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GEOMETRY FROM DONALDSON-THOMAS INVARIANTS

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ABSTRACT. We introduce geometric structures on the space of stability conditions of a three-dimensional Calabi-Yau category which encode the Donaldson-Thomas invariants of the category. We explain in detail a close analogy between these structures, which we call Joyce structures, and Frobenius structures. In the second half of the paper we give explicit calculations of Joyce structures in three interesting classes of examples.

1. INTRODUCTION

The aim of this paper is to use the Donaldson-Thomas (DT) invariants [33, 34] of a CY_3 triangulated category to define a geometric structure on its space of stability conditions [7, 9]. We call the resulting structures Joyce structures, since the most important ingredients already appear in the paper [32]. There is a close analogy between the notion of a Joyce structure and that of a Frobenius structure [19, 20] which will be our main theme. Both structures involve isomonodromic pencils of flat connections on the tangent bundle of a complex manifold M , but in the case of Joyce structures these connections are non-linear.

A key difference between Frobenius and Joyce structures lies in their relationship to the corresponding enumerative invariants. The genus 0 Gromov-Witten invariants of a smooth, complex, projective variety X can be encoded in a Frobenius structure on a small, possibly formal, ball in the vector space $H^*(X, \mathbb{C})$. The pencil of flat connections is given directly by the derivatives of the prepotential, which in turn is the generating function for the invariants. In this paper we shall similarly encode the DT invariants of a CY_3 category in a Joyce structure on its space of stability conditions. But in this context, it is not the flat connections themselves that are described by the invariants, but rather their generalised monodromy or Stokes data.

1.1. BPS structures. Nothing in this paper requires any knowledge of the inner workings of DT theory. The output of DT theory applied to a stability condition on a CY_3 triangulated category was axiomatised in [12] to give the notion of a BPS structure. It consists of the following simple data:

- (a) a free abelian group $\Gamma \cong \mathbb{Z}^{\oplus n}$ equipped with a skew-symmetric integral form $\langle -, - \rangle$;

- (b) a homomorphism of abelian groups $Z: \Gamma \rightarrow \mathbb{C}$ called the central charge;
- (c) a map of sets $\Omega: \Gamma \rightarrow \mathbb{Q}$ encoding the DT invariants;

subject only to the symmetry condition $\Omega(-\gamma) = \Omega(\gamma)$, and a weak finiteness condition known as the support property. The most basic general property of DT invariants is the Kontsevich-Soibelman wall-crossing formula [34], which describes how the invariants change under variations of stability condition. This can be axiomatised in the abstract notion of a variation of BPS structures over a manifold M . These concepts, which are special cases of structures introduced by Kontsevich and Soibelman, will be reviewed in Section 5.

To give a variation of BPS structures should be viewed as being analogous to giving the Stokes factors of a semi-simple Frobenius manifold as a function of the canonical co-ordinates. Reconstructing the Frobenius structure from such data involves inverting the Riemann-Hilbert correspondence for a class of meromorphic connections, which in turn, involves solving a Riemann-Hilbert boundary value problem [19, Lecture 3], [20, Lecture 4] for maps from the complex plane to the group $\mathrm{GL}_n(\mathbb{C})$. The problem of reconstructing a Joyce structure from a variation of BPS structures is entirely analogous, but with the finite-dimensional group $\mathrm{GL}_n(\mathbb{C})$ replaced by the infinite-dimensional group of Poisson automorphisms of an algebraic torus. The corresponding Riemann-Hilbert boundary value problems were studied in detail in [12].

Although we have no general existence or uniqueness results for solutions to the relevant Riemann-Hilbert problems, in the second part of the paper, Sections 8–10, we make some non-trivial calculations in several cases arising naturally in DT theory. We hope that these results, which relate in an interesting way to known structures in mirror symmetry, will provide adequate motivation for the further study of the analogy considered here.

1.2. Joyce structures. Let us now describe in a little more detail the essential features of a Joyce structure. The precise definition can be found in Section 4.4 below. The starting point is a complex manifold M equipped with a flat, torsion-free, holomorphic connection ∇ on its tangent bundle \mathcal{T}_M , and a covariantly constant skew-symmetric pairing

$$\eta: \mathcal{T}_M^* \times \mathcal{T}_M^* \rightarrow \mathbb{C}.$$

The example we have in mind is the space of stability conditions $\mathrm{Stab}(\mathcal{D})$ on a CY_3 triangulated category \mathcal{D} , in which case the central charges of the objects of \mathcal{D} give flat co-ordinates for

the connection ∇ , and the skew-symmetric form $\eta(-, -)$ is induced by the Euler form on the Grothedieck group $K_0(\mathcal{D})$.

Let us choose a local system of flat co-ordinates (z_1, \dots, z_n) on M . We obtain induced linear co-ordinates $(\theta_1, \dots, \theta_n)$ on each tangent space $\mathcal{T}_{M,p}$ by writing a tangent vector X at a point $p \in M$ in the form

$$X = \sum_{i=1}^n \theta_i \cdot \frac{\partial}{\partial z_i}.$$

We can also introduce the constant skew-symmetric matrix $\eta_{ij} = \eta(dz_i, dz_j)$.

A Joyce structure on M involves a pencil of flat, meromorphic Ehresmann connections $\mathcal{A}^{(\epsilon)}$ on the bundle \mathcal{T}_M , whose horizontal subspaces are spanned by vector fields of the form

$$\frac{\partial}{\partial z_i} + \frac{1}{\epsilon} \cdot \frac{\partial}{\partial \theta_i} - \sum_j \eta_{jk} \cdot \frac{\partial^2 J}{\partial \theta_i \partial \theta_j} \cdot \frac{\partial}{\partial \theta_k}, \quad (1)$$

for some meromorphic function $J: \mathcal{T}_M \rightarrow \mathbb{C}$ on the total space of the tangent bundle, which we call the Joyce function.¹ Flatness of the connection (1) involves a partial differential equation

$$\frac{\partial^2 J}{\partial z_i \partial \theta_j} - \frac{\partial^2 J}{\partial z_j \partial \theta_i} = \sum_{p,q} \eta_{pq} \cdot \frac{\partial^2 J}{\partial \theta_i \partial \theta_p} \cdot \frac{\partial^2 J}{\partial \theta_j \partial \theta_q}. \quad (2)$$

The pencil of flat connections (1) should be viewed as analogous to the deformed flat connection of a Frobenius manifold. Just as in the Frobenius case, this pencil extends to a meromorphic connection on the pull-back of the bundle \mathcal{T}_M to the product $M \times \mathbb{P}^1$, which we can then alternatively view as an isomonodromic family of meromorphic Ehresmann connections on the trivial bundles over \mathbb{P}^1 with fibres $\mathcal{T}_{M,x}$. In the Joyce case these connections take the form

$$\frac{\partial}{\partial \epsilon} - \frac{1}{\epsilon^2} \cdot \sum_i z_i \cdot \frac{\partial}{\partial \theta_i} + \frac{1}{\epsilon} \cdot \sum_{i,j,k} z_i \cdot \eta_{jk} \cdot \frac{\partial^2 J}{\partial \theta_i \partial \theta_j} \cdot \frac{\partial}{\partial \theta_k}. \quad (3)$$

It is the Stokes data of the connection (3) at the irregular singularity $\epsilon = 0$ that is encoded by the relevant BPS structure. The reconstruction problem discussed above amounts to finding the Joyce function $J = J(z_i, \theta_j)$ in terms of this Stokes data.

¹In fact, it is the second derivatives of the Joyce function appearing in (1) that are genuine meromorphic functions on \mathcal{T}_M ; in general, the function J itself may acquire logarithmic singularities, and is therefore only well-defined and single-valued on a dense open subset. See the discussion in Section 4.3 below.

1.3. Linearization. Consider again a Joyce structure on a complex manifold M , and let us assume that the Joyce function J is holomorphic in a neighbourhood of the zero-section of \mathcal{T}_M . Although the connections $\mathcal{A}^{(\epsilon)}$ of the pencil (1) are inherently non-linear, we can nonetheless use a Joyce structure to induce linear structures on the tangent bundle \mathcal{T}_M . This construction follows an idea of Joyce [32, Section 6.2]. An interesting observation, which we have no good explanation for at present, is that in some of the examples we discuss below, these linear structures are closely related to certain naturally associated Frobenius structures.

We begin by defining a symmetric bilinear form $g: \mathcal{T}_M \times \mathcal{T}_M \rightarrow \mathcal{O}_M$ by setting

$$g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) = \sum_m z_m \cdot \frac{\partial^3 J}{\partial \theta_i \partial \theta_j \partial \theta_m} \Big|_{\theta=0}. \quad (4)$$

We call g the Joyce form, since an equivalent definition appears in [32, Section 6.2]. This form is not always non-degenerate, but when it is, it follows from the differential equation (2) that the associated Levi-Civita connection is flat. In this situation we can further attempt to define an operation $\diamond: \mathcal{T}_M \times \mathcal{T}_M \rightarrow \mathcal{T}_M$ by the formula

$$g\left(\frac{\partial}{\partial z_i} \diamond \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right) = \frac{\partial^3 J}{\partial \theta_i \partial \theta_j \partial \theta_k} \Big|_{\theta=0} = g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \diamond \frac{\partial}{\partial z_k}\right). \quad (5)$$

We call this operation the diamond product. For an arbitrary Joyce structure it need not be associative, but for at least some of the examples arising naturally in DT theory it is.

There is one other interesting object associated to a Joyce structure, which can be defined only when both sides of the equation (3) identically vanish. This happens for example when the form $\eta = 0$. In this situation there exist locally-defined functions $\mathcal{F}: M \rightarrow \mathbb{C}$ satisfying

$$\frac{\partial^3 \mathcal{F}}{\partial z_i \partial z_j \partial z_k} = \frac{\partial^3 J}{\partial \theta_i \partial \theta_j \partial \theta_k} \Big|_{\theta=0}.$$

We call such a function a prepotential; it is unique up to the addition of quadratic functions in the co-ordinates z_i .

One of the main aims of this paper is to compute the Joyce form g , the diamond product \diamond , and the prepotential \mathcal{F} in some interesting examples. In the next three subsections we will give a brief summary of our results in this direction.

1.4. Finite uncoupled BPS structures. A BPS structure is called finite if there are only finitely many classes $\gamma \in \Gamma$ for which the BPS invariant $\Omega(\gamma)$ is nonzero. It is called uncoupled

if

$$\gamma_1, \gamma_2 \in \Gamma, \quad \Omega(\gamma_i) \neq 0 \implies \langle \gamma_1, \gamma_2 \rangle = 0.$$

This condition implies that the Stokes automorphisms corresponding to different rays commute. It follows that in a variation of uncoupled BPS structures the BPS invariants $\Omega(\gamma)$ are constant.

The Riemann-Hilbert problems defined by BPS structures satisfying these two conditions, and also the integrality property $\Omega(\gamma) \in \mathbb{Z}$, were studied by Barbieri [3], who gave an explicit solution in terms of the gamma function. We review her results in Section 8, and use them to compute the Joyce structure associated to a variation of such BPS structures.

We assume given a variation of finite, uncoupled, integral BPS structures over a manifold M which is also miniversal, in the sense that the locally-defined period map

$$M \rightarrow \text{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}),$$

sending a point $p \in M$ to the corresponding central charge $Z: \Gamma \rightarrow \mathbb{C}$, is a local isomorphism. The derivative of this map then allows us to identify tangent vectors to M with group homomorphisms $\theta: \Gamma \rightarrow \mathbb{C}$, and so the Joyce function can be viewed as depending on the central charge $Z: \Gamma \rightarrow \mathbb{C}$ together with a tangent vector $\theta: \Gamma \rightarrow \mathbb{C}$.

In Section 8 we show that the Joyce function is given explicitly by the simple formula

$$J(Z, \theta) = \frac{1}{12\pi i} \cdot \sum_{\alpha \in \Gamma} \Omega(\alpha) \cdot \frac{\theta(\alpha)^3}{Z(\alpha)}.$$

The Joyce form (4) is therefore

$$g(X_1, X_2) = \frac{1}{2\pi i} \cdot \sum_{\alpha \in \Gamma} \Omega(\alpha) \cdot X_1(\alpha) X_2(\alpha).$$

When this is non-degenerate, the diamond product (5) is defined by

$$g(X_1 \diamond X_2, X_3) = \sum_{\alpha \in \Gamma} \Omega(\alpha) \cdot \frac{X_1(\alpha) X_2(\alpha) X_3(\alpha)}{Z(\alpha)} = g(X_1, X_2 \diamond X_3).$$

Finally, there is a locally-defined prepotential $\mathcal{F}: M \rightarrow \mathbb{C}$ given by the formula

$$\mathcal{F}(Z) = \frac{1}{4\pi i} \cdot \sum_{\alpha \in \Gamma} \Omega(\alpha) \cdot Z(\alpha)^2 \log Z(\alpha).$$

The condition that the product \diamond is associative is non-trivial: it is equivalent to the WDVV equations for the function \mathcal{F} with respect to the metric g . In the special case when $\Omega(\gamma) \in \{0, 1\}$ for all $\gamma \in \Gamma$ this is equivalent to the condition that the set of classes $\gamma \in \Gamma$ satisfying

$\Omega(\gamma) = 1$ form a \vee -system in the sense of Veselov [39]. There is an interesting class of examples of this type which relate to the almost duals [21] of the polynomial Frobenius manifolds.

1.5. Calabi-Yau threefolds without compact divisors. In Section 9 we consider the BPS structures which result from DT theory applied to coherent sheaves on a non-compact Calabi-Yau threefold X containing no compact divisors. Since all compactly supported coherent sheaves $E \in \text{Coh}(X)$ are then supported on curves, the Chern character $(\text{ch}_2(E), \text{ch}_3(E))$ of such a sheaf defines an element of the lattice

$$\Gamma = H_2(X, \mathbb{Z}) \oplus \mathbb{Z}.$$

The Euler form $\langle -, - \rangle$ on this lattice vanishes, since curves do not generically intersect on a threefold, so the resulting BPS structures are uncoupled, although not usually finite.

We assume the well known conjecture that the BPS curve-counting invariants coincide with the genus 0 Gopakumar-Vafa invariants:

$$\Omega(\beta, n) = \text{GV}(0, \beta).$$

We also assume that there are only finitely many homology classes $\beta \in H_2(X, \mathbb{Z})$ for which these invariants are nonzero. We can then solve the corresponding Riemann-Hilbert problem using the methods of [14].

The resulting Joyce function depends on central charge co-ordinates $(v, w) \in H^2(X, \mathbb{C}) \oplus \mathbb{C}$, and fibre co-ordinates $(\theta, \phi) \in H^2(X, \mathbb{C}) \oplus \mathbb{C}$. It is given by

$$J(v, w, \theta, \phi) = \frac{1}{6w^4} \cdot \sum_{\beta \in H_2(X, \mathbb{Z})} \text{GV}(0, \beta) \cdot (v(\beta)\phi - w\theta(\beta))^3 \cdot (1 - e^{-2\pi i v(\beta)/w})^{-1}.$$

In this case the Joyce form vanishes, which is perhaps to be expected, since under our assumptions the intersection form on $H^*(X, \mathbb{C})$ also vanishes. The diamond product is therefore undefined. On the other hand, computing the prepotential gives

$$F(v, w) = \frac{-w^2}{(2\pi i)^3} \cdot \sum_{\beta \in H_2(X, \mathbb{Z})} \text{GV}(0, \beta) \cdot \text{Li}_3(e^{2\pi i v(\beta)/w}),$$

which is precisely the genus 0 part of the Gromov-Witten generating function.

These results should be compared with the main result of [14] which, in the case when X is the resolved conifold, used the solution of the Riemann-Hilbert problem at $(\theta, \phi) = (0, 0)$ to define an analytic function $\tau(v, w; \epsilon)$ whose asymptotic expansion at $\epsilon = 0$ reproduced the genus ≥ 2 terms in the Gromov-Witten generating function of X . Taken together, we view

these results as strong evidence that the Riemann-Hilbert problems we are studying are closely related to topological string theory.

1.6. The A_2 quiver and generalizations. In Section 10 we focus on the variation of BPS structures defined by the DT theory of the A_2 quiver. This is the simplest example of a variation which is not uncoupled, and hence where the wall-crossing formula is non-trivial. The base of the structure is the complex manifold

$$M = \{(a, b) \in \mathbb{C}^2 : 4a^3 + 27b^2 \neq 0\}. \quad (6)$$

It follows from the results of [13] that M can be viewed as the space of stability conditions on the derived category of the three-dimensional Ginzburg algebra associated to the A_2 quiver, quotiented by the group of autoequivalences generated by spherical twists in the two vertex simples.

Each point of M determines a meromorphic quadratic differential on \mathbb{P}^1

$$Q_0(x) dx^{\otimes 2} = (x^3 + ax + b) dx^{\otimes 2},$$

with a single pole of order seven at $x = \infty$. The central charge co-ordinates are given by the period integrals

$$z_i = \int_{\gamma_i} \sqrt{Q_0(x)} dx,$$

where (γ_1, γ_2) is a basis for the first homology group of the affine elliptic curve $y^2 = Q_0(x)$.

Taking inspiration from the work of Gaiotto, Moore and Neitzke [25], we show that the associated Riemann-Hilbert problems can be solved using the monodromy map for a family of differential equations of the form

$$y''(x) = Q(x, \epsilon) \cdot y(x), \quad Q(x, \epsilon) = \epsilon^{-2} \cdot Q_0(x) + \epsilon^{-1} \cdot Q_1(x) + Q_2(x), \quad (7)$$

known as the deformed cubic oscillator. The resulting Joyce structure then encodes the isomonodromy connection for these equations. The details of this are explained in the paper [15].

The Joyce form in this case is

$$g = \frac{1}{5} \cdot da \otimes db.$$

In particular, g is non-degenerate, and we recover the natural co-ordinates (a, b) , which are not at all obvious from the point of view of the space of stability conditions, as the corresponding flat co-ordinates. Furthermore, the diamond product is associative, and coincides, up to a

constant factor, with the product of the dual almost Frobenius structure [21, Section 5.2] associated to the A_2 root system.

This A_2 example fits into a much more general class of examples, which in physical terms correspond to theories of class S with gauge group $SU(2)$. A result of the author with Smith [16] shows that in these cases the space of stability conditions modulo autoequivalences coincides with the moduli space of pairs (S, ϕ) , consisting of a Riemann surface S equipped with a meromorphic quadratic differential ϕ , having poles of fixed orders and simple zeroes. The BPS invariants are obtained by counting finite-length trajectories of the differential ϕ . We expect that also in this more general setting the relevant Joyce structures can be described using the isomonodromy connection for a family of opers with apparent singularities of the form (7).

1.7. Relation to previous work. As well as the theory of generalised DT invariants constructed by Joyce, Kontsevich, Soibelman and others, this paper builds in an essential way on several other works. One of the key underlying ideas is that the wall-crossing formula for DT invariants should be interpreted as the isomonodromy condition for a family of irregular connections. This point-of-view was explained in a slightly different context by the author and Toledano Laredo [10], and also played a prominent role in the work of Gaiotto, Moore and Neitzke [24]. The main results of [10] were themselves inspired by Joyce's remarkable paper [32], which attempts to use weighted sums of DT invariants to define holomorphic generating functions on the space of stability conditions.

The material we present here is closely related to the work of Barbieri, Filipini, Garcia-Fernandez and Stoppa [4, 22]. These authors use DT invariants to construct formal Frobenius-type structures on the space of stability conditions, involving connections taking values in an infinite-dimensional Lie algebra of functions on an algebraic torus. Their approach is close in spirit to that of Joyce [32], and the resulting structures can be viewed as formal versions of the same Joyce structures we consider here.

In the approach of Joyce, and of Stoppa and his collaborators, and also in the approach we advocate here, the non-trivial (and for the most part unsolved) problem is to invert a certain irregular Riemann-Hilbert map. For Joyce and Stoppa *et al* this is done using a formal power-series expansion; the difficult issue is then to prove convergence of this series. In this paper, we instead take an analytic approach using Riemann-Hilbert problems, and the challenge is then to prove the existence of solutions to these problems.

The Riemann-Hilbert problems which allow us to pass from BPS structures to Joyce structures are closely related to those considered by Gaiotto, Moore and Neitzke [24, 25] in the context of quantum field theories with $d = 4$, $\mathcal{N} = 2$ supersymmetry. These authors already remarked on the close analogy between their constructions and the tt^* equations studied by Cecotti and Vafa [17, 18] in the context of theories with $d = 2$, $\mathcal{N} = 2$ supersymmetry. In the conformal limit [23], this analogy becomes the analogy between Joyce structures and Frobenius structures which is the main topic of this paper.

The Riemann-Hilbert problems considered here are also closely-related to the Thermodynamic Bethe Ansatz (TBA) equations although the author is unfortunately not qualified to comment in detail on this. The basic point is that the Riemann-Hilbert problem we discuss here can be reformulated as an integral equation, which after suitable reprocessing becomes the TBA equations. We refer the reader to [?], [23] and [24, Appendix E] for more details on this connection.

1.8. Plan of the paper. The first two sections contain an exposition of the relevant parts of the theory of Frobenius structures. Everything here can be found in some form in Dubrovin's original lecture notes [19, 20]. In Section 2 we give the basic definitions, and explain how a Frobenius structure on a complex manifold can be encoded in an isomonodromic family of irregular connections on \mathbb{P}^1 . In Section 3 we then explain how this family of connections can in turn be encoded by its Stokes data.

In the next two sections we give an analogous treatment of the theory of Joyce structures. The definition of a Joyce structure is introduced in Section 4, guided closely by the analogy with Frobenius structures. We also introduce the associated isomonodromic family of meromorphic Ehresmann connections on \mathbb{P}^1 . In Section 5 we recall from [12] the basic definitions concerning BPS structures, which encode the Stokes data of these connections.

In Section 6 we discuss the general problem of reconstructing pencils of connections from their Stokes data. This amounts to inverting an irregular Riemann-Hilbert correspondence. We consider several approaches to this problem, both in the finite-dimensional Frobenius case, and the infinite-dimensional Joyce case.

Section 7 discusses linear structures on the tangent bundle induced by a Joyce structure. We define the Joyce form, the linearized Joyce connection and the diamond product discussed above. We also define a notion of compatibility for Frobenius and Joyce structures with the same underlying manifold.

The bulk of the second half of the paper, comprising Sections 8–10, is devoted to explicit computations in particular examples. Section 8 deals with uncoupled BPS structures, following work of Barbieri [3]. Section 9 is concerned with the DT theory of coherent sheaves on non-compact Calabi-Yau threefolds, and relies on the computations of [14]. Section 10 focuses on the case of the A_2 quiver, and is essentially a digest of the paper [15].

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2. FROBENIUS STRUCTURES AND PENCILS OF CONNECTIONS

In this section and the next we recall some of the basic theory of Frobenius structures, following Dubrovin [19, 20]. Other useful references are [26, 27, 36, 38]. This material will only be used in what follows as a motivating analogy for the definition of Joyce structures in Section 4, and we tailor our treatment to this purpose. The main point for us is that a Frobenius structure can be encoded in an isomonodromic family of meromorphic connections on \mathbb{P}^1 of a particular kind.

2.1. Definition of a Frobenius structure. We begin with the definition of a Frobenius structure. The holomorphic tangent bundle of a complex manifold M will be denoted by \mathcal{T}_M .

Definition 2.1. A *Frobenius structure* on a complex manifold M consists of the following data

- (i) the *metric*: a holomorphic map $g: \mathcal{T}_M \otimes \mathcal{T}_M \rightarrow \mathcal{O}_M$, inducing a symmetric and non-degenerate bilinear form on each tangent space $\mathcal{T}_{M,p}$;
- (ii) the *product*: a holomorphic map $*$: $\mathcal{T}_M \otimes \mathcal{T}_M \rightarrow \mathcal{T}_M$, inducing on each tangent space $\mathcal{T}_{M,p}$ the structure of a commutative and associative algebra;
- (iii) the *Euler vector field*: a holomorphic section $E: \mathcal{O}_M \rightarrow \mathcal{T}_M$;

subject to the following four axioms:

- (F1) the Levi-Civita connection ∇^{LC} on \mathcal{T}_M defined by the metric g is flat;
- (F2) there exists a covariantly constant vector field $e: \mathcal{O}_M \rightarrow \mathcal{T}_M$ which is a unit for the product;

(F3) in local flat co-ordinates (t_1, \dots, t_n) for the metric g , there exists a locally-defined function $F(t_1, \dots, t_n)$ satisfying

$$g\left(\frac{\partial}{\partial t_i} * \frac{\partial}{\partial t_j}, \frac{\partial}{\partial t_k}\right) = \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k} = g\left(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j} * \frac{\partial}{\partial t_k}\right); \quad (8)$$

(F4) the vector field E is linear: $\nabla^{\text{LC}}(\nabla^{\text{LC}}(E)) = 0$, and satisfies

$$\mathcal{L}ie_E(g) = (2 - d) \cdot g, \quad \mathcal{L}ie_E(*) = *,$$

where $\mathcal{L}ie_E$ denotes the Lie derivative, and $d \in \mathbb{C}$ is some fixed complex number.

By a Frobenius manifold we of course mean a manifold equipped with a Frobenius structure. We commit the usual abuse of labelling a Frobenius manifold by its underlying manifold leaving the metric, product and Euler vector field implicit. The vector field e of (F2) is clearly unique: it is called the *identity vector field*. The number $d \in \mathbb{C}$ appearing in (F4) is called the *conformal dimension* or *charge* of the Frobenius manifold. The complex dimension of the underlying manifold M will usually be denoted n . The function F in condition (F3) is called a *prepotential*: on any given patch it is well-defined up to the addition of quadratic polynomials in the flat co-ordinates t_i . The statement that the product $*$ is associative implies that F satisfies the WDVV equations.

2.2. Tame points and canonical co-ordinates. Let M be a Frobenius manifold. The operation of multiplication by the Euler vector field defines an endomorphism $U(X) = E * X$ of the tangent bundle. A point $p \in M$ is said to be *tame* if U has one-dimensional eigenspaces on the fibre $\mathcal{T}_{M,p}$. The subset of tame points is an open submanifold of M , and we say that M itself is tame if this open subset is the whole of M .

Lemma 2.2. *If a point $p \in M$ is tame then the eigenvalues (u_1, \dots, u_n) of the operator U form a system of co-ordinates in a neighbourhood of $p \in M$. These satisfy*

$$\frac{\partial}{\partial u_i} * \frac{\partial}{\partial u_j} = \delta_{ij} \frac{\partial}{\partial u_i}. \quad (9)$$

Proof. See [20, Theorem 3.1]. □

Let $p \in M$ be a tame point. It follows from Lemma 2.2 that the algebra $(\mathcal{T}_{M,p}, *)$ is semisimple, and that there are expressions

$$e = \sum_i \frac{\partial}{\partial u_i}, \quad E = \sum_i u_i \frac{\partial}{\partial u_i}. \quad (10)$$

The co-ordinates (u_1, \dots, u_n) of Lemma 2.2 are called *canonical co-ordinates*. The compatibility (8) between metric and product ensures that the tangent vectors $\partial/\partial u_i$ define an orthogonal basis of $\mathcal{T}_{M,p}$. We denote by (f_1, \dots, f_n) a corresponding orthonormal basis

$$f_i = \frac{1}{d_i} \frac{\partial}{\partial u_i}, \quad d_i^2 = g\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i}\right). \quad (11)$$

This basis is well-defined up to permutations and sign changes.

2.3. Lie algebra formalities. Let M be a Frobenius manifold with a tame point $p \in M$. In this section we make some formal remarks concerning the Lie algebra $\mathfrak{gl}(\mathcal{T}_{M,p})$, which will be important for the analogy with Joyce structures explained in Section 4 below.

Sending a tangent vector $X \in \mathcal{T}_{M,p}$ to the operation of multiplication by X defines a linear map $m: \mathcal{T}_{M,p} \rightarrow \mathfrak{gl}(\mathcal{T}_{M,p})$. Since the algebra $\mathcal{T}_{M,p}$ is unital, the map m is injective, and since it is commutative, and the elements $m(X)$ are semi-simple, the image coincides with a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{gl}(\mathcal{T}_{M,p})$. Thus we have a diagram

$$m: \mathcal{T}_{M,p} \xrightarrow{\cong} \mathfrak{h} \subset \mathfrak{gl}(\mathcal{T}_{M,p}). \quad (12)$$

The Cartan subalgebra $\mathfrak{h} \subset \mathfrak{gl}(\mathcal{T}_{M,p})$ determines a canonical root decomposition

$$\mathfrak{gl}(\mathcal{T}_{M,p}) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha. \quad (13)$$

Given a vector $X \in \mathcal{T}_{M,p}$ we shall write $X(\alpha) = \langle m(X), \alpha \rangle \in \mathbb{C}$ for the pairing of the vector $m(X) \in \mathfrak{h}$ with a root $\alpha \in \mathfrak{h}^*$. The condition that the point $p \in M$ is tame implies that $U \in \mathfrak{h}^{\text{reg}}$ lies in the complement of the root hyperplanes in \mathfrak{h} .

The metric on $\mathcal{T}_{M,p}$ allows us to define an involution $I(\Psi) = \Psi^{-*}$ of the group $\text{GL}(\mathcal{T}_{M,p})$ which sends an automorphism to its inverse adjoint. This induces an involution

$$\iota: \mathfrak{gl}(\mathcal{T}_{M,p}) \rightarrow \mathfrak{gl}(\mathcal{T}_{M,p}), \quad \iota(\psi) = -\psi^* \quad (14)$$

at the level of Lie algebras, which maps an endomorphism to its negative adjoint. Note that the involution ι acts by -1 on the subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and exchanges the root spaces $\mathfrak{g}_{\pm\alpha}$.

In more down-to-earth terms, the basis (f_1, \dots, f_n) gives an identification

$$\mathfrak{gl}(\mathcal{T}_{M,p}) \cong \mathfrak{gl}_n(\mathbb{C}). \quad (15)$$

Under this identification, the distinguished Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ corresponds to the diagonal matrices, and the root decomposition to the standard decomposition of $\mathfrak{gl}_n(\mathbb{C})$ into

multiples of the elementary matrices E_{ij} . The expression $X(\alpha)$ is just the difference between two eigenvalues of the operator of multiplication by X , and the subset $\mathfrak{h}^{\text{reg}} \subset \mathfrak{h}$ is the set of diagonal matrices with distinct eigenvalues. Since the basis (f_1, \dots, f_n) is orthonormal, the involution ι corresponds under (15) to the usual negative transpose of matrices.

2.4. First structure connection. Let M be a Frobenius manifold, and consider the projection $p: M \times \mathbb{P}^1 \rightarrow M$. Introduce the endomorphism of \mathcal{T}_M

$$V(X) = \frac{1}{2} \cdot (2 - d) \cdot X - \nabla_X^{\text{LC}}(E). \quad (16)$$

The Frobenius structure on M can be encoded in a certain meromorphic connection \mathcal{A} on the bundle $p^*(\mathcal{T}_M)$ known as the *first structure connection* or *deformed flat connection*. It is defined by the formulae

$$\mathcal{A}_X(Y) = \nabla_X^{\text{LC}}(Y) + \frac{1}{\epsilon} \cdot X * Y, \quad \mathcal{A}_{\frac{\partial}{\partial \epsilon}}(Y) = \frac{\partial Y}{\partial \epsilon} - \left(\frac{U}{\epsilon^2} - \frac{V}{\epsilon} \right) Y, \quad (17)$$

where ϵ is a co-ordinate on \mathbb{P}^1 , and we abuse notation by identifying a vector field on M with its lift to a section of $p^*(\mathcal{T}_M)$ which is constant in the ϵ direction. In the literature one usually finds the formulae written in terms of the co-ordinate $z = \epsilon^{-1}$, but the above formulation will be more convenient for us.

Lemma 2.3. *The first structure connection is flat.*

Proof. See for example [20, Proposition 2.1] or [36, Theorem I.1.5]. □

There are two complementary points-of-view on the first structure connection which it is useful to keep in mind. On the one hand, if we forget the derivatives in the ϵ direction, the first structure connection gives a pencil of flat, torsion-free connections

$$\mathcal{A}_X^\epsilon(Y) = \nabla_X^{\text{LC}}(Y) + \frac{1}{\epsilon} \cdot X * Y,$$

on the tangent bundle \mathcal{T}_M , parameterized by $\epsilon \in \mathbb{C}^*$. On the other hand, for each point $p \in M$, the connection \mathcal{A} induces a meromorphic connection \mathcal{A}_p on the trivial vector bundle $\mathcal{T}_{M,p} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1}$ over the complex projective line \mathbb{P}^1 , given by the formula

$$\mathcal{A}_p = d - \left(\frac{U}{\epsilon^2} - \frac{V}{\epsilon} \right) d\epsilon. \quad (18)$$

From this second point-of-view, the manifold M is parameterising a family of meromorphic connections on a trivial bundle over \mathbb{P}^1 .

Remark 2.4. The operators U and V are easily seen to be self-adjoint and skew-adjoint respectively with respect to the metric g . Thus in terms of the involution of Section 2.3 we have $\iota(U) = -U$ and $\iota(V) = V$. It follows that the first structure connection (17) is invariant under the composite operation of applying the involution I and changing the sign of ϵ .

2.5. Canonical connection. Consider a tame Frobenius manifold M . There is a unique connection ∇ on the tangent bundle \mathcal{T}_M in which the local orthonormal bases (f_1, \dots, f_n) defined above are covariantly constant. We refer to ∇ as the *canonical connection*. In general ∇ is flat, but not usually torsion-free. There is a bundle of Lie algebras $\mathfrak{gl}(\mathcal{T}_M)$ over M whose fibre over a point $p \in M$ is the Lie algebra $\mathfrak{gl}(\mathcal{T}_{M,p})$. The canonical connection ∇ induces a flat connection in this bundle, with respect to which the root decomposition (13) and the involution (14) are covariantly constant.

To relate the canonical connection ∇ to the Levi-Civita connection ∇^{LC} we introduce the map of vector bundles

$$\Theta: \mathcal{T}_M \rightarrow \mathfrak{gl}(\mathcal{T}_M), \quad \Theta_X = \sum_{\alpha \in \Phi} \frac{X(\alpha)}{U(\alpha)} \cdot V_\alpha, \quad (19)$$

where we used the canonical decomposition (13) at each point $p \in M$ to write

$$V = \sum_{\alpha \in \Phi} V_\alpha, \quad V_\alpha \in \mathfrak{g}_\alpha.$$

Lemma 2.5. *There is an identity*

$$\nabla_X^{\text{LC}} = \nabla_X - \Theta_X.$$

Proof. This is a consequence of [20, Lemma 3.2]. □

The flatness of the Levi-Civita connection then implies

the following differential equation for the elements V_α as a function of the point $U \in \mathfrak{h}^{\text{reg}}$:

$$dV_\alpha = \sum_{\beta+\gamma=\alpha} [V_\beta, V_\gamma] \cdot d \log U(\beta). \quad (20)$$

It follows from general results on isomonodromic deformations due to Malgrange [35] (see also the discussion in [36, Section II.2.1]), that any local solution to this equation extends to a meromorphic function on the universal cover of the configuration space

$$\mathfrak{h}^{\text{reg}} = \{U \in \mathfrak{h} : U(\alpha) \neq 0 \text{ for all } \alpha \in \Phi\}. \quad (21)$$

The problem of reconstructing of a Frobenius manifold from such a solution is addressed in [19, Proposition 3.5], [20, Lemma 3.3] and [27, Section 4.3].

The fact that the function $V = V(U)$ has poles when extended to the universal cover of $\mathfrak{h}^{\text{reg}}$ is related to the existence of non-trivial vector bundles on \mathbb{P}^1 . In fact the connection (18) on the trivial bundle over \mathbb{P}^1 always extends uniquely to a meromorphic connection on a vector bundle \mathcal{V} over $\mathbb{P}^1 \times \widetilde{\mathfrak{h}^{\text{reg}}}$. In general however, the restrictions of the bundle \mathcal{V} to the slices $\mathbb{P}^1 \times \{U\}$ will not all be trivial, and this leads to poles in the analytic continuation of V .

2.6. First structure connection in canonical co-ordinates. Consider again a tame Frobenius manifold M . In terms of canonical co-ordinates the Frobenius structure can be completely encoded in the pair of maps

$$m: \mathcal{T}_M \rightarrow \mathfrak{gl}(\mathcal{T}_M), \quad \Theta: \mathcal{T}_M \rightarrow \mathfrak{gl}(\mathcal{T}_M),$$

defined in (12) and (19). Their images lie in the $+1$ and -1 eigenspaces of ι respectively. Abusing notation as before, the first structure connection takes the form

$$\mathcal{A}_X = \nabla_X - \Theta_X + \frac{1}{\epsilon} \cdot m_X, \quad \mathcal{A}_{\frac{\partial}{\partial \epsilon}} = \frac{\partial}{\partial \epsilon} + \frac{1}{\epsilon} \cdot \Theta_E - \frac{1}{\epsilon^2} \cdot m_E. \quad (22)$$

The fact that the first structure connection is flat is equivalent to the relations

$$[m_X, m_Y] = 0, \quad [m_X, \Theta_Y] = [m_Y, \Theta_X], \quad (23)$$

$$m_{[X, Y]} = \nabla_X(m_Y) - \nabla_Y(m_X), \quad \nabla_X(m_E) = m_X, \quad (24)$$

$$\Theta_{[X, Y]} + [\Theta_X, \Theta_Y] = \nabla_X(\Theta_Y) - \nabla_Y(\Theta_X), \quad \nabla_X(\Theta_E) + [\Theta_X, \Theta_E] = 0. \quad (25)$$

It is possible to check these equations directly. The first equation of (23) follows because the image of the map m is contained in the bundle of Cartan subalgebras. The second equation of (23) is then immediate from the definition of Θ . It implies that we can write $\Theta_X = [m_X, \Phi]$ and thus encode Θ in the section

$$\Phi: M \rightarrow \mathfrak{gl}_n(\mathcal{T}_M), \quad \Phi = \sum_{\alpha \in \Phi} \frac{V_\alpha}{U(\alpha)}. \quad (26)$$

The first equation of (24) is easily checked by taking $X = \partial/\partial u_i$ and $Y = \partial/\partial u_j$ to be idempotents, and applying both sides of the equation to a ∇ constant section f_k . The second equation follows similarly by taking $X = \partial/\partial u_i$ and using the expression (10) for the Euler vector field. Finally, the equations (25) are equivalent to the differential equation (20): we leave the details to the reader.

3. STOKES DATA AND ISOMONODROMY

In the last section we explained how a Frobenius structure can be encoded in a family of meromorphic connections (18) on a trivial bundle over the complex projective line \mathbb{P}^1 . In this section we explain how this family can in turn be encoded by its Stokes data. As with the contents of the last section, this material will only be used in what follows as a motivating analogy, and we hope the reader will therefore excuse some of the idiosyncrasies of our exposition. We refer the reader to Dubrovin's original notes [19, Section 3] and [20, Section 4]. Other useful references on this material are [2, 6, 10, 38].

3.1. Canonical solutions. Let us begin by considering an abstract connection on the trivial $\mathrm{GL}_n(\mathbb{C})$ bundle over \mathbb{P}^1 of the form

$$\nabla = d - \left(\frac{U}{\epsilon^2} - \frac{V}{\epsilon} \right) d\epsilon, \quad (27)$$

where $U \in \mathfrak{h}^{\mathrm{reg}} \subset \mathfrak{h} \subset \mathfrak{gl}_n(\mathbb{C})$ is a diagonal matrix with distinct eigenvalues, and

$$V \in \mathfrak{g}^{\mathrm{od}} = \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \subset \mathfrak{gl}_n(\mathbb{C})$$

is an arbitrary off-diagonal matrix. This connection has an irregular singularity at $\epsilon = 0$ and a regular singularity at $\epsilon = \infty$.

The *Stokes rays* of the equation (27) at the irregular singularity $\epsilon = 0$ are defined to be the rays $\mathbb{R}_{>0} \cdot U(\alpha) \subset \mathbb{C}^*$, where, as explained in Section 2.3, the points $U(\alpha) \in \mathbb{C}^*$ are the differences $u_i - u_j$ of the eigenvalues of the matrix U . Given an arbitrary ray in the complex plane of the form $r = \mathbb{R}_{>0} \cdot z \subset \mathbb{C}^*$, we denote by

$$\mathbb{H}_r = \{z = uv : u \in r, \mathrm{Re}(v) > 0\} \subset \mathbb{C}^* \quad (28)$$

the half-plane centered on it.

Theorem 3.1. (i) *For any non-Stokes ray $r \subset \mathbb{C}^*$, there is a holomorphic map*

$$\Psi_r : \mathbb{H}_r \rightarrow \mathrm{GL}_n(\mathbb{C}), \quad (29)$$

which is a flat section of the connection (27) and satisfies

$$\Psi_r(\epsilon) \cdot \exp(U/\epsilon) \rightarrow \mathrm{id} \quad (30)$$

as $\epsilon \rightarrow 0$ in the half-plane \mathbb{H}_r .

- (ii) Suppose that $\Delta \subset \mathbb{C}^*$ is a closed, convex sector, bounded in clockwise order by non-Stokes rays r_1, r_2 . Then for any flat sections Ψ_{r_i} as in (i), there is an element

$$S(\Delta) \in \exp \left(\bigoplus_{U(\alpha) \in \Delta} \mathfrak{g}_\alpha \right) \subset \mathrm{GL}_n(\mathbb{C}),$$

such that for all $\epsilon \in \mathbb{H}_{r_1} \cap \mathbb{H}_{r_2}$ one has $\Psi_{r_2}(\epsilon) = \Psi_{r_1}(\epsilon) \cdot S(\Delta)$. \square

Part (i) can be found in [2], although it goes back earlier, and in some form to Birkhoff. Part (ii) is completely elementary: any two flat sections of (18) differ by a constant element $S(\Delta) \in \mathrm{GL}_n(\mathbb{C})$ and the asymptotics (30) ensure that

$$\exp(-U/\epsilon) \cdot S(\Delta) \cdot \exp(U/\epsilon) \rightarrow \mathrm{id}$$

as $\epsilon \rightarrow 0$ in the overlap $\mathbb{H}_{r_1} \cap \mathbb{H}_{r_2}$. This easily implies the given form of S .

Remark 3.2. Part (ii) of Theorem 3.1 immediately gives the following two statements:

- (i) For each non-Stokes ray $r \subset \mathbb{C}^*$, the flat section $\Psi_r: \mathbb{H}_r \rightarrow \mathrm{GL}_n(\mathbb{C})$ of part (i) is unique with the property (30). We call it the *canonical solution* to the connection (27) in the half-plane \mathbb{H}_r .
- (ii) Suppose that two non-Stokes rays r_1 and r_2 are the boundary rays of a convex sector which contains no Stokes rays. Then the corresponding canonical solutions Ψ_{r_1} and Ψ_{r_2} are analytic continuations of one another, in the sense that they glue to give a single analytic function on $\mathbb{H}_{r_1} \cup \mathbb{H}_{r_2}$.

3.2. Stokes data. Let us again consider the meromorphic connection (27). To each Stokes ray

$$\mathbb{R}_{>0} \cdot U(\alpha) = \mathbb{R}_{>0} \cdot (u_i - u_j) \subset \mathbb{C}$$

is associated a *Stokes factor*

$$\mathbb{S}(\ell) = \exp \left(\sum_{\alpha \in \Phi: U(\alpha) \in \ell} D(\alpha) \cdot E_\alpha \right) \in \exp \left(\bigoplus_{\alpha \in \Phi: U(\alpha) \in \ell} \mathfrak{g}_\alpha \right) \subset \mathrm{GL}_n(\mathbb{C}). \quad (31)$$

obtained by taking $\mathbb{S}(\ell) = \mathbb{S}(\Delta)$ as defined in Theorem 3.1(ii), where $\Delta \subset \mathbb{C}^*$ is a closed, convex sector which contains the ray ℓ in its interior and no other Stokes rays. Note that for a generic matrix $U \in \mathfrak{h}^{\mathrm{reg}}$ the Stokes rays $\mathbb{R}_{>0} \cdot U(\alpha)$ are all distinct, and each of the subgroups appearing on the right of (31) is one-dimensional. But for special choices of U several Stokes rays may line up, and the corresponding subgroup is then larger.

The sets $\{\alpha \in \Phi : U(\alpha) \in \ell\}$ partition the set of roots $\Phi \subset \mathfrak{h}^*$ as ℓ ranges over the Stokes rays of ∇ , so we may assemble the elements $D(\alpha) \cdot E_\alpha \in \mathfrak{g}_\alpha$ corresponding to the different Stokes rays and form the sum

$$R = \sum_{\alpha \in \Phi} D(\alpha) \cdot E_\alpha \in \mathfrak{g}^{\text{od}}. \quad (32)$$

Theorem 3.1. *For each element $U \in \mathfrak{h}^{\text{reg}}$ the corresponding map*

$$\mathcal{S}: \mathfrak{g}^{\text{od}} \rightarrow \mathfrak{g}^{\text{od}}, \quad \mathcal{S}(V) = R,$$

sending the equation (27) to its Stokes data, is a local isomorphism of complex manifolds.

Proof. The fact that the map \mathcal{S} is holomorphic follows from a result of Sibuya which proves that the canonical solutions of Theorem 3.1 vary holomorphically with parameters (see e.g. [31, Proposition 3.2]). It remains to show that the derivative of \mathcal{S} is injective. This follows from the description of iso-Stokes deformations in Theorem 3.2 below (see e.g. [31, Theorem 3.3]). \square

The map \mathcal{S} of Theorem 3.1 is usually called the *Stokes map*. Although it is a local isomorphism, it is not usually bijective. We shall return to this point in Section 6 below.

3.3. Iso-Stokes deformations. Let us again return to the meromorphic connection (27), and consider now varying the matrices U and V , always maintaining the given conditions $U \in \mathfrak{h}^{\text{reg}}$ and $V \in \mathfrak{g}^{\text{od}}$. Note that as the matrix $U \in \mathfrak{h}^{\text{reg}}$ varies, the Stokes rays $\mathbb{R}_{>0} \cdot U(\alpha)$ may collide and separate. The resulting family of connections is called *iso-Stokes* if the following condition is satisfied: for any convex sector $\Delta \subset \mathbb{C}^*$, the clockwise product of matrices

$$\mathbb{S}(\Delta) = \prod_{\ell \in \Delta} \mathbb{S}(\ell) \in \text{GL}_n(\mathbb{C}), \quad (33)$$

remains constant, unless and until the boundary rays of Δ become Stokes rays.

Theorem 3.2. *Consider a family of meromorphic connections of the form (18) with $U \in \mathfrak{h}^{\text{reg}}$ and V varying continuously as a function of U . Then the family is iso-Stokes precisely if $V = V(U)$ satisfies the differential equation*

$$dV_\alpha = \sum_{\beta+\gamma=\alpha} [V_\beta, V_\gamma] \cdot d \log U(\beta), \quad (34)$$

where we decompose $V = \sum_{\alpha \in \Phi} V_\alpha$ with $V_\alpha \in \mathfrak{g}_\alpha$.

Proof. This appears in a much more general context in [31, Theorem 3.3]. The explicit form (34) appears for example as [19, Equation 3.107]. \square

As discussed in Section 2.5, any local solution to the equation (34) extends uniquely to a meromorphic function $V = V(U)$ on the universal cover of $\mathfrak{h}^{\text{reg}}$. On the Stokes side, the analogue of this statement is that given a collection of numbers $D(\alpha) \in \mathbb{C}$ as in (32), we can uniquely extend the $D(\alpha)$ to (non-continuous) functions of U on the universal cover of $\mathfrak{h}^{\text{reg}}$, so that the resulting Stokes factors (31) satisfy the iso-Stokes property.

Remark 3.3. When the iso-Stokes property of Theorem 3.2 holds, it follows from [31, Theorem 3.1] that the canonical solutions $\Psi_r : \mathbb{H}_r \rightarrow \text{GL}_n(\mathbb{C})$ of Theorem 3.1 for different $U \in \mathfrak{h}^{\text{reg}}$ assemble to form holomorphic functions $\Psi_r(U, \epsilon)$, which for fixed $\epsilon \in \mathbb{C}^*$ satisfy the differential equations

$$\frac{\partial}{\partial u_i} \Psi_r(U, \epsilon) - \left(\sum_{\alpha \in \Phi} \frac{\alpha_i}{U(\alpha)} \cdot V_\alpha - \frac{1}{\epsilon} \cdot E_{ii} \right) \Psi_r(U, \epsilon) = 0,$$

where E_{ii} denotes the elementary matrix at position (i, i) . Comparing with the formulae in Section 2.6 shows that in the Frobenius setting this is the statement that the functions $\Psi_r(U, \epsilon)$ are flat sections of the first structure connection.

3.4. FS structures. The analysis of the last three subsections applies in particular to the connection (18) defined by a tame point of a Frobenius manifold. For the purposes of comparison with the notion of a BPS structure in Section 4 we will now introduce the notion of a Frobenius Stokes (FS) structure. This consists essentially of the data of the leading term $U \in \mathfrak{h}^{\text{reg}}$ of the connection (18) together with the element $R \in \mathfrak{g}^{\text{od}}$ defined by (32) which encodes the Stokes factors.

Remark 3.4. In the connections (18) arising from Frobenius manifolds, the operator V is skew-adjoint. This results in a symmetry property of the associated Stokes factors. Indeed, according to Remark 2.4, given a canonical solution (29) on the half-plane \mathbb{H}_r , we can define a canonical solution on the opposite half-plane \mathbb{H}_{-r} by setting

$$\Psi_{-r}(\epsilon) = \Psi_r(-\epsilon)^{-T}.$$

It follows that the Stokes factors satisfy $\mathbb{S}(-\ell) = \mathbb{S}(\ell)^{-T}$, or equivalently, in the notation of (31), that $D(-\alpha) = D(\alpha)$ for all $\alpha \in \Phi$.

Given a finite-dimensional complex vector space T equipped with an unordered basis $\{f_1, \dots, f_n\}$, we will denote by $\mathfrak{h} \subset \mathfrak{gl}(T)$ the distinguished Cartan subalgebra consisting of endomorphisms which are diagonal in the given basis, and $\mathfrak{h}^{\text{reg}} \subset \mathfrak{h}$ the open subset of endomorphisms with one-dimensional eigenspaces. The corresponding root system $\Phi \subset \mathfrak{h}^*$ consists of the elements $f_i^* - f_j^*$ for $i \neq j$.

Definition 3.5. An *FS structure* (T, U, D) consists of

- (a) a complex vector space T equipped with an unordered basis $\{f_1, \dots, f_n\}$;
- (b) an endomorphism $U \in \mathfrak{h}^{\text{reg}} \subset \mathfrak{h} \subset \mathfrak{gl}(T)$,
- (c) a map of sets $D: \Phi \rightarrow \mathbb{C}$ satisfying $D(-\alpha) = D(\alpha)$ for all $\alpha \in \Phi$.

The Stokes rays of an FS structure are those of the form $\mathbb{R}_{>0} \cdot U(\alpha)$, and the associated Stokes factors are defined by

$$\mathbb{S}_\ell = \exp \left(\sum_{\alpha \in \Phi: U(\alpha) \in \ell} D(\alpha) \cdot E_\alpha \right) \in \exp \left(\bigoplus_{\alpha \in \Phi: U(\alpha) \in \ell} \mathfrak{g}_\alpha \right) \subset \text{GL}(T), \quad (35)$$

where for each root $\alpha = f_i^* - f_j^*$ we denote by $E_\alpha = E_{ij}$ the distinguished generator of the root space \mathfrak{g}_α defined by the relation $E_{ij}(f_m) = \delta_{im} f_j$.

Remark 3.6. Given a Frobenius manifold M and a tame point $p \in M$, there is an associated FS structure obtained by taking the vector space $T = \mathcal{T}_{M,p}$, the operator U of multiplication by the Euler vector field, and the basis $\{f_1, \dots, f_n\}$ consisting of the normalised idempotents (11). The elements $D(\alpha)$ are determined by the Stokes data of the connection (18). Note however that the definition of the basis elements f_i requires a choice of signs, or in other words a choice of section of a $\{\pm 1\}^n$ torsor over M . We make this remark now, because a very similar issue will arise in Section 5.5 in the context of Joyce structures.

3.5. Variations of FS structures. The following definition is again introduced for the purposes of comparison with the material of Section 4. It is an abstraction of the iso-Stokes condition explained above.

Definition 3.7. A *variation of FS structures* over a complex manifold M is a collection of FS structures (T_p, U_p, D_p) indexed by the points $p \in M$ such that

- (a) the vector spaces T_p form the fibres of a local system over M , and the basis $\{f_1, \dots, f_n\}$ is covariantly constant;

- (b) the endomorphisms $U_p \in \mathfrak{gl}(T_p)$ vary holomorphically;
- (c) for any convex sector $\Delta \subset \mathbb{C}^*$, the clockwise composition

$$\mathbb{S}_p(\Delta) = \prod_{\ell \in \Delta} \mathbb{S}_p(\ell) \in \mathrm{GL}(T_p) \tag{36}$$

is covariantly constant as the point $p \in M$ varies, providing that the boundary rays of Δ are never Stokes rays.

What we mean by (a) more precisely is that there is a holomorphic vector bundle $\pi: T \rightarrow M$ equipped with a flat holomorphic connection ∇ whose fibres can be identified with the vector spaces T_p , and that the unordered bases $\{f_1, \dots, f_n\} \subset T_p$ are induced by locally-defined covariantly constant sections of T . Condition (b) is then the statement that the operators U_p are induced by a holomorphic endomorphism $U \in \mathrm{End}_M(T)$. By covariantly constant in (c) we of course mean with respect to the connection induced by ∇ .

Replacing M by a contractible open subset or cover we can trivialise the local system of vector spaces T_p and obtain a holomorphic period map

$$\pi: M \rightarrow \mathfrak{h} \subset \mathfrak{gl}_n(\mathbb{C}), \quad p \mapsto U_p.$$

We say that a variation of BPS structures is *miniversal* if the derivative of this map is everywhere non-vanishing, so that π is a local isomorphism.

Remark 3.8. Let M be a tame Frobenius manifold. Then the FS structures of Remark 3.6 fit together to form a variation of FS structures over M . The local system is given by the canonical connection ∇ on the tangent bundle \mathcal{T}_M . The only thing to check is the iso-Stokes condition, but this follows from Theorem 3.2 together with the differential equation (20). This variation is miniversal, because the eigenvalues of the operator U are local co-ordinates on M .

4. JOYCE STRUCTURES

In this section we introduce the notion of a Joyce structure. This is the geometric structure we expect to find on the space of stability conditions of a CY_3 triangulated category. The definition is motivated by a strong analogy with Frobenius structures which is summarised in Table 1. The basic idea is to replace the finite-dimensional Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ of Section 2.3 with the infinite-dimensional Lie algebra $\mathrm{vect}_\eta(\mathbb{C}^*)^n$ of vector fields on the algebraic torus $(\mathbb{C}^*)^n$ whose flows preserve a translation-invariant Poisson structure η . The original impetus for this is the form of the wall-crossing formula in DT theory, which is an iso-Stokes condition

$\mathfrak{gl}_n(\mathbb{C})$	$\text{vect}_\eta(\mathbb{C}^*)^n$
Frobenius structure	Joyce structure
Canonical co-ordinates	Central charge co-ordinates
Levi-Civita connection	Joyce connection
Product	Translation
Deformed flat connection	Deformed Joyce connection
Inverse transpose	Conjugation with inverse
Euler vector field	Central charge
Rotation coefficients V_{ij}	Fourier coefficients H_α
Stokes factors	BPS automorphisms
FS structure	BPS structure

TABLE 1. The analogy between Frobenius and Joyce structures.

of precisely the same form as (33), but taking values in the group of Poisson automorphisms of the torus $(\mathbb{C}^*)^n$.

4.1. Lie algebra of vector fields. We begin by discussing the abstract properties of the Lie algebra which will replace the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ in the Frobenius manifold story. The material here should be compared with that of Section 2.3.

Consider a lattice $\Gamma \cong \mathbb{Z}^{\oplus n}$ equipped with a skew-symmetric integral form

$$\langle -, - \rangle : \Gamma \times \Gamma \rightarrow \mathbb{Z}.$$

Introduce the algebraic torus

$$\mathbb{T}_+ = \text{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}^*) \cong (\mathbb{C}^*)^n$$

and its co-ordinate ring, which is also the group ring of the lattice Γ

$$\mathbb{C}[\mathbb{T}_+] = \mathbb{C}[\Gamma] \cong \mathbb{C}[y_1^{\pm 1}, \dots, y_n^{\pm 1}].$$

The character of \mathbb{T}_+ corresponding to an element $\gamma \in \Gamma$ will be denoted y_γ . The skew-symmetric form $\langle -, - \rangle$ induces a translation invariant Poisson structure $\{-, -\}$ on the torus \mathbb{T}_+ , which is given on characters by

$$\{y_\alpha, y_\beta\} = \langle \alpha, \beta \rangle \cdot y_\alpha \cdot y_\beta. \quad (37)$$

Consider the Lie algebra of non-constant algebraic functions on \mathbb{T}_+

$$\mathfrak{g}^{\text{od}} = \bigoplus_{\alpha \in \Gamma \setminus \{0\}} \mathfrak{g}_\alpha = \bigoplus_{\alpha \in \Gamma \setminus \{0\}} \mathbb{C} \cdot y_\alpha \quad (38)$$

equipped with the Lie bracket induced by the Poisson bracket $\{-, -\}$. The abelian Lie algebra

$$\mathfrak{h} = \text{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}),$$

of translation-invariant vector fields on \mathbb{T}_+ acts by derivations on \mathfrak{g}^{od} in the obvious way, and we can form the corresponding semi-direct product. This is the vector space direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}^{\text{od}}$ equipped with the bracket defined on generators by

$$\left[(h_1, y_{\alpha_1}), (h_2, y_{\alpha_2}) \right] = (0, h_1(\alpha_2)y_{\alpha_2} - h_2(\alpha_1)y_{\alpha_1} + \langle \alpha_1, \alpha_2 \rangle \cdot y_{\alpha_1 + \alpha_2}). \quad (39)$$

When the form $\langle -, - \rangle$ is non-degenerate, the subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra, and the decomposition (38) can be viewed as a root decomposition, with the roots being precisely the nonzero elements of Γ .

There is an obvious homomorphism of Lie algebras

$$\rho: \mathfrak{g} \rightarrow \text{vect}_\eta(\mathbb{T}_+), \quad (40)$$

to the Lie algebra of algebraic vector fields on the torus \mathbb{T}_+ whose flows preserve the Poisson structure $\{-, -\}$. It acts by the identity on \mathfrak{h} and sends a function $f \in \mathfrak{g}^{\text{od}}$ to the corresponding Hamiltonian vector field Ham_f . In particular, if we choose a basis $(\gamma_1, \dots, \gamma_n) \subset \Gamma$ and write $y_i = y_{\gamma_i}$, the homomorphism ρ is defined on generators by

$$h \in \mathfrak{h} \mapsto \sum_i h(\gamma_i) \cdot y_i \frac{\partial}{\partial y_i}, \quad y_\alpha \in \mathfrak{g}^{\text{od}} \mapsto \sum_j \langle \alpha, \gamma_j \rangle \cdot y_\alpha \cdot y_j \frac{\partial}{\partial y_j}.$$

In general the map ρ has a large kernel: it is an isomorphism precisely when the form $\langle -, - \rangle$ is non-degenerate. We do not want to make this assumption in general, since it fails in some interesting examples.

There is an involution $\iota: \mathfrak{g} \rightarrow \mathfrak{g}$ induced by the inverse map of the torus \mathbb{T}_+ . It acts on the generators considered above via

$$\iota(h, y_\alpha) = (-h, y_{-\alpha}).$$

In particular, the abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is contained in the -1 eigenspace of ι .

4.2. Structures on the tangent bundle. Consider a complex manifold M equipped with a flat, torsion-free connection ∇ on the tangent bundle \mathcal{T}_M . Suppose also that there is a covariantly constant lattice $\Gamma_M \subset \mathcal{T}_M^*$, and a covariantly constant skew-symmetric form

$$\eta: \mathcal{T}_M^* \times \mathcal{T}_M^* \rightarrow \mathcal{O}_M$$

which takes integral values on Γ_M . Given a vector field X on M there are then two lifts of X to vector fields on the total space of the bundle $\pi: \mathcal{T}_M \rightarrow M$, both of which will be important in what follows.

For the first lift of X , note that the vertical tangent vectors to the space \mathcal{T}_M at a point of the fibre $\pi^{-1}(p) = \mathcal{T}_{M,p}$ are in natural correspondence with the vector space $\mathcal{T}_{M,p}$. Thus there is a vertical vector field m_X on \mathcal{T}_M , obtained by taking at each point of \mathcal{T}_M lying over some point $p \in M$, the vertical tangent vector corresponding to $X_p \in \mathcal{T}_{M,p}$. Clearly the restriction of m_X to each fibre $\mathcal{T}_{M,p}$ is invariant under translations.

The second lift of the vector field X , denoted \mathcal{H}_X , is induced by the connection ∇ . It is the just the ∇ -horizontal lift of X to a vector field on \mathcal{T}_M , and is uniquely defined by the properties

$$\nabla(\mathcal{H}_X) = 0, \quad \pi_*(\mathcal{H}_X) = X.$$

To write all this in co-ordinates, take a basis $(\gamma_1, \dots, \gamma_n) \subset \Gamma_{M,p}$ at some point $p \in M$, and extend to a covariantly constant basis of the bundle of lattices Γ_M using the connection ∇ . In this basis the form η is given by a constant skew-symmetric integral matrix

$$\eta(\gamma_i, \gamma_j) = \eta_{ij} \in \mathbb{Z}.$$

Since the connection ∇ is torsion-free, there are local co-ordinates (z_1, \dots, z_n) on the manifold M satisfying $\gamma_i = dz_i$. We can then define co-ordinates $(\theta_1, \dots, \theta_n)$ on the fibre $\mathcal{T}_{M,p}$ over a point $p \in M$ by writing a vector $X \in \mathcal{T}_{M,p}$ in the form

$$X = \sum_{i=1}^n \theta_i \cdot \frac{\partial}{\partial z_i}.$$

In the resulting local co-ordinate system $(z_1, \dots, z_n, \theta_1, \dots, \theta_n)$ on the manifold \mathcal{T}_M we have

$$m_{\frac{\partial}{\partial z_i}} = \frac{\partial}{\partial \theta_i}, \quad \mathcal{H}_{\frac{\partial}{\partial z_i}} = \frac{\partial}{\partial z_i}.$$

For each point $p \in M$, the skew-symmetric form η_p induces a translation-invariant Poisson structure on the fibre $\mathcal{T}_{M,p}$. These combine to give a Poisson structure on the space \mathcal{T}_M , which

we refer to as the *vertical Poisson structure* induced by the form η . In co-ordinates it is given by

$$\{f, g\} = \sum_{i,j} \eta_{ij} \cdot \frac{\partial f}{\partial \theta_i} \cdot \frac{\partial g}{\partial \theta_j}. \quad (41)$$

At each point $p \in M$ we can apply the construction of the last subsection to the lattice $\Gamma_p \subset \mathcal{T}_{M,p}$ and the skew-symmetric form $\langle -, - \rangle = \eta_p(-, -)$. The corresponding tori are

$$\mathbb{T}_{M,p} = \text{Hom}_{\mathbb{Z}}(\Gamma_{M,p}, \mathbb{C}^*) = \mathcal{T}_{M,p} / \Gamma_{M,p}^*. \quad (42)$$

This results in a family of Lie algebras $\mathfrak{g}_{M,p}$ which fit together to form a bundle of Lie algebras \mathfrak{g}_M over M . The abelian subalgebras $\mathfrak{h}_{M,p} \subset \mathfrak{g}_{M,p}$ form a sub-bundle $\mathfrak{h}_M \subset \mathfrak{g}_M$, and the map m considered above defines a bundle isomorphism

$$m: \mathcal{T}_M \rightarrow \mathfrak{h}_M. \quad (43)$$

Since the connection ∇ preserves the form η and the bundle of lattices $\Gamma_M \subset \mathcal{T}_M^*$, it induces a connection on the bundle of Lie algebras \mathfrak{g}_M , which we also denote by ∇ .

4.3. Heuristics for the definition. In this subsection we explain the heuristics behind the definition of a Joyce structure which will be given in the next subsection. The basic idea is to follow the lead of Section 2.6 and look for maps

$$m: \mathcal{T}_M \rightarrow \mathfrak{g}_M, \quad \Theta: \mathcal{T}_M \rightarrow \mathfrak{g}_M,$$

whose images lie in the $+1$ and -1 eigenspaces of ι respectively, and satisfying the same relations

$$[m_X, m_Y] = 0, \quad [m_X, \Theta_Y] = [m_Y, \Theta_X], \quad (44)$$

$$m_{[X,Y]} = \nabla_X(m_Y) - \nabla_Y(m_X), \quad \nabla_X(m_E) = m_X, \quad (45)$$

$$\Theta_{[X,Y]} + [\Theta_X, \Theta_Y] = \nabla_X(\Theta_Y) - \nabla_Y(\Theta_X), \quad \nabla_X(\Theta_E) + [\Theta_X, \Theta_E] = 0, \quad (46)$$

where $E: \mathcal{O}_M \rightarrow \mathcal{T}_M$ is some holomorphic vector field on M .

The first equation of (44) shows that the image of the map m should be a bundle of abelian subalgebras of \mathfrak{g}_M , and there is then an obvious choice, namely the map (43). The image of the map Θ necessarily lies in the bundle of subalgebras $\mathfrak{g}_M^{\text{od}} \subset \mathfrak{g}_M$. Thus it assigns to each tangent vector $X \in \mathcal{T}_{M,p}$ a non-constant algebraic function $\Theta_{X,p}$ on the torus (42). Pulling back via the quotient map $\mathcal{T}_{M,p} \rightarrow \mathbb{T}_{M,p}$, we can also view $\Theta_{X,p}$ as a holomorphic function on the vector space $\mathcal{T}_{M,p}$. Thus if X is a holomorphic vector field on M we can view Θ_X as a holomorphic function on the total space \mathcal{T}_M .

As in Section 2.6, the second relation of (44) allows us to encode the map Θ more efficiently in the form $\Theta_X = m_X(J)$, where

$$J: \mathcal{T}_M \rightarrow \mathbb{C}$$

is a holomorphic function. Note that if we interpret the map Θ literally as taking values in the bundle of Lie algebras \mathfrak{g}_M , then the restriction of the function J to each fibre $\mathcal{T}_{M,p}$ should descend to an algebraic function on the torus (42).

The connection ∇ on the bundle of Lie algebras \mathfrak{g}_M induces a connection on the sub-bundle $\mathfrak{h}_M \subset \mathfrak{g}_M$, which under the isomorphism (43) reduces to the original connection ∇ on the bundle \mathcal{T}_M . The first relation of (45) is then the condition that the connection ∇ is torsion-free, and the second becomes the statement that $\nabla_X(E) = X$ for all vector fields X on M .

The first relation of (46) implies a non-linear partial differential equation for the function J , namely

$$\mathcal{H}_X(m_Y(J)) - \mathcal{H}_Y(m_X(J)) = m_{[X,Y]}(J) + \{m_X(J), m_Y(J)\}. \quad (47)$$

This is written out in co-ordinates in (51) below. The second relation of (46) follows from this if we also impose the homogeneity relation $\mathcal{H}_E(J) = -J$.

In what follows, this heuristic picture will need to be modified in several ways. Firstly we must allow the function $J: \mathcal{T}_M \rightarrow \mathbb{C}$ to have poles, and drop the condition that the restrictions of J to the fibres $\mathcal{T}_{M,p}$ are induced by algebraic functions on the tori (42). Secondly, it is really the second derivatives of J appearing in the connection (54) below that are the most relevant quantities: the local existence of the function J is just a convenient way to encode the symmetries of its higher derivatives. In particular, the function J should only be considered as being well-defined up to transformations of the form

$$J(z_i, \theta_j) \mapsto J(z_i, \theta_j) + L(z_i, \theta_j), \quad (48)$$

where $L(z_i, \theta_j)$ is a polynomial in the θ_j co-ordinates of degree at most 1. Moreover, since poles in the second derivatives of J can lead to multi-valued logarithmic factors, we should only insist that the function J itself is well-defined on a dense open subset of \mathcal{T}_M .

4.4. Definition of a Joyce structure. After the heuristic discussion of the last subsection we can now proceed to a precise definition of a Joyce structure. In the next subsection we will give a description in terms of co-ordinates, which of course may be more helpful.

Definition 4.1. A *Joyce structure* on a complex manifold M consists of the following data:

- (i) a flat, torsion-free connection ∇ on the tangent bundle \mathcal{T}_M , together with a covariantly constant full rank sublattice $\Gamma_M \subset \mathcal{T}_M^*$;
- (ii) a covariantly constant skew-symmetric form $\eta: \mathcal{T}_M^* \times \mathcal{T}_M^* \rightarrow \mathcal{O}_M$ such that $\eta(\Gamma_M \times \Gamma_M) \subset \mathbb{Z}$;
- (iii) the *Joyce function*: a holomorphic function $J: U \rightarrow \mathbb{C}$ defined on a dense open subset $U \subset \mathcal{T}_M$;
- (iv) the *Euler vector field*: a holomorphic section $E: \mathcal{O}_M \rightarrow \mathcal{T}_M$;

satisfying the following conditions:

- (J1) for any two vector fields X_1, X_2 on M , the function $m_{X_1}m_{X_2}(J)$ extends to a meromorphic function on \mathcal{T}_M , whose restriction to any fibre $\mathcal{T}_{M,p}$ is an odd function;
- (J2) for any three vector fields X_1, X_2, X_3 on M , the function $m_{X_1}m_{X_2}m_{X_3}(J)$ extends to a meromorphic function on \mathcal{T}_M which is periodic with respect to the lattice $\Gamma_{M,p}^* \subset \mathcal{T}_{M,p}$;
- (J3) for any two vector fields X, Y on M , there is a relation

$$\mathcal{H}_X(m_Y(J)) - \mathcal{H}_Y(m_X(J)) = m_{[X,Y]}(J) + \{m_X(J), m_Y(J)\};$$

- (J4) the vector field E satisfies the relations

$$\nabla(E) = \text{id}, \quad \mathcal{H}_E(J) = -J.$$

Here, as in Section 4.2, given a vector field X on M , we denote by m_X the corresponding translation-invariant vertical vector field on \mathcal{T}_M , and by \mathcal{H}_X the ∇ -horizontal lift of X to \mathcal{T}_M . The bracket $\{-, -\}$ appearing in (J3) is the vertical Poisson bracket on \mathcal{T}_M induced by the form η . By an odd function on a vector space in axiom (J1) we mean one that satisfies $f(-v) = -f(v)$.

4.5. Co-ordinate description. Suppose given a Joyce structure as in Definition 4.1, and let us take a co-ordinate system $(z_1, \dots, z_n, \theta_1, \dots, \theta_n)$ on the tangent bundle \mathcal{T}_M as in Section 4.2. In particular the z_i are flat co-ordinates for the connection ∇ . Axiom (J4) shows that after applying a transformation $z_i \mapsto z_i + c_i$, the Euler vector field E takes the form

$$E = \sum_i z_i \cdot \frac{\partial}{\partial z_i}.$$

The conditions on the Joyce function $J(z_1, \dots, z_n, \theta_1, \dots, \theta_n)$ are then:

(i) J has the homogeneity properties

$$J(z_1, \dots, z_n, -\theta_1, \dots, -\theta_n) = -J(z_1, \dots, z_n, \theta_1, \dots, \theta_n), \quad (49)$$

$$J(\lambda \cdot z_1, \dots, \lambda \cdot z_n, \theta_1, \dots, \theta_n) = \lambda^{-1} \cdot J(z_1, \dots, z_n, \theta_1, \dots, \theta_n); \quad (50)$$

(ii) the second partial derivatives of J with respect to the θ_j co-ordinates extend to single-valued meromorphic functions on \mathcal{T}_M , and the third partial derivatives are invariant under $\theta_j \mapsto \theta_j + 2\pi i$;

(iii) J satisfies the differential equation

$$\frac{\partial^2 J}{\partial z_i \partial \theta_j} - \frac{\partial^2 J}{\partial z_j \partial \theta_i} = \sum_{p,q} \eta_{pq} \cdot \frac{\partial^2 J}{\partial \theta_i \partial \theta_p} \cdot \frac{\partial^2 J}{\partial \theta_j \partial \theta_q}. \quad (51)$$

Remark 4.2. In certain situations it is natural to verify the equation (51) by checking that the Hamiltonian vector fields on \mathcal{T}_M defined by the two sides are equal. This means that *a priori* we only know that the equality (51) holds up to the addition of a function independent of the variables θ_i . In fact, it turns out that this function can always be eliminated by applying particular transformations of the form (48). Indeed, if we replace J by the expression

$$J(z_i, \theta_j) - \sum_k \theta_k \cdot \frac{\partial J}{\partial \theta_k}(z_i, 0),$$

then each of the terms on the left-hand side of the expression (51) vanishes along the locus where all $\theta_i = 0$. But the right-hand side also vanishes along this locus, because by part (i) the function J is odd in the co-ordinates θ_j . So if the difference of the two sides of (51) is independent of the co-ordinates θ_j , then after the above modification it must vanish identically.

4.6. Deformed Joyce connection. Suppose given a Joyce structure on a complex manifold M . The formula

$$\mathcal{A}_X^{(\epsilon)} = \nabla_X - \Theta_X + \frac{1}{\epsilon} \cdot m_X \quad (52)$$

defines a pencil of connections with values in the bundle of Lie algebras \mathfrak{g}_M . Applying the homomorphism of Lie algebras $\rho: \mathfrak{g} \rightarrow \text{vect}(\mathbb{T}_+)$ of Section 4.1 then gives a pencil of flat Ehresmann connection on the tangent bundle $\pi: \mathcal{T}_M \rightarrow M$, whose horizontal sections are spanned by the vector fields

$$\mathcal{A}_X^{(\epsilon)} = \mathcal{H}_X - \text{Ham}_{m_X(J)} + \frac{1}{\epsilon} \cdot m_X. \quad (53)$$

In terms of a local co-ordinate system $(z_1, \dots, z_n, \theta_1, \dots, \theta_n)$ as in Section 4.2 we have

$$\mathcal{A}_{\frac{\partial}{\partial z_i}}^{(\epsilon)} = \frac{\partial}{\partial z_i} - \sum_{j,k} \eta_{jk} \cdot \frac{\partial^2 J}{\partial \theta_i \partial \theta_j} \cdot \frac{\partial}{\partial \theta_k} + \frac{1}{\epsilon} \cdot \frac{\partial}{\partial \theta_i}. \quad (54)$$

As in the Frobenius case we actually get more, namely an Ehresmann connection on the bundle $p^*(\mathcal{T}_M)$ over the space $\mathbb{P}^1 \times M$, where $p: \mathbb{P}^1 \times M \rightarrow M$ is the projection map. The space of horizontal sections of this connection is spanned by the vector fields

$$\mathcal{A}_X = \mathcal{H}_X - \text{Ham}_{m_X(J)} + \frac{1}{\epsilon} \cdot m_X, \quad \mathcal{A}_{\frac{\partial}{\partial \epsilon}} = \frac{\partial}{\partial \epsilon} + \frac{1}{\epsilon} \cdot \text{Ham}_{m_E(J)} - \frac{1}{\epsilon^2} \cdot m_E, \quad (55)$$

where we abuse notation by identifying vector fields on the spaces M and \mathcal{T}_M with vector fields on $\mathbb{P}^1 \times M$ and $\mathbb{P}^1 \times \mathcal{T}_M$ respectively, which are constant in the \mathbb{P}^1 direction. We call (55) the *deformed Joyce connection*.

Proposition 4.3. *The deformed Joyce connection is flat.*

Proof. The deformed Joyce connection (55) is obtained by applying the homomorphism of Lie algebras $\rho: \mathfrak{g} \rightarrow \text{vect}(\mathbb{T}_+)$ of Section 4.1 to the \mathfrak{g} -valued connection

$$\mathcal{A}_X = \nabla_X - \Theta_X + \frac{1}{\epsilon} \cdot m_X, \quad \mathcal{A}_{\frac{\partial}{\partial \epsilon}} = \frac{\partial}{\partial \epsilon} + \frac{1}{\epsilon} \cdot \Theta_E - \frac{1}{\epsilon^2} \cdot m_E.$$

Exactly as in Section 2.6, the flatness of this connection is equivalent to the relations (23), (24) and (25). As explained in Section 4.3, these follow directly from the axioms of a Joyce structure. \square

Setting $\epsilon = \infty$ in (53) gives a particular flat Ehresmann connection on \mathcal{T}_M which we call the *Joyce connection*. It is the analogue of the Levi-Civita connection on a Frobenius manifold. In co-ordinates the horizontal sections of the Joyce connection are spanned by the vector fields

$$\mathcal{A}_{\frac{\partial}{\partial z_i}} = \frac{\partial}{\partial z_i} - \sum_{j,k} \eta_{jk} \cdot \frac{\partial^2 J}{\partial \theta_i \partial \theta_j} \cdot \frac{\partial}{\partial \theta_k}. \quad (56)$$

Remark 4.4. It is important to note that if the form η fails to be non-degenerate the homomorphism ρ of (40) has a kernel, and the passage from (52) to (53) therefore loses information. In the extreme case $\eta = 0$, all Hamiltonian vector fields vanish, and the Joyce connection (55) coincides with the linear connection ∇ . We shall return to this point in Section 6.8.

4.7. Isomonodromic family and Stokes data. As with the deformed flat connection on a Frobenius manifold, there are two complementary points-of-view on the deformed Joyce connection (55). In the first, we forget the derivatives in the ϵ -direction, and view (53) as defining a pencil of flat Ehresmann connections on \mathcal{T}_M which deform the Joyce connection (56). In the second, we consider the family of meromorphic connections on \mathbb{P}^1 of the form

$$\nabla = d - \left(\frac{Z}{\epsilon^2} - \frac{H}{\epsilon} \right) d\epsilon, \quad (57)$$

parameterised by the points $p \in M$, and taking values in the Lie algebras $\mathfrak{g} = \mathfrak{g}_{M,p}$. Here $Z = m_E$ is a translation-invariant vector field on the fibre $\mathcal{T} = \mathcal{T}_{M,p}$, and at least heuristically, $H = \Theta_E = m_E(J)$ is a holomorphic function on \mathcal{T} which descends to an algebraic function on the torus $\mathbb{T} = \mathcal{T}/\Gamma^*$. In co-ordinates we have

$$Z = \sum_i z_i \cdot \frac{\partial}{\partial \theta_i} \in \mathfrak{h}, \quad H = \sum_i z_i \cdot \frac{\partial J}{\partial \theta_i} \in \mathfrak{g}^{\text{od}}. \quad (58)$$

As we discussed above, in practice, the function J need not restrict to give an algebraic function on the tori (42), and the same remark applies to H . Nonetheless, it is sometimes useful for heuristic purposes to identify H with its Fourier expansion

$$H = \sum_{\gamma \in \Gamma \setminus \{0\}} H(\gamma) \cdot x^\gamma. \quad (59)$$

Note that the assumption that J restricts to an odd function on the fibre \mathcal{T} means that H is even, and hence $H(-\gamma) = H(\gamma)$. This is the condition that H lies in the $+1$ eigenspace of the involution of Section 4.1.

Recall that the roots of the Lie algebra \mathfrak{g} of Section 4.1 are the non-zero elements of the lattice $\Gamma = \Gamma_{M,p}$. Following the ideas of Section 3, but replacing the Lie algebra $\mathfrak{gl}(\mathcal{T}_{M,p})$ of Section 2.3 with the Lie algebra $\mathfrak{g}_{M,p}$ of Section 4.1, we should expect the Stokes data of the connection (57) at the singularity $\epsilon = 0$ to consist of Stokes rays of the form $\mathbb{R}_{>0} \cdot Z(\gamma) \subset \mathbb{C}^*$ for nonzero elements $\gamma \in \Gamma_p$, and associated Stokes factors

$$\mathbb{S}(\ell) = \exp \left(\sum_{\gamma \in \Gamma: Z(\gamma) \in \ell} \text{DT}(\gamma) \cdot x_\gamma \right), \quad (60)$$

encoded by elements $\text{DT}(\gamma) \in \mathbb{C}$ exactly as in (35), and satisfying the symmetry condition $\text{DT}(-\gamma) = \text{DT}(\gamma)$. In the context of stability conditions on CY_3 triangulated categories, it

is these elements which it is natural to identify with the DT invariants, and we will therefore restrict attention to the case when all $\text{DT}(\gamma) \in \mathbb{Q}$. The resulting Stokes data is encoded by the notion of a BPS structure which is the subject of the next section.

5. BPS STRUCTURES

In this section we recall the basic definitions concerning BPS structures from [12]. A BPS structure axiomatises the output of unrefined Donaldson-Thomas theory applied to a CY_3 triangulated category. A variation of BPS structures describes the way the invariants change under variation of stability condition, the main ingredient being the Kontsevich-Soibelman wall-crossing formula. In the analogy described in this paper, BPS structures correspond to the FS structures of Section 3: they encode the Stokes data of the connection (57) defined by a point $p \in M$ of a complex manifold M equipped with a Joyce structure.

5.1. BPS structures. The following definition is a special case of the notion of stability data on a graded Lie algebra [34, Section 2.1]. It was also studied by Stoppa and his collaborators [4, Section 3], [22, Section 2].

Definition 5.1. A *BPS structure* consists of

- (a) a finite-rank free abelian group $\Gamma \cong \mathbb{Z}^{\oplus n}$, equipped with a skew-symmetric form

$$\langle -, - \rangle: \Gamma \times \Gamma \rightarrow \mathbb{Z};$$

- (b) a homomorphism of abelian groups $Z: \Gamma \rightarrow \mathbb{C}$;

- (c) a map of sets $\Omega: \Gamma \rightarrow \mathbb{Q}$;

satisfying the following properties:

- (i) symmetry: $\Omega(-\gamma) = \Omega(\gamma)$ for all $\gamma \in \Gamma$;
- (ii) support property: fixing a norm $\| \cdot \|$ on the finite-dimensional vector space $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$, there is a constant $C > 0$ such that

$$\Omega(\gamma) \neq 0 \implies |Z(\gamma)| > C \cdot \|\gamma\|. \tag{61}$$

The lattice Γ will be called the *charge lattice*, and the form $\langle -, - \rangle$ is the *intersection form*. The group homomorphism Z is called the *central charge*. The rational numbers $\Omega(\gamma)$ are called *BPS invariants*. A class $\gamma \in \Gamma$ will be called *active* if $\Omega(\gamma) \neq 0$. The *Donaldson-Thomas (DT)*

invariants are defined by the expression

$$\mathrm{DT}(\gamma) = \sum_{\gamma=m\alpha} \frac{1}{m^2} \Omega(\alpha) \in \mathbb{Q}, \quad (62)$$

where the sum is over integers $m > 0$ such that γ is divisible by m in the lattice Γ .

It will be useful to introduce some terminology for describing BPS structures of various special kinds.

Definition 5.2. We say that a BPS structure (Z, Γ, Ω) is

- (a) *finite* if there are only finitely many nonzero BPS invariants $\Omega(\gamma)$;
- (b) *ray-finite*, if for any ray $\ell \subset \mathbb{C}^*$ there are only finitely many active classes $\gamma \in \Gamma$ for which $Z(\gamma) \in \ell$;
- (c) *convergent*, if for some $R > 0$

$$\sum_{\gamma \in \Gamma} |\Omega(\gamma)| \cdot e^{-R|Z(\gamma)|} < \infty; \quad (63)$$

- (d) *uncoupled*, if for any two active classes $\gamma_1, \gamma_2 \in \Gamma$ one has $\langle \gamma_1, \gamma_2 \rangle = 0$;
- (e) *generic*, if for any two active classes $\gamma_1, \gamma_2 \in \Gamma$ one has

$$\mathbb{R}_{>0} \cdot Z(\gamma_1) = \mathbb{R}_{>0} \cdot Z(\gamma_2) \implies \langle \gamma_1, \gamma_2 \rangle = 0;$$

- (f) *integral*, if the BPS invariants $\Omega(\gamma) \in \mathbb{Z}$ are all integers.

5.2. Twisted torus. As well as the algebraic torus \mathbb{T}_+ , we will also consider an associated torsor

$$\mathbb{T}_- = \{g: \Gamma \rightarrow \mathbb{C}^* : g(\gamma_1 + \gamma_2) = (-1)^{\langle \gamma_1, \gamma_2 \rangle} g(\gamma_1) \cdot g(\gamma_2)\},$$

which we call the *twisted torus*. The torus \mathbb{T}_+ acts freely and transitively on the twisted torus \mathbb{T}_- via

$$(f \cdot g)(\gamma) = f(\gamma) \cdot g(\gamma) \in \mathbb{C}^*, \quad f \in \mathbb{T}_+, \quad g \in \mathbb{T}_-.$$

Choosing a base-point $\sigma \in \mathbb{T}_-$ therefore gives a bijection

$$\rho_\sigma: \mathbb{T}_+ \rightarrow \mathbb{T}_-, \quad f \mapsto f \cdot \sigma. \quad (64)$$

We can use the identification ρ_σ to give \mathbb{T}_- the structure of an algebraic variety. The result is independent of the choice of g_0 , since the translation maps on \mathbb{T}_+ are algebraic. The co-ordinate ring of \mathbb{T}_- is spanned as a vector space by the functions

$$x_\gamma: \mathbb{T}_- \rightarrow \mathbb{C}^*, \quad x_\gamma(g) = g(\gamma) \in \mathbb{C}^*,$$

which we refer to as *twisted characters*. Thus

$$\mathbb{C}[\mathbb{T}_-] = \bigoplus_{\gamma \in \Gamma} \mathbb{C} \cdot x_\gamma, \quad x_{\gamma_1} \cdot x_{\gamma_2} = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \cdot x_{\gamma_1 + \gamma_2}. \quad (65)$$

Similarly, the Poisson structure (37) on \mathbb{T}_+ is invariant under translations, and hence can be transferred to \mathbb{T}_- via the map (64). It is given on twisted characters by

$$\{x_\alpha, x_\beta\} = \langle \alpha, \beta \rangle \cdot x_\alpha \cdot x_\beta. \quad (66)$$

There is an involution of \mathbb{T}_- , which we call the *twisted inverse map*, which acts on twisted characters via $x_\gamma \leftrightarrow x_{-\gamma}$. The identification ρ_σ intertwines this twisted inverse map on \mathbb{T}_- with the standard inverse map on \mathbb{T}_+ precisely if the base-point σ lies in the subset

$$\left\{ \sigma : \Gamma \rightarrow \{\pm 1\} \mid \sigma(\gamma_1 + \gamma_2) = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \cdot \sigma(\gamma_1) \cdot \sigma(\gamma_2) \right\} \subset \mathbb{T}_-,$$

whose points are called *quadratic refinements* of the form $\langle -, - \rangle$.

5.3. BPS automorphisms. Associated to any ray $\ell \subset \mathbb{C}^*$ is a formal sum of twisted characters

$$\text{DT}(\ell) = \sum_{\gamma \in \Gamma: Z(\gamma) \in \ell} \text{DT}(\gamma) \cdot x_\gamma. \quad (67)$$

We call the ray active if this sum is nonzero. In general there will be countably many active rays. Naively, we would like to view $\text{DT}(\ell)$ as a well-defined holomorphic function on the twisted torus \mathbb{T}_- , and consider the associated time 1 Hamiltonian flow as a Poisson automorphism

$$\mathbb{S}(\ell) \in \text{Aut}(\mathbb{T}_-).$$

We refer to the resulting automorphism $\mathbb{S}(\ell)$ as the *BPS automorphism* associated to the ray ℓ . In [12] we considered several ways of making sense of $\mathbb{S}(\ell)$ which we briefly summarise here:

- (i) *Formal approach.* If we are only interested in the elements $\mathbb{S}(\ell)$ for rays $\ell \subset \mathbb{C}^*$ lying in a fixed acute sector $\Delta \subset \mathbb{C}^*$, then we can work with a variant of the algebra $\mathbb{C}[\mathbb{T}_-]$ consisting of formal sums of the form

$$\sum_{Z(\gamma) \in \Delta} a_\gamma \cdot x_\gamma, \quad a_\gamma \in \mathbb{C},$$

such that for any $H > 0$ there are only finitely many terms with $|Z(\gamma)| < H$. See [12, Appendix B]. It has the advantage of not requiring any extra assumptions, and gives a rigorous definition of variations of BPS structures (see below).

- (ii) *Analytic approach.* Let us suppose that the BPS structure is convergent. In [12, Appendix B], we associate to each convex sector $\Delta \subset \mathbb{C}^*$, and each sufficiently large real number $R > 0$, a non-empty analytic open subset $U_\Delta(R) \subset \mathbb{T}_-$ defined to be the interior of the subset

$$\{g \in \mathbb{T}_- : Z(\gamma) \in \Delta \text{ and } \Omega(\gamma) \neq 0 \implies |g(\gamma)| < \exp(-R\|\gamma\|)\} \subset \mathbb{T}_-. \quad (68)$$

We then show that for any active ray $\ell \subset \Delta$, the formal series $\text{DT}(\ell)$ is absolutely convergent on $U_\Delta(R) \subset \mathbb{T}_-$, and that the time 1 Hamiltonian flow of the resulting function defines a holomorphic embedding

$$\mathbb{S}(\ell) : U_\Delta(R) \rightarrow \mathbb{T}_-.$$

We can then view this map as being a partially-defined automorphism of \mathbb{T}_- .

- (iii) *Birational approach.* In the case of a generic, integral and ray-finite BPS structure, the partially-defined automorphisms $\mathbb{S}(\ell)$ discussed in (ii) extend to birational transformations of \mathbb{T}_- whose pullback of twisted characters is expressed by the formula

$$\mathbb{S}(\ell)^*(x_\beta) = x_\beta \cdot \prod_{Z(\gamma) \in \ell} (1 - x_\gamma)^{\Omega(\gamma)\langle \beta, \gamma \rangle}. \quad (69)$$

In fact these different approaches will not play a very important role in this paper, and for heuristic purposes we shall continue to regard the elements $\mathbb{S}(\ell)$ as being genuine automorphisms of \mathbb{T}_- . It is worth noting that approaches (ii) and (iii) involve applying the Lie algebra homomorphism (40), and hence lose information when the form $\langle -, - \rangle$ has a kernel. We will return to this point in Section 6.8.

5.4. Variations of BPS structure. The wall-crossing behaviour of BPS invariants in Donaldson-Thomas theory under changes in stability parameters is controlled by the Kontsevich-Soibelman wall-crossing formula. This forms the main ingredient in the following notion of a variation of BPS structures, which is a special case of the notion of a continuous family of stability structures from [34, Section 2.3]. Full details can be found in [12, Appendix B]: here we just give the rough idea.

Definition 5.3. A *variation of BPS structures* over a complex manifold M consists of a collection of BPS structures $(\Gamma_p, Z_p, \Omega_p)$ indexed by the points $p \in M$, such that

- (a) The charge lattices Γ_p form a local system of abelian groups, and the intersection forms $\langle -, - \rangle_p$ are covariantly constant.

- (b) Given a covariantly constant family of elements $\gamma_p \in \Gamma_p$, the central charges $Z_p(\gamma_p) \in \mathbb{C}$ vary holomorphically.
- (c) The constant in the support property (61) can be chosen uniformly on compact subsets.
- (d) For each acute sector $\Delta \subset \mathbb{C}^*$, the anti-clockwise product over active rays in Δ

$$\mathbb{S}_p(\Delta) = \prod_{\ell \subset \Delta} \mathbb{S}_p(\ell) \in \text{Aut}(\mathbb{T}_{p,-}), \tag{70}$$

is covariantly constant as $p \in M$ varies, providing the boundary rays of Δ are never active.

It requires some work to make rigorous sense of the wall-crossing formula, condition (d). This is done in [12, Appendix B] following [34]. This needs no convergence assumptions and completely describes the behaviour of the BPS invariants as the point $p \in M$ varies: once one knows all the invariants $\Omega(\gamma)$ at some point of M , they are determined at all other points.

Remark 5.4. The attentive reader may notice a discrepancy between Definition 3.7 of a variation of FS structures, and the above definition of variations of BPS structures. Namely, the products of Stokes factors over rays are taken in different directions (clockwise or anti-clockwise) in the two cases. The explanation for this is that, for various reasons, it is convenient to take the underlying group G in the case of BPS structures to be the *opposite* of the group of automorphisms of \mathbb{T}_- (at least at the level of heuristics). Thus G is related to the group of automorphisms of the ring of functions $\mathbb{C}[\mathbb{T}_-]$, and has a Lie algebra given by the Poisson bracket on $\mathbb{C}[\mathbb{T}_-]$. Compare [12, Remark 4.6 (i)], and see Section 6.6 below for more details.

Replacing M by a contractible open subset or cover we can use to trivialise the local system of charge lattices and obtain a holomorphic period map

$$\pi: M \rightarrow \text{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}), \quad p \mapsto Z_p. \tag{71}$$

We shall say that a variation of BPS structures is *miniversal* if the derivative of this map is everywhere non-vanishing, so that it is a local isomorphism.

5.5. Quadratic refinements. In the definition of BPS automorphisms we consider Hamiltonian vector fields on the twisted torus \mathbb{T}_- rather than the torus itself \mathbb{T}_+ . This is essential if the notion of a variation of BPS structures is to reproduce the wall-crossing behaviour of DT invariants. On the other hand, in the notion of a Joyce structure we consider vector fields on the family of tori \mathbb{T}_+ , rather than \mathbb{T}_- , and this is also important, since for example, in Section

7 we will define the linear Joyce connection by examining the behaviour of these vector fields near the identity of \mathbb{T}_+ .

To combine these requirements we must choose an identification

$$\rho_\sigma: \mathbb{T}_+ \rightarrow \mathbb{T}_-, \quad f \mapsto f \cdot \sigma \tag{72}$$

of the form (64), defined by a point $\sigma \in \mathbb{T}_-$. Under this identification the twisted and untwisted characters are related via $x_\gamma = \sigma(\gamma) \cdot y_\gamma$. If the symmetry property of the BPS invariants from Definition 5.1 is to correspond to the oddness (49) of the Joyce function, the map (72) needs to intertwine the inverse map on \mathbb{T}_+ and the twisted inverse map on \mathbb{T}_- , so as explained in Section 5.2, the point σ must be a quadratic refinement of the form $\langle -, - \rangle$.

Suppose given a variation of BPS structures over a manifold M . Since the lattices Γ form a local system over M , and the form $\langle -, - \rangle$ is covariantly constant, once a quadratic refinement is chosen at one point $p \in M$, there is at most one way to extend it in a covariantly constant way to the whole of M . But if the answer is to be well-defined we must choose a quadratic refinement which is preserved by parallel transport around loops in M , and in general this may only be possible after passing to a cover of M . Conceptually we can say that we must choose a trivialisation of a $\{\pm 1\}$ -torsor over M . It is interesting to note the analogy between this torsor and the one appearing in Remark 3.6.

6. INVERTING THE STOKES MAP

In Section 2 we explained how a tame Frobenius structure on a manifold M gives rise to an isomonodromic family of meromorphic connections (18) on a trivial vector bundle over \mathbb{P}^1 . We also mentioned that, up to some minor technicalities, the data of this family of connections is enough to reconstruct the Frobenius structure. In Section 3 we explained how the Stokes data of this family of connections in turn defines what we called a variation of FS structures. In this Section we begin by discussing the inverse problem of reconstructing the family of connections (18) from a variation of FS structures. We then move on to the analogous but more difficult problem of reconstructing a Joyce structure from a variation of BPS structures.

6.1. Properties of the Stokes map. Let us again consider connections of the form

$$\nabla = d - \left(\frac{U}{\epsilon^2} - \frac{V}{\epsilon} \right) d\epsilon, \tag{73}$$

as studied in Sections 3. The Lie algebra relevant to this context is $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. Recall from Section 3.2 the definition of the Stokes map

$$\mathcal{S}: \mathfrak{g}^{\text{od}} \rightarrow \mathfrak{g}^{\text{od}}, \quad \mathcal{S}(V) = R, \quad (74)$$

which for fixed $U \in \mathfrak{h}^{\text{reg}}$ sends the connection (73) to its Stokes data

$$R = \sum_{\alpha \in \Phi} D(\alpha) \cdot E_{\alpha} \in \mathfrak{g}^{\text{od}}$$

at the singularity $\epsilon = 0$. To understand the properties of this map it is useful to compare it to a closely-related map, namely the irregular Riemann-Hilbert map.

Rather than studying connections of the form (27), one can consider instead meromorphic connections on the trivial vector bundle over the unit disc $\Delta \subset \mathbb{C}$ of the form

$$\nabla = d - \left(\frac{U}{\epsilon^2} - \frac{V(\epsilon)}{\epsilon} \right) d\epsilon, \quad (75)$$

where $V: \Delta \rightarrow \mathfrak{g}^{\text{od}}$ is an arbitrary holomorphic function. Such connections are naturally considered up to the action of holomorphic gauge transformations $P: \Delta \rightarrow \text{GL}_n(\mathbb{C})$ satisfying the condition $P(0) = 1$. One can define the Stokes data at $\epsilon = 0$ for such connections in exactly the same way as before, and it is immediate from the definitions that gauge-equivalent connections have the same Stokes data.

For fixed $U \in \mathfrak{h}^{\text{reg}}$, the irregular Riemann-Hilbert map sends the gauge equivalence class of the connection (75) to its Stokes data. The results of [2] show that this map is an isomorphism. In contrast, although the Stokes map (74) is a local isomorphism of complex manifolds, it is neither injective nor surjective in general. The reason for this is that not every connection of the form (75) can be put into the constant coefficient form (73) by a gauge transformation, and on the other hand, distinct elements V in (73) can lead to gauge equivalent connections. For more details on this see, for example, [6, Remark 12] and the reference given there.

6.2. Stokes map in the FS structure case. Power series expansions were derived in [10] for the Stokes map \mathcal{S} , and its inverse, in a neighbourhood of the origin in the vector space \mathfrak{g}^{od} . They depend on a fixed element $U \in \mathfrak{h}^{\text{reg}}$. The expansion of \mathcal{S}^{-1} takes the form

$$V_{\alpha} = - \sum_{n \geq 1} \sum_{\substack{\alpha_i \in \Phi \\ \alpha_1 + \dots + \alpha_n = \alpha}} F_n(U(\alpha_1), \dots, U(\alpha_n)) \cdot \prod_{i=1}^n D(\alpha_i) \cdot E_{\alpha_1} \cdot E_{\alpha_2} \cdots E_{\alpha_n}, \quad (76)$$

where the piecewise-holomorphic functions $F_n: (\mathbb{C}^*)^n \rightarrow \mathbb{C}$ are closely related to multi-logarithms. Each term in the sum over n on the right of (76) is a finite sum of elements in the universal enveloping algebra $U(\mathfrak{g})$, but in fact the symmetry properties of the functions F_n ensure that these elements lie in the Lie algebra $\mathfrak{g} \subset U(\mathfrak{g})$.

It was shown in [10] that for sufficiently small $D(\alpha_i) \in \mathbb{C}$ the sum over $n \geq 1$ in (76) is absolutely convergent, and the resulting connection (73) has Stokes factors given by the relation (35). In this case therefore, the sum (76) successfully inverts the Stokes map, and therefore gives a potential method for associating a connection (18) to a FS structure. But since, as we explained above, the Stokes map is not expected to be injective or surjective in general, one should not expect the series (76) to be convergent for arbitrary choices of $D(\alpha_i) \in \mathbb{C}$. And since, for a given $U \in \mathfrak{h}^{\text{reg}}$, nothing is known about how small the $D(\alpha_i)$ need to be to ensure that (76) converges, this approach appears to be only of interest in a formal setting.

Note that when the sum (76) is absolutely convergent, the resulting connections have constant Stokes data (35) as the leading term $U \in \mathfrak{h}^{\text{reg}}$ varies, and therefore the expressions $V_\alpha(U)$ must satisfy the differential equation (20). This is easily seen to be equivalent to the statement that the functions F_n satisfy the differential equation

$$dF_n(u_1, \dots, u_n) = \sum_{1 \leq i < n} F_i(u_1, \dots, u_i) \cdot F_{n-i}(u_{i+1}, \dots, u_n) \cdot d \log \left(\frac{u_{i+1} + \dots + u_n}{u_1 + \dots + u_i} \right) \quad (77)$$

on the dense open subsets of $(\mathbb{C}^*)^n$ where they are holomorphic.

6.3. Riemann-Hilbert problem in the FS structure case. Consider again the problem of reconstructing a connection of the form (73) from its associated FS structure. Rather than working with the generally divergent series (76), we can try to construct the fundamental solutions $X_r(t)$ directly by their defining properties, and then obtain the associated connection by differentiation.

Recall from Section 3.1 the definition of the open half-plane $\mathbb{H}_r \subset \mathbb{C}^*$ centered on a ray $r = \mathbb{R}_{>0} \cdot z \subset \mathbb{C}^*$. Consider an FS structure (T, U, D) , with associated Stokes factors $\mathbb{S}_\ell \in \text{GL}(T)$ as defined in (35). The associated Riemann-Hilbert problem is as follows.

Problem 6.1. *For each non-Stokes ray $r \subset \mathbb{C}^*$, find a holomorphic function*

$$\Psi_r: \mathbb{H}_r \rightarrow \text{GL}(T)$$

satisfying the following conditions:

- (i) let r_1 and r_2 be the two boundary rays, taken in clockwise order, of an acute sector $\Delta \subset \mathbb{C}^*$ containing a single Stokes ray ℓ . Then

$$\Psi_{r_2}(\epsilon) = \Psi_{r_1}(\epsilon) \cdot \mathbb{S}_\ell,$$

for all $\epsilon \in \mathbb{H}_{r_-} \cap \mathbb{H}_{r_+}$;

- (ii) as $\epsilon \rightarrow 0$ in the half-plane \mathbb{H}_r we have $\Psi_r(\epsilon) \cdot \exp(U/\epsilon) \rightarrow \text{id}$;
- (iii) as $\epsilon \rightarrow \infty$ in the half-plane \mathbb{H}_r the norm $\|\Psi_r(\epsilon)\|$ has at most polynomial growth.

Given a connection (73), the fundamental solutions of Section 3.1 satisfy condition (ii) by definition, and condition (i) is the definition of the Stokes factors. Condition (iii) holds because the connection (18) has a regular singularity at $\epsilon = \infty$, which implies that any fundamental solution has moderate growth.

Given a solution to Problem 6.1 we can construct a connection (18) having the Stokes data (35) by writing

$$V = \frac{U}{\epsilon} - \epsilon \cdot \frac{d\Psi}{d\epsilon} \cdot \Psi^{-1},$$

which is independent of the choice of ray ℓ by property (i). As before however, since the Stokes map is not an isomorphism, we cannot expect existence of solutions to the Problem 6.1 for arbitrary choices of FS structure. General results show that when the $D(\alpha) \in \mathbb{C}$ are sufficiently small, so that the Stokes matrices $\mathbb{S}(\ell)$ are close to the identity, Problem 6.1 can be solved using an integral equation [19, Proposition 3.13]. But again, for a given choice of $U \in \mathfrak{h}^{\text{reg}}$, there are no explicit results about what ‘sufficiently small’ means.

When Problem 6.1 has a solution, it need not be unique, since we can obtain another one by pre-multiplying by a holomorphic function $P: \mathbb{C} \rightarrow \text{GL}_n(T)$ satisfying $P(0) = 1$ and such that $\|P(\epsilon)\|$ has moderate growth as $\epsilon \rightarrow \infty$. To eliminate this indeterminacy one can introduce a more complicated Riemann-Hilbert problem which also takes into account the existence of canonical solutions to the connection (18) near $\epsilon = \infty$. Relating these to the canonical solutions near $\epsilon = 0$ leads to new monodromy invariants called central connection matrices. The resulting Riemann-Hilbert problems are discussed in detail in [20, Lecture 4].

6.4. Stokes map in the BPS structure case. Let us now consider the, presumably much more difficult, problem of reconstructing a connection from its Stokes data in the case of the infinite-dimensional Lie algebra \mathfrak{g} of Section 4. More precisely, we would like to be able to

construct Joyce structures for which the associated connection (57) has Stokes data described by a given BPS structure. One possibility is to use the expansion (76). The analogous expression to (76) is

$$H_\alpha = - \sum_{n \geq 1} \sum_{\substack{\gamma_i \in \Gamma \setminus \{0\} \\ \gamma_1 + \dots + \gamma_n = \gamma}} F_n(U(\gamma_1), \dots, U(\gamma_n)) \cdot \prod_{i=1}^n \text{DT}(\gamma_i) \cdot x_{\gamma_1} * x_{\gamma_2} * \dots * x_{\gamma_n}. \quad (78)$$

As with (76) this expression is viewed as living in the universal enveloping algebra $U(\mathfrak{g})$, but due to the symmetry properties of the functions F_n should actually output elements of the Lie algebra \mathfrak{g} . If the sum in (78) is absolutely convergent one can define

$$H = \sum_{\gamma \in \Gamma} H_\gamma \cdot e^{\theta(\gamma)}, \quad J = \sum_{\gamma \in \Gamma} \frac{H_\gamma \cdot e^{\theta(\gamma)}}{Z(\gamma)}, \quad (79)$$

and so obtain the Joyce function.

The formula (78) was first written down by Joyce [32], and was also investigated by Stoppa and collaborators [16, 22]. Given the discussion of Section 6.2 however, we cannot expect that the sum (78) is convergent in general, and in fact the convergence problems are even more acute in the setting of the infinite-dimensional Lie algebra \mathfrak{g} , since even for a fixed value of n the sum in (78) is potentially infinite. One can try to introduce formal parameters as in [16], but since this depends on a non-canonical choice of positive cone $\Gamma^+ \subset \Gamma$, given a variation of BPS structure over a base S , the resulting quantities will not in general vary holomorphically over S .

Remark 6.2. A remarkable feature of the paper [32] is that Joyce wrote down the expansion (78) without any input from the theory of irregular connections or Stokes data. His motivation was simply to write down a generating function involving the invariants $\text{DT}(\alpha)$ which would give holomorphically varying quantities H_α on the space of stability conditions. Postulating the form (78), and making some other natural assumptions, he was able to show that the functions F_n were uniquely determined, and satisfied the differential equation (77), and hence that the resulting quantities H_α satisfied the equation (20). The observation that this equation describes isomonodromic families of connections of the form (57) led directly to the paper [10], which shows that Joyce's expansion can be interpreted as the formal inverse to the power-series expansion of the Stokes map.

6.5. Riemann-Hilbert problem in the BPS structure case. In this paper we will attempt to use the Riemann-Hilbert approach to find Joyce structures whose Stokes data is described by a given BPS structure. Let (Γ, Z, Ω) be a convergent BPS structure, and denote by \mathbb{T}_- the twisted torus. The associated Riemann-Hilbert problem depends also on an element $\xi \in \mathbb{T}_-$ which we call the constant term. It involves meromorphic maps $X_r: \mathbb{H}_r \rightarrow \mathbb{T}_-$ depending on a choice of non-active ray $r \subset \mathbb{C}^*$. Composing with the twisted characters of \mathbb{T}_- we can equivalently consider functions

$$X_{r,\gamma}: \mathbb{H}_r \rightarrow \mathbb{C}^*, \quad X_{r,\gamma}(\epsilon) = x_\gamma(X_r(\epsilon)).$$

The Riemann-Hilbert problem reads as follows; we shall explain the precise analogy with Problem 6.1 in the next subsection.

Problem 6.3. *For each non-active ray $r \subset \mathbb{C}^*$ we seek a meromorphic function*

$$X_r: \mathbb{H}_r \rightarrow \mathbb{T}_-$$

such that the following three conditions are satisfied:

(RH1) *suppose that two non-active rays $r_1, r_2 \subset \mathbb{C}^*$ form the boundary rays of a convex sector $\Delta \subset \mathbb{C}^*$ taken in clockwise order. Then*

$$X_{r_2}(\epsilon) = \mathbb{S}(\Delta)(X_{r_1}(\epsilon)),$$

for all $\epsilon \in \mathbb{H}_{r_1} \cap \mathbb{H}_{r_2}$ with $0 < |\epsilon| \ll 1$;

(RH2) *for each non-active ray $r \subset \mathbb{C}^*$ and each class $\gamma \in \Gamma$ we have*

$$\exp(Z(\gamma)/\epsilon) \cdot X_{r,\gamma}(\epsilon) \rightarrow \xi(\gamma)$$

as $\epsilon \rightarrow 0$ in the half-plane \mathbb{H}_r ;

(RH3) *for any class $\gamma \in \Gamma$ and any non-active ray $r \subset \mathbb{C}^*$, there exists $k > 0$ such that*

$$|\epsilon|^{-k} < |X_{r,\gamma}(\epsilon)| < |\epsilon|^k,$$

for $\epsilon \in \mathbb{H}_r$ satisfying $|\epsilon| \gg 0$.

To make rigorous sense of the condition (RH1) one needs to use the results of [12] which guarantee that the clockwise-product of Stokes factors

$$\mathbb{S}(\Delta) = \prod_{\ell \subset \Delta} \mathbb{S}(\ell)$$

has a well-defined meaning as a partially-defined automorphism of the twisted torus \mathbb{T}_- . See the discussion following [12, Problem 4.3] for more details.

6.6. Riemann-Hilbert problem discussion. At first sight Problem 6.3 looks somewhat different to Problem 6.1. In this section we explain the precise analogy between the two problems. In this discussion we will ignore issues to do with lack of convergence and pretend that the Stokes factors

$$\mathbb{S}(\ell) = \exp \left(\sum_{\gamma \in \Gamma: Z(\gamma) \in \ell} \text{DT}(\gamma) \cdot x_\gamma \right)$$

are well-defined elements of the corresponding group G of Poisson automorphisms of the algebra $\mathbb{C}[\mathbb{T}_-]$. The obvious analogue of Problem 6.1 then involves maps

$$\Psi_r: \mathbb{H}_r \rightarrow G = \text{Aut}_{\{-,-\}}(\mathbb{C}[\mathbb{T}_-]). \quad (80)$$

The first point to note is that instead of the group G , it will be convenient to work with the opposite group G^{op} , which we can view heuristically as the group of Poisson automorphisms of the twisted torus \mathbb{T}_- . There is an obvious anti-homomorphism of groups $G \rightarrow G^{\text{op}}$ which is the identity on the underlying sets. Composing (80) with this gives a map

$$\Psi_r: \mathbb{H}_r \rightarrow G^{\text{op}} = \text{Aut}_{\{-,-\}}(\mathbb{T}_-), \quad (81)$$

which satisfies the analogous conditions to those of Problem 6.1, but with the order of all products reversed. In particular, Stokes automorphisms will act on the left rather than the right.

A second issue is that, rather than considering the infinite-dimensional group G^{op} , we prefer to fix an element $\xi \in \mathbb{T}_-$, and consider the composite of (81) with the evaluation map

$$\text{ev}_\xi: G^{\text{op}} = \text{Aut}_{\{-,-\}}(\mathbb{T}_-) \rightarrow \mathbb{T}_-, \quad \text{ev}_\xi(\Psi) = \Psi(\xi).$$

The maps $X_r(\epsilon)$ considered in Problem 6.3 are then given by

$$X_r(\epsilon) = \Psi_r(\epsilon)(\xi). \quad (82)$$

Finally, although one would initially expect the resulting maps (82) to be holomorphic functions of $\epsilon \in \mathbb{H}_r$, computations in examples show that in general one must allow them to be meromorphic. This has the unfortunate effect of making the solutions of Problem 6.3 highly non-unique. It would be very interesting to understand whether there are some further natural conditions which can be imposed on the solutions to control this lack of uniqueness.

Remark 6.4. One important feature of the heuristic picture involving maps (81) has been lost in the translation to the more concrete Problem 6.3, namely that the automorphism $\Psi_r(\epsilon)$ should preserve the Poisson structure on the torus \mathbb{T}_- . As we explain in the next section, if

we choose a basis $(\gamma_1, \dots, \gamma_n)$ for the lattice Γ we can encode the solutions to Problem 6.3 in terms of functions

$$x_j(z_i, \theta_i, \epsilon) = \log X_{r, \gamma_j}(\epsilon), \quad Z(\gamma_i) = z_i, \quad \xi(\gamma_i) = \sigma(\gamma_i) \cdot \exp(\theta_i),$$

for some quadratic refinement $\sigma: \Gamma \rightarrow \{\pm 1\}$ of the form $\langle -, - \rangle$. The missing condition is then that for fixed $z_i \in \mathbb{C}$ and $\epsilon \in \mathbb{C}^*$, the map $\theta_i \rightarrow x_j$ should be Poisson, so that

$$\sum_{p,q} \eta_{pq} \cdot \frac{\partial x_i}{\partial \theta_p} \cdot \frac{\partial x_j}{\partial \theta_q} = \delta_{ij}.$$

Clearly, imposing this condition will reduce the indeterminacy in the solutions to Problem 6.3 as functions of the constant term $\xi \in \mathbb{T}_-$.

Remark 6.5. In what follows we will not have much to say about the behaviour of the flat sections of the connection (57) near the point $\epsilon = \infty$ where it has a logarithmic singularity. This seems to be an interesting point which deserves further study. The corresponding picture for Frobenius manifolds is explained in detail by Dubrovin [20, Section 2], and leads to the full set of monodromy data for a Frobenius manifold: as well as the Stokes factors at $\epsilon = 0$, there is monodromy data at $\epsilon = \infty$, and a central connection matrix which intertwines them. It would be interesting to understand the analogue of this extra data in the context of Joyce structures.

6.7. Relating BPS structures and Joyce structures. In Section 3.5 we explained that a tame Frobenius structure on a complex manifold M determines a variation of FS structures on M . We similarly expect that a Joyce structure on M , possibly satisfying some additional assumptions, should determine a variation of BPS structures over M , as defined in Section 5.4. To prove such a result one would need an analogue in the context of Joyce structures of the fundamental existence result, Theorem 3.1, which gives canonical flat sections of the connection (57) in suitable half-planes.

What we shall do instead in what follows is to start with a variation of BPS structures on a complex manifold M , and attempt to construct a Joyce structure by solving the Riemann-Hilbert problem of Section 6.5. Given the lack of uniqueness for solutions discussed above, this will necessarily involve a certain amount of guesswork. A more rigorous mathematical result can be obtained by reading the following calculations backwards, and viewing them as proving instances of the missing analogue of Theorem 3.1. In this approach we suppose given

the Joyce structure, and view the solutions to the Riemann-Hilbert problem as giving the required existence of canonical sections of the connection (57).

Let us describe in more detail how to get from a miniversal variation of BPS structures $(\Gamma_p, Z_p, \Omega_p)$ on a complex manifold M to a Joyce structure, assuming the existence of suitably varying solutions to Problem 6.3. Note first that the miniversal condition ensures that the derivative of the period map (71) at a point $p \in M$ gives an isomorphism

$$m: \mathcal{T}_p M \rightarrow \text{Hom}_{\mathbb{Z}}(\Gamma_p, \mathbb{C}).$$

Thus we can identify the cotangent bundle \mathcal{T}_M^* with the bundle of lattices $\Gamma_p \otimes_{\mathbb{Z}} \mathbb{C}$, and the fact that the Γ_p form a local system then gives a flat, torsion-free connection ∇ on \mathcal{T}_M . The skew-symmetric form $\eta(-, -)$ on \mathcal{T}_M^* required in the definition of a Joyce structure corresponds to the form $\langle -, - \rangle$ in the definition of a BPS structure, and we define the Euler vector field E by the requirement that $m(E) = Z$. It just remains to define the Joyce function J .

Let us choose a covariantly constant family of quadratic refinements $\sigma_p: \Gamma_p \rightarrow \{\pm 1\}$ for all $p \in M$. As discussed in Section 5.5 this may only be possible after passing to a cover or an open subset of M . The associated maps (72) then allow us to identify the two tori $\mathbb{T}_{p,\pm}$ at each point $p \in M$. Suppose we can solve the associated Riemann-Hilbert problems for generic choices of constant terms $\xi_p \in \mathbb{T}_{p,-}$. We can then view the solution as a collection of maps (80) which are the analogues of the canonical solutions of Theorem 3.1, and which should therefore, as in Remark 3.3, define sections of the deformed Joyce connection (53). Differentiating then gives a formula for the Joyce function. There is one confusing sign which arises from the fact that, as explained in Section 6.6, we have chosen to work with the opposite group G^{op} .

To carry this out in detail, let us work locally on M , and take a covariantly constant basis $(\gamma_1, \dots, \gamma_n)$ for the lattices Γ_p . We can then identify all the lattices Γ_p , tori $\mathbb{T}_{p,\pm}$, etc., and hence drop the point $p \in M$ from the notation. Problem 6.3 depends on a point $p \in M$ specified locally by the corresponding central charges $z_i = Z(\gamma_i)$, and a constant term $\xi \in \mathbb{T}_-$ specified by writing $\xi(\gamma_i) = \sigma(\gamma_i) \cdot \exp(\theta_i)$. For a given choice of non-active ray r , the solution $X_r: \mathbb{H}_r \rightarrow \mathbb{T}_-$ is then encoded in the functions

$$x_j(z_i, \theta_i, \epsilon) = \log X_{r,\gamma_j}(\epsilon). \tag{83}$$

Note that if we use the quadratic refinement σ to view the solution $X_r: \mathbb{H}_r \rightarrow \mathbb{T}_-$ as taking values in the torus \mathbb{T}_+ , instead of the twisted torus \mathbb{T}_- , the corresponding logarithmic solutions

$y_j(z_i, \theta_i, \epsilon)$ will differ from $x_j(z_i, \theta_i, \epsilon)$ only by constant multiples of π which will vanish upon differentiation, and hence play no role in the calculation.

Substituting the solution (83) into the equation for the extended Joyce connection (54) gives

$$\frac{\partial x_j}{\partial z_i} + \frac{1}{\epsilon} \cdot \frac{\partial x_j}{\partial \theta_i} = \sum_{p,q} \eta_{pq} \cdot \frac{\partial^2 J}{\partial \theta_i \partial \theta_p} \cdot \frac{\partial x_j}{\partial \theta_q}, \tag{84}$$

from which we can extract the second derivatives of the Joyce function as meromorphic functions of z_i and θ_i . Note that J precisely measures the failure of the solution of the Riemann-Hilbert problem to be a function of the combined variables $z_i - \epsilon \theta_i$.

6.8. Doubling construction. Let us return to the setting of Section 4.1 and the definition of the Lie algebra \mathfrak{g} associated to a lattice Γ equipped with an integral skew-symmetric form $\langle -, - \rangle$. When the form $\langle -, - \rangle$ is degenerate, the Lie algebra homomorphism $\rho: \mathfrak{g} \rightarrow \mathbb{C}[\mathbb{T}_+]$ of Section 4.1 has a nonzero kernel. As explained in Remark 4.4, this means that the geometric incarnation of the deformed Joyce connection (53) contains less information than the abstract \mathfrak{g} -valued version (52). Similar remarks apply to a BPS structure (Γ, Z, Ω) with a degenerate intersection form: viewing the wall-crossing automorphisms $\mathbb{S}(\ell)$ as (partially-defined) automorphisms of the torus \mathbb{T}_- , rather than as elements of the abstract group associated to the Lie algebra \mathfrak{g} , loses information. In the extreme case when $\langle -, - \rangle = 0$, the connection (53) is almost trivial, every wall-crossing automorphisms $\mathbb{S}(\ell)$ is the identity, and Problem 6.3 is also uninteresting.

In principle these issues can be dealt with by working directly with the abstract Lie algebra \mathfrak{g} and the associated Lie group. But since these are really only the correct objects at a heuristic level, this would be difficult to make precise. Instead it is easier to reduce to the case when $\langle -, - \rangle$ is non-degenerate using a standard doubling construction [12, Section 2.8], [34, Section 2.6] which we explain here. The reader can also refer to Section 8.2 below, where an explicit example is worked out in detail.

On the monodromy side, suppose given a BPS structure (Γ, Z, Ω) . The *doubled BPS structure* then takes the form

$$(\Gamma \oplus \Gamma^\vee, Z \oplus Z^\vee, \Omega),$$

where $\Gamma^\vee = \text{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z})$ is the dual lattice. We equip the doubled lattice $\Gamma \oplus \Gamma^\vee$ with the non-degenerate skew-symmetric form

$$\langle (\gamma_1, \lambda_1), (\gamma_2, \lambda_2) \rangle = \langle \gamma_1, \gamma_2 \rangle + \lambda_1(\gamma_2) - \lambda_2(\gamma_1). \tag{85}$$

The central charge is defined by $Z(\gamma, \lambda) = Z(\gamma) + Z^\vee(\lambda)$, for some arbitrary group homomorphism $Z^\vee: \Gamma^\vee \rightarrow \mathbb{C}$, and the BPS invariants by

$$\Omega(\gamma, \lambda) = \begin{cases} \Omega(\gamma) & \text{if } \lambda = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Given a miniversal variation of BPS structures over a manifold M , the derivative of the period map (71) at a point $p \in M$ gives an identification between the tangent space $\mathcal{T}_{M,p}$ and the vector space $\Gamma_p^\vee \otimes \mathbb{C}$, and it follows that the doubling construction gives a variation of BPS structures over the total space of the cotangent bundle $P = \mathcal{T}_M^*$.

On the Joyce structure side, consider a Joyce structure on a complex manifold M , given by the data of Definition 4.1. The base of the doubled structure will again be the complex manifold P which is the total space of the cotangent bundle \mathcal{T}_M^* . The projection $q: P \rightarrow M$ induces a projection $q_*: \mathcal{T}_P \rightarrow \mathcal{T}_M$, and the Joyce function on \mathcal{T}_P is just the pullback of the Joyce function $J: \mathcal{T}_M \rightarrow \mathbb{C}$ along this map. The tangent space at a point $p \in P$ is

$$\mathcal{T}_p P = \mathcal{T}_m M \oplus \mathcal{T}_m^* M, \quad m = q(p).$$

We take the lattice $\Gamma_m \oplus \Gamma_m^* \subset \mathcal{T}_p^* P$, equipped with the form (85), and as connection on \mathcal{T}_P the sum of the pullbacks along q of the connections induced by ∇ on the bundles \mathcal{T}_M and \mathcal{T}_M^* . The Euler vector field at a point $p \in P$ is the sum $E_m \oplus V_p$, where E_m is the Euler vector in the fibre $\mathcal{T}_{M,m}$, and V_p is the tautological element of $\mathcal{T}_m^* M$ defined by the point $p \in P = \mathcal{T}_M^*$.

7. LINEAR DATA FROM JOYCE STRUCTURES

The Joyce connection (56) defines an Ehresmann connection on the tangent bundle of a manifold M equipped with a Joyce structure. Joyce pointed out [32, Section 4] that this induces a linear connection on the tangent bundle, which we shall call the linear Joyce connection. We explain the definition of this connection here, together with some closely-related tensors: the Joyce form, and an operator V . In the special case when both sides of the equation (51) individually vanish we also define a holomorphic function $\mathcal{F}: M \rightarrow \mathbb{C}$ which we call the pre-potential. We finish by defining a notion of compatibility for Joyce and Frobenius structures with the same underlying complex manifold.

7.1. Linear Joyce connection. Consider a Joyce structure as in Section 4, and assume that the Joyce function J extends to a holomorphic function in a neighbourhood of the zero-section of \mathcal{T}_M . The assumption that J is an odd function of the fibre co-ordinates means that for any

vector field X on M , the Hamiltonian vector field $\text{Ham}_{m_X(J)}$ on \mathcal{T}_M vanishes along the zero-section. The derivative of this vector field therefore defines an endomorphism of the normal bundle of the zero-section, which can be identified with the tangent bundle to M via the map m . The result is an endomorphism-valued 1-form on the tangent bundle of M

$$C_X(Y) = D \text{Ham}_{m_X(J)}(Y)$$

which we can use to define a linear connection on \mathcal{T}_M

$$\nabla_X^J(Y) = \nabla_X(Y) + D \text{Ham}_{m_X(J)}(Y). \quad (86)$$

We call this the *linear Joyce connection*. Taking co-ordinates as in Section 4.2 we have

$$\nabla_{\frac{\partial}{\partial z_i}}^J \left(\frac{\partial}{\partial z_j} \right) = \sum_{l,m} \eta_{lm} \cdot \frac{\partial^3 J}{\partial \theta_i \partial \theta_j \partial \theta_l} \Big|_{\theta=0} \cdot \frac{\partial}{\partial z_m}. \quad (87)$$

Lemma 7.1. *The linear Joyce connection is flat and torsion-free, and preserves the skew-symmetric form η on the cotangent bundle \mathcal{T}_M^* .*

Proof. These properties can be easily derived from the above geometric description of the linear Joyce connection, but we shall instead give a computational proof based on the formula (87). For flatness we must show that

$$\begin{aligned} 0 &= \nabla_{\frac{\partial}{\partial z_i}}^J \circ \nabla_{\frac{\partial}{\partial z_j}}^J \left(\frac{\partial}{\partial z_k} \right) - \nabla_{\frac{\partial}{\partial z_j}}^J \circ \nabla_{\frac{\partial}{\partial z_i}}^J \left(\frac{\partial}{\partial z_k} \right) \\ &= \sum_{l,m} \eta_{lm} \cdot \left(\frac{\partial^4 J}{\partial z_i \partial \theta_j \partial \theta_k \partial \theta_l} \Big|_{\theta=0} - \frac{\partial^4 J}{\partial z_j \partial \theta_i \partial \theta_k \partial \theta_l} \Big|_{\theta=0} \right) \cdot \frac{\partial}{\partial z_m} \\ &+ \sum_{l,m,p,q} \eta_{lm} \eta_{pq} \cdot \left(\frac{\partial^3 J}{\partial \theta_i \partial \theta_q \partial \theta_l} \Big|_{\theta=0} \cdot \frac{\partial^3 J}{\partial \theta_j \partial \theta_k \partial \theta_p} \Big|_{\theta=0} - \frac{\partial^3 J}{\partial \theta_j \partial \theta_q \partial \theta_l} \Big|_{\theta=0} \cdot \frac{\partial^3 J}{\partial \theta_i \partial \theta_k \partial \theta_p} \Big|_{\theta=0} \right) \cdot \frac{\partial}{\partial z_m}. \end{aligned}$$

Since J is an odd function of the fibre co-ordinates, this follows by differentiating (51) with respect to θ_k and θ_l , and setting all $\theta_i = 0$.

The linear Joyce connection induces a connection on the cotangent bundle \mathcal{T}_M^* given by

$$\nabla_{\frac{\partial}{\partial z_i}}^J (dz_j) = \sum_{k,l} \eta_{jl} \cdot \frac{\partial^3 J}{\partial \theta_i \partial \theta_l \partial \theta_k} \Big|_{\theta=0} \cdot dz_k. \quad (88)$$

The preservation of the form η is then a consequence of

$$\eta \left(\nabla_{\frac{\partial}{\partial z_i}}^J (dz_j), dz_k \right) = \sum_{l,m} \eta_{jl} \cdot \eta_{mk} \cdot \frac{\partial^3 J}{\partial \theta_i \partial \theta_l \partial \theta_m} \Big|_{\theta=0} = \eta \left(dz_j, \nabla_{\frac{\partial}{\partial z_i}}^J (dz_k) \right).$$

The statement that the linear Joyce connection is torsion-free is immediate from (87). \square

Consider the special case when both sides of the equation (51) vanish. This happens for example when the form $\eta = 0$. Differentiating the left-hand side of (51) gives a relation

$$\frac{\partial^4 J}{\partial z_i \partial \theta_j \partial \theta_k \partial \theta_l} \Big|_{\theta=0} = \frac{\partial^4 J}{\partial z_j \partial \theta_i \partial \theta_k \partial \theta_l} \Big|_{\theta=0}.$$

This implies the existence of a locally-defined function $\mathcal{F}: M \rightarrow \mathbb{C}$, well-defined up to the addition of polynomials of degree ≤ 2 in the variables z_i , satisfying

$$\frac{\partial^3 \mathcal{F}}{\partial z_i \partial z_j \partial z_k} = \frac{\partial^3 J}{\partial \theta_i \partial \theta_j \partial \theta_k} \Big|_{\theta=0}. \quad (89)$$

In the notation of Section 4, the function \mathcal{F} satisfies the relations

$$X_1 X_2 X_3(\mathcal{F}) = m_{X_1} m_{X_2} m_{X_3}(J) \Big|_{\theta=0},$$

where X_1, X_2, X_3 are arbitrary vector fields on M . We call a locally-defined function $\mathcal{F}: M \rightarrow \mathbb{C}$ satisfying these conditions a *prepotential*.

7.2. Joyce form. Consider again a Joyce structure on a complex manifold M whose Joyce function extends holomorphically over the zero-section. Recall from (58) the definition of the function $H = m_E(J)$. We can use this function, and the formula

$$g\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right) = \frac{\partial^2 H}{\partial \theta_j \partial \theta_k} \Big|_{\theta=0} = \sum_i z_i \cdot \frac{\partial^3 J}{\partial \theta_i \partial \theta_j \partial \theta_k} \Big|_{\theta=0} \quad (90)$$

to define a symmetric bilinear form on the tangent bundle \mathcal{T}_M . Since it was first considered by Joyce (see Remark 7.4 below) we call $g(-, -)$ the *Joyce form*.

Lemma 7.2. *The Joyce form is preserved by the linear Joyce connection.*

Proof. We must check that

$$\frac{\partial}{\partial z_i} \cdot g\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right) = g\left(\nabla_{\frac{\partial}{\partial z_i}}^J \left(\frac{\partial}{\partial z_j}\right), \frac{\partial}{\partial z_k}\right) + g\left(\frac{\partial}{\partial z_j}, \nabla_{\frac{\partial}{\partial z_i}}^J \left(\frac{\partial}{\partial z_k}\right)\right).$$

This boils down to the identity

$$\frac{\partial^3 H}{\partial z_i \partial \theta_j \partial \theta_k} \Big|_{\theta=0} = \sum_{p,q} \eta_{pq} \left(\frac{\partial^3 J}{\partial \theta_i \partial \theta_j \partial \theta_p} \Big|_{\theta=0} \cdot \frac{\partial^2 H}{\partial \theta_k \partial \theta_q} \Big|_{\theta=0} + \frac{\partial^3 J}{\partial \theta_i \partial \theta_k \partial \theta_p} \Big|_{\theta=0} \cdot \frac{\partial^2 H}{\partial \theta_j \partial \theta_q} \Big|_{\theta=0} \right).$$

To establish this, we sum the relation (51), and use the assumption (58) that J is homogeneous of weight -1 in the z_i variables to obtain

$$\frac{\partial H}{\partial z_i} = \sum_{p,q} \eta_{pq} \cdot \frac{\partial^2 J}{\partial \theta_i \partial \theta_p} \cdot \frac{\partial H}{\partial \theta_q}.$$

The identity then follows by differentiating with respect to θ_j and θ_k , setting all fibre coordinates $\theta_p = 0$, and using the fact that J is an odd function of these co-ordinates. \square

When the Joyce form g is non-degenerate, it defines a holomorphic metric on M , and Lemmas 7.1 and 7.2 then imply that the linear Joyce connection ∇^J is the corresponding Levi-Civita connection, and that g is a flat metric.

7.3. Gradient of the Euler vector field. Let us continue with the set-up of the last two sections. The formula

$$V(X) = \nabla_X^J(E) - X \tag{91}$$

defines an endomorphism of the tangent bundle \mathcal{T}_M . Note the analogy with the definition (16). Explicitly we have

$$V\left(\frac{\partial}{\partial z_i}\right) = \sum_{j,l,m} z_j \cdot \eta_{lm} \cdot \frac{\partial^3 J}{\partial \theta_i \partial \theta_j \partial \theta_l} \Big|_{\theta=0} \cdot \frac{\partial}{\partial z_m} = \sum_{l,m} \eta_{lm} \cdot \frac{\partial^2 H}{\partial \theta_i \partial \theta_l} \Big|_{\theta=0} \cdot \frac{\partial}{\partial z_m}. \tag{92}$$

The following result shows that this operator intertwines the Joyce form g with the skew-symmetric form η .

Lemma 7.3. *The operator V is covariantly constant with respect to the linear Joyce connection, and skew-adjoint with respect to the Joyce form. There is a commutative diagram*

$$\begin{array}{ccc} T & \xrightarrow{V} & T \\ & \searrow^{X \mapsto g(X,-)} & \nearrow^{\gamma \mapsto \eta(\gamma,-)} \\ & & T^* \end{array} \tag{93}$$

Proof. The commutativity of the diagram (93) follows by comparing the expressions (90) and (92). The fact that V is covariantly constant with respect to the linear Joyce connection is then immediate from Lemmas 7.1 and 7.2. For skew-adjointness note that

$$g\left(V\left(\frac{\partial}{\partial z_i}\right), \frac{\partial}{\partial z_j}\right) = \sum_{l,m} \eta_{lm} \cdot \frac{\partial^2 H}{\partial \theta_i \partial \theta_l} \Big|_{\theta=0} \cdot \frac{\partial^2 H}{\partial \theta_m \partial \theta_j} \Big|_{\theta=0},$$

which clearly changes sign when i and j are exchanged. \square

Remark 7.4. In terms of the Fourier expansion (79) we can formally write the linear Joyce connection as

$$\nabla_X^J(Y) = \nabla_X(Y) + \sum_{\gamma \in \Gamma \setminus \{0\}} H(\gamma) \cdot \frac{X(\gamma)Y(\gamma)}{Z(\gamma)} \cdot \eta(\gamma, -). \quad (94)$$

The Joyce form is then given by

$$g(X, Y) = \sum_{\gamma \in \Gamma \setminus \{0\}} H(\gamma) \cdot X(\gamma)Y(\gamma). \quad (95)$$

These expressions were first written down by Joyce, who also gave formal proofs of Lemma 7.1 and Lemma 7.2 in these terms. One can of course write a similar expression

$$V(X) = \sum_{\gamma \in \Gamma \setminus \{0\}} H(\gamma) \cdot X(\gamma) \cdot \eta(\gamma, -),$$

and prove Lemma 7.3 in the same style.

7.4. Compatibility of Joyce and Frobenius structures. There is clear evidence of non-trivial links between Frobenius structures and spaces of stability conditions [9, 8, 13, 28, 29]. The linear data derived from Joyce connections appears to be relevant to this story, although it is perhaps too early to be sure about this. In this section we discuss a compatibility relation for Frobenius and Joyce structures having the same underlying complex manifold, which seems to capture a non-trivial feature of several interesting examples.

Consider again a Joyce structure on a complex manifold M whose Joyce function extends holomorphically over the zero-section. Suppose also that the Joyce form g is non-degenerate. We can then define a bilinear operation $\diamond: \mathcal{T}_M \times \mathcal{T}_M \rightarrow \mathcal{T}_M$ on the tangent space by writing

$$g\left(\frac{\partial}{\partial z_i} \diamond \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right) = \frac{\partial^3 J}{\partial \theta_i \partial \theta_j \partial \theta_k} \Big|_{\theta=0} = g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \diamond \frac{\partial}{\partial z_k}\right). \quad (96)$$

We say that the Joyce structure satisfies the *Frobenius condition* if this operation is associative. This is not the case in general (see Section 8.7 below), but it does seem to hold in a number of interesting examples arising from DT theory. Note that since

$$g(E \diamond X_1, X_2) = m_{X_1} m_{X_2} m_E(J)|_{\theta=0} = m_{X_1} m_{X_2}(H) = g(X_1, X_2),$$

the Euler vector field E defines a unit for the operation \diamond . Thus when the Frobenius condition is satisfied, the pair (g, \diamond) induces the structure of a commutative and unital Frobenius algebra

on each tangent space $\mathcal{T}_{M,p}$. Note that in the special case when we can define a prepotential $\mathcal{F}: M \rightarrow \mathbb{C}$ we can write

$$g(X_1 \diamond X_2, X_3) = X_1 X_2 X_3(\mathcal{F}) = g(X_1, X_2 \diamond X_3).$$

Let us call a Frobenius structure *discriminant-free* if the operator $U(X) = E * X$ is everywhere invertible. Given such a Frobenius structure we can define a twisted multiplication \diamond on the tangent bundle by writing

$$E * (X \diamond Y) = X * Y. \tag{97}$$

It follows immediately from this definition that \diamond is commutative and associative, and that the Euler vector field defines a unit.

Definition 7.5. Suppose that a complex manifold M is equipped with both a discriminant-free Frobenius structure, and a Joyce structure whose Joyce function extends holomorphically over the zero-section. We call the two structures *compatible* if the following conditions hold:

- (i) the Joyce form coincides with the metric of the Frobenius structure up to scale:

$$g(X, Y)_{Joyce} = \lambda \cdot g(X, Y)_{Frob}, \quad \lambda \in \mathbb{C}^*; \tag{98}$$

- (ii) the diamond multiplication of the Joyce structure coincides with the twisted multiplication of the Frobenius structure up to scale:

$$X \diamond_{Joyce} Y = \mu \cdot X \diamond_{Frob} Y, \quad \mu \in \mathbb{C}^*. \tag{99}$$

We shall see some examples where the conditions of Definition 7.5 hold below.

Remark 7.6. An intriguing question is whether there exists some natural construction which, given a tame Frobenius structure, produces a compatible Joyce structure with the same underlying manifold.

7.5. Consequences of compatibility. The following result lists some simple consequences of the compatibility relation of Definition 7.5. Part (e) involves the *odd periods* of a Frobenius manifold [21, Definition 8], which are by definition the flat co-ordinates for the second structure connection of [26, Section 9.2] with parameter $s = \frac{1}{2}$.

Proposition 7.7. *Suppose given a discriminant-free Frobenius structure of conformal dimension d on a complex manifold M , and a compatible Joyce structure. Then $d \neq 2$, and the following conditions hold:*

(a) the scaling factor μ in (99) is $\mu = (2 - d)/2$, so that

$$X \diamond_{\text{Joyce}} Y = \frac{2 - d}{2} \cdot X \diamond_{\text{Frob}} Y;$$

(b) the Euler vector fields of the Frobenius and Joyce structures coincide up to scale:

$$E_{\text{Joyce}} = \frac{2}{2 - d} \cdot E_{\text{Frob}};$$

(c) the operator V defined in Section 7.3 coincides up to scale with the operator V of the Frobenius manifold defined by (16):

$$V_{\text{Joyce}} = \frac{2}{d - 2} \cdot V_{\text{Frob}};$$

(d) the linear Joyce connection coincides with the Levi-Civita connection of the Frobenius structure;

(e) the flat co-ordinates for the connection ∇ of the Joyce structure are the odd periods of the Frobenius structure.

Proof. Since the Euler vector fields E_{Joyce} and E_{Frob} are the identities for the products \diamond_{Joyce} and \diamond_{Frob} respectively, it follows that $E_{\text{Joyce}} = \mu^{-1} \cdot E_{\text{Frob}}$, where μ is the constant of (99). Consider the relations

$$\mathcal{L}ie_{E_{\text{Joyce}}}(g_{\text{Joyce}}) = 2g_{\text{Joyce}}, \quad \mathcal{L}ie_{E_{\text{Frob}}}(g_{\text{Frob}}) = (2 - d) \cdot g_{\text{Frob}}.$$

The first follows from the fact that the homogeneity condition (50), which implies that the function H is homogeneous of degree 0 in the variables z_i , and the second is the definition of the conformal dimension d . Comparing these equations shows that $\mu = (2 - d)/2$ which proves parts (a) and (b).

Definition 7.5(i) ensures that the Joyce form is non-degenerate, so the linear Joyce connection ∇^J is the associated Levi-Civita connection. This gives (d). Part (c) then follows by comparing (16) and (91). Finally, note that using the definition (96) of the diamond product, together with the diagram (93), we can rewrite (87) as

$$\nabla_X^J(Y) = \nabla_X(Y) + V_{\text{Joyce}}(X \diamond_{\text{Joyce}} Y). \tag{100}$$

Part (e) then follows from the observation that this expression coincides

with the difference between the second structure connection and the Levi-Civita connection given in [21, Proposition 3.3] or [26, Definition 9.3]. \square

Remark 7.8. Given a compatible pair as in Definition 7.5, we can attempt to reconstruct the Frobenius structure from the Joyce structure. The main missing ingredient is the identity vector field $e: \mathcal{O}_M \rightarrow \mathcal{T}_M$. It follows from the axioms of a Frobenius manifold that e is an eigenvector of the operator V_{Frob} with eigenvalue $-d/2$, and is flat for the Levi-Civita connection. If we are given e , the formula (97) shows that $U(X) = X \diamond_{Frob} e$, and we can then reconstruct the product of the Frobenius structure by writing $X * Y = U(X \diamond_{Frob} Y)$. For more details see the discussions in [19, Proposition 3.5], [20, Lemma 3.3] and [27, Section 4.3].

Remark 7.9. It will be convenient to extend Definition 7.5 to include slight variations on the notion of a Frobenius structure known as *almost Frobenius structures* [21]. These structures have the same basic ingredients (i)–(iii) appearing in Definition 2.1, and satisfy all the axioms except (F2). Replacing this axiom there is a different condition on a vector field $e: \mathcal{O}_M \rightarrow \mathcal{T}_M$ which will play no role here. See [21, Definition 9] for details. Since an almost Frobenius structure has a metric and a product, we can extend Definition 7.5 in the obvious way to include this case. In fact there are some special features of almost Frobenius manifolds which simplify the results of Proposition 7.7. Firstly, the Euler vector field is the unit of the multiplication, so the non-degeneracy condition is automatic, and the diamond multiplication (97) coincides with the product $*$. Secondly, the gradient of the Euler vector field is a multiple of the identity [21, Equation (3.23)], so the operator V defined by (16) vanishes, and it follows that the odd periods coincide with the flat co-ordinates.

8. THE FINITE UNCOUPLED CASE

In this section we study the Riemann-Hilbert problems of Section 6.5 associated to BPS structures (Γ, Z, Ω) which are finite, uncoupled and integral. Recall that this means that all BPS invariants $\Omega(\gamma)$ are integers, with only finitely many being nonzero, and

$$\Omega(\gamma_1) \neq 0, \Omega(\gamma_2) \neq 0 \implies \langle \gamma_1, \gamma_2 \rangle = 0.$$

These Riemann-Hilbert problems were first solved by Barbieri [3], after a special case was studied in [12]. We shall follow Barbieri's treatment closely in this section.

8.1. The A_1 BPS structure. Let us begin by considering the following basic example of a finite, uncoupled, integral BPS structure, determined by an element $z \in \mathbb{C}^*$:

- (i) the lattice $\Gamma = \mathbb{Z} \cdot \gamma$ has rank one, and thus necessarily $\langle -, - \rangle = 0$;
- (ii) the central charge $Z: \Gamma \rightarrow \mathbb{C}$ is determined by $Z(\gamma) = z \in \mathbb{C}^*$;

(iii) the only non-vanishing BPS invariants are $\Omega(\pm\gamma) = 1$.

This family of BPS structures arises naturally from the DT theory of the A_1 quiver, consisting of one vertex and no arrows. It is easily checked that they fit together to form a miniversal variation of BPS structures over the manifold \mathbb{C}^* . The local system of lattices is the trivial one $\Gamma \times \mathbb{C}^*$.

We will always take the unique quadratic refinement $\sigma: \Gamma \rightarrow \{\pm 1\}$ satisfying $\sigma(\gamma) = -1$, and use the corresponding isomorphism ρ_σ of (72) to relate the two tori \mathbb{T}_\pm . Explicitly this means that given a point $\xi \in \mathbb{T}_-$ we write

$$x_\gamma(\xi) = \exp(\vartheta), \quad y_\gamma(\rho_\sigma^{-1}(\xi)) = \exp(\theta), \quad \theta = \vartheta + \pi i, \quad (101)$$

where $y_\gamma: \mathbb{T}_+ \rightarrow \mathbb{C}^*$ (respectively $x_\gamma: \mathbb{T}_- \rightarrow \mathbb{C}^*$) denotes the character (respectively twisted character) corresponding to the element $\gamma \in \Gamma$.

We would like to construct a Joyce function whose Stokes data is given by this variation of BPS structures. As explained in Section 6.7, most features of this structure are automatic. The underlying manifold is \mathbb{C}^* , with the flat connection ∇ on the tangent bundle \mathcal{T}_M being the one in which the standard co-ordinate $z \in \mathbb{C}^*$ is flat. The bundle of lattices $\Gamma_M \subset \mathcal{T}_M^*$ is spanned by dz , and the form $\eta = 0$. The Euler vector field is $E = z \cdot \frac{\partial}{\partial z}$. What remains is to construct the Joyce function J on the space \mathcal{T}_M as a function of the co-ordinates (z, θ) .

Consider the formula (78) in this case. Since the Lie algebra \mathfrak{g}^{od} is abelian, and the expression on the right is a sum of commutators, all terms with $n > 1$ cancel. Consulting [10] we find that

$$H(\gamma) = -F_1(z) \cdot \text{DT}(\gamma) = -\frac{1}{2\pi i} \cdot \text{DT}(\gamma),$$

for all $\gamma \in \Gamma$. We therefore obtain from the expression (59), and the formula (62) for the DT invariants

$$H(z, \theta) = -\frac{1}{2\pi i} \cdot \sum_{n \geq 1} \frac{1}{n^2} (e^{n\vartheta} + e^{-n\vartheta}) = \frac{1}{2\pi i} \cdot \left(\frac{\theta^2}{2} + \frac{\pi^2}{6} \right), \quad (102)$$

where we used the Fourier series identity

$$\sum_{n \geq 1} \frac{\cos(nt)}{n^2} = \frac{(t + \pi)^2}{4} - \frac{\pi^2}{12},$$

valid for $-\pi < \text{Re}(t + \pi) < \pi$.

The defining relation (58) for the function H then gives the Joyce function

$$J(z, \theta) = \frac{1}{2\pi i} \cdot \frac{\theta^3}{6z},$$

where we neglect functions linear in θ . Note that we made a slightly arbitrary choice to sum the series (102) under the assumption $-\pi < \text{Im}(\theta) < \pi$. If we choose a different interval we will change the Joyce function by a function quadratic in θ . But the choice we made is in a sense the right one, since it results in a Joyce function which is an odd function of θ .

8.2. Riemann-Hilbert problem in the doubled A_1 case. Let us now attempt to calculate the Joyce function for the same BPS structures using the Riemann-Hilbert approach. In fact the Riemann-Hilbert problems associated to the A_1 BPS structures are trivial since the form $\langle -, - \rangle$ vanishes and so all wall-crossing automorphisms are the identity. Following the discussion in Section 6.8 we instead consider the Riemann-Hilbert problems associated to the doubles of these structures

$$(\Gamma_D, Z \oplus Z^\vee, \Omega), \quad \Gamma_D = \Gamma \oplus \Gamma^\vee. \quad (103)$$

Let $\gamma^\vee \in \Gamma^\vee$ be the unique generator satisfying $\gamma^\vee(\gamma) = 1$. Note that the definition (85) of the symplectic form $\langle -, - \rangle$ on the doubled lattice Γ_D implies that $\langle \gamma^\vee, \gamma \rangle = 1$. We set $z^\vee = Z^\vee(\gamma^\vee) \in \mathbb{C}$. We also extend the quadratic refinement $\sigma: \Gamma \rightarrow \{\pm 1\}$ of Section 8.1 to the doubled lattice Γ_D by setting $\sigma(\gamma^\vee) = 1$.

To define the Riemann-Hilbert problem for the doubled BPS structure we must first choose a constant term $\xi_D \in \mathbb{T}_{D,-}$, where $\mathbb{T}_{D,-}$ is the twisted torus defined by the lattice Γ_D and the form $\langle -, - \rangle$. We set

$$\xi_D(\gamma) = \exp(\vartheta), \quad \xi_D(\gamma^\vee) = \exp(\vartheta^\vee).$$

The only active rays are $\ell_\pm = \pm \mathbb{R}_{>0} \cdot z$. Since the functions $X_{r,\gamma}(\epsilon)$ have no discontinuities it is natural to take

$$X_{r,\gamma}(\epsilon) = x(\epsilon) = \exp\left(\vartheta - \frac{z}{\epsilon}\right), \quad (104)$$

for any non-active ray $r \subset \mathbb{C}^*$. If we were looking for holomorphic solutions to the Riemann-Hilbert problem this would be the only possibility [12, Lemma 4.8], but in the meromorphic context there are of course other choices. This is discussed further in Section 8.5 below. Let us also write

$$X_{r,\gamma^\vee}(\epsilon) = \exp\left(\vartheta^\vee - \frac{z^\vee}{\epsilon}\right) \cdot R_r(\epsilon).$$

The functions $R_r(\epsilon)$ for non-active rays $r \subset \mathbb{C}^*$ lying in the same component of $\mathbb{C}^* \setminus (\ell_- \cup \ell_+)$ are necessarily analytic continuations of each other. Thus we obtain just two meromorphic functions R_{\pm} , corresponding to non-active rays lying in the half-planes $\pm \operatorname{Im}(z/\epsilon) > 0$. The Riemann-Hilbert problems for the doubled BPS structures can therefore be restated as follows.

Problem 8.1. *For fixed $z \in \mathbb{C}^*$ and $\vartheta \in \mathbb{C}$, find meromorphic functions*

$$R_{\pm}: \mathbb{C}^* \setminus i\ell_{\pm} \rightarrow \mathbb{C}$$

with the following properties:

(i) *there are relations*

$$R_+(\epsilon) = \begin{cases} R_-(\epsilon) \cdot (1 - x(\epsilon)^{+1}) & \text{if } \epsilon \in \mathbb{H}_{\ell_+}, \\ R_-(\epsilon) \cdot (1 - x(\epsilon)^{-1}) & \text{if } \epsilon \in \mathbb{H}_{\ell_-}, \end{cases} \quad (105)$$

where $x(\epsilon) = \exp(\vartheta - z/\epsilon)$ is as in (104);

(ii) *as $\epsilon \rightarrow 0$ in a closed subsector of $\mathbb{C}^* \setminus i\ell_{\pm}$ we have*

$$R_{\pm}(\epsilon) \rightarrow 1;$$

(iii) *there exists $k > 0$ such that*

$$|\epsilon|^{-k} < |R_{\pm}(\epsilon)| < |\epsilon|^k$$

as $\epsilon \rightarrow \infty$ in $\mathbb{C}^* \setminus i\ell_{\pm}$.

To understand condition (i) note that if $\epsilon \in \mathbb{H}_{\ell_+}$, then ϵ lies in the domains of definition \mathbb{H}_{r_i} of the functions $X_{r_i, \gamma^{\vee}}$ corresponding to sufficiently small deformations r_i of the ray ℓ_+ . Thus (69) applies to the ray ℓ_+ and we obtain the first of the relations (105). The second follows similarly from (69) applied to the opposite ray ℓ_- .

8.3. Solution in the doubled \mathbf{A}_1 case. The formula

$$\Lambda(w, \eta) = \frac{e^w \cdot \Gamma(w + \eta)}{\sqrt{2\pi} \cdot w^{w+\eta-\frac{1}{2}}} \quad (106)$$

defines a meromorphic function of $w \in \mathbb{C}^*$ and $\eta \in \mathbb{C}$, which is multi-valued due to the factor

$$w^{w+\eta-\frac{1}{2}} = \exp\left((w + \eta - \frac{1}{2}) \log w\right). \quad (107)$$

We specify it uniquely for $w \in \mathbb{C}^* \setminus \mathbb{R}_{<0}$ by taking the principal branch of \log . Since $\Gamma(w)$ is meromorphic on \mathbb{C} with poles only at the non-positive integers, it follows that $\Lambda(w, \eta)$ has poles only at the points $w + \eta \in \mathbb{Z}_{\leq 0}$.

The Stirling expansion [41, Section 12.33] gives an asymptotic expansion

$$\log \Lambda(w, \eta) \sim \sum_{k=2}^{\infty} \frac{(-1)^k \cdot B_k(\eta)}{k(k-1)} w^{1-k}, \quad (108)$$

where $B_k(\eta)$ denotes the k th Bernoulli polynomial. This expansion is valid as $w \rightarrow \infty$ in the complement of any closed sector containing the ray $\mathbb{R}_{<0}$. It implies in particular that for fixed $\eta \in \mathbb{C}$ we have $\Lambda(w, \eta) \rightarrow 1$.

Proposition 8.2. *Problem 8.1 has a solution, namely*

$$R_{\pm}(\epsilon) = \Lambda\left(\frac{\pm z}{2\pi i \epsilon}, \frac{\pi i \mp \theta}{2\pi i}\right)^{\mp 1},$$

where we set $\theta = \vartheta + \pi i$ as in (101).

Proof. Note that the function $R_+(\epsilon)$ is well-defined and meromorphic on the domain $\mathbb{C}^* \setminus i\ell_+$ as required, because if we set $w = z/2\pi i \epsilon$ then

$$\epsilon \in i\ell_+ \iff w \in \mathbb{R}_{<0}.$$

Similarly $R_-(\epsilon)$ is well-defined and meromorphic on $\mathbb{C}^* \setminus i\ell_-$. The Euler reflection formula states that

$$\Gamma(x) \cdot \Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \quad x \in \mathbb{C} \setminus \mathbb{Z}.$$

Combined with the formula $\Gamma(n) = (n-1)!$ valid for $n \in \mathbb{Z}_{>0}$, it follows that $\Gamma(x)$ is nowhere vanishing. The same is therefore true of the function $\Lambda(w, \eta)$. The reflection formula also implies that for $w \in \mathbb{C}^* \setminus \mathbb{R}_{<0}$

$$\Lambda(w, \eta) \cdot \Lambda(-w, 1-\eta) = \frac{1}{2\pi} \cdot \frac{\pi}{\sin(\pi(w+\eta))} \cdot e^{-((w+\eta-\frac{1}{2})\log(w) + (-w-\eta+\frac{1}{2})\log(-w))}.$$

The principal branch of $\log(w)$ on the domain $\mathbb{C}^* \setminus \mathbb{R}_{<0}$ satisfies

$$\log(w) = \log(-w) \pm \pi i \text{ when } \pm \operatorname{Im}(w) > 0,$$

so we conclude that when $\pm \operatorname{Im}(w) > 0$

$$\Lambda(w, \eta) \cdot \Lambda(-w, 1 - \eta) = \frac{i \cdot e^{\mp \pi i(w + \eta - \frac{1}{2})}}{e^{\pi i(w + \eta)} - e^{-i\pi(w + \eta)}} = (1 - e^{\pm 2\pi i(w + \eta)})^{-1}. \quad (109)$$

Let us now set $w = z/2\pi i\epsilon$. This implies that

$$\epsilon \in \mathbb{H}_{\ell_{\mp}} \iff \pm \operatorname{Im}(w) > 0.$$

Setting $\eta = (\pi i - \vartheta)/2\pi i = -\vartheta/2\pi i$, and recalling (104), the relation (109) becomes

$$R_+(\epsilon)^{-1} \cdot R_-(\epsilon) = (1 - x(\epsilon)^{\mp 1})^{-1},$$

when $\epsilon \in \mathbb{H}_{\ell_{\mp}}$, which is equivalent to the relations (105).

Condition (ii) in the Problem 8.1 is immediate from the Stirling expansion (108). Condition (iii) is easily checked: for fixed η the term (107) clearly contributes at worst polynomial growth as $w \rightarrow 0$, and since $\Gamma(w)$ is meromorphic at $w = 0$, the same is true of $\Lambda(w, \eta)$. \square

In the special case $\vartheta = 0$ it follows from the identity $\Lambda(w, 0) = \Lambda(w, 1)$ that the solution of Proposition 8.2 reproduces the one given in [12, Section 5].

8.4. Computing the Joyce function. To compute the Joyce function from the solution of Proposition 8.2 we use the formula

$$\frac{\partial x_j}{\partial z_i} + \frac{1}{\epsilon} \cdot \frac{\partial x_j}{\partial \theta_i} = \sum_{p,q} \eta_{pq} \cdot \frac{\partial^2 J}{\partial \theta_i \partial \theta_p} \cdot \frac{\partial x_j}{\partial \theta_q}. \quad (110)$$

derived in Section 6.7. Plugging the trivial part of the solution

$$x_{\gamma, \pm}(\epsilon) = \log X_{\gamma, \pm}(\epsilon) = \vartheta - \frac{z}{\epsilon},$$

into this equation immediately gives

$$\frac{\partial^2 J}{\partial \theta^\vee \partial \theta^\vee} = 0 = \frac{\partial^2 J}{\partial \theta \partial \theta^\vee}.$$

Since we assume that $J(z, z^\vee, \theta, \theta^\vee)$ is an odd function of the variables θ, θ^\vee , and we neglect linear terms in these variables, we can assume that J is independent of θ^\vee .

Note that this fits perfectly with discussion of Section 6.8: we should interpret the solution of the Riemann-Hilbert problem for the doubled BPS structure (103) as describing the double of the Joyce structure corresponding to the original BPS structure (Γ, Z, Ω) . It is therefore to be expected that the Joyce function is independent of the doubling variables z^\vee and θ^\vee .

The non-trivial part of our solution to the Riemann-Hilbert problem is

$$x_{\pm, \gamma^\vee}(\epsilon) = \log X_{\pm, \gamma^\vee}(\epsilon) = \vartheta^\vee - \frac{z^\vee}{\epsilon} \mp \log \Lambda\left(\frac{\pm z}{2\pi i \epsilon}, \frac{\pi i \mp \theta}{2\pi i}\right).$$

Plugging this into (110) gives

$$\frac{\partial^2 J}{\partial \theta^2} = \pm \left(\frac{\partial}{\partial z} + \frac{1}{\epsilon} \cdot \frac{\partial}{\partial \theta} \right) \log \Lambda\left(\frac{\pm z}{2\pi i \epsilon}, \frac{\pi i \mp \theta}{2\pi i}\right) = \frac{1}{2\pi i \epsilon} \left(\frac{\partial}{\partial w} - \frac{\partial}{\partial \eta} \right) \log \Lambda(w, \eta),$$

where, as before, we set $w = z/2\pi i \epsilon$ and $\eta = (\pi i - \theta)/2\pi i = -\vartheta/2\pi i$. By the definition of $\Lambda(w, \eta)$ we therefore obtain

$$\frac{\partial^2 J}{\partial \theta^2} = \frac{1}{2\pi i \epsilon} \left(1 - \frac{w + \eta - \frac{1}{2}}{w} \right) = \frac{\vartheta + \pi i}{2\pi i z} = \frac{\theta}{2\pi i z}.$$

Since the Joyce function is an odd function of θ , we conclude that up to linear terms in θ

$$J(z, \theta) = \frac{\theta^3}{12\pi i z}, \quad (111)$$

in agreement with what we found in Section 8.1.

8.5. Remarks on uniqueness. The solution of Proposition 8.2 is not unique with the stated properties. More precisely, it is unique only up to multiplication of both $R_{\pm}(\epsilon)$ by a meromorphic function $f: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ satisfying $f(0) = 1$. We can further constrain the solution by insisting that the zeroes and poles of $R_{\pm}(\epsilon)$ lie only at the points

$$p_n = \frac{z}{\vartheta + 2\pi i n}, \quad n \in \mathbb{Z}, \quad (112)$$

where the factors appearing in (105) have zeroes or poles. The solution of Proposition 8.2 is then unique with this property up to multiplication of both $R_{\pm}(\epsilon)$ by factors of the form

$$\left(1 - \frac{\epsilon(\vartheta + 2\pi i n)}{z} \right)^{\pm 1}.$$

Note that inserting such factors adds only quadratic functions in θ to the Joyce function, since

$$\left(\frac{\partial}{\partial z} + \frac{1}{\epsilon} \cdot \frac{\partial}{\partial \theta} \right) \log \left(1 - \frac{\epsilon(\vartheta + 2\pi i n)}{z} \right) = -\frac{1}{z}.$$

The particular solution of Proposition 8.2 is distinguished by the property that $R_+(\epsilon)$ is holomorphic and nonzero at the point p_n for all $n > 0$. But note that the labelling of the

p_n depends on a choice of $\vartheta = \log(\xi_D(\gamma))$, and hence cannot be chosen consistently as the constant term $\xi \in \mathbb{T}_{D,-}$ varies.

Remark 8.3. The Riemann-Hilbert problem of [12, Section 5] asks for functions $R_{\pm}(\epsilon)$ exactly as in Problem 8.1, but which are holomorphic and everywhere non-vanishing on their domains. It is easy to see [12, Lemma 4.9] that if a solution to this more constrained problem exists then it is unique. But since the factors appearing in (105) have zeroes or poles at the points (112), such a solution is only possible if all the points (112) lie on the rays $i\ell_{\pm}$, which is only the case if $\vartheta \in i\mathbb{R}$, or equivalently if $|\xi_D(\gamma)| = 1$. Given this, it is easy to see that the unique holomorphic, non-vanishing solution is the one given by Proposition 8.2, with θ chosen to satisfy $-\pi < \text{Im}(\theta) \leq \pi$.

8.6. Compatibility with the \mathbf{A}_1 Frobenius manifold. Let us now consider the linear data of Section 7 in the case of the Joyce structure of Section 8.1. Since the form $\eta = 0$, the Joyce connection coincides with the standard connection ∇ , and hence $z \in \mathbb{C}^*$ is a flat co-ordinate for this connection. The gradient of the Euler vector field $E = z \frac{\partial}{\partial z}$ is then the identity, which implies that the operator V vanishes. The Joyce form is

$$g\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) = \frac{\partial^2 H}{\partial \theta^2} \Big|_{\theta=0} = \frac{1}{2\pi i},$$

which in particular is non-degenerate. The diamond multiplication (96) is then

$$\frac{\partial}{\partial z} \diamond \frac{\partial}{\partial z} = \frac{1}{z} \cdot \frac{\partial}{\partial z}.$$

A possible choice of prepotential is

$$\mathcal{F}(z) = \frac{1}{2} z^2 \log(z).$$

Let us compare these structures with the trivial one-dimensional Frobenius manifold [19, Example I.1.5]. This has flat co-ordinate $t \in \mathbb{C}$ and is defined by the data

$$g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = 1, \quad \frac{\partial}{\partial t} * \frac{\partial}{\partial t} = \frac{\partial}{\partial t}, \quad E = t \cdot \frac{\partial}{\partial t}.$$

Since $\mathcal{L}ie_E(g) = 2g$, the conformal dimension is $d = 0$. The discriminant-free condition holds for $t \in \mathbb{C}^*$, and the corresponding twisted multiplication is

$$\frac{\partial}{\partial t} \diamond \frac{\partial}{\partial t} = \frac{1}{t} \cdot \frac{\partial}{\partial t}.$$

The conditions of Definition 7.5 clearly hold, so this Frobenius structure is compatible with the Joyce structure of Section 8.1. Note that in this case $V = 0$, so that the odd periods coincide with the flat co-ordinates.

For comparison with the material of Section 8.8 it is also worth introducing the almost dual of the above trivial Frobenius structure. The underlying manifold of this almost dual structure is \mathbb{C}^* , and the basic ingredients [21, Example 1] are

$$g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \frac{1}{t}, \quad \frac{\partial}{\partial t} * \frac{\partial}{\partial t} = \frac{1}{t} \cdot \frac{\partial}{\partial t}, \quad E = t \cdot \frac{\partial}{\partial t}.$$

In terms of the flat co-ordinate $x = 2\sqrt{t}$ this takes the form

$$g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = 1, \quad \frac{\partial}{\partial x} * \frac{\partial}{\partial x} = \frac{2}{x} \cdot \frac{\partial}{\partial x}, \quad E = \frac{x}{2} \cdot \frac{\partial}{\partial x}.$$

As explained in Remark 7.9, the Euler vector field is the identity for the multiplication in this almost dual structure, so the diamond product coincides with the ordinary product. It follows that, in the extended sense of Remark 7.9, this almost Frobenius structure is also compatible with the Joyce structure of Section 8.1.

8.7. The general finite uncoupled case. Consider now an arbitrary finite, integral and uncoupled BPS structure (Z, Γ, Ω) . The uncoupled condition ensures that the BPS automorphisms associated to different rays commute, which means that we can solve the Riemann-Hilbert problem by superposing solutions to the A_1 Riemann-Hilbert problem. This is explained in detail in [12, Section 5.3]. There is no canonical choice of quadratic refinement, but this choice does not effect the Joyce function.

We thus conclude that the Joyce function $J(Z, \theta)$ corresponding to (Z, Γ, Ω) is

$$J(Z, \theta) = \frac{1}{12\pi i} \cdot \sum_{\alpha \in \Gamma} \Omega(\alpha) \cdot \frac{\theta(\alpha)^3}{Z(\alpha)}, \quad (113)$$

up to linear terms in θ . The Joyce form is

$$g(X, Y) = \frac{1}{2\pi i} \cdot \sum_{\alpha \in \Gamma} \Omega(\alpha) \cdot X(\alpha)Y(\alpha),$$

and is therefore constant. The Joyce connection is then

$$\nabla_X^J(Y) = \frac{1}{2\pi i} \cdot \sum_{\alpha \in \Gamma} \Omega(\alpha) \cdot \frac{X(\alpha)Y(\alpha)}{Z(\alpha)} \cdot \langle \alpha, - \rangle.$$

The prepotential $\mathcal{F}: M \rightarrow \mathbb{C}$ in the sense of Section 7.1 is given by the formula

$$\mathcal{F}(Z) = \frac{1}{4\pi i} \cdot \sum_{\alpha \in \Gamma} \Omega(\alpha) \cdot Z(\alpha)^2 \log Z(\alpha).$$

When the Joyce form is non-degenerate the diamond product (96) is defined by

$$g(X_1 \diamond X_2, X_3) = \sum_{\alpha \in \Gamma} \Omega(\alpha) \cdot \frac{X_1(\alpha)X_2(\alpha)X_3(\alpha)}{Z(\alpha)} = g(X_1, X_2 \diamond X_3).$$

Recall that the Joyce structure is said to satisfy the Frobenius condition if this operation is associative. Since

$$g(X_1 \diamond X_2, X_3) = X_1 X_2 X_3(\mathcal{F}) = g(X_1, X_2 \diamond X_3)$$

this is equivalent to the condition that the function \mathcal{F} satisfies the WDVV equations with respect to the metric g . In the special case when all $\Omega(\gamma) \in \{0, 1\}$ this happens precisely when the set of active classes $\gamma \in \Gamma$ form a \vee -system in the sense of Veselov (see [39, Theorem 1]). It would be very interesting to understand whether there is some natural class of CY_3 triangulated categories whose associated DT theory always leads to Joyce structures satisfying the Frobenius condition.

8.8. Almost Frobenius manifolds of ADE type. In this section we discuss a particularly interesting class of examples of finite, uncoupled and integral BPS structures. These examples have an ADE classification, and arise from DT theory applied to non-compact Calabi-Yau threefolds which are the total spaces of smoothings of Kleinian surface singularities.

Let us fix a finite-dimensional simply-laced simple Lie algebra \mathfrak{g} with a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and denote by $\mathfrak{h}^{\text{reg}} \subset \mathfrak{h}$ the complement of the complex root hyperplanes. The corresponding BPS structures depend on a point $Z \in \mathfrak{h}^{\text{reg}}$ and consist of the following data:

- (i) the lattice $\Gamma = \mathbb{Z}\Phi$ is the root lattice of \mathfrak{g} , equipped with the zero form $\langle -, - \rangle = 0$;
- (ii) the central charge $Z: \Gamma \rightarrow \mathbb{C}$ corresponds to an element $Z \in \mathfrak{h}^{\text{reg}} \subset \mathfrak{h}$;
- (iii) the BPS invariants are

$$\Omega(\gamma) = \begin{cases} 1 & \text{if } \gamma \in \Phi \subset \mathbb{Z}\Phi \text{ is a root,} \\ 0 & \text{otherwise.} \end{cases}$$

Since the form $\langle -, - \rangle$ vanishes, and the BPS invariants are constant, this defines a variation of BPS structures over the quotient manifold $\mathfrak{h}^{\text{reg}}$. Moreover, since all ingredients are invariant under the action of the Weyl group W , this variation descends to the quotient manifold $\mathfrak{h}^{\text{reg}}/W$.

Applying the formula (113) defines a Joyce structure on $\mathfrak{h}^{\text{reg}}$ which is invariant under W and hence descends to the quotient. The corresponding Joyce form is the constant metric

$$g(X_1, X_2) = \frac{1}{2\pi i} \cdot \sum_{\alpha \in \Phi} X_1(\alpha) X_2(\alpha),$$

which, up to a constant factor, coincides with the Killing form of the Lie algebra \mathfrak{g} restricted to the Cartan subalgebra \mathfrak{h} . In particular g is non-degenerate. The diamond product (96) is defined by

$$g(X_1 \diamond X_2, X_3) = \sum_{\alpha \in \Phi} \frac{X_1(\alpha) X_2(\alpha) X_3(\alpha)}{Z(\alpha)} = g(X_1, X_2 \diamond X_3). \quad (114)$$

In [21, Section 5.2] Dubrovin studies the almost dual Frobenius structure on $\mathfrak{h}^{\text{reg}}$ obtained from the polynomial Frobenius structure on \mathfrak{h} . The flat co-ordinates are the linear co-ordinates on \mathfrak{h} . The metric is constant in these co-ordinates, and invariant under the action of the Weyl group, and hence coincides with the Killing form up to a constant factor. Comparing the formula [21, Equation (5.21)] with (114) shows that the diamond product also coincides with the product of the almost Frobenius structure up to scale. Thus the two structures are compatible in the extended sense of Remark 7.9.

9. THE CASE OF CALABI-YAU THREEFOLDS WITHOUT COMPACT DIVISORS

In this section we consider the case of BPS structures arising from DT theory applied to categories of coherent sheaves on quasi-projective Calabi-Yau threefolds which contain no compact divisors. In this situation all compactly-supported sheaves are supported in dimension at most one, and the DT invariants for such sheaves are conjecturally determined by the genus 0 Gopakumar-Vafa invariants. Assuming this conjecture we can use the methods of [14] to solve the associated Riemann-Hilbert problems, and explicitly describe the corresponding Joyce structures.

9.1. Conifold case. In the finite, uncoupled situation discussed in the previous section it turned out to be enough to understand a single basic example, namely the A_1 BPS structure, all other cases being obtained by superposing copies of this one. In the geometric set-up discussed in this section a similar role is played by the BPS structure associated to the resolved conifold. We shall consider these BPS structures as being parameterised a point in the space

$$M_+ = \{(v, w) \in \mathbb{C}^2 : w \neq 0 \text{ and } \text{Im}(v/w) > 0\} \subset \mathbb{C}^2.$$

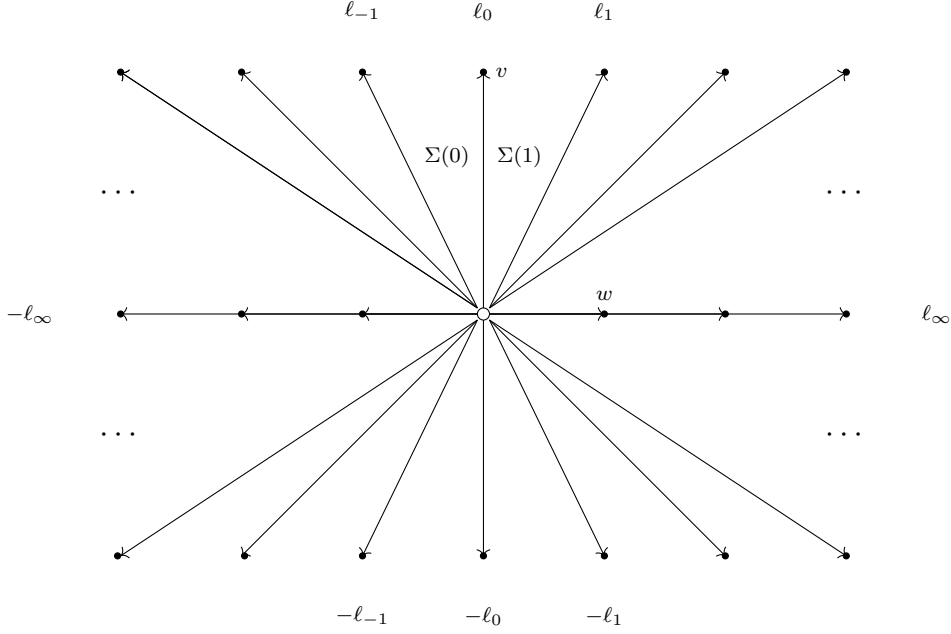


FIGURE 1. The ray diagram associated to a point $(v, w) \in M_+$.

In fact, as explained in [14], we can extend this variation of BPS structures to a larger space, but this will not be important here.

The BPS structure $(\Gamma_{\leq 1}, Z_{\leq 1}, \Omega)$ corresponding to a point $(v, w) \in M_+$ is determined by the following data:

- (i) the lattice $\Gamma_{\leq 1} = \mathbb{Z}\beta \oplus \mathbb{Z}\delta$ with the form $\langle -, - \rangle = 0$;
- (ii) the central charge $Z_{\leq 1}: \Gamma_{\leq 1} \rightarrow \mathbb{C}$ defined by

$$Z_{\leq 1}(a\beta + b\delta) = av + bw;$$

- (iii) the non-zero BPS invariants

$$\Omega(\gamma) = \begin{cases} 1 & \text{if } \gamma = \pm\beta + n\delta \text{ with } n \in \mathbb{Z}, \\ -2 & \text{if } \gamma = k\delta \text{ with } k \in \mathbb{Z} \setminus \{0\}. \end{cases} \quad (115)$$

Together these structures form a miniversal variation of uncoupled BPS structures over M_+ . The BPS invariants are constant because the form $\langle -, - \rangle$ vanishes. Define rays

$$\ell_\infty = \mathbb{R}_{>0} \cdot w, \quad \ell_n = \mathbb{R}_{>0} \cdot (v + nw) \subset \mathbb{C}^*.$$

Then the active rays for the BPS structure $(\Gamma_{\leq 1}, Z_{\leq 1}, \Omega)$ are precisely the rays $\pm\ell_\infty$ and $\pm\ell_n$ for $n \in \mathbb{Z}$, and the associated ray diagram is illustrated in Figure 1. We denote by $\Sigma(n) \subset \mathbb{C}^*$ the convex sector with boundary rays ℓ_{n-1} and ℓ_n .

As in Section 8 we shall also consider the doubles of these BPS structures. We denote these by (Γ, Z, Ω) and use the notation

$$\Gamma = \Gamma_{\leq 1} \oplus \Gamma_{\geq 2}, \quad \Gamma_{\geq 2} := \Gamma_{\leq 1}^{\vee} = \text{Hom}_{\mathbb{Z}}(\Gamma_{\leq 1}, \mathbb{Z}).$$

We denote by $(\beta^{\vee}, \delta^{\vee}) \subset \Gamma_{\geq 2}$ the dual basis to $(\beta, \delta) \subset \Gamma_{\leq 1}$. The map $\Omega: \Gamma \rightarrow \mathbb{Q}$ satisfies $\Omega(\gamma) = 0$ unless $\gamma \in \Gamma_{\leq 1}$. The central charge takes the form

$$Z = Z_{\leq 1} \oplus Z_{\geq 2}: \Gamma \rightarrow \mathbb{C},$$

where the group homomorphism $Z_{\geq 2}: \Gamma_{\geq 2} \rightarrow \mathbb{C}$ is arbitrary. We set

$$Z(\beta^{\vee}) = v^{\vee}, \quad Z(\delta^{\vee}) = w^{\vee}.$$

In this way we obtain a miniversal variation of uncoupled and integral BPS structures over the cotangent bundle \mathcal{T}^*M_+ . These BPS structures are not finite, but are easily seen to be convergent in the sense of Definition 5.2.

The Riemann-Hilbert problem associated to the BPS structure (Z, γ, Ω) depends on a point of the twisted torus \mathbb{T}_- associated to the doubled lattice Γ . We set

$$\xi(\beta) = \exp(\vartheta), \quad \xi(\delta) = \exp(\varphi), \quad \xi(\beta^{\vee}) = \exp(\vartheta^{\vee}), \quad \xi(\delta^{\vee}) = \exp(\varphi^{\vee}).$$

The solution consists of meromorphic functions $X_{r,\gamma}: \mathbb{H}_r \rightarrow \mathbb{C}^*$ for all non-active rays $r \subset \mathbb{C}^*$ and all classes $\gamma \in \Gamma$. The solutions corresponding to different non-active rays contained in the same sector $\Sigma(n)$ are analytic continuations of each other, and hence define a function $X_{n,\gamma}(\epsilon)$ on the region

$$\mathcal{V}(n) = \mathbb{H}_{\ell_{n-1}} \cup \mathbb{H}_{\ell_n} \subset \mathbb{C}^*.$$

The natural symmetries of the problem (see [14, Remark 3.1]) ensure that there is no need to consider the corresponding functions for non-active rays $r \subset \mathbb{C}^*$ lying in the opposite sectors $-\Sigma(n)$.

As in Section 8.2 it is natural to set

$$X_{n,\beta}(\epsilon) = \exp\left(\vartheta - \frac{v}{\epsilon}\right), \quad X_{n,\delta}(\epsilon) = \exp\left(\varphi - \frac{w}{\epsilon}\right), \quad (116)$$

for all integers $n \in \mathbb{Z}$. We then write the non-trivial part of the solution in the form

$$X_{n,\beta^{\vee}}(\epsilon) = \exp\left(\vartheta^{\vee} - \frac{v^{\vee}}{\epsilon}\right) \cdot B_n(\epsilon), \quad X_{n,\delta^{\vee}}(\epsilon) = \exp\left(\varphi^{\vee} - \frac{w^{\vee}}{\epsilon}\right) \cdot D_n(\epsilon), \quad (117)$$

where we drop from the notation the dependence of the functions B_n and D_n on the parameters (z, w) and (ϑ, φ) . Working out the conditions imposed on the functions B_n and D_n we obtain the following explicit version of the Riemann-Hilbert problem:

Problem 9.1. *For each integer $n \in \mathbb{Z}$ find meromorphic functions $B_n(\epsilon)$ and $D_n(\epsilon)$ defined on the region $\mathcal{V}(n)$ with the following properties:*

(i) *as $\epsilon \rightarrow 0$ in any closed subsector of $\mathcal{V}(n)$ one has*

$$B_n(\epsilon) \rightarrow 1, \quad D_n(\epsilon) \rightarrow 1;$$

(ii) *for each $n \in \mathbb{Z}$ there exists $k > 0$ such that for any closed subsector of $\mathcal{V}(n)$*

$$|\epsilon|^{-k} < |B_n(\epsilon)|, |D_n(\epsilon)| < |\epsilon|^k, \quad |\epsilon| \gg 0;$$

(iii) *on the intersection $\mathcal{H}(n) = \mathcal{V}(n) \cap \mathcal{V}(n+1)$ there are relations*

$$B_{n+1}(\epsilon) = B_n(\epsilon) \cdot (1 - xq^n), \quad D_{n+1}(\epsilon) = D_n(\epsilon) \cdot (1 - xq^n)^n;$$

(iv) *note that $\mathcal{V}(0) \cap -\mathcal{V}(0) = i \cdot \Sigma(0) \sqcup -i \cdot \Sigma(0)$; in the region $-i \cdot \Sigma(0)$ there are relations*

$$B_0(\epsilon) \cdot B_0(-\epsilon) = \prod_{n \geq 0} (1 - xq^n)^{-1} \cdot \prod_{n \geq 1} (1 - x^{-1}q^n),$$

$$D_0(\epsilon) \cdot D_0(-\epsilon) = \prod_{n \geq 0} (1 - xq^n)^{-n} \cdot \prod_{n \geq 1} (1 - x^{-1}q^n)^{-n} \cdot \prod_{k \geq 1} (1 - q^k)^{2k},$$

where we used the notation

$$x = \exp\left(\vartheta - \frac{v}{\epsilon}\right), \quad q = \exp\left(\varphi - \frac{w}{\epsilon}\right). \quad (118)$$

In the case when $\vartheta = \varphi = 0$, Problem 9.1 coincides with [14, Problem 3.3] except that we have rescaled the parameters (v, w) by a factor of $2\pi i$, and more importantly, we are now allowing our solutions $B_n(\epsilon)$ and $D_n(\epsilon)$ to have zeroes and poles on the regions $\mathcal{V}(n)$. The argument explained in Remark 8.3 proves the necessity of this change: for general choices of ϑ and φ there can be no solution to Problem 9.1 in which $B_n(\epsilon)$ and $D_n(\epsilon)$ are holomorphic and non-vanishing.

9.2. Difference equations. Our variation of BPS structures carries a free action of \mathbb{Z} . This symmetry will allow us to restate the above Riemann-Hilbert problem as a system of coupled

difference equations. Consider the action of the group \mathbb{Z} on the lattice Γ , preserving the form $\langle -, - \rangle$, in which an element $m \in \mathbb{Z}$ acts via

$$(\beta, \delta) \mapsto (\beta - m\delta, \delta), \quad (\beta^\vee, \delta^\vee) \mapsto (\beta^\vee, \delta^\vee + m\beta^\vee).$$

This induces an action on the manifold \mathcal{T}^*M_+ by

$$(v, w) \mapsto (v + mw, w), \quad (v^\vee, w^\vee) \mapsto (v^\vee, w^\vee - mv^\vee).$$

Then the BPS structure at a point $Z \in \mathcal{T}^*M_+$, and the BPS structure at the point $m \cdot Z$ are related by pullback along the map $m: \Gamma \rightarrow \Gamma$. Moreover a point $\epsilon \in \mathbb{C}^*$ lies in the sector $\Sigma(n)$ for the BPS structure defined by the point (v, w) precisely if it lies in the sector $\Sigma(m+n)$ for the BPS structure defined by $(v - mw, w)$. A similar remark applies to the regions $\mathcal{H}(n)$.

The action of \mathbb{Z} on the lattice Γ induces an action on the corresponding twisted torus \mathbb{T}_- , and hence on the constant terms $\xi \in \mathbb{T}_-$. It is then natural to look for a solution to Problem 9.1 satisfying

$$B_n(v, w, \theta, \phi, \epsilon) = B_0(v + nw, w, \theta + n\phi, \phi, \epsilon), \quad (119)$$

$$D_n(v, w, \theta, \phi, v) = D_0(v + nw, w, \theta + n\phi, \phi, v) \cdot B_0(v + nw, w, \theta + n\phi, \phi, \epsilon)^n. \quad (120)$$

Assuming this ansatz, we can then restate Problem 9.1 in terms of just two functions $B = B_0$ and $D = D_0$.

Problem 9.2. *Find meromorphic functions $B(v, w, \vartheta, \varphi, \epsilon)$ and $D(v, w, \vartheta, \varphi, \epsilon)$ defined for $(v, w) \in M_+$, $(\vartheta, \varphi) \in \mathbb{C}^2$ and $\epsilon \in \mathcal{V}(0)$, such that for fixed $(v, w, \vartheta, \varphi)$ the following properties hold:*

(i) *one has*

$$B(v, w, \vartheta, \varphi, \epsilon) \rightarrow 1, \quad D(v, w, \vartheta, \varphi, \epsilon) \rightarrow 1,$$

as $\epsilon \rightarrow 0$ in any closed subsector of $\mathcal{V}(0)$;

(ii) *there exists $k > 0$ such that*

$$|\epsilon|^{-k} < |B(v, w, \vartheta, \varphi, \epsilon)|, |D(v, w, \vartheta, \varphi, \epsilon)| < |\epsilon|^k,$$

as $\epsilon \rightarrow \infty$ in any closed subsector of $\mathcal{V}(0)$;

(iii) *there are relations*

$$B(v + w, w, \vartheta + \varphi, \varphi, \epsilon) = B(v, w, \vartheta, \varphi, \epsilon) \cdot (1 - \epsilon),$$

$$D(v + w, w, \vartheta + \varphi, \varphi, \epsilon) = D(v, w, \vartheta, \varphi, \epsilon) \cdot B(v + w, w, \vartheta + \varphi, \varphi, \epsilon)^{-1},$$

for $\epsilon \in \mathbb{C}^$ lying in the intersection $\mathcal{V}(0) \cap \mathcal{V}(1) = \mathcal{H}(0)$;*

(iv) for $\epsilon \in -i \cdot \Sigma(0)$ there are relations

$$B(v, w, \vartheta, \varphi, \epsilon) \cdot B(v, w, -\vartheta, -\varphi, -\epsilon) = \prod_{n \geq 0} (1 - xq^n)^{-1} \cdot \prod_{n \geq 1} (1 - x^{-1}q^n),$$

$$D(v, w, \vartheta, \varphi, \epsilon) \cdot D(v, w, -\vartheta, -\varphi, -\epsilon) = \prod_{n \geq 0} (1 - xq^n)^{-n} \cdot \prod_{n \geq 1} (1 - x^{-1}q^n)^{-n} \cdot \prod_{k \geq 1} (1 - q^k)^{2k},$$

where we used the notation (118) as before.

9.3. Special functions. Recall from [14, Section 4] the following two functions defined by the integral representations

$$F(z | \omega_1, \omega_2) = \exp \left(\int_C \frac{e^{zs}}{(e^{\omega_1 s} - 1)(e^{\omega_2 s} - 1)} \frac{ds}{s} \right), \quad (121)$$

$$G(z | \omega_1, \omega_2) = \exp \left(\int_C \frac{-e^{(z+\omega_1)s}}{(e^{\omega_1 s} - 1)^2(e^{\omega_2 s} - 1)} \frac{ds}{s} \right), \quad (122)$$

valid when $\operatorname{Re}(\omega_i) > 0$ and $0 < \operatorname{Re}(z) < \operatorname{Re}(\omega_1 + \omega_2)$. The contour C follows the real axis from $-\infty$ to $+\infty$ avoiding the origin by a small detour in the upper half-plane.

Now consider the combinations

$$F^*(v, w, \vartheta, \varphi, \epsilon) = F(v - \epsilon\vartheta | w - \epsilon\varphi, -2\pi i\epsilon) \cdot \exp(Q_F(v, w, \vartheta, \varphi, \epsilon)), \quad (123)$$

$$G^*(v, w, \vartheta, \varphi, \epsilon) = G(v - \epsilon\vartheta | w - \epsilon\varphi, -2\pi i\epsilon) \cdot \exp(Q_G(v, w, \vartheta, \varphi, \epsilon)), \quad (124)$$

where the exponential factors are

$$Q_F(v, w, \vartheta, \varphi, \epsilon) = \frac{1}{\epsilon} \cdot \frac{w - \epsilon\varphi}{(2\pi i)^2} \cdot \operatorname{Li}_2(e^{2\pi i v/w}) + \left(\frac{v\varphi - w\vartheta}{2\pi i w} - \frac{1}{2} \right) \operatorname{Li}_1(e^{2\pi i v/w}), \quad (125)$$

$$\begin{aligned} Q_G(v, w, \vartheta, \varphi, \epsilon) &= \frac{2}{\epsilon} \cdot \frac{w - \epsilon\varphi}{(2\pi i)^3} \cdot \operatorname{Li}_3(e^{2\pi i v/w}) - \frac{1}{\epsilon} \cdot \frac{v - \epsilon\vartheta}{(2\pi i)^2} \cdot \operatorname{Li}_2(e^{2\pi i v/w}) \\ &+ \frac{1}{\pi i} \left(\frac{v\varphi - w\vartheta}{2\pi i w} - \frac{1}{4} \right) \operatorname{Li}_2(e^{2\pi i v/w}) - \frac{v}{w} \left(\frac{v\varphi - w\vartheta}{2\pi i w} - \frac{1}{2} \right) \operatorname{Li}_1(e^{2\pi i v/w}). \end{aligned} \quad (126)$$

These expressions are uniquely determined by the following result.

Lemma 9.3. *As $\epsilon \rightarrow 0$ in the half-plane $\operatorname{Im}(\epsilon) > 0$ we have*

$$F^*(v, w, \vartheta, \varphi, \epsilon) \rightarrow 1, \quad G^*(v, w, \vartheta, \varphi, \epsilon) \rightarrow 1.$$

Moreover, if ϑ and φ are nonzero these functions are bounded as $\epsilon \rightarrow \infty$ in the same half-plane.

Proof. The first statement follows from the asymptotic expansions given in [14, Proposition 4.6]. As $\epsilon \rightarrow 0$ in the upper half-plane these give

$$\begin{aligned} \log F(v | w, -2\pi i \epsilon) &\sim -\frac{1}{\epsilon} \cdot \frac{w}{(2\pi i)^2} \cdot \text{Li}_2(e^{2\pi i v/w}) + \frac{1}{2} \text{Li}_1(e^{2\pi i v/w}) + O(\epsilon). \\ \log G(v | w, -2\pi i \epsilon) &\sim -\frac{2}{\epsilon} \cdot \frac{w}{(2\pi i)^3} \cdot \text{Li}_3(e^{2\pi i v/w}) + \frac{1}{\epsilon} \cdot \frac{v}{(2\pi i)^2} \cdot \text{Li}_2(e^{2\pi i v/w}) \\ &\quad + \frac{1}{4\pi i} \cdot \text{Li}_2(e^{2\pi i v/w}) - \frac{v}{2w} \cdot \text{Li}_1(e^{2\pi i v/w}) + O(\epsilon), \end{aligned}$$

under the assumption $0 < \text{Re}(v) < \text{Re}(w)$ and $\text{Im}(v/w) > 0$. Substituting $v \mapsto v - \epsilon \vartheta$ and $w \mapsto w - \epsilon \varphi$ and noting that

$$\text{Li}_k \left(e^{\frac{2\pi i(v-\epsilon\vartheta)}{w-\epsilon\varphi}} \right) = \text{Li}_k \left(e^{2\pi i v/w} \right) + 2\pi i \epsilon \left(\frac{v\varphi - w\vartheta}{w^2} \right) \text{Li}_{k-1} \left(e^{2\pi i v/w} \right) + O(\epsilon^2)$$

gives the required relations

$$\begin{aligned} \log F(v - \epsilon \vartheta | w - \epsilon \varphi, -2\pi i \epsilon) &\sim -Q_F(v, w, \vartheta, \varphi, \epsilon) + O(\epsilon), \\ \log G(v - \epsilon \vartheta | w - \epsilon \varphi, -2\pi i \epsilon) &\sim -Q_G(v, w, \vartheta, \varphi, \epsilon) + O(\epsilon). \end{aligned}$$

For the second statement, note that the functions F and G are homogeneous under rescaling all variables, so as $\epsilon \rightarrow \infty$

$$F(v - \epsilon \vartheta | w - \epsilon \varphi, -2\pi i \epsilon) \sim F(\vartheta | \varphi, 2\pi i),$$

and similarly for G . The claim then follows from the form of the expressions Q_F and Q_G . \square

9.4. Solution. Using a mild extension of the techniques of [14] we can solve the Riemann-Hilbert problem for the conifold using the special functions introduced above.

Theorem 9.4. *The functions*

$$B(v, w, \vartheta, \varphi, \epsilon) = F^*(v, w, \vartheta, \varphi, \epsilon)^{-1}, \quad (127)$$

$$D(v, w, \vartheta, \varphi, \epsilon) = G^*(v, w, \vartheta, \varphi, \epsilon)^{-1} \cdot G^*(0, w, 0, \varphi, \epsilon) \quad (128)$$

give a solution to Problem 9.2.

Proof. After Lemma 9.3 we must just check the jumping conditions, parts (iii) and (iv). These follow by the same argument in the proof of [14, Proposition 5.2] which is the case $\vartheta = \varphi = 0$. Note that the exponential factors Q_F and Q_G make no difference in part (iii) because

$$Q_F(v + w, w, \vartheta + \varphi, \varphi, \epsilon) = Q_F(v, w, \vartheta, \varphi, \epsilon) \quad (129)$$

$$Q_G(v+w, w, \vartheta + \varphi, \varphi, \epsilon) - Q_G(v, w, \vartheta, \varphi, \epsilon) = -Q_F(v, w, \vartheta, \varphi, \epsilon). \quad (130)$$

The required identities then follow from the corresponding properties of the functions F and G given in [14, Section 4]. For part (iv) note that the combinations

$$Q_F(v, w, \vartheta, \varphi, \epsilon) + Q_F(v, w, -\vartheta, -\varphi, -\epsilon) = -\text{Li}_1(e^{2\pi iv/w}) = \log(1 - e^{2\pi iv/w}),$$

$$Q_G(v, w, \vartheta, \varphi, \epsilon) + Q_G(v, w, -\vartheta, -\varphi, -\epsilon) = -\frac{d}{dw} \left(\frac{w}{2\pi i} \cdot \text{Li}_2(e^{2\pi iv/w}) \right),$$

are independent of (ϑ, φ) . The required relation then follows from the reflection identities of [14, Proposition 4.3] by the same argument as [14, Proposition 5.2]. \square

The solution of Theorem 9.4 is not unique with the stated properties, and one can make a similar discussion to that of Section 8.5. In particular, when $\vartheta = \varphi = 0$, the solution of Theorem 9.4 reduces to the one studied in [14], which is uniquely determined by the fact that it is holomorphic and non-vanishing on the required half-planes.

9.5. Joyce function. We choose the quadratic refinement $\sigma \in \mathbb{T}_-$ defined by

$$\sigma(\beta) = -1, \quad \sigma(\delta) = 1, \quad \sigma(\beta^\vee) = 1, \quad \sigma(\delta^\vee) = 1,$$

and consider the corresponding isomorphism $\rho_\sigma: \mathbb{T}_+ \rightarrow \mathbb{T}_-$. Setting $\zeta = \rho_\sigma^{-1}(\xi)$ we therefore have

$$y_\beta(\zeta) = \exp(\theta), \quad y_\delta(\zeta) = \exp(\phi), \quad y_{\beta^\vee}(\zeta) = \exp(\theta^\vee), \quad y_{\delta^\vee}(\zeta) = \exp(\phi^\vee),$$

where $\theta = \vartheta + \pi i$ but the other variables are unshifted: $\phi = \varphi$, $\theta^\vee = \vartheta^\vee$ and $\phi^\vee = \varphi^\vee$. Note that these substitutions give a simplification to the expressions (125) and (126).

To compute the Joyce function corresponding to the solution of Theorem 9.4 we again use the formula

$$\frac{\partial x_j}{\partial z_i} + \frac{1}{\epsilon} \cdot \frac{\partial x_j}{\partial \theta_i} = \sum_{p,q} \eta_{pq} \cdot \frac{\partial^2 J}{\partial \theta_i \partial \theta_p} \cdot \frac{\partial x_j}{\partial \theta_q} \quad (131)$$

derived in Section 6.7. Arguing as in Section 8.4 using the trivial part of the solution (116) shows that the Joyce function can be assumed to be independent of the dual variables $\vartheta^\vee, \varphi^\vee$. The non-trivial part (117) of the solution is

$$x_{\beta^\vee, n}(\epsilon) = \theta^\vee - \frac{v^\vee}{\epsilon} + \log B(v + nw, w, \vartheta + n\varphi, \varphi, \epsilon),$$

$$x_{\delta^\vee, n}(\epsilon) = \theta^\vee - \frac{v^\vee}{\epsilon} + \log D(v + nw, w, \vartheta + n\varphi, \varphi, \epsilon) + n \log B(v + nw, w, \vartheta + n\varphi, \varphi, \epsilon).$$

Plugging these solutions into (131) involves applying operators

$$\frac{\partial}{\partial v} + \frac{1}{\epsilon} \cdot \frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial w} + \frac{1}{\epsilon} \cdot \frac{\partial}{\partial \phi}.$$

which kill any functions of the variables $v - \epsilon\theta$, $w - \epsilon\phi$ and ϵ . It therefore follows from the form of the definitions (123) and (124) that we need only deal with the exponential factors Q_F and Q_G . The second factor in (128) makes no contribution because

$$Q_G(0, w, 0, \varphi, \epsilon) = \frac{2}{\epsilon} \cdot \frac{w - \epsilon\varphi}{(2\pi i)^3} \cdot \zeta(3) + \frac{\pi i}{24}.$$

Moreover, the relations (129)-(130) show that the result is independent of n . Differentiating the expressions (125) and (126) now gives

$$\begin{aligned} \frac{\partial^2 J}{\partial \theta \partial \theta} &= - \left(\frac{\partial}{\partial v} + \frac{1}{\epsilon} \frac{\partial}{\partial \theta} \right) \log B(v, w, \theta, \phi, \epsilon) = \frac{1}{w^2} \cdot (v\phi - w\theta) \cdot \text{Li}_0(e^{2\pi i v/w}), \\ \frac{\partial^2 J}{\partial \theta \partial \phi} &= - \left(\frac{\partial}{\partial w} + \frac{1}{\epsilon} \frac{\partial}{\partial \phi} \right) \log B(v, w, \theta, \phi, \epsilon) = \frac{-v}{w^3} \cdot (v\phi - w\theta) \cdot \text{Li}_0(e^{2\pi i v/w}), \\ \frac{\partial^2 J}{\partial \phi \partial \phi} &= - \left(\frac{\partial}{\partial w} + \frac{1}{\epsilon} \frac{\partial}{\partial \phi} \right) \log D(v, w, \theta, \phi, \epsilon) = \frac{v^2}{w^4} \cdot (v\phi - w\theta) \cdot \text{Li}_0(e^{2\pi i v/w}). \end{aligned}$$

It follows that up to linear terms in θ, ϕ the Joyce function is

$$J(v, w, \theta, \phi) = \frac{1}{6w^4} \cdot (v\phi - \theta w)^3 \cdot \text{Li}_0(e^{2\pi i v/w}) = \frac{1}{6w^4} \cdot (v\phi - \theta w)^3 \cdot (1 - e^{-2\pi i v/w})^{-1}.$$

Note that the Joyce form vanishes since

$$H(v, w, \theta, \phi) = v \cdot \frac{\partial J}{\partial \theta} + w \cdot \frac{\partial J}{\partial \phi} = 0.$$

Thus we cannot define a diamond product. The prepotential $\mathcal{F}: M \rightarrow \mathbb{C}$ in the sense of Section 7.1 is given by the formula

$$F(v, w) = -\frac{w^2}{(2\pi i)^3} \cdot \text{Li}_3(e^{2\pi i v/w}),$$

up to quadratic terms in v, w . It satisfies the relations

$$\frac{\partial^3 \mathcal{F}}{\partial v^3} = \frac{\partial^3 J}{\partial \theta^3}, \quad \frac{\partial^3 \mathcal{F}}{\partial v^2 \partial w} = \frac{\partial^3 J}{\partial \theta^2 \partial \phi}, \quad \frac{\partial^3 \mathcal{F}}{\partial v \partial w^2} = \frac{\partial^3 J}{\partial \theta \partial \phi^2}, \quad \frac{\partial^3 \mathcal{F}}{\partial w^3} = \frac{\partial^3 J}{\partial \phi^3}.$$

It coincides with the genus 0 Gromov-Witten generating function for the resolved conifold.

9.6. Threefolds without compact divisors. Let X be a non-compact Calabi-Yau threefold which contains no compact divisors, and introduce the complexified Kähler cone

$$\mathcal{K}_{\mathbb{C}}(X) = \{\omega_{\mathbb{C}} = B + i\omega : \omega \text{ ample}\} \subset H^2(X, \mathbb{C}).$$

The base of our variation of BPS structures will be the complex manifold

$$M = \mathcal{K}_{\mathbb{C}}(X) \times \mathbb{C}^*.$$

We consider a framed variation of BPS structures whose lattice

$$\Gamma_{\leq 1} = H_2(X, \mathbb{Z}) \oplus \mathbb{Z}$$

is the receptacle for Chern characters of compactly-supported coherent sheaves on X . As before, we equip this lattice with the form $\langle -, - \rangle = 0$. The central charge corresponding to a point $(\omega_{\mathbb{C}}, w) \in M$ is the map

$$Z_{\leq 1}: \Gamma_{\leq 1} \rightarrow \mathbb{C}, \quad Z_{\leq 1}(\beta, n) = v(\beta) + nw, \quad v(\beta) = \beta \cdot \omega_{\mathbb{C}}.$$

The relevant BPS invariants $\Omega(\gamma) \in \mathbb{Q}$ were first constructed by Joyce and Song ([33], see particularly Sections 6.3–6.4). These invariants are defined using moduli stacks of semistable objects supported in dimension ≤ 1 , and should not be confused with the ideal sheaf curve-counting invariants appearing in the famous MNOP conjectures [37]. Joyce and Song prove that the numbers $\Omega(\gamma)$ are independent of the point of M . This is to be expected, since wall-crossing is trivial when the form $\langle -, - \rangle$ vanishes. A direct calculation [33, Section 6.3] shows that

$$\Omega(0, n) = -\chi(X), \quad n \in \mathbb{Z} \setminus \{0\}, \quad (132)$$

where $\chi(X)$ denotes the topological Euler characteristic of the manifold X . It is expected [33, Conjecture 6.20] that when $\beta > 0$ is a positive curve class

$$\Omega(\beta, n) = \text{GV}(0, \beta), \quad (133)$$

and in particular, is independent of n . Here $\text{GV}(0, \beta)$ is the genus 0 Gopakumar-Vafa invariant for the class $\beta \in H_2(X, \mathbb{Z})$. We emphasise that the higher genus Gopakumar-Vafa invariants are invisible from the point-of-view of the torsion sheaf invariants $\Omega(\gamma)$.

Assumption 9.5. *We assume the identity (133), and that moreover the invariants $\text{GV}(0, \beta)$ vanish for all but finitely many classes $\beta \in H_2(X, \mathbb{Z})$.*

One can formulate a Riemann-Hilbert problem exactly as above. It depends on a point $\xi \in \mathbb{T}$ and we set

$$\xi(\beta) = \exp(\vartheta(\beta)), \quad \xi(\delta) = \exp(\varphi).$$

We omit the details since the formulae are a little cumbersome. It is clear that a solution can be obtained by superposing copies of the functions B_n and C_n . As in Section 8.7 there is now no canonical quadratic refinement, but since these only effect the Joyce function by the addition of polynomials in the θ_i of degree ≤ 2 we can conclude that the Joyce function must be

$$J(v, w, \theta, \phi) = \frac{1}{6w^4} \cdot \sum_{\beta \in H_2(X, \mathbb{Z})} \text{GV}(0, \beta) \cdot (v(\beta)\phi - w\theta(\beta))^3 \cdot (1 - e^{-2\pi i v(\beta)/w})^{-1},$$

up to the addition of linear terms in the θ_i . The prepotential is

$$F(v, w) = \frac{-w^2}{(2\pi i)^3} \cdot \sum_{\beta \in H_2(X, \mathbb{Z})} \text{GV}(0, \beta) \cdot \text{Li}_3(e^{2\pi i v(\beta)/w}),$$

up to the addition of quadratic functions of the linear variables $v(\beta)$. Note that by the Aspinwall-Morrison covering formula [1], this is precisely the genus 0 part of the Gromov-Witten generating function.

10. THE CASE OF THE A_2 QUIVER

In this final section we summarise the results from [15] on the Riemann-Hilbert problem for the BPS structures arising from DT theory applied to the A_2 quiver. At present this is the only example of a coupled BPS structure where solutions to Problem 6.3 can be constructed. The method we use owes much to the paper of Gaiotto, Moore and Neitzke [25]. The resulting Joyce structure can be described relatively explicitly, and in particular, the Joyce form can be calculated. A remarkable fact is that this Joyce structure is compatible, in the sense of Section 7.4, with the polynomial Frobenius structure associated to the A_2 root system. For further details on the material of this section we refer the reader to [15].

10.1. Quadratic differentials. Let us consider a meromorphic quadratic differential

$$\phi(x) = \varphi(x)dx^{\otimes 2} \tag{134}$$

on the Riemann surface \mathbb{P}^1 having a single pole of order 7 at the point $x = \infty$, and three simple zeroes. It is easy to see [16, Section 12.1] that any such differential can be put in the

form

$$\phi(x) = (x^3 + ax + b)dx^{\otimes 2} \quad (135)$$

by applying an automorphism of \mathbb{P}^1 . Away from the zeroes and poles of $\phi(x)$ there is a distinguished local co-ordinate on \mathbb{P}^1

$$w(x) = \pm \int_*^x \sqrt{\varphi(u)} du \quad (136)$$

in terms of which $\phi(x)$ takes the form $dw^{\otimes 2}$. Such a co-ordinate is uniquely determined up to transformations of the form $w \mapsto \pm w + c$. The *horizontal foliation* determined by $\phi(x)$ then consists of the arcs on which $\text{Im}(w)$ is constant. This foliation has singularities at the zeroes and poles of $\phi(x)$. Local computations summarised in [16, Section 3.3] show that:

- (i) there are three horizontal arcs emanating from each of the three simple zeroes;
- (ii) there are five tangent distinguished directions at the pole $x = \infty$, and an open neighbourhood $\infty \in U \subset \mathbb{P}^1$ such that all horizontal trajectories entering U approach ∞ along one of the distinguished directions.

Following [16, Section 6] we take the real oriented blow-up of the surface \mathbb{P}^1 at the point ∞ corresponding to the unique pole of $\phi(x)$. Topologically the resulting surface \mathbb{S} is a disc. The distinguished directions at the pole determine a subset of five points $\mathbb{M} \subset \partial\mathbb{S}$ of the boundary of this disc. The pair (\mathbb{S}, \mathbb{M}) is an example of a *marked bordered surface*. The horizontal foliation of \mathbb{P}^1 lifts to a foliation on the surface \mathbb{S} , with singularities at the points $\mathbb{M} \subset \partial\mathbb{S}$ and the zeroes of $\phi(x)$.

The quadratic differential (135) determines a double cover

$$\pi: X \rightarrow \mathbb{P}^1, \quad (137)$$

branched at the zeroes and pole of $\phi(x)$, on which there is a well-defined global choice of square-root of $\phi(x)$. This is nothing but the projectivisation of the affine elliptic curve

$$X^\circ = \{(x, y) \in \mathbb{C}^2 : y^2 = x^3 + ax + b\}. \quad (138)$$

Taking the periods of the square-root of $\phi(x)$ defines a group homomorphism

$$Z: H_1(X, \mathbb{Z}) \rightarrow \mathbb{C}, \quad Z(\gamma) = \int_\gamma \sqrt{\phi(x)} \in \mathbb{C}. \quad (139)$$

The differential ϕ is called *generic* if the image of Z is not contained in a one-dimensional real subspace of \mathbb{C} .

A horizontal trajectory of $\phi(x)$ is said to be *finite-length* if it never approaches the pole $x = \infty$. In our situation any such trajectory necessarily connects two distinct simple zeroes of $\phi(x)$, and is known as a *saddle connection*. The inverse image of a saddle connection under the double cover (137) is a cycle γ , which can be canonically oriented by insisting that $Z(\gamma) \in \mathbb{R}_{>0}$. This then defines a homology class in the group $H_1(X, \mathbb{Z})$. See [16, Section 3.2] for more details.²

More generally we can consider trajectories of the differential $\phi(x)$ of some phase $\theta \in \mathbb{R}$. By definition these are arcs which make a constant angle $\pi\theta$ with the horizontal foliation. Alternatively one can view them as horizontal trajectories for the rescaled quadratic differential $e^{-2\pi i\theta} \cdot \phi(x)$. Once again, these finite-length trajectories $\gamma: [a, b] \rightarrow \mathbb{C}$ define homology classes in $H_1(X, \mathbb{Z})$, with the orientation convention being that $Z(\gamma) \in \mathbb{R}_{>0} \cdot e^{\pi i\theta}$.

The quadratic differential (135) is said to be *saddle-free* if it has no finite-length horizontal trajectories. This is an open condition. The horizontal foliation of a saddle-free differential splits the surface \mathbb{P}^1 into a union of domains called *horizontal strips* and *half-planes*. In the present case we obtain five half-planes and two horizontal strips. The resulting trajectory structure on the blown-up surface \mathbb{S} is illustrated in Figure 2. The crosses denote zeroes of the differential, and the black dots are the points of \mathbb{M} .

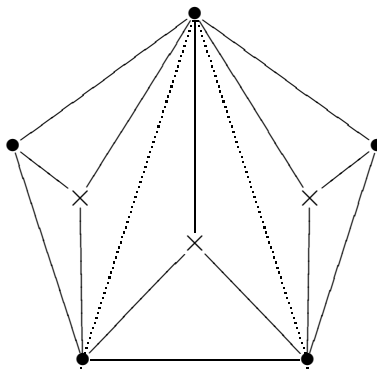


FIGURE 2. The separating trajectories of a saddle-free differential of the form (135).

²For the purposes of comparison with the general situation of [16] involving the hat-homology group $H_1(X_s^\circ, \mathbb{Z})^-$, note that the group $H_1(X_s, \mathbb{Z})$ coincides with its -1 eigenspace under the action of the covering involution of (137); indeed the $+1$ eigenspace can be identified with the homology of the quotient \mathbb{P}^1 , which vanishes; moreover, puncturing X_s at the inverse image of the pole $\infty \in \mathbb{P}^1$ also leaves the first homology group unchanged.

Taking one trajectory from the interior of each horizontal strip defines a triangulation of the marked bordered surface (\mathbb{S}, \mathbb{M}) called the *WKB triangulation* (see [16, Section 10.1] for details). In our case there are exactly two edges, which are the two dashed edges in Figure 2. As explained in [16, Section 3.6], each of the two horizontal strips contains a unique finite-length trajectory with phase in the interval $(0, 1) \subset \mathbb{R}$, and the corresponding classes $\gamma_i \in H_1(X_s, \mathbb{Z})$ determine a basis (γ_1, γ_2) , whose elements are naturally indexed by the edges of the WKB triangulation.

10.2. Variation of BPS structures. The base of our variation will be the complex manifold

$$M = \{(a, b) \in \mathbb{C}^2 : 4a^3 + 27b^2 \neq 0\}. \quad (140)$$

It is proved in [13] that this coincides with the quotient of the space of stability conditions on the derived category of the Ginzburg algebra of the A_2 quiver, by the group of autoequivalences generated by spherical twists in the two vertex simples. This can also be viewed as a special case of the main result of [16].

For each point $p = (a, b) \in M$ there is a meromorphic quadratic differential $\phi_p(x)$ on \mathbb{P}^1 given by the formula (135), and a corresponding double cover $\pi: X_p \rightarrow \mathbb{P}^1$. The Gauss-Manin connection on the family of homology groups

$$\Gamma_p = H_1(X_p, \mathbb{Z}) \cong \mathbb{Z}^{\oplus 2} \quad (141)$$

gives them the structure of a local system of lattices over M . In concrete terms, we can represent classes in $H_1(X_p, \mathbb{Z})$ by the inverse images under the double cover (137) of paths in \mathbb{C} connecting the zeroes of $\phi_p(x)$, and the Gauss-Manin connection is then obtained by holding these paths locally constant as $\phi_p(x)$ varies.

There is a natural variation of BPS structures over the space M determined by the family of quadratic differentials (135). At a point $p \in M$ corresponding to a generic differential the corresponding BPS structure $(\Gamma_p, Z_p, \Omega_p)$ is defined as follows:

- (a) the charge lattice is $\Gamma_p = H_1(X_p, \mathbb{Z})$ with its intersection form $\langle -, - \rangle$;
- (b) the central charge $Z_p: \Gamma_p \rightarrow \mathbb{C}$ is defined by

$$Z_p(\gamma) = \int_{\gamma} \sqrt{\phi_p(x)} dx \in \mathbb{C}; \quad (142)$$

- (c) the BPS invariants $\Omega_p(\gamma) \in \{0, 1\}$ is nonzero precisely if the differential $\phi_p(x)$ has a finite-length trajectory defining the given class $\gamma \in \Gamma_p$.

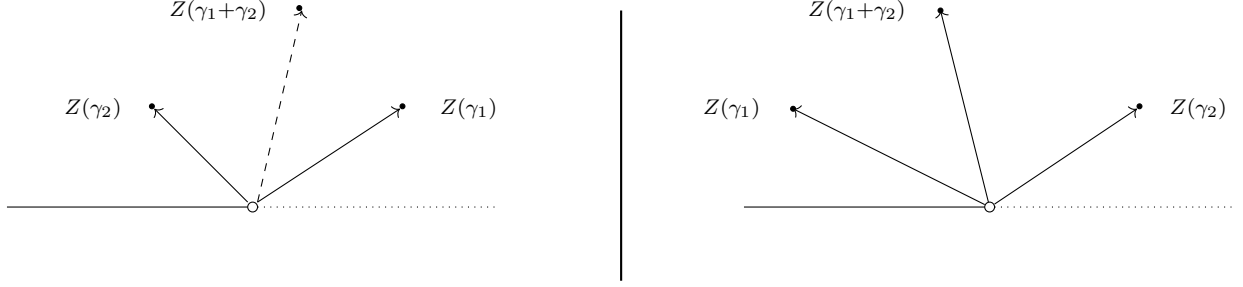


FIGURE 3. The ray diagrams for the BPS structures associated to the A_2 quiver.

Condition (c) needs modification when the differential $\phi_p(x)$ is non-generic, but in fact there is no need to give an explicit description of the BPS invariants $\Omega_p(\gamma)$ in this case, since what is important is the BPS automorphisms $\mathbb{S}_p(\ell)$, and using the wall-crossing formula, these are determined by the BPS invariants at nearby generic points of M .

Suppose that $p \in M$ corresponds to a saddle-free and generic differential $\phi_p(x)$, and let (Γ, Z, Ω) be the associated BPS structure. As explained above, the lattice (141) then has a distinguished basis (γ_1, γ_2) indexed by the edges of the WKB triangulation. We can canonically order these edges by insisting that $\langle \gamma_1, \gamma_2 \rangle = 1$. Define $z_i = Z(\gamma_i) \in \mathbb{C}^*$. The orientation conventions discussed above ensure that these points lie in the upper half-plane. One can then show that the BPS invariants are as follows:

- (a) if $\text{Im}(z_2/z_1) > 0$ then $\Omega(\pm\gamma_1) = \Omega(\pm\gamma_2) = 1$ with all others zero;
- (b) if $\text{Im}(z_2/z_1) < 0$ then $\Omega(\pm\gamma_1) = \Omega(\pm(\gamma_1 + \gamma_2)) = \Omega(\pm\gamma_2) = 1$ with all others zero.

The two cases are illustrated in Figure 3. In both cases there is a distinguished quadratic refinement $\sigma \in \mathbb{T}_-$, uniquely defined by the property that $\sigma(\gamma) = -1$ for every active class $\gamma \in \Gamma$, and it is easy to see that the resulting σ varies continuously as one crosses the wall where $\text{Im} Z(\gamma_2)/Z(\gamma_1) = 0$.

We can use the quadratic refinement σ and the map (64) to identify the twisted torus with the standard torus \mathbb{T}_+ . Under this identification the birational automorphism (69) becomes the birational automorphism of \mathbb{T}_+ defined by

$$\mathbb{S}(\ell)^*(y_\beta) = y_\beta \cdot \prod_{Z(\gamma) \in \ell} (1 + y_\gamma)^{\Omega(\gamma)\langle \beta, \gamma \rangle}. \quad (143)$$

The fact that the above BPS structures fit together to make a variation of BPS structures boils down to the wall-crossing formula

$$C_{\gamma_2} \circ C_{\gamma_1} = C_{\gamma_1} \circ C_{\gamma_1 + \gamma_2} \circ C_{\gamma_2}, \quad (144)$$

where for each class $\gamma \in \Gamma$ we defined a birational automorphism $C_\gamma: \mathbb{T}_+ \dashrightarrow \mathbb{T}_+$ by

$$C_\gamma^*(y_\beta) = y_\beta \cdot (1 + y_\gamma)^{\langle \gamma, \beta \rangle}. \quad (145)$$

The equation (144) is the pentagon identity familiar from cluster theory.

10.3. Associated family of opers. Consider the second-order linear differential equation

$$y''(x) = Q(x, \epsilon) \cdot y(x), \quad Q(x, \epsilon) = \epsilon^{-2} \cdot Q_0(x) + \epsilon^{-1} \cdot Q_1(x) + Q_2(x), \quad (146)$$

where primes denote differentiation with respect to the complex variable $x \in \mathbb{C}$, and the terms in the potential $Q(x, \epsilon)$ are

$$Q_0(x) = x^3 + ax + b, \quad Q_1(x) = \frac{p}{x - q} + r, \quad (147)$$

$$Q_2(x) = \frac{3}{4(x - q)^2} + \frac{r}{2p(x - q)} + \frac{r^2}{4p^2}. \quad (148)$$

We view the equation (146) as being specified by a point of the complex manifold

$$W = \left\{ (a, b, q, p, r) \in \mathbb{C}^5 : p^2 = q^3 + aq + b \text{ and } 27a^2 + 4b^3 \neq 0 \right\}, \quad (149)$$

together with a nonzero complex number $\epsilon \in \mathbb{C}^*$.

The form of (148) is chosen to ensure that the point $x = q$ is an apparent singularity of the equation (146): transporting any solution around this point changes its sign. Thus the generalised monodromy of the equation consists only of the Stokes data at the irregular singularity $x = \infty$. The Stokes sectors are the sectors bounded by the rays through the fifth roots of ϵ^2 . In each such sector there is a distinguished subdominant solution to (146) up to scale, with the property that it decays exponentially as $x \rightarrow \infty$ in the sector. After choosing an initial sector, the collection of these subdominant solutions defines a point of the space

$$V = \left\{ \psi: \mathbb{Z}/5\mathbb{Z} \rightarrow \mathbb{P}^1 : \psi(i+1) \neq \psi(i) \text{ for all } i \in \mathbb{Z}/5\mathbb{Z} \right\} / \text{PGL}_2, \quad (150)$$

which is easily seen to be a two-dimensional complex manifold.

For each $\epsilon \in \mathbb{C}^*$ there is then a holomorphic monodromy map

$$F(\epsilon): W \rightarrow V, \quad (151)$$

sending the equation (147) to its Stokes data. This map depends on a labelling of the Stokes sectors for the equation (146), which in concrete terms amounts to a choice of fifth root of ϵ^2 .

We prove in [15] that this map is invariant under the isomonodromy flows:

$$-\frac{1}{\epsilon} \frac{\partial}{\partial r} + \left(\frac{\partial}{\partial b} + \frac{1}{2p} \frac{\partial}{\partial p} + \frac{r}{2p^2} \frac{\partial}{\partial r} \right), \quad (152)$$

$$-\frac{2p}{\epsilon} \frac{\partial}{\partial q} - \frac{3q^2 + a}{\epsilon} \frac{\partial}{\partial p} + \left(\frac{\partial}{\partial a} - q \frac{\partial}{\partial b} - \frac{r}{p} \frac{\partial}{\partial q} - \frac{r(3q^2 + a)}{2p^2} \frac{\partial}{\partial p} - \frac{r^2}{2p^3} (3q^2 + a) \frac{\partial}{\partial r} \right). \quad (153)$$

In fact the flow (152) leaves the equation (146) itself invariant. In the special case $r = 0$ the second flow (153) reduces to the standard isomonodromy flow often studied in connection with the Painlevé I equation.

10.4. Riemann-Hilbert problem. It is well known that the monodromy manifold V has birational co-ordinate systems

$$X_T: V \dashrightarrow (\mathbb{C}^*)^2 \quad (154)$$

usually referred to as Fock-Goncharov co-ordinates, and indexed by the triangulations T of a regular pentagon. The maps for different triangulations differ by post-composition by explicit birational automorphisms of $(\mathbb{C}^*)^2$.

On the other hand, for generic $\lambda \in \mathbb{C}^*$, as we explained above, the horizontal trajectory structure of the quadratic differential

$$\lambda^{-2} \cdot Q_0(x) dx^{\otimes 2} = \lambda^{-2} \cdot (x^3 + ax + b) dx^{\otimes 2} \quad (155)$$

naturally determines such a triangulation, namely the WKB triangulation. This triangulation is well-defined for all $\lambda \in \mathbb{C}^*$ not lying on finitely many rays, but jumps discontinuously when these rays are crossed.

Fix a point $(a, b, q, p, r) \in W$, and let (Γ, Z, Ω) be the BPS structure defined by the quadratic differential $Q_0(x) dx^{\otimes 2}$ as above. Define an element $\xi: \Gamma \rightarrow \mathbb{C}^*$ of the corresponding algebraic torus \mathbb{T}_+ by setting

$$\xi(\gamma) = \exp \left(\int_{\gamma} \frac{Q_1(x) dx}{2\sqrt{Q_0(x)}} \right) = \exp \left(\int_{\gamma} \left(\frac{p}{x-q} + r \right) \frac{dx}{2y} \right) \in \mathbb{C}^*. \quad (156)$$

The basic claim is that the map which sends $\epsilon \in \mathbb{C}^*$ to the Fock-Goncharov co-ordinates corresponding to the WKB triangulation of the differential (155), applied to the monodromy of the equation (146), gives a solution to the Riemann-Hilbert problem associated to the BPS structure (Γ, Z, Ω) , with constant term given by (156).

To be completely precise, let us fix a non-active ray $r = \mathbb{R}_{>0} \cdot \lambda$ spanned by an element $\lambda \in \mathbb{C}^*$. The non-active condition is equivalent to the statement that the differential (155) is saddle-free. The marked bordered surface $(\mathbb{S}(\lambda), \mathbb{M}(\lambda))$ for this differential can be identified with the disc in \mathbb{C} , with the marked points being the directions of the fifth roots of λ^2 . The saddle-free condition ensures that the differential (155) defines a WKB triangulation $T(\lambda)$ of this surface.

Consider now a point $\epsilon \in \mathbb{H}_r$. Since $\operatorname{Re}(\epsilon/\lambda) > 0$, there is a natural bijection between the fifth roots of λ^2 and ϵ^2 . Thus we can identify the Stokes sectors of the equation (146) with the marked points $\mathbb{M}(\lambda)$. Note that the Fock-Goncharov co-ordinates $(X_1, X_2) \in (\mathbb{C}^*)^2$ for the triangulation $T(\lambda)$ are indexed by the edges of the triangulation, which also index a natural basis for the lattice Γ . Thus we can view the map (154) as taking values in the torus \mathbb{T}_+ , and consider the map

$$X_r: \mathbb{H}_r \rightarrow \mathbb{T}_+, \quad (157)$$

which sends a point $\epsilon \in \mathbb{H}_r$ to the Fock-Goncharov co-ordinates of the equation (146) with respect to the WKB triangulation determined by the differential (155). In [15] we prove

Theorem 10.1. *The maps (157) give a meromorphic solution to the Riemann-Hilbert problem associated to the BPS structure (Γ, Z, Ω) , with constant term given by (156). \square*

This proof of Theorem 10.1 boils down to the following three statements:

- (i) the WKB approximation shows that as $\epsilon \rightarrow 0$ in the half-plane \mathbb{H}_r , the Fock-Goncharov co-ordinates satisfy

$$X_{r,\gamma}(\epsilon) \cdot \exp(Z(\gamma)/\epsilon) \rightarrow \xi(\gamma);$$

- (ii) the homogeneity of the potential (147) under the rescaling of all variables with various weights shows that as $\epsilon \rightarrow \infty$ in the half-plane \mathbb{H}_r , with $\xi \in \mathbb{T}_+$ held fixed

$$X_r(\epsilon) \rightarrow \text{constant};$$

- (iii) given an active ray $\ell = \mathbb{R}_{>0} \cdot \lambda$ for the BPS structure (Γ, Z, Ω) , corresponding to a differential (155) with a horizontal saddle trajectory with class $\gamma \in \Gamma$, the two systems of Fock-Goncharov co-ordinates defined by small clockwise and anti-clockwise perturbations of $\lambda \in \mathbb{C}^*$ differ by post-composition by the birational transformation C_γ of (145).

We unfortunately have no results on the uniqueness of the solution of Theorem 10.1.

10.5. **Joyce structure.** Consider the bundle $\pi: \mathbb{T} \rightarrow M$ whose fibres are the tori

$$\mathbb{T}_{p,+} = \text{Hom}_{\mathbb{Z}}(\Gamma_p, \mathbb{C}^*).$$

There are obvious local co-ordinates on the fibres $\mathbb{T}_{p,+}$ given by

$$\theta_i = \log \xi(\gamma_i) \in \mathbb{C}, \quad \xi \in \mathbb{T}_{p,+},$$

and we therefore obtain local co-ordinates $(z_1, z_2, \theta_1, \theta_2)$ on the total space \mathbb{T} . Let $\pi: W \rightarrow M$ be the obvious projection map. The expression (156) defines a holomorphic map Θ fitting into the diagram

$$\begin{array}{ccc} W & \xrightarrow{\Theta} & \mathbb{T} \\ \pi \searrow & & \swarrow \pi \\ & M & \end{array} \quad (158)$$

This map is in fact an open embedding, and so we can push forward the isomonodromy flows (152) and (153) to obtain a non-linear connection on the bundle of tori $\pi: \mathbb{T} \rightarrow M$.

Theorem 10.2. *In the co-ordinates $(z_1, z_2, \theta_1, \theta_2)$ the push-forward of the isomonodromy flows via the map Θ take the Hamiltonian form*

$$\frac{\partial}{\partial z_i} + \frac{1}{\epsilon} \cdot \frac{\partial}{\partial \theta_i} - \frac{\partial^2 J}{\partial \theta_i \partial \theta_2} \cdot \frac{\partial}{\partial \theta_1} - \frac{\partial^2 J}{\partial \theta_i \partial \theta_1} \cdot \frac{\partial}{\partial \theta_2}, \quad J_i = \frac{\partial J}{\partial \theta_i}, \quad (159)$$

for a single locally-defined function $J: \mathbb{T} \rightarrow \mathbb{C}$.

We can now describe the Joyce structure on M corresponding to the miniversal variation of BPS structures of Section 10.2. As explained in Section 6.7, most of the data carries over easily from the variation of BPS structures. In terms of the local co-ordinates (z_1, z_2) on M considered above, the connection ∇ is the one in which the z_i are flat, and the bundle of lattices $\Gamma_M \subset \mathcal{J}_M^*$ is spanned by dz_1 and dz_2 . Note that by the formula (142), rescaling (a, b) with weights $(4, 6)$ has the effect of rescaling the co-ordinates (z_1, z_2) with weight 5, so the Euler vector field is

$$E = z_1 \cdot \frac{\partial}{\partial z_1} + z_2 \cdot \frac{\partial}{\partial z_2} = \frac{4a}{5} \cdot \frac{\partial}{\partial a} + \frac{6b}{5} \cdot \frac{\partial}{\partial b}. \quad (160)$$

The only non-trivial step is the construction of the Joyce function $J: \mathcal{J}_M \rightarrow \mathbb{C}$. Recall that the solutions to the Riemann-Hilbert problem are flat sections of the deformed Joyce connection. Since these solutions are functions of the monodromy of the equation (146), the deformed Joyce connection must coincide with the isomonodromy connection. Thus the

Joyce function is precisely the function J of Theorem 10.2, and the differential equation (84) coincides with (159). The isomonodromy flows define Ehresmann connections on the two bundles appearing in the commutative diagram (158) which are flat by construction, since they are the pullback via the monodromy map of the trivial connection on the bundle $\pi: V \times M \rightarrow M$. It follows from this that the function J of Theorem 10.2 satisfies the differential equation (51).

10.6. Linearization. To understand the linear data associated to the Joyce structure considered above, it is convenient to change co-ordinates on M from the locally defined co-ordinate system (z_1, z_2) to the global co-ordinates (a, b) . Explicitly we have

$$\frac{\partial}{\partial a} = -\eta_1 \frac{\partial}{\partial z_1} - \eta_2 \frac{\partial}{\partial z_2}, \quad \frac{\partial}{\partial b} = \omega_1 \frac{\partial}{\partial z_1} + \omega_2 \frac{\partial}{\partial z_2}, \quad (161)$$

where ω_i and η_i are the periods and quasi-periods of the Weierstrass elliptic curve (138):

$$\omega_i = \int_{\gamma_i} \frac{dx}{2\sqrt{x^3 + ax + b}}, \quad \eta_i = \int_{\gamma_i} \frac{x dx}{2\sqrt{x^3 + ax + b}}.$$

The fact that the transformation (161) is invertible follows from the Legendre relation

$$\omega_2 \eta_1 - \omega_1 \eta_2 = 2\pi i. \quad (162)$$

It is then also natural to consider new fibre co-ordinates (θ_a, θ_b) defined by the relation

$$\theta_a \cdot \frac{\partial}{\partial a} + \theta_b \cdot \frac{\partial}{\partial b} = \theta_1 \cdot \frac{\partial}{\partial z_1} + \theta_2 \cdot \frac{\partial}{\partial z_2}.$$

Explicitly we have

$$2\pi i \cdot \theta_a = -\omega_2 \theta_1 + \omega_1 \theta_2, \quad 2\pi i \cdot \theta_b = -\eta_2 \theta_1 + \eta_1 \theta_2. \quad (163)$$

In these co-ordinates the deformed Joyce connection takes the form

$$\frac{\partial}{\partial a} + \frac{1}{\epsilon} \cdot \frac{\partial}{\partial \theta_a} + \frac{1}{2\pi i} \cdot \frac{\partial^2 K}{\partial \theta_a \partial \theta_b} \cdot \frac{\partial}{\partial \theta_a} - \frac{1}{2\pi i} \cdot \frac{\partial^2 K}{\partial \theta_a \partial \theta_a} \cdot \frac{\partial}{\partial \theta_b}, \quad (164)$$

$$\frac{\partial}{\partial b} + \frac{1}{\epsilon} \cdot \frac{\partial}{\partial \theta_b} + \frac{1}{2\pi i} \cdot \frac{\partial^2 K}{\partial \theta_b \partial \theta_b} \cdot \frac{\partial}{\partial \theta_a} - \frac{1}{2\pi i} \cdot \frac{\partial^2 K}{\partial \theta_a \partial \theta_b} \cdot \frac{\partial}{\partial \theta_b}, \quad (165)$$

for some function $K(a, b, \theta_a, \theta_b)$. It is easy to see [15, Lemma 4.4] that this differs from the Joyce function $J(z_1, z_2, \theta_1, \theta_2)$ by the addition of terms which are cubic in the fibre directions

$$J(z_1, z_2, \theta_1, \theta_2) = K(a, b, \theta_a, \theta_b) + \sum_{i,j,k} C_{ijk}(z_1, z_2) \cdot \theta_i \theta_j \theta_k.$$

Explicitly these cubic terms are given by

$$C_{ijk} = \omega \left(\nabla_{\frac{\partial}{\partial z_i}}^{\text{LC}} \left(\frac{\partial}{\partial z_j} \right), \frac{\partial}{\partial z_j} \right),$$

where ∇^{LC} denotes the connection on \mathcal{T}_M in which the co-ordinates a, b are flat.

In [15] the following explicit formulae is derived for the third derivatives of K

$$\Theta^* \left(\frac{\partial^3 K}{\partial \theta_a^3} \right) = \frac{3(3q^2 + a)^2 r^2}{2p^4} - \frac{6qr^2}{p^2} + \frac{r}{p}, \quad \frac{\partial^3 K}{\partial \theta_b^3} = 0, \quad (166)$$

$$\Theta^* \left(\frac{\partial^3 K}{\partial \theta_a^2 \partial \theta_b} \right) = -\frac{r}{p^3} (3q^2 + a) + \frac{q}{2p^2}, \quad \Theta^* \left(\frac{\partial^3 K}{\partial \theta_a \partial \theta_b^2} \right) = \frac{1}{2p^2}. \quad (167)$$

It is perhaps interesting to note that these are algebraic functions on the space W .

The point $\theta_a = \theta_b = 0$ corresponds to a limit whereby $u \rightarrow 0$ with $r = \zeta(u)$. In this limit we have $q \sim u^{-2}$, $p \sim u^{-3}$ and $r \sim u^{-1}$ which means that all the expressions (166)-(167) vanish. It follows that the linear Joyce connection coincides with ∇^{LC} . It then follows immediately from (160) that

$$V \left(\frac{\partial}{\partial a} \right) = -\frac{1}{5} \cdot \frac{\partial}{\partial a}, \quad V \left(\frac{\partial}{\partial b} \right) = \frac{1}{5} \cdot \frac{\partial}{\partial b},$$

and hence the Joyce form is

$$g = \frac{1}{5} \cdot da \otimes db. \quad (168)$$

Recall now that there is a Frobenius structure on the vector space \mathbb{C}^2 , with linear flat co-ordinates (a, b) , and metric and Euler vector field

$$g_{\text{Frob}} = \frac{1}{6} da \otimes db, \quad E_{\text{Frob}} = \frac{2a}{3} \cdot \frac{\partial}{\partial a} + b \cdot \frac{\partial}{\partial b}. \quad (169)$$

The conformal dimension is $d = \frac{1}{3}$. The prepotential is the polynomial function

We refer to this as the A_2 Frobenius structure, since it relates the unfolding space of the A_2 singularity $x^3 = 0$. The discriminant locus is the submanifold cut out by the equation $27a^2 + 4b^3 = 0$. Thus by restriction we obtain a discriminant-free Frobenius structure on the open submanifold $M \subset \mathbb{C}^2$.

Theorem 10.3. *The Joyce structure on M considered above is compatible with the A_2 Frobenius structure n restricted to M .*

Proof. Comparing the expressions (169) with (160) and (168) shows that the metrics and Euler fields associated to the Joyce and Frobenius structures agree up to scale. The same is therefore true of the corresponding operators V . By a result of Dubrovin [21, Proposition 5.1] the odd periods of the Frobenius structure are given by the co-ordinates z_i . Reversing the argument we gave for Proposition 7.7(e) it therefore follows that the twisted product of this Frobenius structure must also be proportional to the diamond product of the Joyce structure. This completes the proof. \square

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