_{22} \\ f_3 \to p_{33} \end{cases}$$

Since $f_2 > f_3$, there is no equilibrium point.

Sub-region 2.3:

Equations for case 1 are:

$$\begin{cases} r_{23}x_3 \to 0 \\ r_{33}x_3 \to \infty \end{cases} \Rightarrow \begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{12}x_2p_{12} + r_{13}x_3p_{12}}{r_{11}x_1 + r_{12}x_2 + r_{13}x_3} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{22}x_2p_{22}}{r_{21}x_1 + r_{22}x_2} \\ f_3 \to p_{33} \end{cases}$$

Since $f_2 \in (p_{22}, p_{21})$, $f_2 > f_3$ and no equilibrium point is identi.

Step 3:

The condition in which $\dot{x}_3 < 0$ (where $x_3 = 1-\varepsilon$) is :

$$\begin{cases} x_3 = 1-\varepsilon \\ x_1 + x_2 = \varepsilon \end{cases} \quad \text{if } : \dot{x}_3 < 0 \quad \Rightarrow \quad f_3 < x_1 f_1 + x_2 f_2 + x_3 f_3 \quad \Rightarrow \quad \varepsilon f_3 < x_1 f_1 + x_2 f_2 \tag{18}$$

So, recommend propositions are made for the equation to be active.

Sub-region 2.1:

In this sub-region:

$$\begin{cases} x_1 \to \varepsilon & , \quad f_1 \to \dfrac{r_{11}\varepsilon p_{11} + r_{13} p_{13}(1-\varepsilon)}{r_{11}\varepsilon + r_{13}(1-\varepsilon)} \\ x_2 \to 0 & , \quad f_2 \to p_{21} \\ x_3 = 1-\varepsilon & , \quad f_3 \to p_{33} \end{cases} \Rightarrow \begin{cases} x_1 f_1 + x_2 f_2 \to \varepsilon \left(\dfrac{r_{11}\varepsilon p_{11} + r_{13} p_{13}(1-\varepsilon)}{r_{11}\varepsilon + r_{13}(1-\varepsilon)} \right) \\ \varepsilon f_3 \to \varepsilon p_{33} \end{cases} \Rightarrow \varepsilon f_3 < x_1 f_1 + x_2 f_2$$

It is obvious that the equation is active.

Sub-region 2.2:

In this region:

$$\begin{cases} x_1 \to 0 & , \quad f_1 \to \dfrac{r_{12} x_2}{r_{12} x_2 + r_{13} x_3} p_{12} + \dfrac{r_{13} x_3}{r_{12} x_2 + r_{13} x_3} p_{13} \\ x_2 \to \varepsilon & , \quad f_2 \to p_{22} \\ x_3 = 1-\varepsilon & , \quad f_3 \to p_{33} \end{cases} \Rightarrow \begin{cases} x_1 f_1 + x_2 f_2 \to \varepsilon p_{22} \\ \varepsilon f_3 \to \varepsilon p_{33} \end{cases} \Rightarrow \varepsilon f_3 < x_1 f_1 + x_2 f_2$$

Similar to previous sub-region, the equation is active.

Sub-region 2.3:

In this region:

$$\begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{12}(\varepsilon - x_1)p_{12} + r_{13}(1-\varepsilon)p_{13}}{r_{11}x_1 + r_{12}(\varepsilon - x_1) + r_{13}(1-\varepsilon)} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{22}(\varepsilon - x_1)p_{22}}{r_{21}x_1 + r_{22}(\varepsilon - x_1)} \\ f_3 \to p_{33} \end{cases}$$

If: $x_1f_1 + x_2f_2 > \varepsilon f_3 \Rightarrow r_{11}x_1(x_1\, p_{11} - M_4) > r_{12}(\varepsilon - x_1)(M_4 - x_1p_{12}) + r_{13}(1-\varepsilon)(M_4 - x_1c)$

Where: $M_4 = \varepsilon f_3 - x_2f_2$

If $M_4 < x_1p_{11}$, then the following situation is our proposition to make equation () active:

$$\text{Our proposition: } r_{11} > \frac{r_{12}(\varepsilon - x_1)(M_4 - x_1p_{12}) + r_{13}(1-\varepsilon)(M_4 - x_1p_{13})}{x_1(x_1\, p_{11} - M_4)} \quad (19)$$

Therefore, we check if the following equation is active:

$$M_4 < x_1p_{11} \quad (20)$$

$$M_4 - x_1p_{11} = \varepsilon i - (\varepsilon - x_1)\left(\frac{r_{21}x_1p_{21} + r_{22}(\varepsilon - x_1)p_{22}}{r_{12}x_1 + r_{22}(\varepsilon - x_1)}\right) - x_1p_{11}$$

$$= \frac{r_{12}x_1(\varepsilon p_{33} - (\varepsilon - x_1)p_{21} - x_1\, p_{11}) - r_{22}(\varepsilon - x_1)(x_1p_{11} + (\varepsilon - x_1)p_{22} - \varepsilon p_{33})}{r_{12}x_1 + r_{22}(\varepsilon - x_1)}$$

And: $\begin{cases} (\varepsilon p_{33} - (\varepsilon - x_1)p_{21} - x_1p_{11}) < 0 \\ (x_1p_1 + (\varepsilon - x_1)p_2 - \varepsilon p_{33}) > 0 \end{cases} \Rightarrow M_4 - x_1p_{11} < 0 \Rightarrow M_4 < x_1p_{11}$

Therefore, the equation () is always active and there is no need to extra proposition for that.

Step 4:

In this step we want to recommend propositions so that $\dot{x}_3 < 0$, where $x_3 = \sigma$. Therefore, it would be similar to step 3, and we just replace $1-\varepsilon$ with σ.

$$
\begin{cases}
f_1 \to \dfrac{r_{11} x_1 p_{11} + r_{12}(1-\sigma-x_1) p_{12} + r_{13}\sigma p_{13}}{r_{11} x_1 + r_{12}(1-\sigma-x_1) + r_{13}\sigma} \\
f_2 \to \dfrac{r_{21} x_1 p_{21} + r_{22}(1-\sigma-x_1) p_{22}}{r_{21} x_1 + r_{22}(1-\sigma-x_1)} \\
f_3 \to p_{33}
\end{cases}
$$

$$\text{Our proposition: } r_{11} > \frac{r_{12}(1-\sigma-x_1)(N_4 - x_1 p_{12}) + r_{13}\sigma(N_4 - x_1 p_{13})}{x_1(x_1 p_{11} - N_4)} \quad (21)$$

Where: $N_4 = (1-\sigma) f_3 - x_2 f_2$

5. **Case A2B1**

Taking into account the previous parts, our proposition for this case is:

$$\frac{r_{22} r_{33}}{r_{23}^2} > k_1 \quad \text{and} \quad \begin{cases} r_{11} r_{33} \to \infty \\ r_{13} \to 0 \end{cases}$$

We propose an extra proposition to keep the left equation active:

$$\text{Extra proposition:} \begin{cases} r_{33} \to \infty \\ \dfrac{r_{22}}{r_{23}^2} \gg 0 \end{cases} \quad (22)$$

Step 2:

In this step we will present propositions to have zero equilibrium points in region 2. Again, region 2 is consisted of 3 sub-regions.

Sub-region 2.1:

In this sub-region, $x_2 \to 0$ and considering the following proposed condition:

$$\begin{cases} r_{12}x_2 \to 0 \quad , \quad r_{13}x_3 \to 0 \\ r_{22}x_2 \to 0 \quad , \quad r_{33}x_3 \to \infty \end{cases} \Rightarrow \begin{cases} f_1 \to p_{11} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{23}x_3p_{23}}{r_{21}x_1 + r_{23}x_3} \\ f_3 \to p_{33} \end{cases}$$

$f_1 > f_3$ and there are no equilibrium points in this sub-region for this case.

Sub-region 2.2:

In this sub-region, $x_1 \to 0$ and indicates that:

$$\begin{cases} r_{11}x_1 \to 0 \quad , \quad r_{13}x_3 \to 0 \\ r_{21}x_1 \to 0 \quad , \quad r_{33}x_{3\infty} \to 0 \end{cases} \Rightarrow \begin{cases} f_1 \to p_{12} \\ f_2 \to \dfrac{r_{22}x_2p_{22} + r_{23}x_3p_{23}}{r_{21}x_1 + r_{23}x_3} \\ f_3 \to p_{33} \end{cases}$$

Again, $f_2 > f_3$ and no equilibrium point exists.

Sub-region 2.3:

In this sub-region for case 1 equations are as bellow:

$$\begin{cases} r_{13}x_3 \to 0 \\ r_3x_3 \to \infty \end{cases} \Rightarrow \begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{12}x_2p_{12}}{r_{11}x_1 + r_{12}x_2} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{22}x_2p_{22} + r_{23}x_3p_{23}}{r_{21}x_1 + r_{22}x_2 + r_{23}x_3} \\ f_3 \to p_{33} \end{cases}$$

Since $f_2 \in (p_{22}, p_{21})$, $f_2 > f_3$ and there is no equilibrium point.

Step 3:

Propositions are made in a way that $\dot{x}_3 < 0$ (where $x_3 = 1-\varepsilon$)

$$\begin{cases} x_3 = 1-\varepsilon \\ x_1 + x_2 = \varepsilon \end{cases} \quad \text{if } : \dot{x}_3 < 0 \quad \Rightarrow \quad f_3 < x_1f_1 + x_2f_2 + x_3f_3 \quad \Rightarrow \quad \varepsilon f_3 < x_1f_1 + x_2f_2$$

Therefore, we will make propositions so that equation becomes active.

Sub-region 2.1:

In this region:

$$\begin{cases} x_1 \to \varepsilon \quad , & f_1 \to p_{11} \\ x_2 \to 0 \quad , & f_2 \to \dfrac{r_{21}\varepsilon p_{21} + r_{23}(1-\varepsilon)p_{23}}{r_{21}\varepsilon + r_{23}(1-\varepsilon)} \Rightarrow \begin{cases} x_1f_1 + x_2f_2 \to \varepsilon p_{11} \\ \varepsilon f_3 \to \varepsilon p_{33} \end{cases} \Rightarrow \varepsilon f_3 < x_1f_1 + x_2f_2 \\ x_3 = 1-\varepsilon \quad , & f_3 \to p_{33} \end{cases}$$

Therefore, in this region the equation is always active and we don’t need any proposition.

Sub-region 2.2:

In this region:

$$\begin{cases} x_1 \to 0 & , \quad f_1 \to p_{12} \\ x_2 \to \varepsilon & , \quad f_2 \to \dfrac{r_{22}\varepsilon p_{22} + r_{23}(1-\varepsilon)p_{23}}{r_{22}\varepsilon + r_{23}(1-\varepsilon)} \\ x_3 = 1-\varepsilon & , \quad f_3 \to p_{33} \end{cases} \Rightarrow \begin{cases} x_1 f_1 + x_2 f_2 \to \varepsilon(\dfrac{r_{22}x_2 p_{22} + r_{23}x_3 p_{23}}{r_{21}x_1 + r_{23}x_3}) \\ \varepsilon f_3 \to \varepsilon p_{33} \end{cases}$$

Since, $p_{22}, p_{23} > p_{33} \quad \Rightarrow x_1 f_1 + x_2 f_2 > \varepsilon f_3$

It is clear that the equation is always active and there is no necessity for proposition.

Sub-region 2.3:

In this region:

$$\begin{cases} f_1 \to \dfrac{r_{11}x_1 p_{11} + r_{12}(\varepsilon - x_1)p_{12}}{r_{11}x_1 + r_{12}(\varepsilon - x_1)} \\ f_2 \to \dfrac{r_{21}x_1 p_{21} + r_{22}x_2 p_{22}}{r_{21}x_1 + r_{22}(\varepsilon - x_1)} \\ f_3 \to p_{33} \end{cases}$$

If: $x_1 f_1 + x_2 f_2 > \varepsilon f_3 \Rightarrow r_{11}x_1(x_1 p_{11} - M_5) > r_{12}(\varepsilon - x_1)(M_5 - x_1 p_{12})$

Where: $M_5 = \varepsilon f_3 - x_2 f_2$

If $M_5 < x_1 p_{11}$, then the following condition is our proposition to make equation () active:

$$\text{Our proposition: } r_{11} > \frac{r_{12}(\varepsilon - x_1)(M_5 - x_1 p_{12})}{x_1(x_1 p_{11} - M_5)} \quad (24)$$

Moreover, the activeness of the following equation is verified:

$$M_5 < x_1 p_{11} \ (25)$$

$$M_5 - x_1 p_{11} = \varepsilon \mathrm{i} - (\varepsilon - x_1)\left(\frac{r_{21}x_1 p_{21} + r_{22}(\varepsilon - x_1)p_{22} + r_{23}(1-\varepsilon)p_{23}}{r_{12}x_1 + r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)}\right) - x_1 p_{11}$$

$$= \frac{r_{12}x_1(\varepsilon p_{33} - (\varepsilon - x_1)p_{21} - x_1 p_{11}) - r_{22}(\varepsilon - x_1)(x_1 p_{11} + (\varepsilon - x_1)p_{22} - \varepsilon p_{33}) - r_{23}(1-\varepsilon)(x_1 p_{11} + (\varepsilon - x_1)p_{23} - \varepsilon p_{33})}{r_{12}x_1 + r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)}$$

$$\text{And: } \begin{cases} (\varepsilon p_{33} - (\varepsilon - x_1)p_{21} - x_1 p_{11}) < 0 \\ (x_1 p_1 + (\varepsilon - x_1)p_2 - \varepsilon p_{33}) > 0 \\ (x_1 p_{11} + (\varepsilon - x_1)p_{23} - \varepsilon p_{33}) > 0 \end{cases} \Rightarrow M_5 - x_1 p_{11} < 0 \Rightarrow M_5 < x_1 p_{11}$$

Therefore, the equation (25) is always active and no propositions are needed.

Step 4:

In this step, the intention is to maintain $\dot{x}_3 < 0$ (where $x_3 = \sigma$). Therefore, we just replace $1-\varepsilon$ with σ as we did in step 3.

$$\begin{cases} f_1 \to \dfrac{r_{11}x_1 p_{11} + r_{12}(1-\sigma - x_1)p_{12}}{r_{11}x_1 + r_{12}(1-\sigma - x_1)} \\ f_2 \to \dfrac{r_{21}x_1 p_{21} + r_{22}x_2 p_{22}}{r_{21}x_1 + r_{22}(1-\sigma - x_1)} \\ f_3 \to p_{33} \end{cases}$$

$$\text{Our proposition: } r_{11} > \frac{r_{12}(1-\sigma - x_1)(N_5 - x_1 p_{12})}{x_1(x_1 p_{11} - N_5)} \quad (26)$$

Where: $N_5 = (1-\sigma)f_3 - x_2 f_2$

6. **Case A2B2**

According to preceding parts, our proposition for this case is:

$$\frac{r_{22}r_{33}}{r_{23}^2} > k_1 \quad \text{and} \quad \frac{r_{11}r_{33}}{r_{13}^2} > k_2$$

An extra condition is applied as follows to keep the equation active:

$$\text{Extra proposition:} \begin{cases} r_{33} \to \infty \\ \dfrac{r_{22}}{r_{23}^2} \gg 0 \\ \dfrac{r_{11}}{r_{13}^2} \gg 0 \end{cases} \qquad (27)$$

Step 2:

Region 2 is divided into 3 sub-regions and the idea is to eliminate equilibrium points in this region.

Sub-region 2.1:

In this sub-region, $x_2 \to 0$ and the proposed conditions are:

$$\begin{cases} r_{12}x_2 \to 0 \\ r_{22}x_2 \to 0 \\ r_{33}x_3 \to \infty \end{cases} \Rightarrow \begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{13}x_3p_{13}}{r_{11}x_1 + r_{13}x_3} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{23}x_3p_{23}}{r_{21}x_1 + r_{23}x_3} \\ f_3 \to p_{33} \end{cases}$$

Since $f_2 > f_3$, there are no equilibrium point in this sub-region for this case.

Sub-region 2.2:

In this sub-region, $x_1 \to 0$ and:

$$\begin{cases} r_{11}x_1 \to 0 \\ r_{21}x_1 \to 0 \\ r_{33}x_3 \to \infty \end{cases} \Rightarrow \begin{cases} f_1 \to \dfrac{r_{12}x_2p_{12} + r_{13}x_3p_{13}}{r_{12}x_2 + r_{13}x_3} \\ f_2 \to \dfrac{r_{22}x_2p_{22} + r_{23}x_3p_{23}}{r_{22}x_2 + r_{23}x_3} \\ f_3 \to p_{33} \end{cases}$$

$f_2 > f_3$ and there would be no equilibrium points.

Sub-region 2.3:

Equations for case 1 are as follows:

$$r_3x_3 \to \infty \Rightarrow \begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{12}x_2p_{12} + r_{13}x_3p_{13}}{r_{11}x_1 + r_{12}x_2 + r_{13}x_3} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{22}x_2p_{22} + r_{23}x_3p_{23}}{r_{21}x_1 + r_{22}x_2 + r_{23}x_3} \\ f_3 \to p_{33} \end{cases}$$

$f_2 \in (p_{23}, p_{21})$ and $f_2 > f_3$ which implies the existence of no equilibrium points.

Step 3:

Conditions are applied to maintain $\dot{x}_3 < 0$, where $x_3 = 1-\varepsilon$.

$$\begin{cases} x_3 = 1-\varepsilon \\ x_1 + x_2 = \varepsilon \end{cases} \text{if } : \dot{x}_3 < 0 \Rightarrow f_3 < x_1f_1 + x_2f_2 + x_3f_3 \Rightarrow \varepsilon f_3 < x_1f_1 + x_2f_2 \quad (28)$$

Thus, the suggestions are in a way to keep the above equations active.

Sub-region 2.1:

In this sub-region:

$$\begin{cases} x_1 \to \varepsilon \quad , \quad f_1 \to \dfrac{r_{11}\varepsilon p_{11} + r_{13}(1-\varepsilon)p_{13}}{r_{11}\varepsilon + r_{13}(1-\varepsilon)} \\ x_2 \to 0 \quad , \quad f_2 \to \dfrac{r_{21}\varepsilon p_{21} + r_{23}(1-\varepsilon)p_{23}}{r_{21}\varepsilon + r_{23}(1-\varepsilon)} \\ x_3 = 1-\varepsilon \quad , \quad f_3 \to p_{33} \end{cases} \Rightarrow \begin{cases} x_1 f_1 + x_2 f_2 \to \varepsilon\left(\dfrac{r_{11}\varepsilon p_{11} + r_{13}(1-\varepsilon)p_{13}}{r_{11}\varepsilon + r_{13}(1-\varepsilon)}\right) \\ \varepsilon f_3 \to \varepsilon p_{33} \end{cases}$$

Since, $p_{11}, p_{13} > p_{33} \quad \Rightarrow x_1 f_1 + x_2 f_2 > \varepsilon f_3$

Therefore, in this sub-region the equation is always active and we don't need any proposition.

Sub-region 2.2:

In this region:

$$\begin{cases} x_1 \to 0 \quad , \quad f_1 \to \dfrac{r_{12}\varepsilon p_{12} + r_{13}(1-\varepsilon)p_{13}}{r_{12}\varepsilon + r_{13}(1-\varepsilon)} \\ x_2 \to \varepsilon \quad , \quad f_2 \to \dfrac{r_{22}\varepsilon p_{22} + r_{23}(1-\varepsilon)p_{23}}{r_{22}\varepsilon + r_{23}(1-\varepsilon)} \\ x_3 = 1-\varepsilon \quad , \quad f_3 \to p_{33} \end{cases} \Rightarrow \begin{cases} x_1 f_1 + x_2 f_2 \to \varepsilon(\dfrac{r_{22}x_2 p_{22} + r_{23}x_3 p_{23}}{r_{22}x_1 + r_{23}x_3}) \\ \varepsilon f_3 \to \varepsilon p_{33} \end{cases}$$

Since, $p_{22}, p_{23} > p_{33} \quad \Rightarrow x_1 f_1 + x_2 f_2 > \varepsilon f_3$

As a result the equation is active and no extra condition is required.

Sub-region 2.3:

In this sub-region:

$$\begin{cases} f_1 \rightarrow \dfrac{r_{11}x_1p_{11} + r_{12}(\varepsilon - x_1)p_{12} + r_{13}(1-\varepsilon)p_{13}}{r_{11}x_1 + r_{12}(\varepsilon - x_1) + r_{13}(1-\varepsilon)} \\ f_2 \rightarrow \dfrac{r_{21}x_1p_{21} + r_{22}(\varepsilon - x_1)p_{22} + r_{13}(1-\varepsilon)p_{23}}{r_{21}x_1 + r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)} \\ f_3 \rightarrow p_{33} \end{cases}$$

If:

$$x_1f_1 + x_2f_2 > \varepsilon f_3 \Rightarrow r_{11}x_1(x_1 p_{11} - M_6) > r_{12}(\varepsilon - x_1)(M_6 - x_1p_{12}) + r_{13}(1-\varepsilon)(M_6 - x_1p_{13})$$

Where: $M_6 = \varepsilon f_3 - x_2f_2$

If $M_5 < x_1p_{11}$, then the proposed condition to make the equation 27 an active equation is:

$$r_{11} > \frac{r_{12}(\varepsilon - x_1)(M_6 - x_1p_{12}) + r_{13}(1-\varepsilon)(M_6 - x_1p_{13})}{x_1(x_1 p_{11} - M_6)} \quad (28)$$

The following equation must be active: :

$$M_6 < x_1p_{11} \quad (29)$$

$$M_6 - x_1p_{11} = \varepsilon i - (\varepsilon - x_1)\left(\frac{r_{21}x_1p_{21} + r_{22}(\varepsilon - x_1)p_{22} + r_{23}(1-\varepsilon)p_{23}}{r_{12}x_1 + r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)}\right) - x_1p_{11}$$

$$= \frac{r_{12}x_1(\varepsilon p_{33} - (\varepsilon - x_1)p_{21} - x_1 p_{11}) - r_{22}(\varepsilon - x_1)(x_1p_{11} + (\varepsilon - x_1)p_{22} - \varepsilon p_{33}) - r_{23}(1-\varepsilon)(x_1p_{11} + (\varepsilon - x_1)p_{23} - \varepsilon p_{33})}{r_{12}x_1 + r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)}$$

And: $$\begin{cases} (\varepsilon p_{33} - (\varepsilon - x_1)p_{21} - x_1p_{11}) < 0 \\ (x_1p_{11} + (\varepsilon - x_1)p_{22} - \varepsilon p_{33}) > 0 \\ (x_1p_{11} + (\varepsilon - x_1)p_{23} - \varepsilon p_{33}) > 0 \end{cases} \Rightarrow M_6 - x_1p_{11} < 0 \Rightarrow M_6 < x_1p_{11}$$

Therefore, the equation (29) is always active and no added condition is required.

Step 4:

In this step we want to recommend propositions to make $\dot{x}_3 < 0$, where $x_3 = \sigma$. Similar to step 3 $1-\varepsilon$ is replaced with σ in equations.

$$\begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{12}(1-\sigma-x_1)p_{12} + r_{13}\sigma p_{13}}{r_{11}x_1 + r_{12}(1-\sigma-x_1) + r_{13}\sigma} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{22}(1-\sigma-x_1)p_{22} + r_{13}(\sigma p_{23}}{r_{21}x_1 + r_{22}(1-\sigma-x_1) + r_{23}\sigma} \\ f_3 \to p_{33} \end{cases}$$

$$\text{Our proposition: } r_{11} > \frac{r_{12}(1-\sigma-x_1)(N_6 - x_1p_{12}) + r_{13}\sigma(N_6 - x_1p_{13})}{x_1(x_1p_{11} - N_6)} \quad (30)$$

Where: $N_6 = (1-\sigma)f_3 - x_2f_2$

7. **Case A2B3**

Recalling the previous sections, our suggestion for this case is::

$$\frac{r_{22}r_{33}}{r_{23}^2} > k_1 \quad \text{and} \quad r_{11} \to \infty$$

Applying the following extra condition to this case makes the left equation active:

$$\text{Extra proposition:} \begin{cases} r_{23} \to 0 \\ r_{22}r_{33} \gg 0 \end{cases} \quad (31)$$

Step 2:

Propositions are made to eliminate equilibrium points in region 2. We assume region 2 is consist of 3 sub-regions.

Sub-region 2.1:

In this sub-region, $x_2 \to 0$ and considering our propositions we have:

$$\begin{cases} r_{11}x_1 \to \infty, & r_{22}x_2 \to 0 \\ r_{23}x_3 \to 0, & r_{32}x_2 \to 0 \end{cases} \Rightarrow \begin{cases} f_1 \to p_{11} \\ f_2 \to p_{21} \\ f_3 \to \dfrac{r_{31}x_1p_{31} + r_{33}x_3p_{33}}{r_{31}x_1 + r_{33}x_3} \end{cases}$$

$f_1 > f_3$ and consequently there are no equilibrium points.

Sub-region 2.2:

In this sub-region, $x_1 \to 0$, similar to sub-region 1, we have:

$$\begin{cases} r_{21}x_1 \to 0, & r_{23}x_3 \to 0 \\ r_{32}x_2 \to \infty, & r_{33}x_3 \to 0 \end{cases} \Rightarrow \begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{12}x_2p_{12} + r_{13}x_3p_{13}}{r_{11}x_1 + r_{12}x_2 + r_{13}x_3} \\ f_2 \to p_{22} \\ f_3 \to p_{33} \end{cases}$$

Since $f_2 > f_3$ no equilibrium points exists.

Sub-region 2.3:

Equations for case 1are:

$$\begin{cases} r_{11}x_1 \to \infty \\ r_{23}x_3 \to 0 \\ r_{32}x_2 \to 0 \end{cases} \Rightarrow \begin{cases} f_1 \to p_{11} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{22}x_2p_{22}}{r_{21}x_1 + r_{22}x_2} \\ f_3 \to \dfrac{r_{31}x_1p_{31} + r_{33}x_3p_{33}}{r_{31}x_1 + r_{33}x_3} \end{cases}$$

Since, $f_2 \in (p_{22}, p_{21})$, always $f_2 > f_3$ and there are no equilibrium points.

Step 3:

The objective is to have $\dot{x}_3 < 0$ (where $x_3 = 1-\varepsilon$)

$$\begin{cases} x_3 = 1-\varepsilon \\ x_1 + x_2 = \varepsilon \end{cases} \quad \text{if } : \dot{x}_3 < 0 \quad \Rightarrow \quad f_3 < x_1f_1 + x_2f_2 + x_3f_3 \quad \Rightarrow \quad \varepsilon f_3 < x_1f_1 + x_2f_2 \quad (32)$$

Therefore, propositions are recommended to activate the above equation.

Sub-region 2.1:

In this region:

$$\begin{cases} x_1 \to \varepsilon \quad , \quad f_1 \to p_{11} \\ x_2 \to 0 \quad , \quad f_2 \to p_{21} \\ x_3 = 1-\varepsilon \quad , \quad f_3 \to \dfrac{r_{31}\varepsilon p_{31} + r_{33}(1-\varepsilon)p_{33}}{r_{31}\varepsilon + r_{33}(1-\varepsilon)} \end{cases} \Rightarrow \begin{cases} x_1f_1 + x_2f_2 \to \varepsilon p_{11} \\ \varepsilon f_3 \to \varepsilon\left(\dfrac{r_{31}\varepsilon p_{31} + r_{33}(1-\varepsilon)p_{33}}{r_{31}\varepsilon + r_{33}(1-\varepsilon)}\right) \end{cases}$$

$p_{11} > p_{31}, p_{33} \quad \Rightarrow x_1f_1 + x_2f_2 > \varepsilon f_3$.

Therefore, in this region the equation is always active and we don’t need any proposition.

Sub-region 2.2:

In this region:

$$\begin{cases} x_1 \to 0 & , \quad f_1 \to \dfrac{r_{11}x_1p_{11} + r_{12}(\varepsilon - x_1)p_{12} + r_{13}(1-\varepsilon)p_{13}}{r_{11}x_1 + r_{12}(\varepsilon - x_1) + r_{13}(1-\varepsilon)} \\ x_2 \to \varepsilon & , \quad f_2 \to p_{22} \\ x_3 = 1-\varepsilon & , \quad f_3 \to p_{33} \end{cases} \Rightarrow \begin{cases} x_1f_1 + x_2f_2 \to \varepsilon p_{22} \\ \varepsilon f_3 \to \varepsilon p_{33} \end{cases}$$

Therefore, in this region, the equation 32 is always active and we don't need any proposition.

Sub-region 2.3:

$$\begin{cases} f_1 \to p_{11} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{22}(\varepsilon - x_1)p_{22}}{r_{21}x_1 + r_{22}(\varepsilon - x_1)} \\ f_3 \to \dfrac{r_{31}x_1p_{31} + r_{33}(1-\varepsilon)p_{33}}{r_{31}x_1 + r_{33}(1-\varepsilon)} \end{cases}$$

If: $x_1f_1 + x_2f_2 > \varepsilon f_3 \Rightarrow r_{22}(\varepsilon - x_1)((\varepsilon - x_1)p_{22} - M_7) > r_{21}x_1(M_7 - (\varepsilon - x_1)p_{12})$

Where: $M_7 = \varepsilon f_3 - x_1f_1$

If $M_7 < (\varepsilon - x_1)p_{21}$, then the following situation is our proposition to make equation (32) active:

$$\text{Our proposition: } r_{22} > \frac{r_{21}x_1(M_7 - (\varepsilon - x_1)p_{21})}{(\varepsilon - x_1)((\varepsilon - x_1)p_{22} - M_7)} \qquad (33)$$

Therefore, we check if the following equation is active:

$$M_7 < (\varepsilon - x_1)p_{21} \qquad (34)$$

$$M_7 - (\varepsilon - x_1)p_{21} = \varepsilon\left(\frac{r_{31}x_1p_{31} + r_{33}(1-\varepsilon)p_{33}}{r_{31}x_1 + r_{33}(1-\varepsilon)}\right) - x_1p_{11} - (\varepsilon - x_1)p_{21}$$

$$= \frac{r_{31}x_1(\varepsilon p_{31} - x_1p_{11} - (\varepsilon - x_1)p_{21}) - r_{33}(1-\varepsilon)((\varepsilon - x_1)p_{21} + x_1p_{11} - \varepsilon p_{33})}{r_{31}x_1 + r_{33}(1-\varepsilon)}$$

Since, $((\varepsilon - x_1)p_{21} + x_1p_{11} - \varepsilon p_{33}) > 0$ by applying the following extra proposition the equation is always active and no extra condition is required.:

$$\text{Our extra proposition: } r_{33} > \frac{r_{13}x_1(\varepsilon p_{31} - x_1p_{11} - (\varepsilon - x_1)p_{21})}{(1-\varepsilon)((\varepsilon - x_1)p_{21} + x_1p_{11} - \varepsilon p_{33})} \quad (35)$$

Step 4:

Propositions to have $\dot{x}_3 < 0$ (where $x_3 = \sigma$) are made by replacing $1-\varepsilon$ with σ..

$$\begin{cases} f_1 \to p_{11} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{22}(1-\sigma-x_1)p_{22}}{r_{21}x_1 + r_{22}(1-\sigma-x_1)} \\ f_3 \to \dfrac{r_{31}x_1p_{31} + r_{33}\sigma p_{33}}{r_{31}x_1 + r_{33}\sigma} \end{cases}$$

$$\text{Our proposition: } r_{22} > \frac{r_{21}x_1(N_7 - (1-\sigma-x_1)p_{21})}{(1-\sigma-x_1)((1-\sigma-x_1)p_{22} - N_7)} \quad (36)$$

$$\text{extra proposition: } r_{33} > \frac{r_{13}x_1((1-\sigma)p_{31} - x_1p_{11} - (1-\sigma-x_1)p_{21})}{\sigma((1-\sigma-x_1)p_{21} + x_1p_{11} - (1-\sigma)p_{33})} \quad (37)$$

8. **Case A4B4**

Proposition for this are:

$$\frac{r_{22}r_{33}}{r_{23}^{2}} > k_1 \quad \text{and} \quad r_{33} \to \infty$$

:

Step 2:

To prevent locating equilibrium points in region 2 the following analysis are carried out in 3 different sub-regions. **Sub-region 2.1:**

In this sub-region, $x_2 \to 0$ and::

$$\begin{cases} r_{12}x_2 \to 0 \\ r_{22}x_2 \to 0 \\ r_{33}x_3 \to \infty \end{cases} \Rightarrow \begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{13}x_3p_{13}}{r_{11}x_1 + r_{13}x_3} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{23}x_3p_{23}}{r_{21}x_1 + r_{23}x_3} \\ f_3 \to p_{33} \end{cases}$$

Again $f_2 > f_3$ and there are no equilibrium points in this sub-region for this case.

Sub-region 2.2:

In this sub-region, $x_1 \to 0$, similar to sub-region 1 we may write:

$$\begin{cases} r_{11}x_1 \to 0 \\ r_{21}x_1 \to 0 \\ r_{33}x_3 \to \infty \end{cases} \Rightarrow \begin{cases} f_1 \to \dfrac{r_{12}x_2p_{12} + r_{13}x_3p_{13}}{r_{12}x_2 + r_{13}x_3} \\ f_2 \to \dfrac{r_{22}x_2p_{22} + r_{23}x_3p_{23}}{r_{22}x_2 + r_{23}x_3} \\ f_3 \to p_{33} \end{cases}$$

Since $f_2 > f_3$ no equilibrium point is located in this sub-region.

Sub-region 2.3:

For case 1, the following conditions are proposed:

$$r_{33}x_3 \to \infty \quad \Rightarrow \quad \begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{12}x_2p_{12} + r_{13}x_3p_{13}}{r_{11}x_1 + r_{12}x_2 + r_{13}x_3} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{22}x_2p_{22} + r_{23}x_3p_{23}}{r_{21}x_1 + r_{22}x_2 + r_{23}x_3} \\ f_3 \to p_{33} \end{cases}$$

$f_2 \in (p_{23}, p_{21})$ that means $f_2 > f_3$ and there is no equilibrium point here.

Step 3:

In this step we attempt to keep a $\dot{x}_3 < 0$ (where $x_3 = 1-\varepsilon$)

$$\begin{cases} x_3 = 1-\varepsilon \\ x_1 + x_2 = \varepsilon \end{cases} \quad \text{if } : \dot{x}_3 < 0 \quad \Rightarrow \quad f_3 < x_1f_1 + x_2f_2 + x_3f_3 \quad \Rightarrow \quad \varepsilon f_3 < x_1f_1 + x_2f_2 \quad (38)$$

So, we will recommend propositions to make the equation active.

Sub-region 2.1:

In this sub-region:

$$\begin{cases} x_1 \to \varepsilon \quad , \quad f_1 \to \dfrac{r_{11}x_1p_{11} + r_{13}(1-\varepsilon)p_{13}}{r_{11}x_1 + r_{13}(1-\varepsilon)} \\ x_2 \to 0 \quad , \quad f_2 \to \dfrac{r_{21}x_1p_{21} + r_{23}(1-\varepsilon)p_{23}}{r_{21}x_1 + r_{23}(1-\varepsilon)} \\ x_3 = 1-\varepsilon \quad , \quad f_3 \to p_{33} \end{cases} \Rightarrow \begin{cases} x_1f_1 + x_2f_2 \to \varepsilon\left(\dfrac{r_{11}x_1p_{11} + r_{13}(1-\varepsilon)p_{13}}{r_{11}x_1 + r_{13}(1-\varepsilon)}\right) \\ \varepsilon f_3 \to \varepsilon p_{33} \end{cases}$$

$$p_{11}, p_{13} > p_{33} \quad \Rightarrow x_1 f_1 + x_2 f_2 > \varepsilon f_3 \ .$$

Therefore, the equation is always active and no proposal is needed..

Sub-region 2.2:

In this sub-region:

$$\begin{cases} x_1 \to 0 \quad , \quad f_1 \to \dfrac{r_{12}(\varepsilon - x_1)p_{12} + r_{13}(1-\varepsilon)p_{13}}{r_{12}(\varepsilon - x_1) + r_{13}(1-\varepsilon)} \\ x_2 \to \varepsilon \quad , \quad f_2 \to \dfrac{r_{22}(\varepsilon - x_1)p_{22} + r_{23}(1-\varepsilon)p_{23}}{r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)} \\ x_3 = 1-\varepsilon \quad , \quad f_3 \to p_{33} \end{cases} \Rightarrow \begin{cases} x_1 f_1 + x_2 f_2 \to \varepsilon \left(\dfrac{r_{22}(\varepsilon - x_1)p_{22} + r_{23}(1-\varepsilon)p_{23}}{r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)} \right) \\ \varepsilon f_3 \to \varepsilon p_{33} \end{cases}$$

Thus, the equation (38) is active and we don’t need any proposition as well.

Sub-region 2.3:

In this sub-region:

$$\begin{cases} f_1 \to \dfrac{r_{11}x_1 p_{11} + r_{12}(\varepsilon - x_1)p_{12} + r_{13}(1-\varepsilon)p_{13}}{r_{11}x_1 + r_{12}(\varepsilon - x_1) + r_{13}(1-\varepsilon)} \\ f_2 \to \dfrac{r_{21}x_1 p_{21} + r_{22}(\varepsilon - x_1)p_{22} + r_{23}(1-\varepsilon)p_{23}}{r_{21}x_1 + r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)} \\ f_3 \to p_{33} \end{cases}$$

If:

$$x_1 f_1 + x_2 f_2 > \varepsilon f_3 \Rightarrow r_{11}x_1(x_1 p_{11} - M_8) > r_{12}(\varepsilon - x_1)(M_8 - x_1 b) + r_{13}(1-\varepsilon)(M_8 - x_1 p_{13})$$

Where: $M_8 = \varepsilon f_3 - x_2 f_2$

If $M_8 < x_1 p_{11}$, then the following condition is our proposition to make equation (38) active:

Our proposition: $r_{11} > \dfrac{r_{12}(\varepsilon - x_1)(M_8 - x_1 p_{12}) + r_{13}(1-\varepsilon)(M_8 - x_1 p_{13})}{x_1(x_1 p_{11} - M_8)}$ (39)

Therefore, we evaluate the activeness of the following equation::

$$M_8 < x_1 p_{11} \quad (40)$$

$$M_8 - x_1 p_{11} = \varepsilon p_{33} - (\varepsilon - x_1)\left(\frac{r_{21}x_1 p_{21} + r_{22}(\varepsilon - x_1)p_{22} + r_{23}(1-\varepsilon)p_{23}}{r_{21}x_1 + r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)}\right) - x_1 p_{11}$$

$$= \frac{r_{21}x_1(\varepsilon p_{33} - (\varepsilon - x_1)p_{21} - x_1 p_{11}) - r_{22}(\varepsilon - x_1)(x_1 p_{11} + (\varepsilon - x_1)p_{22} - \varepsilon p_{33}) - r_{23}(1-\varepsilon)(x_1 p_{11} + (\varepsilon - x_1)p_{23} - \varepsilon p_{33})}{r_{21}x_1 + r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)}$$

$$\begin{cases} (\varepsilon p_{33} - (\varepsilon - x_1)p_{21} - x_1 p_{11}) < 0 \\ (x_1 p_{11} + (\varepsilon - x_1)p_{22} - \varepsilon p_{33}) > 0 \\ (x_1 p_{11} + (\varepsilon - x_1)p_{23} - \varepsilon p_{33}) > 0 \end{cases} \Rightarrow M_8 - x_1 p_{11} < 0 \Rightarrow M_8 < x_1 p_{11}$$

It is apparent that the equation is active and no propositions are recommended.

Step 4:

In this step we want to recommend propositions so that $\dot{x}_3 < 0$ (where $x_3 = \sigma$). It is exactly similar to step 3, and we just replace $1-\varepsilon$ with σ.

$$\begin{cases} f_1 \rightarrow \dfrac{r_{11}x_1 p_{11} + r_{12}(1-\sigma - x_1)p_{12} + r_{13}\sigma p_{13}}{r_{11}x_1 + r_{12}(1-\sigma - x_1) + r_{13}\sigma} \\ f_2 \rightarrow \dfrac{r_{21}x_1 p_{21} + r_{22}(1-\sigma - x_1)p_{22} + r_{23}\sigma p_{23}}{r_{21}x_1 + r_{22}(1-\sigma - x_1) + r_{23}\sigma} \\ f_3 \rightarrow p_{33} \end{cases}$$

Our proposition: $r_{11} > \dfrac{r_{12}(1-\sigma - x_1)(N_8 - x_1 p_{12}) + r_{13}\sigma(N_8 - x_1 p_{13})}{x_1(x_1 p_{11} - N_8)}$

Where: $N_8 = (1-\sigma)f_3 - x_2 f_2$

9. **Case A3B1**

According to previous parts, our proposition for this case is :

$$r_{22} \to \infty \quad \text{and} \quad \begin{cases} r_{11}r_{33} \to \infty \\ r_{13} \to 0 \end{cases}$$

Step 2:

In this step we will present propositions to have zero equilibrium points in region 2. Again, region 2 is consisted of 3 sub-regions

Sub-region 2.1:

In this sub-region, $x_2 \to 0$ and taking into account the proposed conditions.:

$$\begin{cases} r_{12}x_2 \to 0, \quad r_{13}x_3 \to 0 \\ r_{31}x_1 \to 0, \quad r_{32}x_2 \to 0 \end{cases} \Rightarrow \begin{cases} f_1 \to p_{11} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{22}x_2p_{22} + r_{23}x_3p_{23}}{r_{21}x_1 + r_{22}x_2 + r_{23}x_3} \\ f_3 \to p_{33} \end{cases}$$

Since, $f_1 > f_3$, there are no equilibrium point in this sub-region for this case.

Sub-region 2.2:

$x_1 \to 0$ and:

$$\begin{cases} r_{11}x_1 \to 0, \quad r_{13}x_3 \to 0 \\ r_{22}x_2 \to \infty, \quad r_{31}x_1 \to 0 \end{cases} \Rightarrow \begin{cases} f_1 \to p_{12} \\ f_2 \to p_{22} \\ f_3 \to \dfrac{r_{32}x_2 p_{32} + r_{33}x_3 p_{33}}{r_{32}x_2 + r_{33}x_3} \end{cases}$$

$f_2 > f_3$, there is no equilibrium point in this sub-region.

Sub-region 2.3:

In this sub-region for case 1 equations are as bellow:

$$\begin{cases} r_{13}x_3 \to 0 \\ r_{22}x_2 \to \infty \\ r_{31}x_1 \to 0 \end{cases} \Rightarrow \begin{cases} f_1 \to \dfrac{r_{11}x_1 p_{11} + r_{12}x_2 p_{12}}{r_{11}x_1 + r_{12}x_2} \\ f_2 \to p_{22} \\ f_3 \to \dfrac{r_{32}x_2 p_{32} + r_{33}x_3 p_{33}}{r_{32}x_2 + r_{33}x_3} \end{cases}$$

Since, $f_2 \in (p_{23}, p_{21})$, $f_2 > f_3$ there is no equilibrium point is located in this region.

Step 3:

In this step we attempt to keep a $\dot{x}_3 < 0$ (where $x_3 = 1 - \varepsilon$)

$$\begin{cases} x_3 = 1 - \varepsilon \\ x_1 + x_2 = \varepsilon \end{cases} \quad \text{if } : \dot{x}_3 < 0 \quad \Rightarrow \quad f_3 < x_1 f_1 + x_2 f_2 + x_3 f_3 \quad \Rightarrow \quad \varepsilon f_3 < x_1 f_1 + x_2 f_2 \quad (41)$$

So, we will recommend propositions to make the equation active

Sub-region 2.1:

In this sub-region:

$$\begin{cases} x_1 \to \varepsilon \quad , & f_1 \to p_{11} \\ x_2 \to 0 \quad , & f_2 \to \dfrac{r_{21}x_1p_{21} + r_{22}(\varepsilon - x_1)p_{22} + r_{23}(1-\varepsilon)p_{23}}{r_{21}x_1 + r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)} \\ x_3 = 1-\varepsilon \quad , & f_3 \to p_{33} \end{cases} \Rightarrow \begin{cases} x_1f_1 + x_2f_2 \to \varepsilon p_{11} \\ \varepsilon f_3 \to \varepsilon p_{33} \end{cases} \Rightarrow x_1f_1 + x_2f_2 > \varepsilon f_3$$

Therefore, in this region the equation is always active and we don't need any proposition.

Sub-region 2.2:

In this sub-region:

$$\begin{cases} x_1 \to 0 \quad , & f_1 \to p_{12} \\ x_2 \to \varepsilon \quad , & f_2 \to p_{22} \\ x_3 = 1-\varepsilon \quad , & f_3 \to \dfrac{r_{32}(\varepsilon - x_1)p_{32} + r_{33}(1-\varepsilon)p_{33}}{r_{32}(\varepsilon - x_1) + r_{33}(1-\varepsilon)} \end{cases} \Rightarrow \begin{cases} x_1f_1 + x_2f_2 \to \varepsilon p_{22} \\ \varepsilon f_3 \to \varepsilon p_{33} \end{cases} \Rightarrow x_1f_1 + x_2f_2 > \varepsilon f_3$$

Therefore, the equation is always active and no proposal is needed.

Sub-region 2.3:

In this sub-region:

$$\begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{12}(\varepsilon - x_1)p_{12}}{r_{11}x_1 + r_{12}(\varepsilon - x_1)} \\ f_2 \to p_{22} \\ f_3 \to \dfrac{r_{32}(\varepsilon - x_1)p_{32} + r_{33}(1-\varepsilon)p_{33}}{r_{32}(\varepsilon - x_1) + r_{33}(1-\varepsilon)} \end{cases}$$

If: $x_1f_1 + x_2f_2 > \varepsilon f_3 \Rightarrow r_{11}x_1(x_1 p_{11} - M_9) > r_{12}(\varepsilon - x_1)(M_9 - x_1 b)$

Where: $M_9 = \varepsilon f_3 - x_2f_2$

If $M_9 < x_1p_{11}$, the following situation would be our proposition to make equation (41) active:

$$\text{Our proposition: } r_{11} > \frac{r_{12}(\varepsilon - x_1)(M_9 - x_1 p_{12})}{x_1(x_1 p_{11} - M_9)} \quad (42)$$

Therefore, we check if the following equation is active:

$$M_9 < x_1 p_{11} \quad (43)$$

$$M_9 - x_1 p_{11} = \varepsilon\left(\frac{r_{32}(\varepsilon - x_1)p_{32} + r_{33}(1-\varepsilon)p_{33}}{r_{32}(\varepsilon - x_1) + r_{33}(1-\varepsilon)}\right) - (\varepsilon - x_1)p_{22} - x_1 p_{11}$$

$$= \frac{r_{32}(\varepsilon - x_1)(\varepsilon p_{32} - (\varepsilon - x_1)p_{22} - x_1 p_{11}) - r_{33}(1-\varepsilon)(-\varepsilon p_{33} + (\varepsilon - x_1)p_{22} + x_1 p_{11})}{r_{21}x_1 + r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)}$$

Since $(-\varepsilon p_{33} + (\varepsilon - x_1)p_{22} + x_1 p_{11}) > 0$, With the following extra proposition the equation is always active:

$$\text{Our extra proposition: } r_{23} > \frac{r_{33}(1-\varepsilon)(\varepsilon p_{31} - (\varepsilon - x_1)p_{22} + x_1 p_{11})}{(\varepsilon - x_1)(\varepsilon p_{32} - (\varepsilon - x_1)p_{22} - x_1 p_{11})} \quad (44)$$

Therefore, the equation () becomes active and there is no need for extra conditions to be applied.

Step 4:

In this step we attempt to keep a $\dot{x}_3 < 0$ (where $x_3 = 1-\varepsilon$) by replacing $1-\varepsilon$ with σ.

$$\begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{12}(1-\sigma-x_1)p_{12}}{r_{11}x_1 + r_{12}(1-\sigma-x_1)} \\ f_2 \to p_{22} \\ f_3 \to \dfrac{r_{32}(1-\sigma-x_1)p_{32} + r_{33}\sigma p_{33}}{r_{32}(1-\sigma-x_1) + r_{33}\sigma} \end{cases}$$

Our proposition: $r_{11} > \dfrac{r_{12}(1-\sigma-x_1)(N_9 - x_1p_{12})}{x_1(x_1\, p_{11} - N_9)}$ (45)

Our extra proposition: $r_{23} > \dfrac{r_{33}\sigma((1-\sigma)p_{31} - ((1-\sigma)-x_1)p_{22} + x_1\, p_{11})}{(1-\sigma-x_1)((1-\sigma)p_{32} - (1-\sigma-x_1)p_{22} - x_1\, p_{11})}$ (46)

Where: $N_9 = (1-\sigma)f_3 - x_2 f_2$

10. **Case A3B2**

According to previous parts, our proposition for this case is as bellow:

$$r_{22} \to \infty \quad \text{and} \quad \frac{r_{11}r_{33}}{r_{13}^2} > k_2$$

We propose an extra proposition to this case so that the left equation above always be active:

$$\begin{cases} r_{13} \to 0 \\ r_{11}r_{33} \gg 0 \end{cases} \qquad (47)$$

Step 2:

In this step we will present propositions to have zero equilibrium points in region 2. Again, region 2 is consisted of 3 sub-regions.

Sub-region 2.1:

In this sub-region, $x_2 \to 0$ and the propositions are::

$$\begin{cases} r_{12}x_2 \to 0, \quad r_{13}x_3 \to 0 \\ r_{31}x_1 \to 0, \quad r_{32}x_2 \to 0 \end{cases} \Rightarrow \begin{cases} f_1 \to p_{11} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{22}x_2p_{22} + r_{23}x_3p_{23}}{r_{21}x_1 + r_{22}x_2 + r_{23}x_3} \\ f_3 \to p_{33} \end{cases}$$

Since in equations above $f_1 > f_3$, there are no equilibrium points case.

Sub-region 2.2:

In this sub-region, $x_1 \to 0$ and we recommend the following trend:

$$\begin{cases} r_{11}x_1 \to 0, \quad r_{13}x_3 \to 0 \\ r_{22}x_2 \to \infty, \quad r_{31}x_1 \to 0 \end{cases} \Rightarrow \begin{cases} f_1 \to p_{12} \\ f_2 \to p_{22} \\ f_3 \to \dfrac{r_{32}x_2p_{32} + r_{33}x_3p_{33}}{r_{32}x_2 + r_{33}x_3} \end{cases}$$

Since $f_2 > f_3$, no equilibrium point exists.

Sub-region 2.3:

In this sub-region equations for case 1 are:

$$\begin{cases} r_{13}x_3 \to 0 \\ r_{22}x_2 \to \infty \\ r_{31}x_1 \to 0 \end{cases} \Rightarrow \begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{12}x_2p_{12}}{r_{11}x_1 + r_{12}x_2} \\ f_2 \to p_{22} \\ f_3 \to \dfrac{r_{32}x_2p_{32} + r_{33}x_3p_{33}}{r_{32}x_2 + r_{33}x_3} \end{cases}$$

$f_2 \in (p_{23}, p_{21})$ that results in $f_2 > f_3$ which means there is no equilibrium point here.

Step 3:

In this step, propositions are made in way that $\dot{x}_3 < 0$ (where $x_3 = 1-\varepsilon$)

$$\begin{cases} x_3 = 1-\varepsilon \\ x_1 + x_2 = \varepsilon \end{cases} \quad \text{if } : \dot{x}_3 < 0 \quad \Rightarrow \quad f_3 < x_1 f_1 + x_2 f_2 + x_3 f_3 \quad \Rightarrow \quad \varepsilon f_3 < x_1 f_1 + x_2 f_2 \quad (47)$$

So, we will recommend propositions to make the equation active

Sub-region 2.1:

In this sub-region:

$$\begin{cases} x_1 \to \varepsilon \quad , & f_1 \to p_{11} \\ x_2 \to 0 \quad , & f_2 \to \dfrac{r_{21}x_1 p_{21} + r_{22}(\varepsilon - x_1)p_{22} + r_{23}(1-\varepsilon)p_{23}}{r_{21}x_1 + r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)} \Rightarrow \begin{cases} x_1 f_1 + x_2 f_2 \to \varepsilon p_{11} \\ \varepsilon f_3 \to \varepsilon p_{33} \end{cases} \Rightarrow x_1 f_1 + x_2 f_2 > \varepsilon f_3 \\ x_3 = 1-\varepsilon \quad , & f_3 \to p_{33} \end{cases}$$

It is apparent that the equation is active and no propositions are recommended

Sub-region 2.2:

In this sub-region:

$$\begin{cases} x_1 \to 0 \quad , & f_1 \to p_{12} \\ x_2 \to \varepsilon \quad , & f_2 \to p_{22} \\ x_3 = 1-\varepsilon \quad , & f_3 \to \dfrac{r_{32}(\varepsilon - x_1)p_{32} + r_{33}(1-\varepsilon)p_{33}}{r_{32}(\varepsilon - x_1) + r_{33}(1-\varepsilon)} \end{cases} \Rightarrow \begin{cases} x_1 f_1 + x_2 f_2 \to \varepsilon p_{22} \\ \varepsilon f_3 \to \varepsilon p_{33} \end{cases} \Rightarrow x_1 f_1 + x_2 f_2 > \varepsilon f_3$$

Thus, the equation is active and we don’t need any proposition as well.

Sub-region 2.3:

In this sub-region:

$$\begin{cases} f_1 \rightarrow \dfrac{r_{11}x_1 p_{11} + r_{12}(\varepsilon - x_1)p_{12}}{r_{11}x_1 + r_{12}(\varepsilon - x_1)} \\ f_2 \rightarrow p_{22} \\ f_3 \rightarrow \dfrac{r_{32}(\varepsilon - x_1)p_{32} + r_{33}(1-\varepsilon)p_{33}}{r_{32}(\varepsilon - x_1) + r_{33}(1-\varepsilon)} \end{cases}$$

If: $x_1 f_1 + x_2 f_2 > \varepsilon f_3 \Rightarrow r_{11}x_1(x_1 p_{11} - M_{10}) > r_{12}(\varepsilon - x_1)(M_{10} - x_1 p_{12})$

Where: $M_{10} = \varepsilon f_3 - x_2 f_2$

If $M_{10} < x_1 p_{11}$, then the following condition is applied to make equation (47) active:

$$\text{Our proposition: } r_{11} > \frac{r_{12}(\varepsilon - x_1)(M_{10} - x_1 p_{12})}{x_1(x_1 p_{11} - M_{10})} \quad (48)$$

Therefore, we check if the following equation is active:

$$M_{10} < x_1 p_{11} \quad (49)$$

$$M_{10} - x_1 p_{11} = \varepsilon\left(\frac{r_{32}(\varepsilon - x_1)p_{32} + r_{33}(1-\varepsilon)p_{33}}{r_{32}(\varepsilon - x_1) + r_{33}(1-\varepsilon)}\right) - (\varepsilon - x_1)p_{22} - x_1 p_{11}$$

$$= \frac{r_{32}(\varepsilon - x_1)(\varepsilon p_{32} - (\varepsilon - x_1)p_{22} - x_1 p_{11}) - r_{33}(1-\varepsilon)(-\varepsilon p_{33} + (\varepsilon - x_1)p_{22} + x_1 p_{11})}{r_{21}x_1 + r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)}$$

Since, $(-\varepsilon p_{33} + (\varepsilon - x_1)p_{22} + x_1 p_{11}) > 0$, by applying the following extra proposition the equation becomes active

$$\text{Our extra proposition: } r_{23} > \frac{r_{33}(1-\varepsilon)(\varepsilon p_{31} - (\varepsilon - x_1)p_{22} + x_1 p_{11})}{(\varepsilon - x_1)(\varepsilon p_{32} - (\varepsilon - x_1)p_{22} - x_1 p_{11})} \quad (50)$$

Step 4:

In this step we want to recommend propositions so that $\dot{x}_3 < 0$ (where $x_3 = \sigma$). It is exactly similar to step 3, and we just replace $1-\varepsilon$ with σ.

$$\begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{12}(1-\sigma-x_1)p_{12}}{r_{11}x_1 + r_{12}(1-\sigma-x_1)} \\ f_2 \to p_{22} \\ f_3 \to \dfrac{r_{32}(1-\sigma-x_1)p_{32} + r_{33}\sigma p_{33}}{r_{32}(1-\sigma-x_1) + r_{33}\sigma} \end{cases}$$

$$\text{Our proposition: } r_{11} > \frac{r_{12}(1-\sigma-x_1)(N_{10}-x_1p_{12})}{x_1(x_1\, p_{11} - N_{10})} \qquad (51)$$

$$\text{Our extra proposition: } r_{23} > \frac{r_{33}\sigma(\varepsilon\, p_{31} - (1-\sigma-x_1)p_{22} + x_1\, p_{11})}{\sigma(\varepsilon\, p_{32} - (1-\sigma-x_1)p_{22} - x_1\, p_{11})} \qquad (52)$$

Where: $N_{10} = (1-\sigma)f_3 - x_2 f_2$

11. **Case A3B3**

Taking into account the preceding discussions, our proposition for this case is:

$$r_{22} \to \infty \quad \text{and} \quad r_{11} \to \infty$$

Step 2:

To prevent locating equilibrium points in region 2 the following analysis are carried out in 3 different sub-regions.

Sub-region 2.1:

In this sub-region, $x_2 \to 0$ and our propositions are::

$$
\begin{cases} r_{11}x_1 \to \infty \\ r_{22}x_2 \to \infty \\ r_{32}x_2 \to 0 \end{cases} \Rightarrow \begin{cases} f_1 \to p_{11} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{22}x_2p_{22} + r_{23}x_3p_{23}}{r_{21}x_1 + r_{22}x_2 + r_{23}x_3} \\ f_3 \to \dfrac{r_{31}x_1p_{31} + r_{33}x_3p_{33}}{r_{31}x_1 + r_{33}x_3} \end{cases}
$$

Since $f_1 > f_3$, there are no equilibrium points in this sub-region for this case.

Sub-region 2.2:

In this sub-region, $x_1 \to 0$ and it is apparent that:

$$
\begin{cases} r_{22}x_2 \to \infty \\ r_{31}x_1 \to 0 \end{cases} \Rightarrow \begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{12}x_2p_{12} + r_{13}x_3p_{13}}{r_{11}x_1 + r_{12}x_2 + r_{13}x_3} \\ f_2 \to p_{22} \\ f_3 \to \dfrac{r_{32}x_2p_{32} + r_{33}x_3p_{33}}{r_{32}x_2 + r_{33}x_3} \end{cases}
$$

Since $f_2 > f_3$, no equilibrium points exists..

Sub-region 2.3:

In this sub-region equations are written as follows for case 1:

$$
\begin{cases} r_{11}x_1 \to \infty \\ r_{22}x_2 \to \infty \end{cases} \Rightarrow \begin{cases} f_1 \to p_{11} \\ f_2 \to p_{22} \\ f_3 \to \dfrac{r_{31}x_1p_{31} + r_{32}x_2p_{32} + r_{33}x_3p_{33}}{r_{31}x_1 + r_{32}x_2 + r_{33}x_3} \end{cases}
$$

Considering the following condition which is highly possible, $f_1 \neq f_2$ and consequently, there would be no equilibrium points in this region.

$$p_{11} \neq p_{22} \quad (53)$$

Step 3:

In this step we attempt to keep a $\dot{x}_3 < 0$(where $x_3 = 1-\varepsilon$).

$$\begin{cases} x_3 = 1-\varepsilon \\ x_1 + x_2 = \varepsilon \end{cases} \quad \text{if } : \dot{x}_3 < 0 \quad \Rightarrow \quad f_3 < x_1 f_1 + x_2 f_2 + x_3 f_3 \quad \Rightarrow \quad \varepsilon f_3 < x_1 f_1 + x_2 f_2 \quad (54)$$

So, we will recommend propositions to make the equation active

Sub-region 2.1:

In this sub-region:

$$\begin{cases} x_1 \to \varepsilon \quad , & f_1 \to p_{11} \\ x_2 \to 0 \quad , & f_2 \to \dfrac{r_{21}\varepsilon p_{21} + r_{22}(\varepsilon - x_1) p_{22} + r_{23}(1-\varepsilon) p_{23}}{r_{21}\varepsilon + r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)} \\ x_3 = 1-\varepsilon \quad , & f_3 \to \dfrac{r_{31}\varepsilon p_{31} + r_{33}(1-\varepsilon) p_{33}}{r_{31}\varepsilon + r_{33}(1-\varepsilon)} \end{cases} \Rightarrow \begin{cases} x_1 f_1 + x_2 f_2 \to \varepsilon p_{11} \\ \varepsilon f_3 \to \varepsilon \left(\dfrac{r_{31} x_1 p_{31} + r_{33}(1-\varepsilon) p_{33}}{r_{31} x_1 + r_{33}(1-\varepsilon)} \right) \end{cases} \Rightarrow x_1 f_1 + x_2 f_2 > \varepsilon f_3$$

Therefore, the equation is always active and no proposal is needed.

Sub-region 2.2:

In this sub-region:

$$\begin{cases} x_1 \to 0 \quad , & f_1 \to \dfrac{r_{11} x_1 p_{11} + r_{12}\varepsilon p_{12} + r_{13}(1-\varepsilon) p_{13}}{r_{11} x_1 + r_{12}\varepsilon + r_{13}(1-\varepsilon)} \\ x_2 \to \varepsilon \quad , & f_2 \to p_{22} \\ x_3 = 1-\varepsilon \quad , & f_3 \to \dfrac{r_{32}\varepsilon p_{32} + r_{33}(1-\varepsilon) p_{33}}{r_{32}\varepsilon + r_{33}(1-\varepsilon)} \end{cases} \Rightarrow \begin{cases} x_1 f_1 + x_2 f_2 \to \varepsilon p_{22} \\ \varepsilon f_3 \to \varepsilon \left(\dfrac{r_{32}\varepsilon p_{32} + r_{33}(1-\varepsilon) p_{33}}{r_{32}\varepsilon + r_{33}(1-\varepsilon)} \right) \end{cases} \Rightarrow x_1 f_1 + x_2 f_2 > \varepsilon f_3$$

Thus, the equation is active and we don't need any proposition as well.

Sub-region 2.3:

In this sub-region:

$$\begin{cases} f_1 \to p_{11} \\ f_2 \to p_{22} \\ f_3 \to \dfrac{r_{31}x_1p_{31} + r_{32}(\varepsilon - x_1)p_{32} + r_{33}(1-\varepsilon)p_{33}}{r_{31}x_1 + r_{32}(\varepsilon - x_1) + r_{33}(1-\varepsilon)} \end{cases}$$

If: $x_1f_1 + x_2f_2 > \varepsilon f_3 \Rightarrow r_{11}x_1(x_1 p_{11} - M_{11}) > r_{12}(\varepsilon - x_1)(M_{11} - x_1 p_{12}) + r_{13}(1-\varepsilon)(M_{11} - x_1p_{13})$

Where: $M_{11} = \varepsilon f_3 - x_2f_2$

If $M_{11} < x_1p_{11}$, then the following condition is our suggestion to make equation (54) active:

Our proposition: $$r_{11} > \frac{r_{12}(\varepsilon - x_1)(M_{11} - x_1p_{12}) + r_{13}(1-\varepsilon)(M_{11} - x_1p_{13})}{x_1(x_1 p_{11} - M_{11})} \quad (55)$$

Therefore, we check if the following equation to find out if it is active:

$$M_{11} < x_1p_{11} \quad (56)$$

$$M_{11} - x_1 p_{11} = \varepsilon p_{33} - (\varepsilon - x_1)p_{22} - x_1p_{11}$$

It is apparent that the equation is active and no propositions are recommended

Step 4:

Propositions to have $\dot{x}_3 < 0$ (where $x_3 = \sigma$) are made by replacing $1-\varepsilon$ with σ.

$$\begin{cases} f_1 \to p_{11} \\ f_2 \to p_{22} \\ f_3 \to \dfrac{r_{31}x_1p_{31} + r_{32}(1-\sigma-x_1)p_{32} + r_{33}\sigma p_{33}}{r_{31}x_1 + r_{32}(1-\sigma-x_1) + r_{33}\sigma} \end{cases}$$

Our proposition: $r_{11} > \dfrac{r_{12}(1-\sigma-x_1)(N_{11}-x_1p_{12}) + r_{13}\sigma(N_{11}-x_1p_{13})}{x_1(x_1 p_{11} - N_{11})}$ (57)

Where: $N_{11} = \varepsilon f_3 - x_2 f_2$

12. **Case A4B4**

According to previous parts, our proposition for this case is:

$$r_{22} \to \infty \quad \text{and} \quad r_{33} \to \infty$$

Step 2:

To prevent locating equilibrium points in region 2 the following analysis are carried out in 3 different sub-regions.

Sub-region 2.1:

In this sub-region, $x_2 \to 0$ and based on our propositions we may write::

$$\begin{cases} r_{12}x_2 \to 0 \\ r_{33}x_3 \to \infty \end{cases} \Rightarrow \begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{13}x_3p_{13}}{r_{11}x_1 + r_{13}x_1} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{22}x_2p_{22} + r_{23}x_3p_{23}}{r_{21}x_1 + r_{22}x_2 + r_{23}x_3} \\ f_3 \to p_{33} \end{cases}$$

$f_1 > f_3$ and there are no equilibrium point in this sub-region for this case.

Sub-region 2.2:

In this sub-region, $x_1 \to 0$, similar to the procedure in sub-region 1 we may suggest:

$$\begin{cases} r_{11}x_1 \to 0 \\ r_{22}x_2 \to \infty \\ r_{33}x_3 \to \infty \end{cases} \Rightarrow \begin{cases} f_1 \to \dfrac{r_{12}x_2p_{12} + r_{13}x_3p_{13}}{r_{12}x_2 + r_{13}x_3} \\ f_2 \to p_{22} \\ f_3 \to p_{33} \end{cases}$$

Since $f_2 > f_3$, there is no equilibrium point there as well..

Sub-region 2.3:

In this sub-region, equations for case 1:

$$\begin{cases} r_{22}x_2 \to \infty \\ r_{33}x_3 \to \infty \end{cases} \Rightarrow \begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{12}x_2p_{12} + r_{13}x_3p_{13}}{r_{11}x_1 + r_{12}x_2 + r_{13}x_3} \\ f_2 \to p_{22} \\ f_3 \to p_{33} \end{cases}$$

$f_2 > f_3$ and no equilibrium point exists..

Step 3:

In this step we attempt to keep a $\dot{x}_3 < 0$ (where $x_3 = 1 - \varepsilon$)

$$\begin{cases} x_3 = 1-\varepsilon \\ x_1 + x_2 = \varepsilon \end{cases} \quad \text{if } : \dot{x}_3 < 0 \quad \Rightarrow \quad f_3 < x_1 f_1 + x_2 f_2 + x_3 f_3 \quad \Rightarrow \quad \varepsilon f_3 < x_1 f_1 + x_2 f_2 \quad (58)$$

So, we will recommend propositions to make the equation active

Sub-region 2.1:

In this sub-region:

$$\begin{cases} x_1 \to \varepsilon \quad , & f_1 \to \dfrac{r_{11} x_1 p_{11} + r_{13}(1-\varepsilon) p_{13}}{r_{11} x_1 + r_{13} x_1} \\ x_2 \to 0 \quad , & f_2 \to \dfrac{r_{21} x_1 p_{21} + r_{22}(\varepsilon - x_1) p_{22} + r_{23}(1-\varepsilon) p_{23}}{r_{21} x_1 + r_{22}(\varepsilon - x_1) + (1-\varepsilon) x_3} \\ x_3 = 1-\varepsilon \quad , & f_3 \to p_{33} \end{cases} \Rightarrow \begin{cases} x_1 f_1 + x_2 f_2 \to \varepsilon \left(\dfrac{r_{11} x_1 p_{11} + r_{13}(1-\varepsilon) p_{13}}{r_{11} x_1 + r_{13} x_1} \right) \\ \varepsilon f_3 \to \varepsilon p_{33} \end{cases} \Rightarrow x_1 f_1 + x_2 f_2 > \varepsilon f_3$$

Therefore, the equation is always active and no proposal is needed.

Sub-region 2.2:

In this sub-region:

$$\begin{cases} x_1 \to 0 \quad , & f_1 \to \dfrac{r_{12} \varepsilon p_{12} + r_{13}(1-\varepsilon) p_{13}}{r_{12} \varepsilon + r_{13}(1-\varepsilon)} \\ x_2 \to \varepsilon \quad , & f_2 \to p_{22} \\ x_3 = 1-\varepsilon \quad , & f_3 \to p_{33} \end{cases} \Rightarrow \begin{cases} x_1 f_1 + x_2 f_2 \to \varepsilon p_{22} \\ \varepsilon f_3 \to \varepsilon p_{33} \end{cases} \Rightarrow x_1 f_1 + x_2 f_2 > \varepsilon f_3$$

Therefore, the equation () is always active and we don’t need any proposition.

Sub-region 2.3:

In this sub-region:

$$\begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{12}(\varepsilon - x_1)p_{12} + r_{13}(1-\varepsilon)p_{13}}{r_{11}x_1 + r_{12}(\varepsilon - x_1) + r_{13}(1-\varepsilon)} \\ f_2 \to p_{22} \\ f_3 \to p_{33} \end{cases}$$

If: $x_1f_1 + x_2f_2 > \varepsilon f_3 \Rightarrow r_{11}x_1(x_1 p_{11} - M_{12}) > r_{12}(\varepsilon - x_1)(M_{12} - x_1 p_{12}) + r_{13}(1-\varepsilon)(M_{12} - x_1p_{13})$

Where: $M_{12} = \varepsilon f_3 - x_2f_2$

If $M_{12} < x_1p_{11}$, then the following conditions our proposition to make equation () active:

$$\text{Our proposition: } r_{11} > \frac{r_{12}(\varepsilon - x_1)(M_{12} - x_1p_{12}) + r_{13}(1-\varepsilon)(M_{12} - x_1p_{13})}{x_1(x_1 p_{11} - M_{12})} \qquad (59)$$

Therefore, we check if the following equation is active:

$$M_{12} < x_1p_{11} \qquad (60)$$

$$M_{12} - x_1 p_{11} = \varepsilon p_{33} - (\varepsilon - x_1)p_{22} - x_1p_{11}$$

It is apparent that the equation is active and no propositions are recommended

Step 4:

In this step we recommend propositions so that $\dot{x}_3 < 0$ (where $x_3 = \sigma$). It is exactly similar to step 3, and we just replace $1-\varepsilon$ with σ.

$$\begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{12}(1-\sigma - x_1)p_{12} + r_{13}\sigma p_{13}}{r_{11}x_1 + r_{12}(1-\sigma - x_1) + r_{13}\sigma} \\ f_2 \to p_{22} \\ f_3 \to p_{33} \end{cases}$$

$$\text{Our proposition: } r_{11} > \frac{r_{12}(1-\sigma-x_1)(N_{12}-x_1p_{12})+r_{13}\sigma(N_{12}-x_1p_{13})}{x_1(x_1 p_{11}-N_{12})} \qquad (61)$$

Where: $N_{12} = \varepsilon f_3 - x_2 f_2$

Therefore, the equation is always active and no proposal is needed.

13. **Case A4B1**

According to preceding parts, our proposition for this case is:

$$r_{33} \to \infty \quad \text{and} \quad \begin{cases} r_{11}r_{33} \to \infty \\ r_{13} \to 0 \end{cases}$$

Step 2:

In this step we will present propositions to have zero equilibrium points in region 2. Again, region 2 is consisted of 3 sub-regions.

Sub-region 2.1:

In this sub-region, $x_2 \to 0$ and our proposition is:

$$\begin{cases} r_{12}x_2 \to 0, \quad r_{13} \to 0 \\ r_{22}x_2 \to 0, \quad r_{33}x_3 \to \infty \end{cases} \Rightarrow \begin{cases} f_1 \to p_{11} \\ f_2 \to \dfrac{r_{21}x_1p_{21}+r_{23}x_3p_{23}}{r_{21}x_1+r_{23}x_3} \\ f_3 \to p_{33} \end{cases}$$

$f_1 > f_3$, there are no equilibrium point in this sub-region for this case.

Sub-region 2.2:

In this sub-region, $x_1 \to 0$ and:

$$\begin{cases} r_{11}x_1 \to 0, & r_{13}x_3 \to 0 \\ r_{21}x_1 \to 0, & r_{33}x_3 \to \infty \end{cases} \Rightarrow \begin{cases} f_1 \to p_{12} \\ f_2 \to \dfrac{r_{22}x_2p_{22} + r_{23}x_3p_{23}}{r_{22}x_2 + r_{23}x_3} \\ f_3 \to p_{33} \end{cases}$$

Since $f_2 > f_3$, there is no equilibrium point there.

Sub-region 2.3:

In this sub-region::

$$\begin{cases} r_{13}x_3 \to 0 \\ r_{33}x_3 \to \infty \end{cases} \Rightarrow \begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{13}x_3p_{13}}{r_{11}x_1 + r_{13}x_3} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{22}x_2p_{22} + r_{23}x_3p_{23}}{r_{21}x_1 + r_{22}x_2 + r_{23}x_3} \\ f_3 \to p_{33} \end{cases}$$

$f_2 > f_3$ andthere is no equilibrium point there.

Step 3:

In this step we want to recommend propositions to have $\dot{x}_3 < 0$ (where $x_3 = 1 - \varepsilon$)

$$\begin{cases} x_3 = 1 - \varepsilon \\ x_1 + x_2 = \varepsilon \end{cases} \quad \text{if } : \dot{x}_3 < 0 \quad \Rightarrow \quad f_3 < x_1f_1 + x_2f_2 + x_3f_3 \quad \Rightarrow \quad \varepsilon f_3 < x_1f_1 + x_2f_2 \quad (62)$$

So, we will recommend propositions to make the equation active

Sub-region 2.1:

In this sub-region:

$$\begin{cases} x_1 \to \varepsilon & , \quad f_1 \to p_{11} \\ x_2 \to 0 & , \quad f_2 \to \dfrac{r_{21}x_1p_{21} + r_{23}(1-\varepsilon)p_{23}}{r_{21}x_1 + r_{23}(1-\varepsilon)} \Rightarrow \begin{cases} x_1f_1 + x_2f_2 \to \varepsilon p_{11} \\ \varepsilon f_3 \to \varepsilon p_{33} \end{cases} \quad \Rightarrow x_1f_1 + x_2f_2 > \varepsilon f_3 \\ x_3 = 1-\varepsilon & , \quad f_3 \to p_{33} \end{cases}$$

It is apparent that the equation is active and no propositions are recommended

Sub-region 2.2:

In this sub-region:

$$\begin{cases} x_1 \to 0 & , \quad f_1 \to p_{12} \\ x_2 \to \varepsilon & , \quad f_2 \to \dfrac{r_{22}\varepsilon p_{22} + r_{23}(1-\varepsilon)p_{23}}{r_{22}\varepsilon + (1-\varepsilon)x_3} \Rightarrow \begin{cases} x_1f_1 + x_2f_2 \to \varepsilon\left(\dfrac{r_{22}\varepsilon p_{22} + r_{23}(1-\varepsilon)p_{23}}{r_{22}\varepsilon + (1-\varepsilon)x_3}\right) \\ \varepsilon f_3 \to \varepsilon p_{33} \end{cases} \quad \Rightarrow x_1f_1 + x_2f_2 > \varepsilon f_3 \\ x_3 = 1-\varepsilon & , \quad f_3 \to p_{33} \end{cases}$$

Thus, the equation is active and we don’t need any proposition as well.

Sub-region 2.3:

In this sub-region:

$$\begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{13}(1-\varepsilon)p_{13}}{r_{11}x_1 + r_{13}(1-\varepsilon)} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{22}(\varepsilon - x_1)p_{22} + r_{23}(1-\varepsilon)p_{23}}{r_{21}x_1 + r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)} \\ f_3 \to p_{33} \end{cases}$$

If: $x_1f_1 + x_2f_2 > \varepsilon f_3 \Rightarrow r_{11}x_1(x_1 p_{11} - M_{13}) > r_{12}(\varepsilon - x_1)(M_{13} - x_1 p_{12})$

Where: $M_{13} = \varepsilon f_3 - x_2f_2$

If $M_{13} < x_1p_{11}$, then the following situation is our proposition to make equation (62) active:

$$\text{Our proposition: } r_{11} > \frac{r_{12}(\varepsilon - x_1)(M_{13} - x_1 p_{12})}{x_1(x_1 p_{11} - M_{13})} \quad (63)$$

Therefore, we check if the following equation is active:

$$M_{13} < x_1 p_{11} \quad (64)$$

$$M_{13} - x_1 p_{11} = \varepsilon p_{33} - (\varepsilon - x_1)\left(\frac{r_{21}x_1 p_{21} + r_{22}(\varepsilon - x_1)p_{22} + r_{23}(1-\varepsilon)p_{23}}{r_{21}x_1 + r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)}\right) - x_1 p_{11}$$

$$= \frac{r_{21}x_1(\varepsilon p_{33} - (\varepsilon - x_1)p_{21} - x_1 p_{11}) - r_{22}(\varepsilon - x_1)(x_1 p_{11} + (\varepsilon - x_1)p_{22} - \varepsilon p_{33}) - r_{23}(1-\varepsilon)(x_1 p_{11} + (\varepsilon - x_1)p_{23} - \varepsilon p_{33})}{r_{21}x_1 + r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)}$$

$$\begin{cases} (\varepsilon p_{33} - (\varepsilon - x_1)p_{21} - x_1 p_{11}) < 0 \\ (x_1 p_{11} + (\varepsilon - x_1)p_{22} - \varepsilon p_{33}) > 0 \\ (x_1 p_{11} + (\varepsilon - x_1)p_{23} - \varepsilon p_{33}) > 0 \end{cases} \Rightarrow M_{13} - x_1 p_{11} < 0 \Rightarrow M_{13} < x_1 p_{11}$$

It is apparent that the equation is active and no propositions are recommended

Step 4:

Propositions to have $\dot{x}_3 < 0$ (where $x_3 = \sigma$) are made by replacing $1-\varepsilon$ with σ.

$$\begin{cases} f_1 \to \dfrac{r_{11}x_1 p_{11} + r_{13}\sigma p_{13}}{r_{11}x_1 + r_{13}\sigma} \\ f_2 \to \dfrac{r_{21}x_1 p_{21} + r_{22}(1-\sigma - x_1)p_{22} + r_{23}\sigma p_{23}}{r_{21}x_1 + r_{22}(1-\sigma - x_1) + r_{23}\sigma} \\ f_3 \to p_{33} \end{cases}$$

$$\text{Our proposition: } r_{11} > \frac{r_{12}(1-\sigma - x_1)(N_{13} - x_1 p_{12})}{x_1(x_1 p_{11} - N_{13})} \quad (65)$$

Where: $N_{13} = (1-\sigma)f_3 - x_2 f_2$

14. **Case A4B2**

According to previous parts:

$$r_{33} \to \infty \quad \text{and} \quad \frac{r_{11}r_{33}}{r_{13}^2} > k_2$$

Step 2:

To prevent locating equilibrium points in region 2 the following analysis are carried out in 3 different sub-regions.

Sub-region 2.1:

In this sub-region, $x_2 \to 0$ and:

$$\begin{cases} r_{12}x_2 \to 0 \\ r_{22}x_2 \to 0 \\ r_{33}x_3 \to \infty \end{cases} \Rightarrow \begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{13}x_3p_{13}}{r_{11}x_1 + r_{13}x_3} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{23}x_3p_{23}}{r_{21}x_1 + r_{23}x_3} \\ f_3 \to p_{33} \end{cases}$$

Since $f_2 > f_3$, there would be no equilibrium point in this sub-region.

Sub-region 2.2:

In this sub-region, $x_1 \to 0$ and similar to sub-region 1 the following approach is proposed:

$$\begin{cases} r_{11}x_1 \to 0 \\ r_{21}x_1 \to 0 \\ r_{33}x_3 \to \infty \end{cases} \Rightarrow \begin{cases} f_1 \to \dfrac{r_{12}x_2p_{12} + r_{13}x_3p_{13}}{r_{12}x_2 + r_{13}x_3} \\ f_2 \to \dfrac{r_{22}x_2p_{22} + r_{23}x_3p_{23}}{r_{22}x_2 + r_{23}x_3} \\ f_3 \to p_{33} \end{cases}$$

$f_2 > f_3$ and there is no equilibrium point there.

Sub-region 2.3:

equations are:

$$r_{33}x_3 \to \infty \quad \Rightarrow \quad \begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{12}x_2p_{12} + r_{13}x_3p_{13}}{r_{11}x_1 + r_{12}x_2 + r_{13}x_3} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{22}x_2p_{22} + r_{23}x_3p_{23}}{r_{21}x_1 + r_{22}x_2 + r_{23}x_3} \\ f_3 \to p_{33} \end{cases}$$

Since $f_2 > f_3$, no equilibrium point exists.

Step 3:

In this step we attempt to keep $\dot{x}_3 < 0$ (where $x_3 = 1 - \varepsilon$)

$$\begin{cases} x_3 = 1 - \varepsilon \\ x_1 + x_2 = \varepsilon \end{cases} \quad \text{if } : \dot{x}_3 < 0 \quad \Rightarrow \quad f_3 < x_1f_1 + x_2f_2 + x_3f_3 \quad \Rightarrow \quad \varepsilon f_3 < x_1f_1 + x_2f_2 \quad (66)$$

Therefore, propositions are required for the equationto be active.

Sub-region 2.1:

In thissub- region:

$$\begin{cases} x_1 \to \varepsilon \quad , \quad f_1 \to \dfrac{r_{11}x_1p_{11} + r_{13}(1-\varepsilon)p_{13}}{r_{11}x_1 + r_{13}(1-\varepsilon)} \\ x_2 \to 0 \quad , \quad f_2 \to \dfrac{r_{21}x_1p_{21} + r_{23}(1-\varepsilon)p_{23}}{r_{21}x_1 + r_{23}(1-\varepsilon)} \\ x_3 = 1-\varepsilon \quad , \quad f_3 \to p_{33} \end{cases} \Rightarrow \begin{cases} x_1f_1 + x_2f_2 \to \varepsilon\left(\dfrac{r_{11}x_1p_{11} + r_{13}(1-\varepsilon)p_{13}}{r_{11}x_1 + r_{13}(1-\varepsilon)}\right) \\ \varepsilon f_3 \to \varepsilon p_{33} \end{cases}$$

Therefore, the equation is always active and no proposal is needed.

Sub-region 2.2:

In this sub-region:

$$\begin{cases} x_1 \to 0 \quad , \quad f_1 \to \dfrac{r_{12}(\varepsilon - x_1)p_{12} + r_{13}(1-\varepsilon)p_{13}}{r_{12}(\varepsilon - x_1) + r_{13}(1-\varepsilon)} \\ x_2 \to \varepsilon \quad , \quad f_2 \to \dfrac{r_{22}(\varepsilon - x_1)p_{22} + r_{23}(1-\varepsilon)p_{23}}{r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)} \\ x_3 = 1-\varepsilon \quad , \quad f_3 \to p_{33} \end{cases} \Rightarrow \begin{cases} x_1f_1 + x_2f_2 \to \varepsilon\left(\dfrac{r_{22}(\varepsilon - x_1)p_{22} + r_{23}(1-\varepsilon)p_{23}}{r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)}\right) \\ \varepsilon f_3 \to \varepsilon p_{33} \end{cases}$$

Thus, the equation is active and we don’t need any proposition as well.

Sub-region 2.3:

In this sub-region:

$$\begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{12}(\varepsilon - x_1)p_{12} + r_{13}(1-\varepsilon)p_{13}}{r_{11}x_1 + r_{12}(\varepsilon - x_1) + r_{13}(1-\varepsilon)} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{22}(\varepsilon - x_1)p_{22} + r_{23}(1-\varepsilon)p_{23}}{r_{21}x_1 + r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)} \\ f_3 \to p_{33} \end{cases}$$

If:

$$x_1f_1 + x_2f_2 > \varepsilon f_3 \Rightarrow r_{11}x_1(x_1 p_{11} - M_{14}) > r_{12}(\varepsilon - x_1)(M_{14} - x_1 b) + r_{13}(1-\varepsilon)(M_{14} - x_1p_{13})$$

Where: $M_{14} = \varepsilon f_3 - x_2 f_2$

If $M_{14} < x_1 p_{11}$, then the following condition is proposed to make equation (66) active:

$$\text{Our proposition: } r_{11} > \frac{r_{12}(\varepsilon - x_1)(M_{14} - x_1 p_{12}) + r_{13}(1-\varepsilon)(M_{14} - x_1 p_{13})}{x_1(x_1 p_{11} - M_{14})} \qquad (67)$$

Therefore, we check if the following equation is active:

$$M_{14} < x_1 p_{11} \qquad (68)$$

$$M_{14} - x_1 p_{11} = \varepsilon p_{33} - (\varepsilon - x_1)\left(\frac{r_{21}x_1 p_{21} + r_{22}(\varepsilon - x_1)p_{22} + r_{23}(1-\varepsilon)p_{23}}{r_{21}x_1 + r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)}\right) - x_1 p_{11}$$

$$= \frac{r_{21}x_1(\varepsilon p_{33} - (\varepsilon - x_1)p_{21} - x_1 p_{11}) - r_{22}(\varepsilon - x_1)(x_1 p_{11} + (\varepsilon - x_1)p_{22} - \varepsilon p_{33}) - r_{23}(1-\varepsilon)(x_1 p_{11} + (\varepsilon - x_1)p_{23} - \varepsilon p_{33})}{r_{21}x_1 + r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)}$$

$$\begin{cases} (\varepsilon p_{33} - (\varepsilon - x_1)p_{21} - x_1 p_{11}) < 0 \\ (x_1 p_{11} + (\varepsilon - x_1)p_{22} - \varepsilon p_{33}) > 0 \\ (x_1 p_{11} + (\varepsilon - x_1)p_{23} - \varepsilon p_{33}) > 0 \end{cases} \Rightarrow M_{14} - x_1 p_{11} < 0 \Rightarrow M_{14} < x_1 p_{11}$$

It is apparent that the equation is active and no propositions are recommended

Step 4:

In this step Propositions to have $\dot{x}_3 < 0$ (where $x_3 = \sigma$) are made by replacing $1-\varepsilon$ with σ.

$$\begin{cases} f_1 \to \dfrac{r_{11}x_1 p_{11} + r_{12}(1-\sigma - x_1)p_{12} + r_{13}\sigma p_{13}}{r_{11}x_1 + r_{12}(1-\sigma - x_1) + r_{13}\sigma} \\ f_2 \to \dfrac{r_{21}x_1 p_{21} + r_{22}(1-\sigma - x_1)p_{22} + r_{23}\sigma p_{23}}{r_{21}x_1 + r_{22}(1-\sigma - x_1) + r_{23}\sigma} \\ f_3 \to p_{33} \end{cases}$$

Our proposition: $r_{11} > \dfrac{r_{12}(1-\sigma-x_1)(N_{14}-x_1p_{12})+r_{13}\sigma(N_{14}-x_1p_{13})}{x_1(x_1\, p_{11}-N_{14})}$ (69)

Where: $N_{14}=(1-\sigma)f_3-x_2f_2$

15. **Case A4B3**

According to preceding discussions: :

$$r_{33}\to\infty \quad \text{and} \quad r_{11}\to\infty$$

Step 2:

In this step we present propositions to have zero equilibrium points in region 2. Again, region 2 is consisted of 3 sub-regions.

Sub-region 2.1:

In this sub-region, $x_2\to 0$ and based on our propositions:

$$\begin{cases} r_{11}x_1\to\infty \\ r_{22}x_2\to 0 \\ r_{33}x_3\to\infty \end{cases} \Rightarrow \begin{cases} f_1\to p_{11} \\ f_2\to \dfrac{r_{21}x_1p_{21}+r_{23}x_3p_{23}}{r_{21}x_1+r_{23}x_3} \\ f_3\to p_{33} \end{cases}$$

$f_1 > f_3$, there are no equilibrium point in this sub-region for this case.

Sub-region 2.2:

In this sub-region, $x_1\to 0$ and:

$$\begin{cases} r_{21}x_1 \to 0 \\ r_{33}x_3 \to \infty \end{cases} \Rightarrow \begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{12}x_2p_{12} + r_{13}x_3p_{13}}{r_{11}x_1 + r_{12}x_2 + r_{13}x_3} \\ f_2 \to \dfrac{r_{22}x_2p_{22} + r_{23}x_3p_{23}}{r_{22}x_2 + r_{23}x_3} \\ f_3 \to p_{33} \end{cases}$$

Since $f_2 > f_3$, there is no equilibrium point.

Sub-region 2.3:

In this sub-region::

$$\begin{cases} r_{11}x_1 \to \infty \\ r_{33}x_3 \to \infty \end{cases} \Rightarrow \begin{cases} f_1 \to p_{11} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{22}x_2p_{22} + r_{23}x_3p_{23}}{r_{21}x_1 + r_{22}x_2 + r_{23}x_3} \\ f_3 \to p_{33} \end{cases}$$

$f_1 > f_3$ no equilibrium point is located in this sub-region.

Step 3:

In this step we want to recommend propositions so that $\dot{x}_3 < 0$ (where $x_3 = 1-\varepsilon$).

$$\begin{cases} x_3 = 1-\varepsilon \\ x_1 + x_2 = \varepsilon \end{cases} \quad \text{if } : \dot{x}_3 < 0 \quad \Rightarrow \quad f_3 < x_1f_1 + x_2f_2 + x_3f_3 \quad \Rightarrow \quad \varepsilon f_3 < x_1f_1 + x_2f_2 \quad (70)$$

So, we will recommend propositions to make the equation active

Sub-region 2.1:

In this sub-region:

$$\begin{cases} x_1 \to \varepsilon & , \quad f_1 \to p_{11} \\ x_2 \to 0 & , \quad f_2 \to \dfrac{r_{21}x_1p_{21} + r_{23}(1-\varepsilon)p_{23}}{r_{21}x_1 + r_{23}(1-\varepsilon)} \\ x_3 = 1-\varepsilon & , \quad f_3 \to p_{33} \end{cases} \Rightarrow \begin{cases} x_1f_1 + x_2f_2 \to \varepsilon p_{11} \\ \varepsilon f_3 \to \varepsilon p_{33} \end{cases}$$

It is apparent that the equation is active and no propositions are recommended.

Sub-region 2.2:

In this sub-region:

$$\begin{cases} x_1 \to 0 & , \quad f_1 \to \dfrac{r_{11}x_1p_{11} + r_{12}(\varepsilon - x_1)p_{12} + r_{13}(1-\varepsilon)p_{13}}{r_{11}x_1 + r_{12}(\varepsilon - x_1) + r_{13}(1-\varepsilon)} \\ x_2 \to \varepsilon & , \quad f_2 \to \dfrac{r_{22}(\varepsilon - x_1)p_{22} + r_{23}(1-\varepsilon)p_{23}}{r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)} \\ x_3 = 1-\varepsilon & , \quad f_3 \to p_{33} \end{cases} \Rightarrow \begin{cases} x_1f_1 + x_2f_2 \to \varepsilon \left(\dfrac{r_{22}(\varepsilon - x_1)p_{22} + r_{23}(1-\varepsilon)p_{23}}{r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)} \right) \\ \varepsilon f_3 \to \varepsilon p_{33} \end{cases}$$

Thus, the equation is active and we don’t need any proposition as well.

Sub-region 2.3:

In this sub-region:

$$\begin{cases} f_1 \to p_{11} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{22}(\varepsilon - x_1)p_{22} + r_{23}(1-\varepsilon)p_{23}}{r_{21}x_1 + r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)} \\ f_3 \to p_{33} \end{cases}$$

Since: $f_3 < f_2, \quad f_3 < f_1 \Rightarrow \varepsilon f_3 < x_1f_1 + x_2f_2$

Therefore, the equation is always active and no proposal is needed.

Step 4:

In this step we recommend propositions so that $\dot{x}_3 < 0$ (where $x_3 = \sigma$). It is exactly similar to step 3, and we just replace $1-\varepsilon$ with σ.

16. **Case A4B4**

According to previous parts::

$$r_{33} \to \infty$$

Step 2:

To prevent locating equilibrium points in region 2 the following analysis are carried out in 3 different sub-regions.

Sub-region 2.1:

In this sub-region, $x_2 \to 0$ and based on our propositions:

$$\begin{cases} r_{12}x_2 \to 0 \\ r_{22}x_2 \to 0 \\ r_{33}x_3 \to \infty \end{cases} \Rightarrow \begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{13}x_3p_{13}}{r_{11}x_1 + r_{13}x_3} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{23}x_3p_{23}}{r_{21}x_1 + r_{23}x_3} \\ f_3 \to p_{33} \end{cases}$$

Since $f_2 > f_3$, there are no equilibrium point in this sub-region for this case.

Sub-region 2.2:

In this sub-region, $x_1 \to 0$ and we may write:::

$$\begin{cases} r_{11}x_1 \to 0 \\ r_{21}x_1 \to 0 \\ r_{33}x_3 \to \infty \end{cases} \Rightarrow \begin{cases} f_1 \to \dfrac{r_{12}x_2p_{12} + r_{13}x_3p_{13}}{r_{12}x_2 + r_{13}x_3} \\ f_2 \to \dfrac{r_{22}x_2p_{22} + r_{23}x_3p_{23}}{r_{22}x_2 + r_{23}x_3} \\ f_3 \to p_{33} \end{cases}$$

$f_2 > f_3$ and no equilibrium points exists.

Sub-region 2.3:

In this sub-region equations for case 1 are::

$$r_{33}x_3 \to \infty \quad \Rightarrow \quad \begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{12}x_2p_{12} + r_{13}x_3p_{13}}{r_{11}x_1 + r_{12}x_2 + r_{13}x_3} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{22}x_2p_{22} + r_{23}x_3p_{23}}{r_{21}x_1 + r_{22}x_2 + r_{23}x_3} \\ f_3 \to p_{33} \end{cases}$$

Since $f_2 > f_3$, no equilibrium point is located in this sub-region.

Step 3:

Propositions to have $\dot{x}_3 < 0$ (where $x_3 = \sigma$) are made by replacing $1-\varepsilon$ with σ.

$$\begin{cases} x_3 = 1-\varepsilon \\ x_1 + x_2 = \varepsilon \end{cases} \quad \text{if } : \dot{x}_3 < 0 \quad \Rightarrow \quad f_3 < x_1f_1 + x_2f_2 + x_3f_3 \quad \Rightarrow \quad \varepsilon f_3 < x_1f_1 + x_2f_2 \quad (71)$$

So, we will recommend propositions to make the equation active

Sub-region 2.1:

In this sub-region:

$$\begin{cases} x_1 \to \varepsilon \quad , \quad f_1 \to \dfrac{r_{11}x_1p_{11} + r_{13}(1-\varepsilon)p_{13}}{r_{11}x_1 + r_{13}(1-\varepsilon)} \\ x_2 \to 0 \quad , \quad f_2 \to \dfrac{r_{21}x_1p_{21} + r_{23}(1-\varepsilon)p_{23}}{r_{21}x_1 + r_{23}(1-\varepsilon)} \\ x_3 = 1-\varepsilon \quad , \quad f_3 \to p_{33} \end{cases} \Rightarrow \begin{cases} x_1f_1 + x_2f_2 \to \varepsilon\left(\dfrac{r_{11}x_1p_{11} + r_{13}(1-\varepsilon)p_{13}}{r_{11}x_1 + r_{13}(1-\varepsilon)}\right) \\ \varepsilon f_3 \to \varepsilon p_{33} \end{cases}$$

Therefore, the equation is always active and no proposal is needed.

Sub-region 2.2:

In this sub-region:

$$\begin{cases} x_1 \to 0 \quad , \quad f_1 \to \dfrac{r_{12}(\varepsilon - x_1)p_{12} + r_{13}(1-\varepsilon)p_{13}}{r_{12}(\varepsilon - x_1) + r_{13}(1-\varepsilon)} \\ x_2 \to \varepsilon \quad , \quad f_2 \to \dfrac{r_{22}(\varepsilon - x_1)p_{22} + r_{23}(1-\varepsilon)p_{23}}{r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)} \\ x_3 = 1-\varepsilon \quad , \quad f_3 \to p_{33} \end{cases} \Rightarrow \begin{cases} x_1f_1 + x_2f_2 \to \varepsilon\left(\dfrac{r_{22}(\varepsilon - x_1)p_{22} + r_{23}(1-\varepsilon)p_{23}}{r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)}\right) \\ \varepsilon f_3 \to \varepsilon p_{33} \end{cases}$$

Thus, the equation is active and we don’t need any proposition as well.

Sub-region 2.3:

In this sub-region:

$$\begin{cases} f_1 \to \dfrac{r_{11}x_1p_{11} + r_{12}(\varepsilon - x_1)p_{12} + r_{13}(1-\varepsilon)p_{13}}{r_{11}x_1 + r_{12}(\varepsilon - x_1) + r_{13}(1-\varepsilon)} \\ f_2 \to \dfrac{r_{21}x_1p_{21} + r_{22}(\varepsilon - x_1)p_{22} + r_{23}(1-\varepsilon)p_{23}}{r_{21}x_1 + r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)} \\ f_3 \to p_{33} \end{cases}$$

If:

$$x_1f_1 + x_2f_2 > \varepsilon f_3 \Rightarrow r_{11}x_1(x_1p_{11} - M_{16}) > r_{12}(\varepsilon - x_1)(M_{16} - x_1b) + r_{13}(1-\varepsilon)(M_{16} - x_1p_{13})$$

Where: $M_{16} = \varepsilon f_3 - x_2 f_2$

If $M_{16} < x_1 p_{11}$, the proposed condition to make equation (71) active is:

$$\text{Our proposition: } r_{11} > \frac{r_{12}(\varepsilon - x_1)(M_{16} - x_1 p_{12}) + r_{13}(1-\varepsilon)(M_{16} - x_1 p_{13})}{x_1(x_1 p_{11} - M_{16})} \quad (72)$$

Therefore, we check if the following equation is active:

$$M_{16} < x_1 p_{11} \quad (73)$$

$$M_{16} - x_1 p_{11} = \varepsilon p_{33} - (\varepsilon - x_1)\left(\frac{r_{21}x_1 p_{21} + r_{22}(\varepsilon - x_1)p_{22} + r_{23}(1-\varepsilon)p_{23}}{r_{21}x_1 + r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)}\right) - x_1 p_{11}$$

$$= \frac{r_{21}x_1(\varepsilon p_{33} - (\varepsilon - x_1)p_{21} - x_1 p_{11}) - r_{22}(\varepsilon - x_1)(x_1 p_{11} + (\varepsilon - x_1)p_{22} - \varepsilon p_{33}) - r_{23}(1-\varepsilon)(x_1 p_{11} + (\varepsilon - x_1)p_{23} - \varepsilon p_{33})}{r_{21}x_1 + r_{22}(\varepsilon - x_1) + r_{23}(1-\varepsilon)}$$

$$\begin{cases} (\varepsilon p_{33} - (\varepsilon - x_1)p_{21} - x_1 p_{11}) < 0 \\ (x_1 p_{11} + (\varepsilon - x_1)p_{22} - \varepsilon p_{33}) > 0 \\ (x_1 p_{11} + (\varepsilon - x_1)p_{23} - \varepsilon p_{33}) > 0 \end{cases} \Rightarrow M_{16} - x_1 p_{11} < 0 \Rightarrow M_{16} < x_1 p_{11}$$

It is apparent that the equation is active and no propositions are recommended.

Step 4:

In this step we recommend propositions so that $\dot{x}_3 < 0$ (where $x_3 = \sigma$). It is exactly similar to step 3, and we just replace $1-\varepsilon$ with σ.

$$\begin{cases} f_1 \to \dfrac{r_{11}x_1 p_{11} + r_{12}(1-\sigma - x_1)p_{12} + r_{13}\sigma p_{13}}{r_{11}x_1 + r_{12}(1-\sigma - x_1) + r_{13}\sigma} \\ f_2 \to \dfrac{r_{21}x_1 p_{21} + r_{22}(1-\sigma - x_1)p_{22} + r_{23}\sigma p_{23}}{r_{21}x_1 + r_{22}(1-\sigma - x_1) + r_{23}\sigma} \\ f_3 \to p_{33} \end{cases}$$

Our proposition: $$r_{11} > \frac{r_{12}(1-\sigma-x_1)(N_{16}-x_1p_{12})+r_{13}\sigma(N_{16}-x_1p_{13})}{x_1(x_1\,p_{11}-N_{16})} \quad (74)$$

Where: $N_{16}=(1-\sigma)f_3-x_2f_2$