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Classification of Simple Modules of the Ore Extension $K[X][Y; f\frac{d}{dX}]$

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Abstract For the algebras Λ in the title of the paper, a classification of simple modules is given, an explicit description of the prime and completely prime spectra is obtained, the global and the Krull dimensions of Λ are computed.

Keywords A skew polynomial ring · A prime ideal · A completely prime ideal · A simple module · The global dimension · The Krull dimension · A normal element

Mathematics Subject Classification 16D60 · 13N10 · 16S32

1 Introduction

Let D be a ring and $A = D[x; \sigma, \delta]$ be a skew polynomial ring where σ is an automorphism of D and δ is a σ-derivation of D (for all $a, b \in D$, $\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$). The ring A is generated by D and x subject to the defining relations $xa = \sigma(a)x + \delta(a)$ for all elements $a \in D$. When D is a Dedekind domain, a classification of simple A-modules is given in [4]. This is a large class of rings. A machinery is developed in [4] to cover all possible situations (non-commutative valuations, etc).

The algebra

$$\Lambda = K[X]\left[Y; \delta := f\frac{d}{dX}\right] = \bigoplus_{i\geq 0} K[X]Y^i$$

is a particular example of the ring A where $\sigma = \text{id}$ is the identity automorphism of the polynomial ring $K[X]$, $f \in K[X]$ and $\delta = f\frac{d}{dX}$ is a K-derivation of $K[X]$ ($\delta(X) = f$). If $f = 1$ (or, more generally, $f \in K^{\times}\backslash\{0\}$) then the algebra $\Lambda(1)$ is the *Weyl algebra*

$$A_1 = K\langle X, \partial \,|\, \partial X - X\partial = 1\rangle \simeq K[X]\left[Y; \frac{d}{dX}\right].$$

In 1981, a classification of simple A_1-modules was obtained by Block (over the field of complex numbers) in [9] (see also [2,3] for an alternative approach via generalized Weyl algebras in a more general situation).

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Recently, classifications of simple *weight* modules are obtained for some classical algebras (the *Euclidean* algebra, the *Schrödinger* algebra, the universal enveloping algebra $U(\mathrm{sl}_2 \ltimes V_2)$), see [5–7]. In these classifications, classifications of *all simple* modules over certain subalgebras of the Weyl algebra A_1 that contain the polynomial algebra $K[X]$ (the, so-called, *polyonic* algebras) play a crucial role. The polyonic algebras are investigated in [8]. Each polyonic algebra contains the algebra $\Lambda = \Lambda(f)$ for some non-scalar polynomial $f \in K[X]$ which play an important role in studying of its properties. This is the main reason why we decided to collect main properties of the algebras Λ in this paper. In particular, a classification of simple Λ-modules is given in Sect. 2 (Lemma 2.1 and Theorem 2.10). This classification can be derived from [4] but we give different and simpler proofs which are based on generalized Weyl algebras rather than skew polynomial rings.

An ideal $\mathfrak{p}$ of a ring R is called a *completely prime ideal* if the factor ring $R/\mathfrak{p}$ is a domain. A completely prime ideal is a prime ideal. The sets of prime and completely prime ideals of the ring R are denoted by $\mathrm{Spec}(R)$ and $\mathrm{Spec}_c(R)$, respectively.

In Theorem 1.1, a classification of prime and completely prime ideals of the algebra Λ is given, the Krull and global dimensions of the algebra Λ are found. The algebra Λ is a Noetherian domain of Gelfand-Kirillov dimension 2.

Theorem 1.1 *Let K be a field of characteristic zero, $\Lambda = K[X][Y; \delta := f\frac{d}{dX}]$ where $f \in K[X]\backslash K$. Let $f = p_1^{n_1} \cdots p_s^{n_s}$ be a unique (up to permutation) product of irreducible polynomials of $K[X]$. Then*

1. *The Krull dimension of Λ is* $\mathrm{Kdim}(\Lambda) = 2$.
2. *The global dimension of Λ is* $\mathrm{gldim}(\Lambda) = 2$.
3. *The elements $p_1, \ldots, p_s$ are regular normal elements of the algebra Λ (i.e. p_i is a non-zero-divisor of Λ and $p_i\Lambda = \Lambda p_i$).*
4. $\mathrm{Spec}(\Lambda) = \mathrm{Spec}_c(\Lambda) = \{0, \Lambda p_i, (p_i, q_i) \,|\, i = 1, \ldots, s;\ q_i \in \mathrm{Irr}_m(F_i[Y])\}$ *where $F_i := K[X]/(p_i)$ is a field and $\mathrm{Irr}_m(F_i[Y])$ is the set of monic irreducible polynomials of the polynomial algebra $F_i[Y]$ over the field F_i in the variable Y. If, in addition, the field K is an algebraically closed and $\lambda_1, \ldots, \lambda_s$ are the roots of the polynomial f then* $\mathrm{Spec}(\Lambda) = \{0, \Lambda(X - \lambda_i), (X - \lambda_i, Y - \mu) \,|\, i = 1, \ldots, s;\ \mu \in K\}$.

The proof of Theorem 1.1 is given in Sect. 3.

2 Classification of Simple Λ-Modules

In this section, 'module' means left module, K is an algebraically closed field of characteristic zero, $\Lambda = K[X][Y, \delta = f\frac{d}{dX}]$ where $f \in K[X]\backslash K$. The algebra Λ is a Noetherian domain. The aim of the section is to give a classification of simple Λ-modules (Lemma 2.1 and Theorem 2.10).

The element f is a regular normal element of Λ. It follows from

$$fY = Yf - f'f = (Y - f')f, \quad \text{where} \quad f' = \frac{df}{dX},$$

that the element f is a *normal* element of Λ (i.e. $\Lambda f = f\Lambda$). It determines a K-automorphism ω_f of the algebra Λ:

$$fu = \omega_f(u)f, \quad u \in \Lambda,$$
$$\omega_f : X \mapsto X, \quad Y \mapsto Y - f'.$$

The algebra Λ can be identified with a subalgebra of the first Weyl algebra A_1 by the map

$$\Lambda \to A_1, \quad X \mapsto X, \quad Y \mapsto f\partial. \tag{1}$$

The Weyl algebra A_1 is a generalized Weyl algebra. The Weyl algebra A_1 is a simple Noetherian domain with *restricted minimum condition*, i.e. any proper left or right factor module of A_1 has finite length, [10].

Definition, [1,2]. Let D be a ring, σ be an automorphisms of D and a be a central element of D. A **generalized Weyl algebra** (GWA) $A = D(\sigma, a)$ of degree 1, is the ring generated by D and by two indeterminates X and Y subject to the relations [1,2]: For all $\alpha \in D$,

$$X\alpha = \sigma(\alpha)X \quad \text{and} \quad Y\alpha = \sigma^{-1}(\alpha)Y, \quad YX = a \quad \text{and} \quad XY = \sigma(a).$$

The algebra

$$A = \bigoplus_{n\in\mathbb{Z}} A_n$$

is a $\mathbb{Z}$-graded algebra, where $A_n = Dv_n,\ v_n = X^n\ (n > 0),\ v_n = Y^{-n}\ (n < 0),\ v_0 = 1$.

The Weyl algebra A_1 is a GWA,

$$A_1 = D(\sigma, a = H),\ X \leftrightarrow X,\ \partial \leftrightarrow Y,\ \partial X \leftrightarrow H,\quad D = K[H],$$

with coefficients from a polynomial ring $K[H]$ where $\sigma \in \mathrm{Aut}_K\, K[H]$ and $\sigma : H \to H - 1$.

We denote by Λ_f (resp., $A_{1,f}$) the localization of the ring Λ (resp., A_1) at the powers of the element f, i.e.

$$\Lambda_f = S_f^{-1}\Lambda, \quad (\text{resp.,} \quad A_{1,f} = S_f^{-1}A_1) \quad \text{where} \quad S_f = \{f^i, i \geq 0\}.$$

By (1), Λ_f is a subalgebra of $A_{1,f}$ such that

$$\Lambda_f = A_{1,f}. \tag{2}$$

The algebras Λ and A_1 can be considered as subalgebras of $A_{1,f}$,

$$\Lambda \subseteq A_1 \subseteq \Lambda_f = A_{1,f}. \tag{3}$$

The algebra $A_{1,f}$ is a simple Noetherian domain with restricted minimum condition.

$\hat{\Lambda}(f - \text{torsion})$. The sets of isoclasses of Λ-modules $\hat{\Lambda}$ and of A_1-modules $\hat{A}_1$ are disjoint unions of f-*torsion* ($M_f = 0$) and f-*torsionfree* ($M_f \neq 0$) simple Λ-modules and A_1-modules, respectively,

$$\hat{\Lambda} = \hat{\Lambda}(f - \text{torsion}) \coprod \hat{\Lambda}(f - \text{torsionfree}), \tag{4}$$

$$\hat{A}_1 = \hat{A}_1(f - \text{torsion}) \coprod \hat{A}_1(f - \text{torsionfree}). \tag{5}$$

Lemma 2.1 is a classification of simple f-torsion Λ-modules.

Lemma 2.1 *Let $\lambda_1, \ldots, \lambda_s$ be the roots of the polynomial f. Then*

$$\hat{\Lambda}(f - \text{torsion}) = \{[\Lambda/\Lambda(X - \lambda_i, Y - \mu)] \mid i = 1, \ldots, s;\ \mu \in K\}.$$

All these Λ-modules are 1-dimensional and they are the only simple finite dimensional Λ-modules (by Theorem 2.10).

Proof Each simple f-torsion Λ-module M is annihilated by the (normal) element f ($fM = 0$). So, in fact, the Λ-module M is a simple module over the factor algebra

$$\Lambda/(f) = K[X, Y]/(f)$$

which is isomorphic to the factor algebra of the polynomial ring $K[X, Y]$ in two variables at the ideal (f) generated by f, and the equality in the lemma follows.

Let N be a simple Λ-module. Then the map

$$f_N : N \to N,\ n \mapsto fn$$

is either 0 or a bijection (f is normal in Λ). In the second case N is, in fact, a simple ($\Lambda_f \equiv A_{1,f}$)-module, so $\dim_K N = \infty$, since $A_{1,f}$ is a simple infinite dimensional algebra. If the module N is finite dimensional, then $fN = 0$, i.e. $[N] \in \hat{\Lambda}(f - \text{torsion})$. □

$\hat{\Lambda}(f - \text{torsionfree})$. The sum of all simple submodules of a Λ-module M is called the *socle* of M which is denoted by $\text{soc}_{\Lambda} M$. It is the largest semisimple submodule of M. A Λ-module N is called Λ-*socle* (or, *socle*, for short) provided $\text{soc}_{\Lambda} N \neq 0$. Denote by $\hat{A}_1(\Lambda\text{-socle})$ the set of isoclasses of simple Λ-socle A_1-modules. The proof of the following lemma is evident (see [2, Lemma 3.4] for details).

Lemma 2.2 1. *The canonical map*

$$(\cdot)_f : \hat{\Lambda}(f - \text{torsionfree}) \to \hat{A}_{1,f}(\Lambda - \text{socle}), \ [M] \mapsto [M_f := A_{1,f} \otimes_{\Lambda} M]$$

is a bijection with inverse $[N] \to [\text{soc}_{\Lambda}(N)]$.

2. *Each simple* f*-torsionfree* Λ*-module has the form*

$$M_{\mathfrak{m}} := \Lambda/\Lambda \cap \mathfrak{m} \tag{6}$$

for some maximal left ideal $\mathfrak{m}$ *of the ring* $A_{1,f}$. *Two such modules are isomorphic,* $M_{\mathfrak{m}} \simeq M_{\mathfrak{n}}$, *iff the* $A_{1,f}$*-modules* $A_{1,f}/\mathfrak{m}$ *and* $A_{1,f}/\mathfrak{n}$ *are isomorphic.* □

Lemma 2.3 *Let* $\lambda_1, \dots, \lambda_s$ *be the roots of the polynomial* f. *Then*

$$\hat{A}_1(f - \text{torsion}) = \{[M_i := A_1/A_1(X - \lambda_i)] \mid i = 1, \dots, s\}.$$

Proof As a vector space the module M_i can be identified with the polynomial ring $K[y]$ in a variable $y = \partial + A_1(X - \lambda_i)$ and

$$\partial y^j = y^{j+1}, \quad X y^j = \lambda_i y^j + \cdots \quad \text{for} \quad j \geq 0$$

where by three dots we denote the sum of elements of smaller degree in the variable y. Thus the linear operator

$$X - \mu : M_i \to M_i, \quad m \mapsto (X - \mu)m, \quad m \in M_i,$$

is nilpotent if and only if $\mu = \lambda_i$; otherwise, $X - \mu$ is an isomorphism of the vector space M_i. From this fact it follows that the A_1-modules $\{M_i\}$ are simple and non-isomorphic.

Now, let $[M] \in \hat{A}_1(f - \text{torsion})$. Then there exists i such that M is an epimorphic image of M_i, hence $M \simeq M_i$. □

Theorem 2.4 *The map*

$$\hat{A}_1(f - \text{torsionfree}) = \hat{A}_1 \backslash \{[M_1], \dots, [M_s]\} \to \hat{A}_{1,f}, \quad [M] \mapsto [M_f]$$

is bijective.

Proof The map above is well defined and injective.

Let $[N] \in \hat{A}_{1,f}$. Then $N \simeq A_{1,f}/J$ for some nonzero maximal left ideal J of $A_{1,f}$. Then $I = J \cap A_1 \neq 0$ and A_1/I is a A_1-submodule of N. The A_1-module A_1/I has finite length [10], thus it contains a simple A_1-submodule, say M. Then $N \simeq M_f$ which means that the map above is surjective. □

Recall that $D = K[H]$. The localization $B = S^{-1}A_1$ of the Weyl algebra A_1 at the Ore set $S = D\backslash\{0\}$ is a *skew Laurent polynomial ring*

$$B = K(H)[X, X^{-1}; \sigma], \quad \sigma(H) = H - 1,$$

with coefficients from the field $K(H)$ of rational functions. The algebra B is a right and left *Euclidean* domain with respect to the 'length' function

$$l : B\backslash\{0\} \to \mathbb{N} := \{0, 1, 2, \dots\}, \quad l(\alpha X^m + \beta X^{m+1} + \cdots + \gamma X^n) = n - m, \quad \alpha \neq 0, \ \gamma \neq 0 \in K(H),$$

hence, it is a right and left principal ideal domain.

We have

$$\hat{A}_1 = \hat{A}_1(D - \text{torsion}) \coprod \hat{A}_1(D - \text{torsionfree})$$

where a simple A_1-module M belongs to the first (resp., second) set if $S^{-1}M = 0$ (resp., $S^{-1}M \neq 0$).

For $\lambda \in K$ set $\mathcal{O}(\lambda) := \lambda + \mathbb{Z}$. We say that scalars λ and μ are *equivalent*, $\lambda \sim \mu$, if either $\mathcal{O}(\lambda) = \mathcal{O}(\mu) \neq \mathbb{Z}$ or both λ and μ belong either to $(-\infty, 0] := \{i \in \mathbb{Z} \mid i \leq 0\}$ or to $[1, \infty) := \{i \in \mathbb{Z} \mid i \geq 1\}$. Then $\sim$ is an equivalence relation on K. Let $K/\sim$ be the set of equivalence classes of K under $\sim$. So, the elements of the set $K/\sim$ are distinct sets $\lambda + \mathbb{Z}$ where $\lambda \notin \mathbb{Z}$ and the two sets $(-\infty, 0]$ and $[1, \infty)$. Notice that $\mathbb{Z} = (-\infty, 0] \coprod [1, \infty)$.

Proposition 2.5 ([2, Theorem 3.1]) *The map*

$$K/\sim \,\to\, \hat{A}_1(D - \text{torsion}), \quad [\Gamma] \mapsto [L(\Gamma)],$$

is a bijection, where

1. *If* $\Gamma = \mathcal{O}(\lambda) \neq \mathbb{Z}$, *then* $L(\Gamma) = A_1/A_1(H - \lambda)$.
2. *If* $\Gamma = (-\infty, 0]$, *then* $L(\Gamma) = A_1/A_1X$.
3. *If* $\Gamma = [1, \infty)$, *then* $L(\Gamma) = A_1/A_1(H - 1, Y)$. □

Corollary 2.6

$$\hat{A}_1(D - \text{torsion}, f - \text{torsion}) = \begin{cases} \{[L((-\infty, 0]) = A_1/A_1X]\} & \text{if } 0 \text{ is a root of } f(X), \\ \emptyset & \text{if } 0 \text{ is not a root of } f(X). \end{cases}$$

Proof Straightforward. □

Corollary 2.7 *Let* $[M] \in \hat{A}_1(D - \text{torsion}, f - \text{torsionfree})$.

1. *If* $M = A_1/A_1X$ *(i.e.* 0 *is not a root of* f, *by Corollary 2.6) then* M *is a simple* f*-torsionfree* Λ*-module with* $M = M_f$.
2. *If* $M \neq A_1/A_1X$ *then* $\mathrm{soc}_\Lambda M = \mathrm{soc}_\Lambda M_f = 0$. *The set* $\hat{\Lambda}(D - \text{torsion}, f - \text{torsionfree})$ *is equal to* $\{A_1/A_1/X\}$ *if 0 is not a root of f and* $\emptyset$, *otherwise.*

Proof 1. As a vector space the module $M = A_1/A_1X$ has the basis $\{y^i = \partial^i + A_1X, \ i \geq 0\}$, and

$$\partial y^i = y^{i+1}, \quad Xy^i = -iy^{i-1} \ \text{ and } \ Yy^i = f(0)y^{i+1} + \sum_{0 \leq j \leq i} \mu_j y^j,$$

for some scalars $\mu_j \in K$. Now, it is obvious that the Λ-module M is a simple f-torsionfree Λ-module ($f(0) \neq 0$). Moreover, the linear map $f_M : M \to M, m \mapsto fm$ is a bijection, hence, $M = M_f$.

2. Since $\Lambda_f = A_{1,f}$, $\mathrm{soc}_\Lambda M = \mathrm{soc}_\Lambda M_f$. Let M belongs to the first (resp., third) class of modules from Proposition 2.5 , i.e.

$$M = L(\Gamma), \quad \Gamma = \mathcal{O}(\lambda) \neq \mathbb{Z} \ \text{ (resp., } \ \Gamma = [1, \infty)).$$

The element $\bar{1} = 1 + A_1(H - \lambda)$ (resp., $\bar{1} = 1 + A_1(H - 1, Y)$) is a canonical generator of the A_1-module M. In both cases, for $i \geq 0$, set $x^i = X^i\bar{1}$. In the first case, for $i < 0$, set $x^i = \mu_i \partial^{-i}\bar{1}$, $\mu_i \in K$. The scalars μ_i can be chosen in such a way that (in both cases) $Xx^i = x^{i+1}$ for all possible i. Degree argument shows that the module M contains a strictly descending chain of Λ-submodules

$$M \supset fM \supset \cdots \supset f^nM \supset \cdots \ \text{ with } \ \bigcap_{n \geq 0} f^nM = 0.$$

Suppose that $N := \mathrm{soc}_\Lambda M \neq 0$, then, in a view of Lemma 2.2 and Theorem 2.4, N is an essential simple Λ-submodule of both M_f and M, hence $0 \neq N \subseteq \cap_{n \geq 0} f^nM = 0$, a contradiction. □

An element of a ring is called *regular* if it is not a zero divisor. Given a ring A and a multiplicatively closed subset S of A which consists of regular normal elements. Let $B = S^{-1}A$ be the localization of A at S.

Theorem 2.8 *Let A, B, and S be as above and let $\mathfrak{m}$ be a maximal left ideal of B. The following are equivalent.*

1. *The A-module $M_\mathfrak{m} := A/A\cap \mathfrak{m}$ is simple.*
2. *The socle* $\mathrm{soc}_A(M_\mathfrak{m}) \neq 0$.
3. $A = As + A\cap\mathfrak{m}$ *for all* $s \in S$.

□

Remark. If $S = \{f^n, n \geq 0\}$ for some regular normal element f of A, then the last condition of this lemma is equivalent to $A = Af + A\cap\mathfrak{m}$. We shall use this fact in what follows. In general situation, it suffices to check whether the third condition holds only for generators of the monoid S.

Proof The implications $(1 \Rightarrow 2)$ and $(1 \Rightarrow 3)$ are obvious.

$(2 \Rightarrow 1)$ If $\mathrm{soc}_A(M_\mathbf{m}) \neq 0$ then it is a simple A-module which for some $s \in S$ is equal to

$$(As + A\cap\mathfrak{m})/A\cap\mathfrak{m} \simeq As/As\cap\mathfrak{m} \simeq A/A\cap\mathfrak{m}s^{-1} = \omega_s(A)/\omega_s(A\cap\mathfrak{m}) \simeq {}^{\omega_s^{-1}}M_\mathfrak{m},$$

where ${}^{\omega_s^{-1}}M_\mathfrak{m}$ is the twisted A-module $M_\mathfrak{m}$ by the automorphism ω_s^{-1} of A (the element s is regular and normal). Since the A-module ${}^{\omega_s^{-1}}M_\mathfrak{m}$ is simple, so is $M_\mathfrak{m}$.

$(3 \Rightarrow 1)$ If J is a left ideal of A which contains $A\cap\mathfrak{m}$ but does not coincide with it, then, by the maximality of $\mathfrak{m}$, $S^{-1}J = B$. Therefore $J\cap S \neq \emptyset$. Let $s \in J\cap S$. Then $J \supseteq As + A\cap\mathfrak{m} = A$, that is $M_\mathfrak{m}$ is a simple A-module. □

$\hat{A}_1(D - \text{torsionfree})$. Let us recall a description of $\hat{A}_1(D - \text{torsionfree})$ from [2]. In the set $S = K[H]\backslash\{0\}$ consider the relation $<$: $\alpha < \beta$ if there are no roots λ and μ of the polynomial α and β respectively and such that $\lambda - \mu$ is non-negative integer.

Definition, [2]. An element $b = Y^m\beta_{-m} + \cdots + \beta_0 \in A_1, m > 0$, all $\beta_i \in D$, is called **l-normal** if $\beta_0 < \beta_{-m}$ and $\beta_0 < H$, (i.e. the polynomial β_0 has no root from $\{0, 1, 2, \ldots\}$ and there are no roots λ and μ of the polynomials β_0 and β_m respectively with $\lambda - \mu \in \{0, 1, 2, \ldots\}$).

Theorem 2.9 ([2, Theorem 3.8]) *Let $b = Y^m\beta_{-m} + \cdots + \beta_0 \in A_1$, $m > 0$, all $\beta_i \in D$, be an l-normal and irreducible element in B. Then*

$$\mathcal{M}_b := A_1/A_1 \cap Bb$$

is a simple D-torsionfree A_1-module. Two such A_1-modules are isomorphic, $\mathcal{M}_b \simeq \mathcal{M}_c$, iff $B/Bb \simeq B/Bc$ as B-modules. Each simple D-torsionfree A_1-module is isomorphic to some $\mathcal{M}_b$. □

Set

$$B_f := S_f^{-1}B = A_{1,f} \otimes_\Lambda B = \Lambda_f \otimes_\Lambda B$$

for the *localization of the (left) Λ-module B* at S_f. Then the algebra $A_{1,f} = \Lambda_f$ can be considered as a $(A_{1,f} = \Lambda_f)$-submodules of B_f. For any nonzero $b \in B$, $(Bb)_f = B_f b$.

Theorem 2.10 is a classification of simple f-torsionfree Λ-modules.

Theorem 2.10 *Let $b = Y^m\beta_{-m} + \cdots + \beta_0 \in A_1$, $m > 0$, all $\beta_i \in D$, be an l-normal and irreducible element in B such that*

1. $\Lambda = \Lambda f + \Lambda \cap B_f b$ $(= \Lambda f + \Lambda \cap Bb)$, *and*
2. *the simple B-module B/Bb is not isomorphic to any of modules $B/B(X-\lambda)$ where λ runs through the nonzero roots of f.*

Then

$\mathcal{M}_b := \Lambda/\Lambda \cap Bb \ \ (= \Lambda/\Lambda \cap B_f b)$

is a simple f-torsionfree Λ-module. Two such Λ-modules are isomorphic, $\mathcal{M}_b \simeq \mathcal{M}_c$, iff $B/Bb \simeq B/Bc$ as B-modules.

Each simple f-torsionfree Λ-module is isomorphic either to some $\mathcal{M}_b$ or to the module $M = A_1/A_1X$ from Corollary 2.7, if 0 is not a root of f (the Λ-module M is not isomorphic to any $\mathcal{M}_b$). The condition 1 above is equivalent to the condition that $\Lambda = \Lambda(X - \lambda_i) + \Lambda \cap B_f b$ $(= \Lambda(X - \lambda_i) + \Lambda \cap Bb)$ for all roots λ_i of the polynomial f.

Each simple f-torsionfree Λ-module is infinite dimensional.

Proof By Lemma 2.2,

$[M] \in \hat{\Lambda}(f - \text{torsionfree}) \Leftrightarrow [M_f] \in \hat{A}_{1,f}(\Lambda - \text{socle})$

and $M = \text{soc}_\Lambda(M_f) \simeq \Lambda/\Lambda \cap \mathfrak{m}$ for some maximal left ideal $\mathfrak{m}$ of $A_{1,f}$. By Corollary 2.7, either $M_f \simeq A_1/A_1X$ (0 is not a root of f) or $M_f \in \hat{A}_{1,f}(D - \text{torsionfree}, \Lambda - \text{socle})$. In the first case, $M = \text{soc}_\Lambda(M_f) = M_f = A_1/A_1X$ (Corollary 2.7).

In the second case, by Theorems 2.4 and 2.9,

$M_f \simeq (\mathcal{M}_b)_f = A_{1,f}/A_{1,f} \cap B_f b$

for some l-normal irreducible element b from Theorem 2.9. Note that the left ideal $\mathfrak{m} = A_{1,f} \cap B_f b$ of $A_{1,f}$ is maximal. By Lemma 2.3 and Theorem 2.9, $[\mathcal{M}_b] \in \hat{A}_1(D - \text{torsionfree}, f - \text{torsionfree})$ iff the second condition of the theorem holds. Now,

$$\text{soc}_\Lambda(M_f) = \text{soc}_\Lambda(\mathcal{M}_b)_f = \text{soc}_\Lambda(\Lambda/\Lambda \cap A_{1,f} \cap B_f b) = \text{soc}_\Lambda(\Lambda/\Lambda \cap B_f b). \quad (7)$$

By Theorem 2.8 and by the Remark after it,

$\text{soc}_\Lambda(M_f) \neq 0 \ \text{ iff } \ \Lambda = \Lambda f + \Lambda \cap (A_{1,f} \cap B_f b) = \Lambda f + \Lambda \cap B_f b.$

In this case,

$\text{soc}_\Lambda(M_f) = \Lambda/\Lambda \cap (A_{1,f} \cap B_f b) = \Lambda/\Lambda \cap B_f b.$

Let us show that (in this case) the natural Λ-module epimorphism

$\varphi : M_b = \Lambda/\Lambda \cap Bb \to \Lambda/\Lambda \cap B_f b, \ \ \lambda + \Lambda \cap Bb \to \lambda + \Lambda \cap B_f b,$

is an isomorphism. Note that

$\ker \varphi = \Lambda \cap B_f b/\Lambda \cap Bb.$

The A_1-module $\mathcal{M}_b$ is a submodule of $(\mathcal{M}_b)_f \simeq M_f$. So,

$\text{soc}_\Lambda(M_f) = \text{soc}_\Lambda(\mathcal{M}_b) = \text{soc}_\Lambda(\Lambda/\Lambda \cap Bb).$

By assumption $\text{soc}_\Lambda(M_f) \neq 0$, then it is a simple essential f-torsionfree Λ-submodule of M_f. If $\ker \varphi \neq 0$, then $\text{soc}_\Lambda(M_f) \subseteq \ker \varphi$, but $\ker \varphi$ is an f-torsion Λ-module, a contradiction.

Let $\mathcal{M}_b$ and $\mathcal{M}_c$ be as in the theorem. By Lemma 2.2, $\mathcal{M}_b \simeq \mathcal{M}_c$ as Λ-modules $\Leftrightarrow A_{1,f} \otimes_\Lambda \mathcal{M}_b \simeq A_{1,f} \otimes_\Lambda \mathcal{M}_c$ as $A_{1,f}$-modules. Since

$A_{1,f} \otimes_\Lambda \mathcal{M}_b \simeq A_{1,f}/A_{1,f} \cap B_f b \simeq (\mathcal{M}_b)_f,$

by Theorem 2.4, the above $A_{1,f}$-modules are isomorphic iff $\mathcal{M}_b \simeq \mathcal{M}_c$ as A_1-modules, so, by Theorem 2.9, $B/Bb \simeq B/Bc$ as B-modules.

The condition 1 of the theorem is equivalent to the condition that $\Lambda = \Lambda(X-\lambda_i) + \Lambda \cap B_f b$ $(= \Lambda(X-\lambda_i) + \Lambda \cap Bb)$ for all roots λ_i of the polynomial f (since the elements $X-\lambda_i$ are regular normal elements of Λ and λ_i are the roots of f).

By Lemma 2.1, each simple f-torsionfree Λ-module is infinite dimensional. If 0 is not a root of f, then the modules $M = A_1/A_1X$ and $\mathcal{M}_b$ (from the theorem) are not isomorphic, since the linear map $X_M : M \to M$, $m \mapsto Xm$ is locally nilpotent but $\ker X_{\mathcal{M}_b} = 0$. □

3 The Prime Ideals, the Krull and Global Dimensions of the Algebra Λ

In this section, K is a field of characteristic zero (not necessarily algebraically closed) and $f = p_1^{n_1} \cdots p_s^{n_s}$ is a nonscalar polynomial of $K[X]$ where $p_1, \dots, p_s$ are irreducible, co-prime divisors of f (i.e. $K[X]p_i + K[X]p_j = K[X]$ for all $i \neq j$). The aim of this section is to give a proof of Theorem 1.1.

Proof of Theorem 1.1 3. The elements $p_1, \dots, p_s$ are regular normal elements of the algebra Λ since

$$Yp_i = p_i(Y - p_i^{-1}f) \quad \text{and} \quad Xp_i = p_iX.$$

4. The algebra Λ is a domain, hence $0 \in \mathrm{Spec}_c(\Lambda)$.

Since

$$\Lambda/\Lambda p_i \simeq F_i[Y] \tag{8}$$

is a polynomial algebra with coefficients in the field F_i (since $YX - XY = f \in \Lambda p_i$), the ideal Λp_i is a completely prime ideal of Λ.

By (3), $\Lambda_f = A_{1,f}$ is a simple algebra (as a localization of a simple Noetherian algebra). If $\mathfrak{p}$ is a nonzero prime ideal of the algebra Λ then $f^n \in \mathfrak{p}$ for some natural number $n \geq 1$. Hence, $p_i \in \mathfrak{p}$ for some i, by statement 3. By (8), $\mathfrak{p} = (p_i, g_i)$ for some monic irreducible polynomial g_i of the polynomial algebra $F_i[Y]$.

1. By [11, Theorem 6.5.4.(i)], $\mathrm{Kdim}(\Lambda) \leq \mathrm{Kdim}(K[X]) + 1 = 1 + 1 = 2$.

Since p_i is a regular normal element of the algebra Λ,

$$\mathrm{Kdim}(\Lambda) \geq \mathrm{Kdim}(\Lambda/\Lambda p_i) + 1 \overset{(8)}{=} \mathrm{Kdim}(F_i[Y]) + 1 = 1 + 1 = 2,$$

by [11, Theorem 6. 5.9]. Therefore, $\mathrm{Kdim}(\Lambda) = 2$.

2. By [11, Theorem 7.5.3.(i)], $\mathrm{gldim}(\Lambda) \leq \mathrm{gldim}(K[X]) + 1 = 1 + 1 = 2$.

By (8), $\mathrm{gldim}(\Lambda/\Lambda p_i) = \mathrm{gldim}(F_i[Y]) = 1 < \infty$. Now, by [11, Theorem 7.3.5.(i)],

$$\mathrm{gldim}(\Lambda) \geq \mathrm{gldim}(\Lambda/\Lambda p_i) + 1 \overset{(8)}{=} \mathrm{gldim}(F_i[Y]) + 1 = 1 + 1 = 2.$$

Therefore, $\mathrm{gldim}(\Lambda) = 2$. □

References

1. Bavula, V.V.: The finite-dimensionality of Ext^n's and Tor_n's of simple modules over a class of algebras. Funktion. Anal. i Prilozhen. **25**(3), 80–82 (1991)

2. Bavula, V.V.: Generalized Weyl algebras and their representations. Algebra i Analiz **4**(1), 75–97 (1992). English transl. in St.Petersburg Math. J. **4**(1) (1993), 71–92
3. Bavula, V.V.: Generalized Weyl algebras, kernel and tensor-simple algebras, their simple modules, representations of algebras. In: Dlab, V., Lenzing, H. (eds.) Sixth International Conference, August 19–22, 1992. CMS Conference proceedings, vol. 14, pp. 83–106 (1993)
4. Bavula, V.V.: The simple modules of the ore extensions with coefficients from Dedekind ring. Commun. Algebra **27**(6), 2665–2699 (1999)
5. Bavula, V.V., Lu, T.: Prime ideals of the enveloping algebra of the Euclidean algebra and a classification of its simple weight modules. J. Math. Phys. **58**(1), 011701 (2017). https://doi.org/10.1063/1.4973378
6. Bavula, V.V., Lu, T.: Classification of simple weight modules over the Schrödinger algebra. Canad. Math. Bull. **61**(1), 16–39 (2018)
7. Bavula, V.V., Lu, T.: The universal enveloping algebra $U(_2 \ltimes V_2)$, its prime spectrum and a classification of its simple weight modules. J. Lie Theory **28**(2), 525–560 (2018)
8. Bavula, V.V., Cai, Y., Lü, R., Tan, H., Zhao, K.: The Polyonic Algebras and their simple modules (work in progress)
9. Block, R.E.: The irreducible representations of the Lie algebra $sl(2)$ and of the Weyl algebra. Adv. Math. **39**, 69–110 (1981)
10. Dixmier, J.: Sur les algébres de Weyl, II. Bull. Sci. Math. **94**(2), 289–301 (1970)
11. McConnell, J.C., Robson, J.C.: Noncommutative Noetherian rings. With the cooperation of L. W. Small. Revised edition. In; Graduate Studies in Mathematics, 30. American Mathematical Society, Providence, RI (2001)