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Dependence of synchronization frequency of Kuramoto oscillators on symmetry of intrinsic frequency in ring network

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Abstract. Kuramoto oscillators have been proposed earlier as a model for interacting systems that exhibit synchronization. In this article, we study the difference between networks with symmetric and asymmetric distribution of natural frequencies. We first indicate that synchronization frequency of oscillators in a completely connected network is always equal to the mean of the natural frequency distribution. In particular, shape of the natural frequency distribution does not affect the synchronization frequency in this case. Then, we analyse the case of oscillators in a directed ring network, where asymmetry in the natural frequency distribution is seen to shift the synchronization frequency of the network. We also present an estimate of the shift in the frequencies for slightly asymmetric distributions.

Keywords. Kuramoto; synchronization; asymmetry; frequency distribution.

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1. Introduction

The phenomenon of synchronization is seen in interacting oscillatory systems in nature. Most striking examples include the regular flashing of light by fireflies [1], simultaneous clapping by the audience in a theatre during an applause [2,3], Josephson junctions [4,5] and chemical oscillations [6]. Collective synchronization was first studied mathematically by Wiener [7,8]. He realized the ubiquity of the phenomenon and speculated its involvement in the generation of alpha rhythms in the brain. Unfortunately, Wiener's mathematical approach based on Fourier integrals [7] has turned out to be a dead end [9]. In 1975, Kuramoto introduced a model, which took into consideration oscillators, which were coupled to each other and showed the phenomenon of synchronization for sufficiently large coupling strengths.

The dynamics of a general ith oscillator in a system of N Kuramoto oscillators is given as

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} K_{ij} \sin(\theta_j - \theta_i) \qquad \forall i = 1, 2, \ldots, N, \tag{1}$$

where θ_i and ω_i are the phase and natural frequencies of the ith oscillator, respectively, and K_{ij} is the coupling strength between the ith and jth oscillators. The coupling depends on the sine of the phase difference between the oscillators and hence, is nonlinear. For simplicity, we set $K_{ij} = K/N$ for all $1 \leq i, j \leq N$.

In this paper, we analyse the Kuramoto model with an aim to study the difference in dynamics of the oscillators, as the distribution from which natural frequencies of the oscillators are chosen, changes from symmetric to asymmetric form. In particular, we study the change in synchronization frequency as the symmetry is changed under the limit of large N. We first analyse a complete network of oscillators and show that symmetry of natural frequency distribution has no effect on the synchronization frequency. We then consider a network of oscillators connected in a directed ring and obtain qualitative differences between the dynamics of the system, when natural frequencies are chosen from symmetric and asymmetric distributions, respectively. These differences are presented as numerical results. We also analyse the differences analytically and present an estimate of the shift in synchronization frequency.

2. Synchronization in completely connected network

To show the independence of synchronization frequency on the symmetry of natural frequency distribution, we consider the dynamical eq. (1) for the completely synchronized state of oscillators under the mean-field approximation. In such a scenario, all oscillators have the same effective frequency, $\dot{\theta}_i = \Omega$, and the equation transforms as

$$\Omega = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) \qquad \forall i = 1, 2, \ldots, N. \tag{2}$$

Adding the equations for all i's together gives

$$\Omega = \bar{\omega}_i + \frac{K}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i). \tag{3}$$

The second term in the equation drops out due to its antisymmetry in i and j and leads to the synchronization frequency Ω to be equal to the mean frequency $\bar{\omega}_i$. Hence, we conclude that the synchronization frequency is determined solely by the mean of the distribution and not its shape.

3. Synchronization in directed ring network

To construct a general system whose synchronization frequency can be changed by altering the shape of the distribution, we consider a ring topology in the oscillators, where each oscillator interacts in a unidirectional fashion, with its immediate neighbour only and the last oscillators interacts with the first oscillator. This is described mathematically as

$$\dot{\theta}_i = \omega_i + K \sin(\theta_{i+1} - \theta_i) \qquad \forall i = 1, 2, \ldots, N, \tag{4}$$

with the boundary condition $\theta_{N+1} = \theta_N$.

As in the previous case, the natural frequencies ω_i's are chosen from a general distribution $g(\omega)$. From here on, we assume without any loss of generality that the mean of $g(\omega)$ is zero because if the mean is different (say Ω), we can always apply a change of variables $\theta' = \theta - \Omega t$. In such a rotating frame, the apparent frequencies would seem to have zero as the mean.

3.1 *Numerical results on synchronization condition*

To analyse synchronization phenomenon in the directed ring network, numerical simulations are performed by RK4 method for numerical integration at double precision. Ensembles of oscillators with random natural frequencies are constructed [10,11]. Natural frequencies are chosen from symmetric (Gaussian and uniform) and asymmetric (χ^2 and log-normal) distributions. For the simulations, the distributions are chosen to have mean zero and sampling is performed using inverse Fourier and rejection methods [12,13].

For each simulation, the time evolution of the order parameter [6]

$$\mathcal{O} = r\mathrm{e}^{\iota\psi} = \frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{\iota\theta_j}, \tag{5}$$

is also computed. Here the coherence parameter

$$r(t) = \frac{1}{N}\sqrt{\left(\sum_{i=1}^{N}\cos\theta_i\right)^2 + \left(\sum_{i=1}^{N}\sin\theta_i\right)^2}, \tag{6}$$

lies between 0 and 1 and measures the phase coherence. If all the oscillators move in a tight clump, the phases are almost the same and $r \approx 1$, whereas if all the oscillators are scattered, then $r \approx 0$.

The results obtained for symmetric $g(\omega)$ are similar to those of the mean-field Kuramoto oscillators (figure 1). For very low K values, we obtain $r(t) \to 0$ as $t \to \infty$. For this case, the value of $\dot{\theta}$ averaged over all oscillators fluctuates with time; and we see no synchronization. For a large value of K, $r(t) \to 1$ as $t \to \infty$ and the system gets synchronized to the mean of the natural frequency distribution, which is zero in this case.

For very low and very high values of K, the behaviour of the asymmetric system is much like that of the symmetric one. The oscillators are unsynchronized for low values of K and are synchronized for very high K values (figure 2).

However, for intermediate values of K, the asymmetric oscillators become phase locked even for very low value asymptotic value of $r(t)$ (values less than 0.1) (figure 3a). This can be clearly seen from the fact that variance of $\dot{\theta}_i$ tends to zero with time (figure 3b) and that the difference between the phases of the oscillators becomes approximately

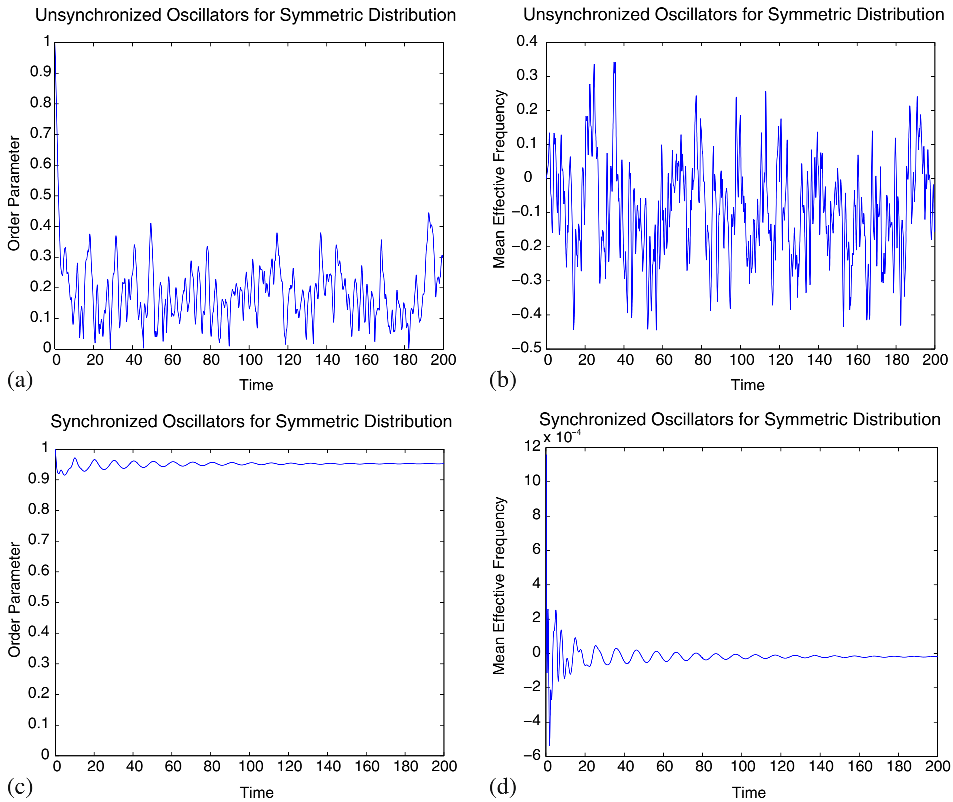


Figure 1. Unsynchronized and synchronized states of oscillators for symmetric natural frequency distribution. (**a**) shows the variation of coherence parameter with time and (**b**) shows the corresponding mean frequency variation for an unsynchronized state. (**c**) and (**d**) show the counterparts for a synchronized state.

constant. What is more interesting is that in this state, the effective synchronization frequency of the oscillators shifts from the mean of their natural frequencies (figure 3c).

The phenomenon of synchronization at low values of r for asymmetric distribution points towards phases being spread out as the oscillators get synchronized. While the possibility of such 'spread-out synchronized states' cannot be denied in symmetric distributions, our simulations suggest that such a state is seen only in networks with asymmetric natural frequency distribution. More explicitly, the only form of synchronization observed in oscillators with symmetric natural frequency distribution is that where $r(t)$ is close to one (figures 1c and 1d). Numerical explorations do not yield cases, where synchronization occurs for $r(t)$ close to zero for symmetric natural frequency distributions. Such cases are obtained only for asymmetric natural frequency distributions (figures 3a and 3c).

Another notable feature of synchronization in asymmetric natural frequency distributions is the dependence of synchronization frequency on the strength of coupling. For symmetric distributions, whenever the oscillators get synchronized, the synchronization

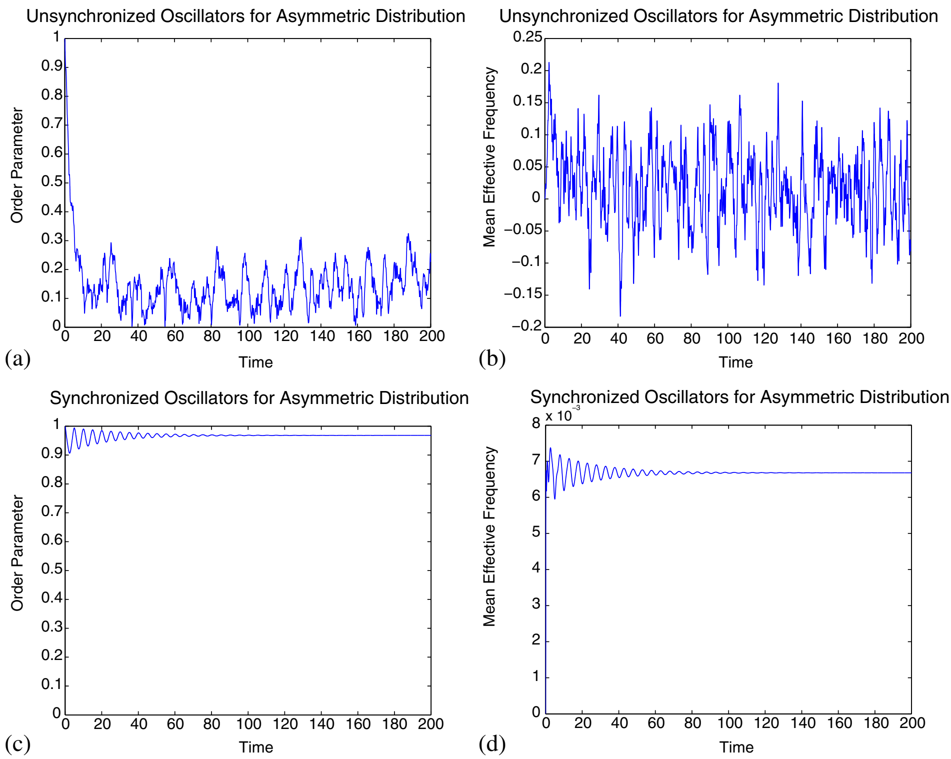


Figure 2. Unsynchronized and synchronized states of oscillators for asymmetric natural frequency distribution. (**a**) shows the variation of coherence parameter with time and (**b**) shows the corresponding mean frequency variation for an unsynchronized state. (**c**) and (**d**) show the counterparts for a synchronized state.

frequency is always numerically found to be at $\Omega = 0$ (the mean of the distribution) irrespective of the synchronization frequency. In stark contrast, for the synchronization in the intermediate coupling ranges, the synchronization frequency is observed to be away from the mean. Moreover, the synchronization frequency changes as coupling strength K is varied (figure 4b). Additionally, the synchronization frequency asymptotically converges to the mean for very high K values (figure 4c).

3.2 *Theoretical analysis of synchronization condition*

To obtain the condition for synchronization, we assume that each oscillator moves with an effective frequency, $\dot{\theta}_i = \Omega$. We then invert eq. (4) to obtain

$$\sin^{-1}\left(\frac{\Omega - \omega_i}{K}\right) = \theta_{i+1} - \theta_i \qquad \forall i = 1, 2, \ldots, N. \tag{7}$$

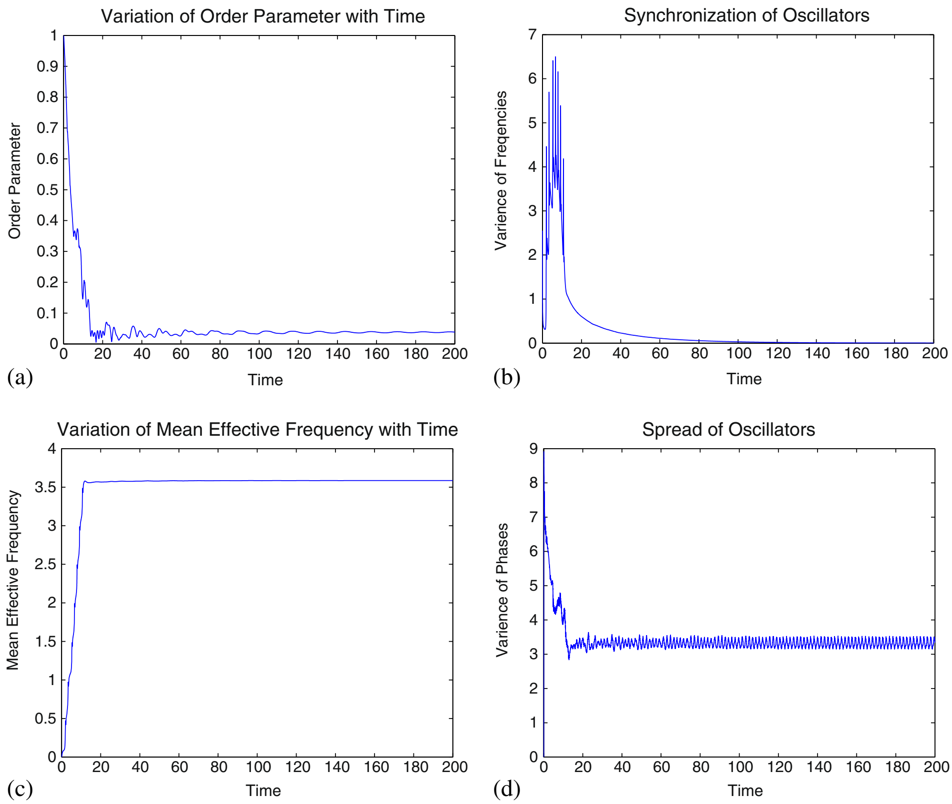


Figure 3. Numerical results of simulations with asymmetric distribution of natural frequencies. The network gets synchronized as seen by the variance of frequency decaying down to zero (in (**b**)). For the same simulation, the coherence parameter (in (**a**)) attains a very low value and the mean effective frequency (in (**c**)) – which is the common synchronization frequency for asymptotically low variance – settles to a value away from the mean of the natural frequencies. (**d**) shows the variance of the phases which stabilizes to a finite value showing the spread of oscillators in the synchronized state.

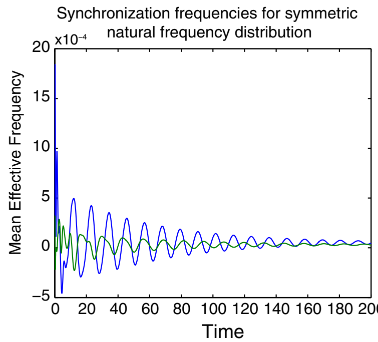
Summing over all oscillators cancels out the right-hand side completely to yield the condition for synchronization to be

$$\sum_{i=1}^{N} \sin^{-1}\left(\frac{\Omega - \omega_i}{K}\right) = 0. \tag{8}$$

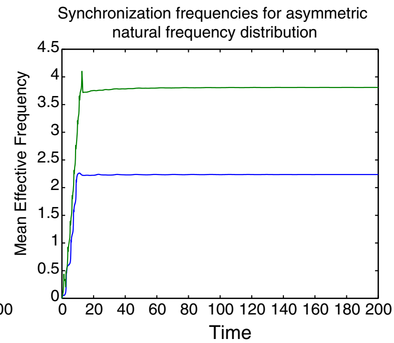
We now Taylor expand the sine inverses around $(\Omega - \omega_i)/K = 0$, and then by applying binomial expansion on each of the resulting terms, we have

$$\sum_{n=0}^{\infty} c_{2n+1} \sum_{j=0}^{2n+1} (-1)^j \, {}^{2n+1}C_j \left(\frac{\Omega}{K}\right)^{2n+1-j} \sum_{i=1}^{N} \left(\frac{\omega_i}{K}\right)^j = 0, \tag{9}$$

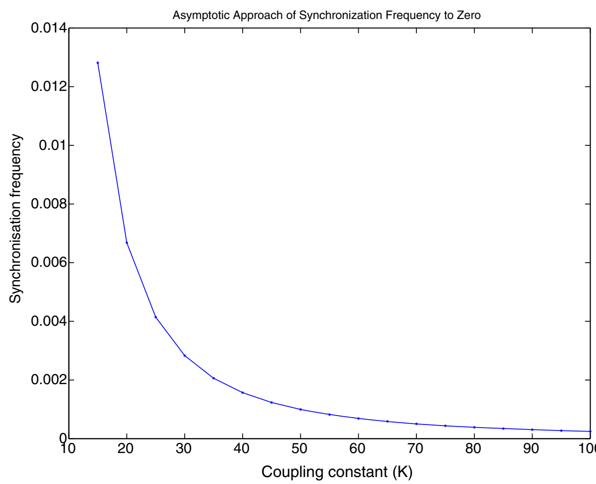
where c_{2n+1} is the coefficient of x^{2n+1} in the expansion of $\sin^{-1} x$.



(a) Symmetric natural frequency distribution



(b) Asymmetric natural frequency distribution



(c) Synchronization frequency for very high coupling

Figure 4. Effect of coupling strength on synchronization frequency. (**a**) and (**b**) show the time evolution of mean frequency for $K = 8$ (green) and $K = 9$ (blue) for symmetric and asymmetric natural frequency distributions. Synchronization occurs in both the cases. However, the synchronization frequency is unaffected (remains at zero) for the symmetric case, whereas it changes for its counterpart. As shown in (**c**), the synchronization frequency reaches zero for very high coupling strength.

Now collecting the terms containing the same powers of Ω/K together, we get

$$-\sum_{j=\text{even}}\left(\sum_{n=0}^{\infty} c_{2n+1+j}\;\; \mu_{2n+1}\;{}^{2n+1+j}C_{2n+1}\right)\left(\frac{\Omega}{K}\right)^{j} + \sum_{j=\text{odd}}\left(\sum_{n=0}^{\infty} c_{2n+j}\;\mu_{2n}\;{}^{2n+j}C_{2n}\right)\left(\frac{\Omega}{K}\right)^{j} = 0, \tag{10}$$

where

$$\mu_j = \sum_{i=1}^{N}\left(\frac{\omega_i}{K}\right)^{j}. \tag{11}$$

Writing it explicitly, we have

$$
\begin{aligned}
&-\left(c_1\mu_1 + c_3\mu_3 + \cdots\right) + \left(c_1\mu_0 + c_3\mu_2\, {}^3C_2 + \cdots\right)\frac{\Omega}{K} \\
&\quad -\left(c_3\mu_1\, {}^3C_1 + c_5\mu_3\, {}^5C_3 + \cdots\right)\left(\frac{\Omega}{K}\right)^2 + \cdots = 0.
\end{aligned} \tag{12}
$$

From eq. (12), we note that for a symmetric distribution, all the odd-μ_i's are zeroes; all the terms of the form $(\Omega/K)^{2j}$ vanish. In particular, the first term of the LHS vanishes. This results in $\Omega = 0$ being a trivial solution for the equation. Hence, a set of oscillators with any symmetric distribution of natural frequencies can synchronize to the mean of the natural frequencies.

For a asymmetric natural frequency distribution, however, at least one of the odd moments is non-zero. In that case, the first term of eq. (12) does not vanish. Hence, $\Omega = 0$ does not in general satisfy the equation. Hence, a set of oscillators with an asymmetric distribution of natural frequencies cannot synchronize to the mean of the natural frequencies.

To estimate the synchronization frequency for a slightly asymmetric natural frequency distribution, we can use the expansion in eq. (12). We introduce the asymmetry in the distribution by adding some small non-zero odd k-moments to the symmetric distribution. Let S_k be the set of all such odd k's. Let μ_k's be the terms corresponding to the small non-zero odd moments in the slightly asymmetric distribution.

For such a slightly asymmetric natural frequency distribution, we expect the synchronization frequency to be slightly shifted from $\Omega = 0$, obtained for symmetric distribution. Hence, we keep the expansion to the linear term in Ω/K and neglect the terms of higher orders to obtain

$$
-\sum_{k\in S_K} c_k\mu_k + \sum_{j=0}^{\infty}(2j+1)\, c_{2j+1}\, \mu_{2j}\left(\frac{\Omega}{K}\right) = 0. \tag{13}
$$

Hence, the synchronization frequency for the asymmetric distribution is given as

$$
\Omega = K\frac{\sum_{k\in S_K} c_k\mu_k}{\sum_{j=0}^{\infty}(2j+1)\, c_{2j+1}\, \mu_{2j}}, \tag{14}
$$

where c_j is the coefficient of x^j in the expansion of $\sin^{-1} x$.

Some conclusions can be immediately drawn from eq. (14). First, as the values of c_j decreases with increasing j, the effect of asymmetry on shifting the synchronization frequency is maximum, if asymmetry is introduced by increasing μ_3 or the skewness of $g(\omega)$. Secondly, as the denominator of the expression contains weighted sum of even moments of the distribution, we can conclude that a greater shift in synchronization frequency will be observed for a sharper distribution of natural frequencies.

4. Conclusion

In this paper, we study the effect of symmetry of natural frequency distribution on the synchronization frequency of Kuramoto oscillators. After establishing that synchronization frequency of a complete network of oscillators is unaffected by symmetry of the distribution of natural frequencies, we construct a ring network of oscillators to observe a shift in synchronization frequency for asymmetric natural frequency distribution and an intermediate range of coupling strength. We analytically show that oscillators can synchronize to the mean of the frequency distribution only for symmetric distributions. As asymmetry is introduced in a synchronized symmetric system, the synchronization frequency gradually shifts away from the mean. An estimate of the shift for small asymmetry is also given. The results are also qualitative and predict an increase in the shift with reduction in coupling strength which is also seen in numerical simulations.

Although we have been able to come up with a network of Kuramoto oscillators whose synchronization frequency can be changed by introducing asymmetries in the natural frequency distribution, a few questions still remain open and unanswered. These include estimation of synchronization frequency for a more general asymmetric distribution and characterization of phase difference between the oscillators for the non-collapsed phase-locked state. We hope that future research will throw some light on the unaddressed issues and help us characterize the effect of asymmetries in a more efficient way.

References

[1] J Buck, *Quart. Rev. Biol.* **63(3)**, 265 (1988)
[2] Z Néda, E Ravasz, T Vicsek, Y Brechet and A L Barabási, *Phys. Rev. E* **61(6)**, 6987 (2000)
[3] W Gerstner, *Phys. Rev. E* **51(1)**, 738 (1995)
[4] Kurt Wiesenfeld, Pere Colet and Steven H Strogatz, *Phys. Rev. E* **57**, 1563 (1998)
[5] Kurt Wiesenfeld, Pere Colet and Steven H Strogatz, *Phys. Rev. Lett.* **76(3)**, 404 (1996)
[6] Y Kuramoto, *Chemical oscillations, waves, and turbulence* (Dover Publications, Mineola, New York, 2003)
[7] N Wiener, *Nonlinear problem in random theory* edited by Norbert Wiener, ISBN 0-262-73012-X (The MIT Press, Cambridge, Massachusetts, USA, 1966) p. 142
[8] N Wiener, *Scient. Amer.* **179(5)**, 14 (1948)
[9] S H Strogatz, *Phys. D: Nonlinear Phenom.* **143(1)**, 1 (2000)
[10] Radford M Neal, *Ann. Stat.* **31(3)**, 705 (2003)
[11] Christian P Robert and George Casella, *Monte Carlo statistical methods* (Citeseer, 2004) Vol. 319
[12] S M Ross, *Simulation* (Elsevier Academic Press, Amsterdam, 2006)
[13] J Aitchison and J A C Brown, *The lognormal distribution* (University of Cambridge, Department of Applied Economics, Monograph No. 5, 1957)