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EXTENDED r-SPIN THEORY AND THE MIRROR SYMMETRY FOR THE A_{r-1}-SINGULARITY

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Abstract. By a famous result of K. Saito, the parameter space of the miniversal deformation of the A_{r-1}-singularity carries a Frobenius manifold structure. The Landau–Ginzburg mirror symmetry says that, in the flat coordinates, the potential of this Frobenius manifold is equal to the generating series of certain integrals over the moduli space of r-spin curves. In this paper we show that the parameters of the miniversal deformation, considered as functions of the flat coordinates, also have a simple geometric interpretation using the extended r-spin theory, first considered by T. J. Jarvis, T. Kimura and A. Vaintrob [JKV01b], and studied in a recent paper of E. Clader, R. J. Tessler and the author [BCT19]. We prove a similar result for the singularity D_4 and present conjectures for the singularities E_6 and E_8.

1. Introduction

The Landau–Ginzburg mirror symmetry conjecture originates from an old physical construction of P. Berglund and T. Hübsch [BH93]. Let us very briefly recall the general statement.

Let $N \geq 1$ and let us fix a matrix $A = (a_{ij})_{1\leq i,j\leq N}$ with non-negative integer entries a_{ij}. Consider the polynomial $W(x_1,\ldots,x_N)$ and its mirror partner $W^T(x_1,\ldots,x_N)$, defined by

$$W(x_1,\ldots,x_N) := \sum_{i=1}^N \prod_{j=1}^N x_j^{a_{ij}}, \qquad W^T(x_1,\ldots,x_N) := \sum_{i=1}^N \prod_{j=1}^N x_j^{a_{ji}}.$$

Suppose that the polynomial W is quasihomogeneous, has an isolated critical point at the origin and $\det A \neq 0$. Quasihomogeneity means that there exist positive rational numbers $q_1,\ldots,q_N$ such that

$$W(\lambda^{q_1}x_1,\lambda^{q_2}x_2,\ldots,\lambda^{q_N}x_N) = \lambda W(x_1,\ldots,x_N),$$

for each $\lambda \in \mathbb{C}^*$. There are two theories, associated to the polynomial W. They are usually called the A-model and the B-model.

The A-model is the Fan–Jarvis–Ruan–Witten (FJRW) theory ([FJR13, FJR07, Wit93]) of the pair (W, G_W), where G_W is the maximal group of diagonal symmetries of the polynomial W:

$$G_W := \left\{ (\lambda_1,\ldots,\lambda_N) \in (\mathbb{C}^*)^N \middle| W(\lambda_1 x_1,\ldots,\lambda_N x_N) = W(x_1,\ldots,x_N) \right\}.$$

The main object in this theory is the moduli space of W-orbicurves. Recall that an orbifold curve C with marked points $p_1,\ldots,p_n$ is a (possibly nodal) Riemann surface C with orbifold structure at each p_i and each node. Moreover, we require that the local picture at each node is $\{xy=0\}/\mathbb{Z}_m$, for some $m \geq 1$, where the action of the group $\mathbb{Z}_m$ of m-th roots of unity is given by $\zeta_m\cdot(x,y) = (\zeta_m x, \zeta_m^{-1} y)$, $\zeta_m = e^{\frac{2\pi i}{m}}$. For an orbifold curve C denote by $\rho\colon C \to |C|$ the forgetful map to the underlying (coarse, or non-orbifold) curve $|C|$. A W-orbicurve is a marked

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orbifold curve $(C;p_1,\ldots,p_n)$ together with a collection of orbifold line bundles $L_1,\ldots,L_N$ over C and isomorphisms

$$\phi_i\colon \bigotimes_{j=1}^N L_j^{\otimes a_{ij}} \xrightarrow{\sim} \rho^*\left(\omega_{|C|}\left(\sum\nolimits_{i=1}^n p_i\right)\right), \quad 1\le i\le N,$$

where $\omega_{|C|}$ is the canonical line bundle on $|C|$. Suppose that the local group at a marked point p_i of C is $\mathbb{Z}_{m_i}$, $m_i\ge 1$. Then the line bundles $L_1,\ldots,L_N$ induce a representation $\theta_i\colon \mathbb{Z}_{m_i}\to(\mathbb{C}^*)^N$. Our W-orbicurve is called stable if the underlying marked curve $(|C|;p_1,\ldots,p_n)$ is stable and if for each marked point p_i the representation $\theta_i\colon\mathbb{Z}_{m_i}\to(\mathbb{C}^*)^N$ is faithful.

In [FJR13] the authors proved that the moduli space of stable W-orbicurves of genus g with n marked points is a smooth compact orbifold. It is denoted by $\overline{\mathcal{W}}_{g,n}$. The moduli space $\overline{\mathcal{W}}_{g,n}$ is not connected. Numerical invariants of the representations θ_i, $1\le i\le n$, give a decomposition of the moduli space $\overline{\mathcal{W}}_{g,n}$ into open and closed components. Consider now the case $g=0$. In [FJR07] the authors constructed a virtual fundamental class on each component of $\overline{\mathcal{W}}_{0,n}$ and defined the corresponding intersection number. All these intersection numbers for all components of $\overline{\mathcal{W}}_{0,n}$ and for all n can be naturally written as the coefficients of a generating series, which is a formal power series in variables $t_0,\ldots,t_{\mu^T-1}$, $\mu^T\ge 1$, with rational coefficients. Here the number μ^T is equal to the dimension of the local algebra

$$\mathcal{A}_{W^T}:=\mathcal{O}_{\mathbb{C}^N,0}\Big/\left(\frac{\partial W^T}{\partial x_1},\ldots,\frac{\partial W^T}{\partial x_N}\right)=\mathbb{C}[x_1,\ldots,x_N]\Big/\left(\frac{\partial W^T}{\partial x_1},\ldots,\frac{\partial W^T}{\partial x_N}\right)$$

of the singularity of W^T at the origin, where by $\mathcal{O}_{\mathbb{C}^N,0}$ we denote the ring of germs of holomorphic functions on $\mathbb{C}^N$ at the origin. The generating series of the intersection numbers is denoted by

$$\mathcal{F}^{\mathrm{FJRW}}_{0,W}(t_0,\ldots,t_{\mu^T-1})\in\mathbb{Q}[[t_0,\ldots,t_{\mu^T-1}]].$$

In [FJR07] the authors proved that the function $\mathcal{F}^{\mathrm{FJRW}}_{0,W}$ satisfies the WDVV equations and, therefore, defines a Frobenius manifold structure in a formal neighbourhood of $0\in\mathbb{C}^{\mu^T}$. Frobenius manifolds were introduced and studied in detail by B. Dubrovin in [Dub96]. For a more detailed introduction to the FJRW theory, we refer a reader to the original papers [FJR13, FJR07].

The B-model is the Saito Frobenius manifold structure on the parameter space of a miniversal deformation of the singularity of the polynomial W. A miniversal deformation (also called a universal unfolding) of the singularity of W is a deformation

$$W_s(x_1,\ldots,x_N)=W(x_1,\ldots,x_N)+s_0+\sum_{i=1}^{\mu-1}s_i\phi_i(x_1,\ldots,x_N), \tag{1.1}$$
$$\phi_i(x_1,\ldots,x_N)\in\mathbb{C}[x_1,\ldots,x_N],\quad s_i\in\mathbb{C},$$

where the polynomials $\phi_0:=1,\phi_1,\ldots,\phi_{\mu-1}$ form a basis of the local algebra $\mathcal{A}_W$ of W at the origin and μ is the dimension of $\mathcal{A}_W$.

The Frobenius manifold structure on the parameter space $\mathbb{C}^\mu=\{(s_0,\ldots,s_{\mu-1})|s_i\in\mathbb{C}\}$ of the miniversal deformation (1.1) is constructed in the following way. Consider the deformation $W_s(x_1,\ldots,x_N)$, as a function on $\mathbb{C}^N\times\mathbb{C}^\mu$,

$$W_s(x_1,\ldots,x_N)\in\mathcal{O}_{\mathbb{C}^N\times\mathbb{C}^\mu,0},$$

and consider the ring

$$\widetilde{\mathcal{A}}_W:=\mathcal{O}_{\mathbb{C}^N\times\mathbb{C}^\mu}\Big/\left(\frac{\partial W_s}{\partial x_1},\ldots,\frac{\partial W_s}{\partial x_N}\right).$$

Via the natural projection $\mathbb{C}^N\times\mathbb{C}^\mu\to\mathbb{C}^\mu$ the ring $\widetilde{\mathcal{A}}_W$ becomes an $\mathcal{O}_{\mathbb{C}^\mu,0}$-algebra. Moreover, it is a free $\mathcal{O}_{\mathbb{C}^\mu,0}$-module with the basis $\phi_0(x),\ldots,\phi_{\mu-1}(x)$. Denote by $\mathcal{T}_{\mathbb{C}^\mu,0}$ the space of germs of sections of the holomorphic tangent bundle $T\mathbb{C}^\mu$ to $\mathbb{C}^\mu$ at the origin. It is also a free $\mathcal{O}_{\mathbb{C}^\mu,0}$-module with the basis $\frac{\partial}{\partial s_0},\dots,\frac{\partial}{\partial s_{\mu-1}}$. Let us identify the $\mathcal{O}_{\mathbb{C}^\mu,0}$-modules $\widetilde{\mathcal{A}}_W$ and $\mathcal{T}_{\mathbb{C}^\mu,0}$ by identifying the basis elements ϕ_i and $\frac{\partial}{\partial s_i}$, for each i. Since $\widetilde{\mathcal{A}}_W$ is an $\mathcal{O}_{\mathbb{C}^\mu,0}$-algebra, this construction endows the tangent bundle $T\mathbb{C}^\mu$ with a multiplication in a neighbourhood of the origin.

A metric $\frac{1}{2}\sum_{0\le i,j\le \mu-1} g_{ij}(s)ds_ids_j$ on $\mathbb{C}^\mu$ is defined in the following way. Define a bilinear form $\langle\cdot,\cdot\rangle$ on the local algebra $\mathcal{A}_W$ by

$$\langle p(x),q(x)\rangle := \frac{1}{(2\pi i)^N}\int_{\bigcap_{i=1}^N\left\{\left|\frac{\partial W}{\partial x_i}\right|=\varepsilon\right\}}\frac{p(x)q(x)dx_1\wedge\dots\wedge dx_N}{\prod_{i=1}^N\frac{\partial W}{\partial x_i}},\quad p(x),q(x)\in\mathcal{A}_W,$$

where ε is a sufficiently small positive number. This bilinear form is symmetric and non-degenerate [AGV85, Section 5.11]. K. Saito [Sai83] introduced the notion of a primitive form, which is a nowhere vanishing holomorphic form of top degree on $\mathbb{C}^\mu$ in a neighbourhood of the origin with certain properties. He proved that for such a form ζ the metric $g_{ij}(s)$ on $\mathbb{C}^\mu$, defined by

$$g_{ij}(s) := \frac{1}{(2\pi i)^N}\int_{\bigcap_{i=1}^N\left\{\left|\frac{\partial W_s}{\partial x_i}\right|=\varepsilon\right\}}\frac{\phi_i(x)\phi_j(x)\zeta}{\prod_{i=1}^N\frac{\partial W_s}{\partial x_i}},$$

for sufficiently small s_i's, is flat. Together with the multiplication in $T\mathbb{C}^\mu$, constructed above, this metric defines an analytical Frobenius manifold structure on $\mathbb{C}^\mu$ in a neighbourhood of the origin. The vector field $\frac{\partial}{\partial s_0}$ is the unit of it. Let us call this Frobenius manifold the Saito Frobenius manifold. The existence of a primitive form was proved in [Sai89]. The primitive forms for the simple singularities

$$\begin{array}{ll} A_r & W(x)=x^{r+1},\\ D_r & W(x_1,x_2)=x_1^{r-1}+x_1x_2^2,\\ E_6 & W(x_1,x_2)=x_1^4+x_2^3,\\ E_7 & W(x_1,x_2)=x_1^3x_2+x_2^3,\\ E_8 & W(x_1,x_2)=x_1^5+x_2^3,\end{array}$$

are given by $\lambda dx_1\wedge\dots\wedge dx_N$, $\lambda\in\mathbb{C}^*$. For a more detailed introduction to theory of the Saito Frobenius manifolds, we refer to the paper [ST08] and to the book [Hert02].

The Landau–Ginzburg mirror symmetry conjecture says that there exists a primitive form ζ such that the Saito Frobenius manifold, corresponding to the polynomial W, is isomorphic to the Frobenius manifold given by the function $\mathcal{F}^{\mathrm{FJRW}}_{0,W^T}$. A precise description of the necessary primitive form together with the isomorphism is given, for example, in [HLSW15]. The conjecture is proved in certain cases [JKV01a, FJR13, KS11, MS16, LLSS17]. A step towards a proof of the conjecture in the general case was made in [HLSW15], where the authors managed to prove the conjecture, assuming that certain small set of correlators in the A- and B-models agree (see Theorem 1.2 in [HLSW15] and the paragraph after it).

The Landau–Ginzburg mirror symmetry conjecture provides a beautiful link between the singularity theory and the geometry of the moduli spaces of curves. However, one can see that the relation between the A- and B-models, which this conjecture describes, is still not complete. Consider the parameters $s_i(t_*)$ of the miniversal deformation expressed as functions of the flat coordinates. As far as we know, a description of the functions $s_i(t_*)$ and also of the primitive form ζ in terms of the A-model are not known. Therefore, it is natural to ask the following question.

Question 1. *How to describe the functions $s_i(t_*)$ and the primitive form ζ in terms of the A-model?*

In this note we answer this question in the case of the A_{r-1}-singularity, where $N=1$ and $W(x)=x^r$, $r\ge 2$, $W^T=W$. The mirror symmetry conjecture in this case was proved in [JKV01a]. The primitive form is trivial, so our question is only about the functions $s_i(t_*)$. We have $\mu=r-1$ and the function $\mathcal{F}^{\mathrm{FJRW}}_{0,W}(t_0,\ldots,t_{r-2})$ can be described as the generating series of the so-called r-spin intersection numbers. The r-spin theory possesses a certain extension, which was first considered in [JKV01b] and then studied in [BCT19] from the point of view of integrable hierarchies. The generating series $\mathcal{F}^{\mathrm{ext}}(t_0,\ldots,t_{r-1})$ of the extended r-spin intersection numbers depends on the old variables $t_0,\ldots,t_{r-2}$ and also on an additional variable t_{r-1}. We prove that, up to certain rescaling parameters, the function $s_i(t_0,\ldots,t_{r-2})$ is equal to the coefficient of $(t_{r-1})^i$ in the series $\frac{\partial\mathcal{F}^{\mathrm{ext}}}{\partial t_{r-1}}$.

In [BCT19] E. Clader, R. J. Tessler and the author derived a topological recursion relation for the generating series of the extended r-spin intersection numbers with descendents. This equation immediately implies certain WDVV type equations for the function $\mathcal{F}^{\mathrm{ext}}$. We show that the mirror symmetry for the A_{r-1}-singularity together with the Saito formulas for the Frobenius manifold structure in the coordinates s_i can be simply derived from these equations.

We also answer Question 1 for the singularity D_4 and propose conjectural answers for the singularities E_6 and E_8.

Remark. *In a work in preparation* [GKT]*, M. Gross, T. L. Kelly and R. J. Tessler study open FJRW invariants and provide a similar interpretation of the flat coordinates for the Frobenius manifold for the Landau–Ginzburg models* $(\mathbb{C},\mathbb{Z}_r,x^r)$ *and* $(\mathbb{C}^2,\mathbb{Z}_r\times\mathbb{Z}_s,x^r+y^s)$ *and their mirrors. In the former case, their results using open r-spin invariants are analogous to the results appearing here.*

Plan of the paper. In Section 2 we formulate precisely the statement of the Landau-Ginzburg mirror symmetry for the A_{r-1}-singularity. The main result of the paper, Theorem 3.1, which describes the geometric interpretation of the functions $s_i(t_*)$, is contained in Section 3. In Section 4 we show how to derive the mirror symmetry for the A_{r-1}-singularity from the WDVV type equations for the function $\mathcal{F}^{\mathrm{ext}}$. In Section 5 we answer Question 1 for the singularity D_4 and propose conjectural answers for the singularities E_6 and E_8.

2. Landau-Ginzburg mirror symmetry for the A_{r-1}-singularity

In this section we present a more detailed description of the Landau–Ginzburg mirror symmetry for the singularity A_{r-1}: $W(x)=x^r$, $r\ge 2$. We would also like to fix a notation for the r-th root of -1, $\theta_r:=e^{\frac{\pi i}{r}}$, which we will often use in the rest of the paper.

2.1. **A-model.** The FJRW theory of the singularity $W(x)=x^r$ can be equivalently described using the r-spin theory ([Chi08, JKV01a], see also [BCT19, Section 2]). An orbifold curve $(C;p_1,\ldots,p_n)$ is called r-stable, if the coarse underlying marked curve $|C|$ is stable and the isotropy group is $\mathbb{Z}_r$ at every marked point and node. Consider a list of integers $0\le\alpha_1,\ldots,\alpha_n\le r-1$. An r-spin structure with the twists $\alpha_1,\ldots,\alpha_n$ on an r-stable orbifold curve $(C;p_1,\ldots,p_n)$ is an orbifold line bundle L over C together with an isomorphism

$$\phi\colon L^{\otimes r}\xrightarrow{\sim}\rho^*\omega_{|C|}\left(-\sum_{i=1}^n\alpha_ip_i\right),$$

and such that the isotropy groups at all markings act trivially on the fiber of L. Recall that by $\rho\colon C\to|C|$ we denote the forgetful map to the underlying coarse curve $|C|$. The moduli space of r-stable orbifold curves of genus g with an r-spin structure with the twists $\alpha_1,\ldots,\alpha_n$ is denoted by $\overline{\mathcal{M}}^{1/r}_{g;\alpha_1,\ldots,\alpha_n}$. It is non-empty if and only if $2g-2-\sum\alpha_i$ is divisible by r, and in this case it is a smooth compact orbifold of complex dimension $3g-3+n$.

Let us describe now the construction of the virtual fundamental class on $\overline{\mathcal{M}}^{1/r}_{g;\alpha_1,\ldots,\alpha_n}$ in the genus 0 case. We assume that

$$r \mid \left(\sum \alpha_i + 2\right). \tag{2.1}$$

Denote by $\mathcal{C} \to \overline{\mathcal{M}}^{1/r}_{0;\alpha_1,\ldots,\alpha_n}$ the universal curve and by $\mathcal{L} \to \mathcal{C}$ the universal line bundle. It is straightforward to check that for any r-stable curve $(C; p_1, \ldots, p_n)$ and an r-spin structure

$$\left(L \to C, \phi\colon L^{\otimes r} \xrightarrow{\sim} \rho^*\omega_{|C|}\left(-\sum \alpha_i p_i\right)\right)$$

on C the cohomology group $H^0(C, L)$ vanishes and, therefore, the cohomology group $H^1(C, L)$ has dimension $\frac{\sum_{i=1}^n \alpha_i - (r-2)}{r}$. This implies that $R^1\pi_*\mathcal{L}$ is a vector bundle over $\overline{\mathcal{M}}^{1/r}_{0;\alpha_1,\ldots,\alpha_n}$ and we denote the dual to it by $\mathcal{W}$,

$$\mathcal{W} := (R^1\pi_*\mathcal{L})^\vee.$$

It is called the Witten bundle. The top Chern class of it,

$$c_W := e(\mathcal{W}) \in H^{\deg c_W}\left(\overline{\mathcal{M}}^{1/r}_{0;\alpha_1,\ldots,\alpha_n}, \mathbb{Q}\right), \qquad \deg c_W = 2\frac{\sum \alpha_i - (r-2)}{r}, \tag{2.2}$$

is called the Witten class. It satisfies an important vanishing property which is called the Ramond vanishing: $c_W = 0$, if $\alpha_i = r-1$, for some i.

The FJRW intersection numbers for the singularity A_{r-1} are also called r-spin intersection numbers or r-spin correlators. They are obtained by integrating Witten's class against ψ-classes on the moduli space $\overline{\mathcal{M}}^{1/r}_{0;\alpha_1,\ldots,\alpha_n}$. Denote by $\mathbb{L}_i$ the line bundle over $\overline{\mathcal{M}}^{1/r}_{0;\alpha_1,\ldots,\alpha_n}$ whose fiber over an r-stable curve C is the cotangent space to the coarse curve $|C|$ at the i-th marked point. The r-spin correlators in genus 0 are defined by

$$\left\langle \prod_{i=1}^n \tau_{\alpha_i,d_i} \right\rangle^{r\text{-spin}} := r\int_{\overline{\mathcal{M}}^{1/r}_{0;\alpha_1,\ldots,\alpha_n}} c_W \prod_{i=1}^n \psi_i^{d_i}, \quad d_1, \ldots, d_n \geq 0. \tag{2.3}$$

Because of the Ramond vanishing, this correlator is equal to zero, if $\alpha_i = r-1$, for some i. A correlator $\langle \prod \tau_{\alpha_i,d_i}\rangle^{r\text{-spin}}$ is defined to be zero, if the divisibility condition (2.1) is not satisfied. Correlators $\langle \prod \tau_{\alpha_i,0}\rangle^{r\text{-spin}}$ are called primary correlators and also denoted by $\langle \prod \tau_{\alpha_i}\rangle^{r\text{-spin}}$. To be precise, the FJRW intersection numbers for the singularity A_{r-1} coincide with the primary r-spin correlators $\langle \prod \tau_{\alpha_i}\rangle^{r\text{-spin}}$, where $0 \leq \alpha_i \leq r-2$.

The FJRW generating series $\mathcal{F}^{\mathrm{FJRW}}_{0,W}$ in our case is also denoted by $\mathcal{F}^{r\text{-spin}}$ and defined by

$$\mathcal{F}^{r\text{-spin}}(t_0, \ldots, t_{r-2}) := \sum_{n\geq 3} \sum_{0\leq \alpha_1,\ldots,\alpha_n \leq r-2} \left\langle \prod_{i=1}^n \tau_{\alpha_i} \right\rangle^{r\text{-spin}} \frac{\prod_{i=1}^n t_{\alpha_i}}{n!},$$

where $t_0, \ldots, t_{r-2}$ are formal variables. From formula (2.2) for the degree of Witten's class it follows that the series $\mathcal{F}^{r\text{-spin}}$ is a polynomial in $t_0, \ldots, t_{r-2}$ which satisfies the following homogeneity condition:

$$\mathcal{F}^{r\text{-spin}}(\lambda^r t_0, \lambda^{r-1} t_1, \ldots, \lambda^2 t_{r-2}) = \lambda^{2r+2}\mathcal{F}^{r\text{-spin}}(t_0, t_1, \ldots, t_{r-2}), \quad \lambda \in \mathbb{C}^*. \tag{2.4}$$

The polynomial $\mathcal{F}^{r\text{-spin}}$ satisfies the property $\frac{\partial^3 \mathcal{F}^{r\text{-spin}}}{\partial t_0 \partial t_\alpha \partial t_\beta} = \delta_{\alpha+\beta,r-2}$ and also the following system of equations:

$$\sum_{\mu+\nu=r-2} \frac{\partial^3 \mathcal{F}^{r\text{-spin}}}{\partial t_\alpha \partial t_\beta \partial t_\mu} \frac{\partial^3 \mathcal{F}^{r\text{-spin}}}{\partial t_\nu \partial t_\gamma \partial t_\delta} = \sum_{\mu+\nu=r-2} \frac{\partial^3 \mathcal{F}^{r\text{-spin}}}{\partial t_\alpha \partial t_\gamma \partial t_\mu} \frac{\partial^3 \mathcal{F}^{r\text{-spin}}}{\partial t_\nu \partial t_\beta \partial t_\delta}, \quad 0 \leq \alpha, \beta, \gamma, \delta \leq r-2, \tag{2.5}$$

which are called the WDVV equations. Therefore, the function $\mathcal{F}^{r\text{-spin}}$ defines a Frobenius manifold structure on $\mathbb{C}^{r-1}$ in the coordinates $t_0, \ldots, t_{r-2}$ with the metric $\eta = (\eta_{\alpha\beta})_{0\leq\alpha,\beta\leq r-2}$, given by $\eta_{\alpha\beta} = \delta_{\alpha+\beta,r-2}$, and the unit vector field $\frac{\partial}{\partial t_0}$.

It is worth to mention that the polynomial $\mathcal{F}^{r\text{-spin}}$ is uniquely determined by the homogeneity condition (2.4), the WDVV equations (2.5) and the following initial conditions:

$$\langle\tau_\alpha\tau_\beta\tau_\gamma\rangle^{r\text{-spin}}=\delta_{\alpha+\beta+\gamma,r-2},\qquad \left\langle\tau_{r-2}^2\tau_1^2\right\rangle^{r\text{-spin}}=\frac{1}{r}. \tag{2.6}$$

This was already shown by E. Witten in [Wit93], but we would also like to mention the paper [PPZ19, Section 1.2], which contains a very short and clear proof of this fact.

2.2. **B-model.** A miniversal deformation of the singularity $W(x)=x^r$ is given by

$$W_s(x)=x^r+\sum_{i=0}^{r-2}s_ix^i,\quad s_i\in\mathbb{C}.$$

Let us choose $\zeta=-\theta_r^2rdx$ to be the primitive form in the Saito construction. So we get the following formula for the metric:

$$g_{ij}(s)=\frac{1}{2\pi i}\int_{\left|\frac{\partial W_s}{\partial x}\right|=\varepsilon}\frac{x^{i+j}}{\frac{\partial W_s}{\partial x}}(-\theta_r^2r)dx=\theta_r^2r\operatorname{Res}_{x=\infty}\frac{x^{i+j}}{\frac{\partial W_s}{\partial x}},\quad 0\le i,j\le r-2.$$

Flat coordinates for the metric $g_{ij}(s)$ can be explicitly constructed in the following way (see e.g. [Dub03, page 112]). Consider the series

$$k(x):=W_s(x)^{\frac{1}{r}}=x+O(x^{-1}).$$

Introduce functions $T^\alpha(s_0,\ldots,s_{r-2})$, $1\le\alpha\le r-1$, as the first non-trivial coefficients of the expansion of x in terms of $k(x)$:

$$x=k+\frac{1}{r}\left(\frac{T^{r-1}(s)}{k}+\frac{T^{r-2}(s)}{k^2}+\ldots+\frac{T^1(s)}{k^{r-1}}\right)+O(k^{-r}).$$

It is not hard to see that the functions $T^\alpha(s)$ are polynomials in the variables $s_0,\ldots,s_{r-2}$. They are flat coordinates for the metric $g_{ij}(s)$.

The Landau–Ginzburg mirror symmetry conjecture for the singularity A_{r-1} was proved in [JKV01a]. It says that the change of variables

$$t_\alpha(T^*)=\theta_r^{r-\alpha}T^{\alpha+1},\quad 0\le\alpha\le r-2,$$

defines an isomorphism between the Saito Frobenius manifold and the Frobenius manifold, given by the potential $\mathcal{F}^{r\text{-spin}}$ and the unit vector field $\frac{\partial}{\partial t_0}$.

3. Extended r-spin theory and the functions $s_i(t_*)$

In this section we describe a certain extension of the r-spin theory and prove that the functions $s_i(t_*)$ are given by the generating series of extended r-spin intersection numbers.

The moduli space $\overline{\mathcal{M}}^{1/r}_{g;\alpha_1,\ldots,\alpha_n}$ is actually well defined for all integers $\alpha_1,\ldots,\alpha_n$ and there are canonical isomorphisms $\overline{\mathcal{M}}^{1/r}_{g;\alpha_1,\ldots,\alpha_n}\cong\overline{\mathcal{M}}^{1/r}_{g;\beta_1,\ldots,\beta_n}$, if all differences $\alpha_i-\beta_i$ are divisible by r. In [JKV01b] the authors noticed that the construction of Witten's class on $\overline{\mathcal{M}}^{1/r}_{0;\alpha_1,\ldots,\alpha_n}$, described in the previous section, works in the case, when $\alpha_i=-1$ for some i, and $0\le\alpha_j\le r-1$ for $j\ne i$. Following [BCT19], we refer to this theory as the extended r-spin theory. So for all $n\ge 2$ and integers $0\le\alpha_1,\ldots,\alpha_n\le r-1$, satisfying the divisibility condition

$$r\mid\left(\sum\alpha_i+1\right), \tag{3.1}$$

there is well-defined Witten's class

$$c_W\in H^{\deg c_W}\left(\overline{\mathcal{M}}^{1/r}_{0;-1,\alpha_1,\ldots,\alpha_n},\mathbb{Q}\right),\qquad \deg c_W=2\frac{\sum\alpha_i-(r-1)}{r}. \tag{3.2}$$

Extended r-spin correlators are defined by

$$\left\langle \tau_{-1}\prod_{i=1}^{n}\tau_{\alpha_i,d_i}\right\rangle^{r\text{-spin}} := r\int_{\overline{\mathcal{M}}^{1/r}_{0;-1,\alpha_1,\ldots,\alpha_n}} c_W\prod_{i=1}^{n}\psi_{i+1}^{d_i},\quad d_1,\ldots,d_n\geq 0. \tag{3.3}$$

The correlator (3.3) is defined to be zero, if the divisibility condition (3.1) is not satisfied. The Ramond vanishing doesn't hold in the extended theory: for example, in [BCT19, Lemma 3.8] the authors showed that $\left\langle\tau_{-1}\tau_1\tau_{r-1}^2\right\rangle^{r\text{-spin}} = -\frac{1}{r}$.

Let t_{r-1} be a formal variable and introduce the generating series

$$\mathcal{F}^{\text{ext}}(t_0,\ldots,t_{r-1}) := \sum_{n\geq 2}\sum_{0\leq\alpha_1,\ldots,\alpha_n\leq r-1}\left\langle\tau_{-1}\prod_{i=1}^{n}\tau_{\alpha_i}\right\rangle^{r\text{-spin}}\frac{\prod_{i=1}^{n}t_{\alpha_i}}{n!}.$$

From formula (3.2) it follows that the series $\mathcal{F}^{\text{ext}}$ is a polynomial in $t_0,\ldots,t_{r-1}$ satisfying the following homogeneity property:

$$\mathcal{F}^{\text{ext}}(\lambda^r t_0,\lambda^{r-1}t_1,\ldots,\lambda t_{r-1}) = \lambda^{r+1}\mathcal{F}^{\text{ext}}(t_0,t_1,\ldots,t_{r-1}),\quad \lambda\in\mathbb{C}^*. \tag{3.4}$$

Let us now consider the polynomial $\mathcal{F}^{\text{ext}}$ as a polynomial in t_{r-1} with the coefficients from $\mathbb{Q}[t_0,\ldots,t_{r-2}]$. Consider also the parameters of the miniversal deformation $s_i(t_*)$, expressed as functions of the flat coordinates $t_0,\ldots,t_{r-2}$. The main result of the paper is the following theorem.

Theorem 3.1. *We have*

$$s_i(t_*) = (-r\theta_r)^i\mathrm{Coef}_{t_{r-1}^i}\frac{\partial\mathcal{F}^{\text{ext}}}{\partial t_{r-1}},\quad 0\leq i\leq r-2.$$

Before we prove the theorem, let us present an alternative description of the flat coordinates for the Saito Frobenius manifold for the A_{r-1}-singularity. Define functions $v_\alpha(s_0,\ldots,s_{r-2})$, $1\leq\alpha\leq r-1$, by

$$v_\alpha(s) := -\operatorname{Res}_{x=\infty}\left(W_s(x)^{\frac{\alpha}{r}}\right).$$

It is not hard to see that $v_\alpha(s)$ is a polynomial in $s_0,s_1,\ldots,s_{r-2}$ of the form $v_\alpha(s) = \frac{\alpha}{r}s_{r-\alpha-1} + O(s^2)$. Therefore, the functions $v_1(s),\ldots,v_{r-1}(s)$ can serve as coordinates on $\mathbb{C}^{r-1}$ in a neighbourhood of the origin.

Lemma 3.2. *We have*

$$v_\alpha(s) = -\frac{\alpha}{r}T^{r-\alpha}(s),\quad 1\leq\alpha\leq r-1.$$

Proof. We only have to check the following identity:

$$x = k - \sum_{\alpha=1}^{r-1}\frac{v_\alpha(s)}{\alpha}\frac{1}{k^\alpha} + O(x^{-r}). \tag{3.5}$$

For a Laurent series $A = \sum_{i=-\infty}^{m}a_i(s)x^i$, $a_i(s)\in\mathbb{C}[s_0,\ldots,s_{r-2}]$, denote

$$A_+ := \sum_{i=0}^{m}a_i x^i,\qquad A_- := A - A_+,\qquad \operatorname{res}A := a_{-1}.$$

We want to use the results from [BCT19, Section 4]. In [BCT19, Lemma 4.2] we considered the polynomial $\widehat{L}_0 = z^r + \sum_{i=0}^{r-2}f_i^{[0]}z^i$ and in [BCT19, page 147] we introduced the variables $v_i = \operatorname{res}\left(\widehat{L}_0^{i/r}\right)$, $1\leq i\leq r-1$. We can identify $f_i^{[0]} = s_i$ and $z = x$, then we get $\widehat{L}_0 = W_s$. By [BCT19, equation (4.22)], we have

$$\frac{r+1}{r}\left(W_s^{\frac{1}{r}}\right)_- - \sum_{\alpha=1}^{r-1}\frac{\alpha}{r}\frac{\partial}{\partial v_\alpha}\operatorname{res}\left(W_s^{\frac{r+1}{r}}\right)W_s^{\frac{\alpha-r}{r}} = O(x^{-r}). \tag{3.6}$$

From the equation before equation (4.20) in [BCT19] and [BCT19, Lemma 4.2] it follows that

$$v_{r-\alpha}=\frac{\alpha(r-\alpha)}{r+1}\frac{\partial}{\partial v_\alpha}\operatorname{res}\left(W_s^{\frac{r+1}{r}}\right),\quad 1\le\alpha\le r-1. \tag{3.7}$$

Combining (3.6) and (3.7), we obtain

$$\left(W_s^{\frac{1}{r}}\right)_--\sum_{\alpha=1}^{r-1}\frac{v_{r-\alpha}}{r-\alpha}\frac{1}{k^{r-\alpha}}=O(x^{-r}).$$

It remains to note that $\left(W_s^{\frac{1}{r}}\right)_-=k-x$, and identity (3.5) becomes clear. The lemma is proved. □

Proof of Theorem 3.1. The theorem is a consequence of the following stronger statement:

$$W_s(x)=\left.\frac{\partial\mathcal{F}^{\mathrm{ext}}}{\partial t_{r-1}}\right|_{t_{r-1}=-r\theta_rx}. \tag{3.8}$$

Let us prove it.

Similarly to the proof of Lemma 3.2, we want to use the results from [BCT19, Section 4]. We again identify $f_i^{[0]}=s_i$, $z=x$ and $\widehat{L}_0=W_s$. In [BCT19, Section 4] we have the variables T_i, $i\ge1$, and t^α_d, $0\le\alpha\le r-1$, $d\ge0$. Let us set $T_{\ge r+1}=0$, $t^\alpha_{\ge1}=0$, and denote $t^\alpha:=t^\alpha_0$. We have the following relation [BCT19, equation (4.7)]:

$$t^\alpha=(\alpha+1)(-r)^{\frac{3(\alpha+1)}{2(r+1)}-\frac12}T_{\alpha+1},\quad 0\le\alpha\le r-2.$$

Let us use the following notations:

$$\widetilde{\mathcal{F}}^{r\text{-spin}}:=\left.\mathcal{F}^{r\text{-spin}}\right|_{t_\alpha\mapsto t^\alpha},\qquad \widetilde{\mathcal{F}}^{\mathrm{ext}}:=\left.\mathcal{F}^{\mathrm{ext}}\right|_{t_\alpha\mapsto t^\alpha}.$$

Since $\frac{\partial^2\widetilde{\mathcal{F}}^{r\text{-spin}}}{\partial T_1\partial T_a}=v_a$, $1\le a\le r-1$ [BCT19, Lemma 4.2], we get

$$\frac{\partial^2\widetilde{\mathcal{F}}^{r\text{-spin}}}{\partial t^0\partial t^{a-1}}a(-r)^{\frac{3(a+1)}{2(r+1)}-1}=v_a,\quad 1\le a\le r-1.$$

Together with the formulas $\frac{\partial^2\widetilde{\mathcal{F}}^{r\text{-spin}}}{\partial t^0\partial t^{a-1}}=t^{r-1-a}$ and $v_a=-\frac{a}{r}T^{r-a}$ this implies that

$$t^a=\frac{1}{\lambda_r^{r-a}}t_a,\quad\text{where}\quad 0\le a\le r-2\quad\text{and}\quad\lambda_r=\theta_r(-r)^{\frac{3}{2(r+1)}}.$$

Proposition 4.10 in [BCT19] says that

$$\frac{\partial\widetilde{\mathcal{F}}^{\mathrm{ext}}}{\partial t^{r-1}}=\frac{1}{r(-r)^{\frac{r-2}{2(r+1)}}}W_s\left(\frac{\lambda_r}{-r\theta_r}t^{r-1}\right).$$

We compute

$$\left.\frac{\partial\widetilde{\mathcal{F}}^{\mathrm{ext}}}{\partial t^{r-1}}\right|_{t^{r-1}=\frac{x}{\lambda_r}}=\frac{\partial\widetilde{\mathcal{F}}^{\mathrm{ext}}}{\partial t^{r-1}}\left(\frac{t_0}{\lambda_r^r},\frac{t_1}{\lambda_r^{r-1}},\ldots,\frac{t_{r-2}}{\lambda_r^2},\frac{x}{\lambda_r}\right)\overset{\text{by (3.4)}}{=}\frac{1}{\lambda_r^r}\left.\frac{\partial\mathcal{F}^{\mathrm{ext}}}{\partial t_{r-1}}\right|_{t_{r-1}=x}.$$

Therefore,

$$\frac{1}{\lambda_r^r}\left.\frac{\partial\mathcal{F}^{\mathrm{ext}}}{\partial t_{r-1}}\right|_{t_{r-1}=x}=\frac{1}{r(-r)^{\frac{r-2}{2(r+1)}}}W_s\left(\frac{x}{-r\theta_r}\right)\Rightarrow\left.\frac{\partial\mathcal{F}^{\mathrm{ext}}}{\partial t_{r-1}}\right|_{t_{r-1}=x}=W_s\left(\frac{x}{-r\theta_r}\right),$$

which proves equation (3.8). □

4. Mirror symmetry as a consequence of the WDVV type equations

In the previous section we saw that the functions $s_i(t_*)$ appear as the coefficients of the polynomial $\frac{\partial \mathcal{F}^{\mathrm{ext}}}{\partial t_{r-1}}$. Now we want to show that the Saito formulas for the multiplication and the metric in the coordinates s_i can be naturally deduced from the WDVV type equations for the polynomial $\mathcal{F}^{\mathrm{ext}}$.

Introduce formal variables $t_{\alpha,d}$, $0 \le \alpha \le r-1$, $d \ge 0$, and let $t_{\alpha,0} := t_\alpha$. Consider the generating series

$$F^{r\text{-spin}}(t_{*,*}) := \sum_{n\ge 3}\sum_{\substack{0\le\alpha_1,\dots,\alpha_n\le r-2\\ d_1,\dots,d_n\ge 0}} \left\langle \prod_{i=1}^n \tau_{\alpha_i,d_i}\right\rangle^{r\text{-spin}} \frac{\prod_{i=1}^n t_{\alpha_i,d_i}}{n!},$$

$$F^{\mathrm{ext}}(t_{*,*}) := \sum_{n\ge 2}\sum_{\substack{0\le\alpha_1,\dots,\alpha_n\le r-1\\ d_1,\dots,d_n\ge 0}} \left\langle \tau_{-1}\prod_{i=1}^n \tau_{\alpha_i,d_i}\right\rangle^{r\text{-spin}} \frac{\prod_{i=1}^n t_{\alpha_i,d_i}}{n!}.$$

In [BCT19, equation (4.26)] the authors proved the following topological recursion relation:

$$\frac{\partial^2 F^{\mathrm{ext}}}{\partial t_{\alpha,p+1}\partial t_{\beta,q}} = \sum_{\mu+\nu=r-2}\frac{\partial^2 F^{r\text{-spin}}}{\partial t_{\alpha,p}\partial t_{\mu,0}}\frac{\partial^2 F^{\mathrm{ext}}}{\partial t_{\nu,0}\partial t_{\beta,q}} + \frac{\partial F^{\mathrm{ext}}}{\partial t_{\alpha,p}}\frac{\partial^2 F^{\mathrm{ext}}}{\partial t_{r-1,0}\partial t_{\beta,q}}, \quad 0\le\alpha,\beta\le r-1, \quad p,q\ge 0.$$

These relations can be equivalently written as the following equations between differential 1-forms:

$$d\left(\frac{\partial F^{\mathrm{ext}}}{\partial t_{\alpha,p+1}}\right) = \sum_{\mu+\nu=r-2}\frac{\partial^2 F^{r\text{-spin}}}{\partial t_{\alpha,p}\partial t_{\mu,0}} d\left(\frac{\partial F^{\mathrm{ext}}}{\partial t_{\nu,0}}\right) + \frac{\partial F^{\mathrm{ext}}}{\partial t_{\alpha,p}} d\left(\frac{\partial F^{\mathrm{ext}}}{\partial t_{r-1,0}}\right), \quad 0\le\alpha\le r-1, \quad p\ge 0.$$

Taking the exterior derivative of the both sides and setting $p=0$ and $t_{*,\ge 1}=0$, we obtain the following relation:

$$\sum_{\mu+\nu=r-2} d\left(\frac{\partial^2\mathcal{F}^{r\text{-spin}}}{\partial t_\alpha\partial t_\mu}\right)\wedge d\left(\frac{\partial\mathcal{F}^{\mathrm{ext}}}{\partial t_\nu}\right) + d\left(\frac{\partial\mathcal{F}^{\mathrm{ext}}}{\partial t_\alpha}\right)\wedge d\left(\frac{\partial\mathcal{F}^{\mathrm{ext}}}{\partial t_{r-1}}\right) = 0, \quad 0\le\alpha\le r-1,$$

or, equivalently,

(4.1)
$$\sum_{\mu+\nu=r-2}\frac{\partial^3\mathcal{F}^{r\text{-spin}}}{\partial t_\alpha\partial t_\beta\partial t_\mu}\frac{\partial^2\mathcal{F}^{\mathrm{ext}}}{\partial t_\nu\partial t_\gamma} + \frac{\partial^2\mathcal{F}^{\mathrm{ext}}}{\partial t_\alpha\partial t_\beta}\frac{\partial^2\mathcal{F}^{\mathrm{ext}}}{\partial t_{r-1}\partial t_\gamma} = \sum_{\mu+\nu=r-2}\frac{\partial^3\mathcal{F}^{r\text{-spin}}}{\partial t_\alpha\partial t_\gamma\partial t_\mu}\frac{\partial^2\mathcal{F}^{\mathrm{ext}}}{\partial t_\nu\partial t_\beta} + \frac{\partial^2\mathcal{F}^{\mathrm{ext}}}{\partial t_\alpha\partial t_\gamma}\frac{\partial^2\mathcal{F}^{\mathrm{ext}}}{\partial t_{r-1}\partial t_\beta},$$

where $0\le\alpha,\beta,\gamma\le r-1$. We call these equations the WDVV type equations for the function $\mathcal{F}^{\mathrm{ext}}$. They already appeared in literature before in the context of open Gromov–Witten theory [HS12]. For $\beta=r-1$, equation (4.1) looks as follows:

$$\frac{\partial^2\mathcal{F}^{\mathrm{ext}}}{\partial t_\alpha\partial t_{r-1}}\frac{\partial^2\mathcal{F}^{\mathrm{ext}}}{\partial t_{r-1}\partial t_\gamma} = \sum_{\mu+\nu=r-2}\frac{\partial^3\mathcal{F}^{r\text{-spin}}}{\partial t_\alpha\partial t_\gamma\partial t_\mu}\frac{\partial^2\mathcal{F}^{\mathrm{ext}}}{\partial t_\nu\partial t_{r-1}} + \frac{\partial^2\mathcal{F}^{\mathrm{ext}}}{\partial t_\alpha\partial t_\gamma}\frac{\partial^2\mathcal{F}^{\mathrm{ext}}}{\partial t_{r-1}^2}, \tag{4.2}$$

and is non-trivial only if $0\le\alpha,\gamma\le r-2$.

Define functions $\widetilde{s}_i(t_0,\dots,t_{r-2})$, $0\le i\le r-2$, by

$$\widetilde{s}_i(t_0,\dots,t_{r-2}) := (-r\theta_r)^i\mathrm{Coef}_{t_{r-1}^i}\left(\frac{\partial\mathcal{F}^{\mathrm{ext}}}{\partial t_{r-1}}\right).$$

Of course, we already know that $\widetilde{s}_i(t_*) = s_i(t_*)$, but we don't want to use it in this section and want to show that the Saito formulas for the Frobenius manifold structure in the coordinates $\widetilde{s}_i$ can be derived directly from the WDVV type equations (4.2).

Let us begin with the metric. We have to prove that

$$-\operatorname{Res}_{t_{r-1}=\infty}\left(\frac{\frac{\partial^2\mathcal{F}^{\mathrm{ext}}}{\partial t_\alpha\partial t_{r-1}}\frac{\partial^2\mathcal{F}^{\mathrm{ext}}}{\partial t_\beta\partial t_{r-1}}}{\frac{\partial^2\mathcal{F}^{\mathrm{ext}}}{\partial t_{r-1}^2}}\right)=-r\delta_{\alpha+\beta,r-2},\quad 0\le\alpha,\beta\le r-2. \tag{4.3}$$

For this we compute

$$\begin{aligned}-\operatorname{Res}_{t_{r-1}=\infty}\left(\frac{\frac{\partial^2\mathcal{F}^{\mathrm{ext}}}{\partial t_\alpha\partial t_{r-1}}\frac{\partial^2\mathcal{F}^{\mathrm{ext}}}{\partial t_\beta\partial t_{r-1}}}{\frac{\partial^2\mathcal{F}^{\mathrm{ext}}}{\partial t_{r-1}^2}}\right)&\overset{\text{by (4.2)}}{=}-\operatorname{Res}_{t_{r-1}=\infty}\left(\frac{\partial^2\mathcal{F}^{\mathrm{ext}}}{\partial t_\alpha\partial t_\beta}\right)-\\&-\sum_{\mu+\nu=r-2}\frac{\partial^3\mathcal{F}^{r\text{-spin}}}{\partial t_\alpha\partial t_\beta\partial t_\mu}\operatorname{Res}_{t_{r-1}=\infty}\left(\frac{\frac{\partial^2\mathcal{F}^{\mathrm{ext}}}{\partial t_\nu\partial t_{r-1}}}{\frac{\partial^2\mathcal{F}^{\mathrm{ext}}}{\partial t_{r-1}^2}}\right).\end{aligned}\tag{4.4}$$

The first term on the right-hand side of this equation is equal to zero, because $\frac{\partial^2\mathcal{F}^{\mathrm{ext}}}{\partial t_\alpha\partial t_\beta}$ is a polynomial in t_{r-1}. In order to compute the second term, note that the homogeneity condition (3.4) implies that the polynomial $\mathcal{F}^{\mathrm{ext}}$ has the form

$$\mathcal{F}^{\mathrm{ext}}=\left\langle\tau_{-1}\tau_{r-1}^{r+1}\right\rangle^{r\text{-spin}}\frac{t_{r-1}^{r+1}}{(r+1)!}+\left\langle\tau_{-1}\tau_{r-2}\tau_{r-1}^{r-1}\right\rangle^{r\text{-spin}}t_{r-2}\frac{t_{r-1}^{r-1}}{(r-1)!}+O(t_{r-1}^{r-2}).$$

Since [BCT19, proof of Theorem 4.6]

$$\left\langle\tau_{-1}\tau_\gamma\tau_{r-1}^{\gamma+1}\right\rangle^{r\text{-spin}}=\frac{\gamma!}{(-r)^\gamma},\quad 0\le\gamma\le r-1, \tag{4.5}$$

we obtain

$$\frac{\partial\mathcal{F}^{\mathrm{ext}}}{\partial t_{r-1}}=-\frac{1}{(-r)^r}t_{r-1}^r+\frac{t^{r-2}}{(-r)^{r-2}}t_{r-1}^{r-2}+O(t_{r-1}^{r-3}).$$

So the second term on the right-hand side of (4.4) is equal to

$$\begin{aligned}-\frac{\partial^3\mathcal{F}^{r\text{-spin}}}{\partial t_\alpha\partial t_\beta\partial t_0}\operatorname{Res}_{t_{r-1}=\infty}\left(\frac{\frac{\partial^2\mathcal{F}^{\mathrm{ext}}}{\partial t_{r-2}\partial t_{r-1}}}{\frac{\partial^2\mathcal{F}^{\mathrm{ext}}}{\partial t_{r-1}^2}}\right)&=-\delta_{\alpha+\beta,r-2}\operatorname{Res}_{t_{r-1}=\infty}\left(\frac{\frac{t_{r-1}^{r-2}}{(-r)^{r-2}}+O(t_{r-1}^{r-3})}{\frac{t_{r-1}^{r-1}}{(-r)^{r-1}}+O(t_{r-1}^{r-3})}\right)=\\&=-r\delta_{\alpha+\beta,r-2}.\end{aligned}$$

Thus, equation (4.3) is proved.

Let us now show that the multiplication in the variables $\widetilde{s}_i$ is given by the Saito construction. For this we have to check that

$$\sum_{k=0}^{r-2}x^kc_{ij}^k=x^{i+j}\mod\frac{\partial Q}{\partial x}, \tag{4.6}$$

where by c_{ij}^k, $0\le i,j,k\le r-2$, we denote the structure constants of the multiplication in the variables $\widetilde{s}_*$, and $Q(t_*,x):=\left.\frac{\partial\mathcal{F}^{\mathrm{ext}}}{\partial t_{r-1}}\right|_{t_{r-1}=-r\theta_rx}$. Denote by $c_{\alpha\beta}^\gamma$, $0\le\alpha,\beta,\gamma\le r-2$, the structure constants of the multiplication in the variables t_*. Clearly, $c_{\alpha\beta}^\gamma=\frac{\partial^3\mathcal{F}^{r\text{-spin}}}{\partial t_\alpha\partial t_\beta\partial t_{r-2-\gamma}}$. We compute

(we follow the convention of sum over repeated Greek indices)

$$\begin{aligned}\sum_{k=0}^{r-2} x^k c_{ij}^k =& \frac{\partial t_\alpha}{\partial \widetilde{s}_i}\frac{\partial t_\beta}{\partial \widetilde{s}_j} c_{\alpha\beta}^{\gamma} \sum_{k=0}^{r-2}\frac{\partial \widetilde{s}_k}{\partial t_\gamma} x^k = \frac{\partial t_\alpha}{\partial \widetilde{s}_i}\frac{\partial t_\beta}{\partial \widetilde{s}_j} c_{\alpha\beta}^{\gamma}\frac{\partial Q}{\partial t_\gamma} = \\ =& \frac{\partial t_\alpha}{\partial \widetilde{s}_i}\frac{\partial t_\beta}{\partial \widetilde{s}_j}\sum_{\mu+\nu=r-2}\frac{\partial^3 \mathcal{F}^{r\text{-spin}}}{\partial t_\alpha \partial t_\beta \partial t_\mu}\left.\frac{\partial^2 \mathcal{F}^{\mathrm{ext}}}{\partial t_\nu \partial t_{r-1}}\right|_{t_{r-1}=-r\theta_r x} \overset{\text{by (4.2)}}{=} \\ =& \frac{\partial t_\alpha}{\partial \widetilde{s}_i}\frac{\partial t_\beta}{\partial \widetilde{s}_j}\left(\frac{\partial Q}{\partial t_\alpha}\frac{\partial Q}{\partial t_\beta} + \frac{1}{r\theta_r}\frac{\partial Q}{\partial x}\left.\frac{\partial^2 \mathcal{F}^{\mathrm{ext}}}{\partial t_\alpha \partial t_\beta}\right|_{t_{r-1}=-r\theta_r x}\right) = \\ =& \frac{\partial Q}{\partial \widetilde{s}_i}\frac{\partial Q}{\partial \widetilde{s}_j} + \frac{\partial Q}{\partial x}\left(\frac{1}{r\theta_r}\frac{\partial t_\alpha}{\partial \widetilde{s}_i}\frac{\partial t_\beta}{\partial \widetilde{s}_j}\left.\frac{\partial^2 \mathcal{F}^{\mathrm{ext}}}{\partial t_\alpha \partial t_\beta}\right|_{t_{r-1}=-r\theta_r x}\right) = \\ =& x^{i+j} \mod \frac{\partial Q}{\partial x},\end{aligned}$$

which proves formula (4.6).

5. Singularity D_4

In this section we give an answer to Question 1 for the singularity $D_4 = x_1^3 + x_1x_2^2$. We also propose conjectural answers for the singularities E_6 and E_8.

The Landau–Ginzburg mirror symmetry for the polynomial $x_1^3 + x_1x_2^2$ is equivalent to the mirror symmetry for the polynomial $x_1^3 + x_2^3$. In order to see it, note, first of all, that the singularity $\{x_1^3 + x_1x_2^2 = 0\} \subset \mathbb{C}^2$ is just three lines intersecting at the origin. Therefore, a linear change of variables transforms this singularity to the singularity $\{x_1^3 + x_2^3 = 0\} \subset \mathbb{C}^2$. Thus, the Saito Frobenius manifolds, corresponding to the polynomials $x_1^3 + x_1x_2^2$ and $x_1^3 + x_2^3$, are isomorphic. The mirror partner to the polynomial $D_4 = x_1^3 + x_1x_2^2$ is the polynomial $D_4^T = x_1^3x_2 + x_2^2$. The potential $\mathcal{F}^{\mathrm{FJRW}}_{0,D_4^T}$ was computed in [FFJMR16] and in Section 5.1 we show that it is equivalent to the FJRW potential for the polynomial $x_1^3 + x_2^3$.

In Section 5.2 we describe explictly the B-model for the polynomial $x_1^3 + x_2^3$ together with the Landau–Ginzburg mirror symmetry in this case. Then in Section 5.3 we answer Question 1 for the polynomial $x_1^3 + x_2^3$ and in Section 5.4 propose conjectures for the singularities E_6 and E_8.

5.1. **A-model.** Let us consider the more general case of the polynomial

$$W_{r_1,r_2}(x_1,x_2) := x_1^{r_1} + x_2^{r_2}, \quad r_1, r_2 \geq 3.$$

The dimension of the corresponding local algebra $\mathcal{A}_{W_{r_1,r_2}}$ is equal to $(r_1-1)(r_2-1)$. The FJRW theory for the polynomial W_{r_1,r_2} can be described using the r-spin theory in the following way. Consider the moduli space $\overline{\mathcal{M}}_{0,n}$ of stable curves of genus 0 with n marked points and denote by $\mathrm{st}\colon \overline{\mathcal{M}}^{1/r}_{0;\alpha_1,\ldots,\alpha_n} \to \overline{\mathcal{M}}_{0,n}$ the forgetful map, which forgets an r-spin structure together with an orbifold structure on an orbifold curve. For $0 \leq \alpha_1, \ldots, \alpha_n \leq r-1$, satisfying the divisibility condition 2.1, define the cohomology class

$$c^{r\text{-spin}}(\alpha_1,\ldots,\alpha_n) := r \cdot \mathrm{st}_*(c_W) \in H^*(\overline{\mathcal{M}}_{0,n},\mathbb{Q}) \tag{5.1}$$

of degree

$$\deg c^{r\text{-spin}}(\alpha_1,\ldots,\alpha_n) = 2\frac{\sum \alpha_i - (r-2)}{r}. \tag{5.2}$$

The class $c^{r\text{-spin}}(\alpha_1,\ldots,\alpha_n)$ is defined to be zero, if the divisibility condition (2.1) is not satisfied.

The FJRW intersection numbers for the polynomial W_{r_1,r_2} are given by [FJR13, Theorem 4.2.2])

(5.3)
$$\left\langle\prod_{i=1}^n\tau_{\alpha_i,\beta_i}\right\rangle^{W_{r_1,r_2}}=\int_{\overline{\mathcal{M}}_{0,n}}c^{r_1\text{-spin}}(\alpha_1,\ldots,\alpha_n)c^{r_2\text{-spin}}(\beta_1,\ldots,\beta_n),\quad 0\le\alpha_i\le r_1-2,\quad 0\le\beta_i\le r_2-2.$$

By (5.2) and the divisibility condition (2.1), this intersection number is equal to zero unless the following constraints are satisfied:

(5.4)
$$\sum_{i=1}^n\left(1-\frac{\alpha_i}{r_1}-\frac{\beta_i}{r_2}\right)=1+\frac{2}{r_1}+\frac{2}{r_2},\qquad \sum_{i=1}^n\alpha_i=r_1-2\bmod r_1,\qquad \sum_{i=1}^n\beta_i=r_2-2\bmod r_2.$$

The FJRW generating series is equal to

$$\mathcal{F}^{\mathrm{FJRW}}_{0,W_{r_1,r_2}}=\sum_{n\ge 3}\sum_{\substack{0\le\alpha_1,\ldots,\alpha_n\le r_1-2\\ 0\le\beta_1,\ldots,\beta_n\le r_2-2}}\left\langle\prod_{i=1}^n\tau_{\alpha_i,\beta_i}\right\rangle^{W_{r_1,r_2}}\frac{\prod_{i=1}^n t_{\alpha_i,\beta_i}}{n!},$$

where $t_{\alpha,\beta}$, $0\le\alpha\le r_1-2$, $0\le\beta\le r_2-2$, are formal variables. The unit vector of the corresponding Frobenius manifold is $\frac{\partial}{\partial t_{0,0}}$ and the metric is

$$\eta_{\alpha_1,\beta_1;\alpha_2,\beta_2}=\frac{\partial^3\mathcal{F}^{\mathrm{FJRW}}_{0,W_{r_1,r_2}}}{\partial t_{0,0}\partial t_{\alpha_1,\beta_1}\partial t_{\alpha_2,\beta_2}}=\delta_{\alpha_1+\alpha_2,r_1-2}\delta_{\beta_1+\beta_2,r_2-2}.$$

Consider now the case $r_1=r_2=3$. The function $\mathcal{F}^{\mathrm{FJRW}}_{0,W_{3,3}}$ depends on four variables $t_{0,0},t_{1,0},t_{0,1},t_{1,1}$. The metric is antidiagonal in these variables. From constraints (5.4) it follows that there are only two non-trivial 3-point correlators and only two non-trivial 4-point correlators:

$$\langle\tau_{1,1}\tau_{0,0}\tau_{0,0}\rangle^{W_{3,3}}=\langle\tau_{0,1}\tau_{1,0}\tau_{0,0}\rangle^{W_{3,3}}=1,\qquad \left\langle\tau_{1,1}\tau_{1,0}^3\right\rangle^{W_{3,3}}=\left\langle\tau_{1,1}\tau_{0,1}^3\right\rangle^{W_{3,3}}=\frac{1}{3},\tag{5.5}$$

which are computed using equations (5.3) and (2.6). Using the argument, similar to the one, presented in [PPZ19, Section 1.2], one can show that the function $\mathcal{F}^{\mathrm{FJRW}}_{0,W_{3,3}}$ is uniquely determined by the WDVV equations, the first constraint in (5.4) and the initial conditions (5.5). As a result, we obtain

$$\mathcal{F}^{\mathrm{FJRW}}_{0,W_{3,3}}=\frac{1}{2}t_{0,0}^2t_{1,1}+t_{0,0}t_{1,0}t_{0,1}+\frac{1}{18}t_{1,0}^3t_{1,1}+\frac{1}{18}t_{0,1}^3t_{1,1}+\frac{1}{54}t_{1,0}t_{0,1}t_{1,1}^3+\frac{t_{1,1}^7}{68040}.\tag{5.6}$$

Let us compare the potential (5.6) with the FJRW potential for the singularity D_4^T. According to [FFJMR16, page 183], we have

$$\mathcal{F}^{\mathrm{FJRW}}_{0,D_4^T}=\frac{1}{12}t_X^2t_1-\frac{1}{4}t_Y^2t_1+\frac{1}{12}t_{X^2}t_1^2+\frac{a}{6}t_X^3t_{X^2}+\frac{3a}{2}t_Xt_Y^2t_{X^2}+a^2t_X^2t_{X^2}^3-3a^2t_Y^2t_{X^2}^3+\frac{36a^4}{35}t_{X^2}^7,$$

where t_1,t_X,t_Y,t_{X^2} are formal variables and a is an explicitly computed non-zero constant. The unit vector field is $\frac{\partial}{\partial t_1}$. One can directly compute that after the change of variables

$$t_{0,0}=t_1,\qquad t_{1,0}=\beta t_X+\gamma t_Y,\qquad t_{0,1}=\beta t_X-\gamma t_Y,\qquad t_{1,1}=\delta t_{X^2},$$

with

$$\beta=3^{2/3}a^{1/3},\qquad \gamma=3\cdot 3^{1/6}a^{1/3},\qquad \delta=6\cdot 3^{1/3}a^{2/3},$$

we get

$$\mathcal{F}^{\mathrm{FJRW}}_{0,D_4^T}=\frac{1}{36\cdot 3^{1/3}a^{2/3}}\mathcal{F}^{\mathrm{FJRW}}_{0,W_{3,3}}.$$

Therefore, after a rescaling of the metric the FJRW Frobenius manifolds for the polynomials D_4^T and $W_{3,3}$ become isomorphic.

5.2. **B-model.** Consider the following miniversal deformation of the singularity $W_{3,3}$:

$$W_{3,3;s}(x_1,x_2)=x_1^3+x_2^3+s_{1,1}x_1x_2+s_{0,1}x_2+s_{1,0}x_1+s_{0,0},\qquad s_{i,j}\in\mathbb{C}.$$

Let us choose the form $\zeta=9dx_1\wedge dx_2$ to be the primitive form, defining the metric $g_{i_1,j_1;i_2,j_2}(s)$:

$$g_{i_1,j_1;i_2,j_2}(s)=\frac{1}{(2\pi i)^2}\int_{\left|\frac{\partial W_{3,3;s}}{\partial x_1}\right|=\left|\frac{\partial W_{3,3;s}}{\partial x_2}\right|=\varepsilon}\frac{x_1^{i_1+i_2}x_2^{j_1+j_2}}{\frac{\partial W_{3,3;s}}{\partial x_1}\frac{\partial W_{3,3;s}}{\partial x_2}}9dx_1\wedge dx_2,\quad 0\le i_1,i_2,j_1,j_2\le 1.$$

Then the matrix of the metric in the variables $s_{0,0},s_{1,0},s_{0,1},s_{1,1}$ is

$$g(s)=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&\frac{s_{1,1}^2}{9}\end{pmatrix}.$$

A direct computation shows that the change of coordinates

$$t_{0,0}(s)=s_{0,0}+\frac{s_{1,1}^3}{54},\qquad t_{1,0}(s)=-s_{1,0},\qquad t_{0,1}(s)=-s_{0,1},\qquad t_{1,1}(s)=s_{1,1},$$

defines an isomorphism between the Saito Frobenius manifold for the singularity $W_{3,3}$ and the Frobenius manifold given by the potential $\mathcal{F}^{\mathrm{FJRW}}_{0,W_{3,3}}$ and the unit $\frac{\partial}{\partial t_{0,0}}$.

5.3. **Answer to Question 1.** In the extended r-spin theory, by the same formula (5.1), we can define the class

$$c^{r\text{-spin}}(-1,\alpha_1,\ldots,\alpha_n)\in H^*(\overline{\mathcal{M}}_{0,n+1},\mathbb{Q}),\quad 0\le\alpha_1,\ldots,\alpha_n\le r-1,$$

of degree $2\frac{\sum\alpha_i-(r-1)}{r}$. Note also that we have the property [BCT19, Lemma 3.5]

$$c^{r\text{-spin}}(-1,\alpha_1,\ldots,\alpha_n,0)=\pi^*c^{r\text{-spin}}(-1,\alpha_1,\ldots,\alpha_n),$$

where $\pi\colon\overline{\mathcal{M}}_{0,n+2}\to\overline{\mathcal{M}}_{0,n+1}$ is the forgetful map, which forgets the last marked point.

Let us define extended intersection numbers for the singularity $W_{3,3}$ by the formula

(5.7)
$$\left\langle\tau_{-1,-1}\prod_{i=1}^n\tau_{\alpha_i,\beta_i}\right\rangle^{W_{3,3}}:=\int_{\overline{\mathcal{M}}_{0,n+1}}c^{3\text{-spin}}(-1,\alpha_1,\ldots,\alpha_n)c^{3\text{-spin}}(-1,\beta_1,\ldots,\beta_n),\quad 0\le\alpha_i,\beta_i\le 2.$$

This correlator is equal to zero unless

$$\sum_{i=1}^n\left(1-\frac{\alpha_i+\beta_i}{3}\right)=\frac{2}{3},\qquad\sum_{i=1}^n\alpha_i=2\bmod 3,\qquad\sum_{i=1}^n\beta_i=2\bmod 3.\tag{5.8}$$

Define a series $\mathcal{P}^{W_{3,3}}(t_{0,0},t_{1,0},t_{0,1},t_{1,1},x_1,x_2)$ by

$$\mathcal{P}^{W_{3,3}}(t,x):=\sum_{\substack{n,k_1,k_2\ge0\\n+k_1+k_2\ge1}}\frac{x_1^{k_1}}{k_1!}\frac{x_2^{k_2}}{k_2!}\sum_{\substack{0\le\alpha_1,\ldots,\alpha_n\le1\\0\le\beta_1,\ldots,\beta_n\le1}}\left\langle\tau_{-1,-1}\tau_{2,2}\tau_{2,0}^{k_1}\tau_{0,2}^{k_2}\prod_{i=1}^n\tau_{\alpha_i,\beta_i}\right\rangle^{W_{3,3}}\frac{\prod_{i=1}^n t_{\alpha_i,\beta_i}}{n!}.$$

From (5.8) it follows that the series $\mathcal{P}^{W_{3,3}}$ is actually a polynomial in the variables $t_{\alpha,\beta},x_1,x_2$. The following result is an analog of Theorem 3.1 for the singularity $W_{3,3}$.

Theorem 5.1. *The function $s_{i,j}(t_{*,*})$ is equal to the coefficient of $x_1^ix_2^j$ in the polynomial $\mathcal{P}^{W_{3,3}}(t_{*,*},3x_1,3x_2)$.*

Proof. We will actually prove a bit stronger statement:

$$W_{3,3;s}(x_1,x_2)=\mathcal{P}^{W_{3,3}}(t_{*,*},3x_1,3x_2).\tag{5.9}$$

The proof is by direct computation. For the left-hand side we have

$$W_{3,3;s}(x)=x_1^3+x_2^3+t_{1,1}x_1x_2-t_{1,0}x_1-t_{0,1}x_2+\left(t_{0,0}-\frac{t_{1,1}^3}{54}\right).$$

Let us compute the polynomial $\mathcal{P}^{W_{3,3}}(t,x)$. From (5.8) it follows that it has the form

$$\mathcal{P}^{W_{3,3}}(t,x)=\alpha_1\frac{x_1^3}{6}+\alpha_2\frac{x_2^3}{6}+\alpha_3t_{1,1}x_1x_2+\alpha_4t_{1,0}x_1+\alpha_5t_{0,1}x_2+\left(\alpha_6t_{0,0}+\alpha_7\frac{t_{1,1}^3}{6}\right),\quad \alpha_1,\ldots,\alpha_7\in\mathbb{Q}.$$

We compute

$$\begin{aligned}\alpha_1=\alpha_2=\left\langle\tau_{-1,-1}\tau_{2,2}\tau_{2,0}^3\right\rangle^{W_{3,3}}&=\int_{\overline{\mathcal{M}}_{0,5}}c^{\text{3-spin}}(-1,2,2,2,2)c^{\text{3-spin}}(-1,2,0,0,0)=\\&=\left\langle\tau_{-1}\tau_2^4\right\rangle^{\text{3-spin}}\overset{\text{by (4.5)}}{=}\frac{2}{9},\\ \alpha_3=\left\langle\tau_{-1,-1}\tau_{2,2}\tau_{2,0}\tau_{0,2}\tau_{1,1}\right\rangle^{W_{3,3}}&=\int_{\overline{\mathcal{M}}_{0,5}}c^{\text{3-spin}}(-1,2,2,0,1)c^{\text{3-spin}}(-1,2,0,2,1).\end{aligned}$$

Denote by $\pi_3\colon\overline{\mathcal{M}}_{0,5}\to\overline{\mathcal{M}}_{0,4}$ and $\pi_4\colon\overline{\mathcal{M}}_{0,5}\to\overline{\mathcal{M}}_{0,4}$ the forgetful maps which forget the third and fourth marked points, respectively. Then

$$c^{\text{3-spin}}(-1,2,2,0,1)=\pi_4^*c^{\text{3-spin}}(-1,2,2,1)=-\frac{1}{3}\pi_4^*\psi_1=-\frac{1}{3}(\psi_1-D_{1,4}),$$

where $D_{i,j}$ denotes the cohomology class, Poincaré dual to the divisor in $\overline{\mathcal{M}}_{0,5}$, whose generic point is a nodal curve made of two bubbles containing the marked points labeled by $\{i,j\}$ and $\{1,2,3,4,5\}\backslash\{i,j\}$, respectively. Similarly,

$$c^{\text{3-spin}}(-1,2,0,2,1)=-\frac{1}{3}\pi_3^*\psi_1=-\frac{1}{3}(\psi_1-D_{1,3}).$$

As a result,

$$\alpha_3=\int_{\overline{\mathcal{M}}_{0,5}}c^{\text{3-spin}}(-1,2,2,0,1)c^{\text{3-spin}}(-1,2,0,2,1)=\frac{1}{9}\int_{\overline{\mathcal{M}}_{0,5}}(\psi_1-D_{1,4})(\psi_1-D_{1,3})=\frac{1}{9}.$$

For the constants α_4 and α_5 we get

$$\alpha_4=\alpha_5=\left\langle\tau_{-1,-1}\tau_{2,2}\tau_{2,0}\tau_{1,0}\right\rangle^{W_{3,3}}=\left\langle\tau_{-1}\tau_2^2\tau_1\right\rangle^{\text{3-spin}}\overset{\text{by (4.5)}}{=}-\frac{1}{3}.$$

We continue with α_6:

$$\alpha_6=\left\langle\tau_{-1,-1}\tau_{2,2}\tau_{0,0}\right\rangle^{W_{3,3}}=1.$$

For the constant α_7 we compute

$$\alpha_7=\left\langle\tau_{-1,-1}\tau_{2,2}\tau_{1,1}^3\right\rangle^{W_{3,3}}=\int_{\overline{\mathcal{M}}_{0,5}}c^{\text{3-spin}}(-1,2,1,1,1)^2.$$

Consider the S_3-action on $\overline{\mathcal{M}}_{0,5}$ induced by permutations of the last three marked points. Clearly,

$$c^{\text{3-spin}}(-1,2,1,1,1)\in H^2(\overline{\mathcal{M}}_{0,5},\mathbb{Q})^{S_3}.$$

Let us compute this class. The basis of the group $H^2(\overline{\mathcal{M}}_{0,5},\mathbb{Q})^{S_3}$ is given by

$$D_1=D_{1,2},\qquad D_2=D_{3,4}+D_{3,5}+D_{4,5},\qquad D_3=D_{2,3}+D_{2,4}+D_{2,5}.$$

The intersection matrix $\left(\int_{\overline{\mathcal{M}}_{0,5}}\psi_iD_j\right)_{1\le i,j\le 3}$ is non-degenerate. Using also the integrals

$$\int_{\overline{\mathcal{M}}_{0,5}}\psi_ic^{\text{3-spin}}(-1,2,1,1,1)=-\frac{1}{3}\delta_{i,3},\quad i=1,2,3,$$

computed using the topological recursion relations in the extended r-spin theory [BCT19, Lemma 3.6], we conclude that $c^{\text{3-spin}}(-1,2,1,1,1) = -\frac{1}{3}D_1 = -\frac{1}{3}D_{1,2}$. Therefore,

$$\alpha_7 = \int_{\overline{\mathcal{M}}_{0,5}} c^{\text{3-spin}}(-1,2,1,1,1)^2 = \frac{1}{9}\int_{\overline{\mathcal{M}}_{0,5}} D_{1,2}^2 = -\frac{1}{9}.$$

Thus, the polynomial $\mathcal{P}^{W_{3,3}}(t,x)$ is equal to

$$\mathcal{P}^{W_{3,3}}(t,x) = \frac{x_1^3}{27} + \frac{x_2^3}{27} + t_{1,1}\frac{x_1x_2}{9} - t_{1,0}\frac{x_1}{3} - t_{0,1}\frac{x_2}{3} + \left(t_{0,0} - \frac{t_{1,1}^3}{54}\right),$$

and we can immediately see that the polynomial $W_{3,3;s}(x)$ is obtained from it by the rescaling $x_i \mapsto 3x_i$. The theorem is proved. $\square$

5.4. **Conjecture for the singularities E_6 and E_8.** The singularities E_6 and E_8 coincide with the singularities

$$W_{4+m,3}(x_1,x_2) = x_1^{4+m} + x_2^3,$$

for $m=0$ and $m=1$, respectively. Similarly to the case of the singularity $W_{3,3}$, let us define functions $\mathcal{P}^{E_{6+2m}}(t,x)$, $m=0,1$, by

$$\mathcal{P}^{E_{6+2m}}(t,x) := \sum_{\substack{n,k_1,k_2\ge 0\\ n+k_1+k_2\ge 1}} \frac{x_1^{k_1}}{k_1!}\frac{x_2^{k_2}}{k_2!} \sum_{\substack{0\le\alpha_1,\ldots,\alpha_n\le 2+m\\ 0\le\beta_1,\ldots,\beta_n\le 1}} \left\langle \tau_{-1,-1}\tau_{3+m,2}\tau_{3+m,0}^{k_1}\tau_{0,2}^{k_2}\prod_{i=1}^n \tau_{\alpha_i,\beta_i}\right\rangle^{W_{4+m,3}} \frac{\prod_{i=1}^n t_{\alpha_i,\beta_i}}{n!}.$$

The Landau–Ginzburg mirror symmetry for the singularities E_6 and E_8, as well as for all simple singularities, was proved in [FJR13]. Denote by $s_{i,j}^{E_{6+2m}}(t)$, $0\le i\le 2+m$, $0\le j\le 1$, the corresponding transformation between the flat coordinates and the parameters of the miniversal deformation.

Conjecture 5.2. *We have*

$$s_{i,j}^{E_{6+2m}}(t_{*,*}) = \mathrm{Coef}_{x_1^ix_2^j}\mathcal{P}^{E_{6+2m}}(t_{*,*}, -(4+m)\theta_{4+m}x_1, -3\theta_3x_2).$$

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