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ITERATED CHROMATIC LOCALISATION

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ABSTRACT. We study a certain monoid of endofunctors of the stable homotopy category that includes localizations with respect to finite unions of Morava K -theories. We work in an axiomatic framework that can also be applied to analogous questions in equivariant stable homotopy theory. Our results should be helpful for the study of transchromatic phenomena, including the Chromatic Splitting Conjecture. The combinatorial parts of this work have been formalised in the Lean proof assistant.

1. INTRODUCTION

Fix a prime p , and let \mathcal{B} denote the category of p -local spectra.

The Bousfield localisation functors $L_{K(n)} : \mathcal{B} \rightarrow \mathcal{B}$ and $L_n = L_{K(0) \vee \dots \vee K(n)}$ play a central role in chromatic homotopy theory. It is a well-known and useful fact that $L_n L_m = L_{\min(n,m)}$ and $L_n L_{K(n)} = L_{K(n)}$. It is not hard to see that $L_{K(n)} L_{K(m)} = L_{K(n)} L_m = 0$ when $n > m$. Two versions of the Chromatic Splitting Conjecture of Hopkins involve the functors $L_{n-1} L_{K(n)}$ and $L_{K(n-1)} L_{K(n)}$, and the latter can naturally be compared with $L_{K(n-1) \vee K(n)}$. Spectra such as $L_{K(m)} \widehat{E(n)} = L_{K(m)} L_{K(n)} E(n)$ (for $m < n$) occur naturally in the transchromatic character theory of Stapleton, and also in work of Torii. To encompass all these examples, we make the following definitions:

Definition 1.1. Given a finite subset $A \subset \mathbb{N}$, we put

$$K(A) = \bigvee_{a \in A} K(a) \in \mathcal{B},$$

and we let $\lambda_A : \mathcal{B} \rightarrow \mathcal{B}$ denote the Bousfield localisation functor with respect to $K(A)$. Fix $n^* \geq 1$, and put $N = \{0, \dots, n^* - 1\}$. Let Λ denote the monoid of endofunctors of \mathcal{B} generated by all the λ_A for $A \subseteq N$.

Our original goal was to describe the structure of Λ . It turns out to be natural to consider instead a certain monoid \mathbb{Q} that acts on \mathcal{B} and includes all the functors λ_A . This differs from Λ in that (a) we do not know whether the map $\mathbb{Q} \rightarrow \pi_0 \text{End}(\mathcal{B})$ is injective, and (b) the image of \mathbb{Q} is strictly larger than Λ .

We will take an axiomatic approach, which will also cover some non-chromatic examples. However, in this introduction we will focus on the chromatic case. As an example, consider the functor

$$F = \lambda_{013} \lambda_{023} = L_{K(0) \vee K(1) \vee K(3)} L_{K(0) \vee K(2) \vee K(3)},$$

and the submonoid $\langle F \rangle \leq \Lambda$ that it generates. We do not know a very easy way to see that $|\langle F \rangle|$ is even finite, but we will show that in fact $|\langle F \rangle| = 3$.

In order to motivate our general approach, we recall the theory of chromatic fracture squares. As a special case of a fact that was already known to Bousfield, for all $m < n$ and $X \in \mathcal{B}$, there is a homotopy cartesian square

$$\begin{array}{ccc} L_{K(m) \vee K(n)} X & \longrightarrow & L_{K(n)} X \\ \downarrow & & \downarrow \\ L_{K(m)} X & \longrightarrow & L_{K(m)} L_{K(n)} X, \end{array}$$

which exhibits $L_{K(m) \vee K(n)} X$ as the homotopy inverse limit of a certain subdiagram of $\psi(X)$. By the same methods, if $|A| = d$ then one can exhibit a homotopy cartesian hypercube of dimension d , which expresses $\lambda_A X$ as a homotopy inverse limit of terms of the form $L_{K(t_1)} \cdots L_{K(t_r)} X$. We will again call this phenomenon *chromatic fracture*.

We now give an initial version of our main definitions and results. To make them precise, we will need a significant amount of foundational work, as will be discussed below.

Definition 1.2. Let \mathbb{P} be the set of subsets of N , ordered by inclusion. For $A, B \in \mathbb{P}$, we write $A \angle B$ if $a \leq b$ for all $a \in A$ and $b \in B$. If $A = \{a_1, \dots, a_r\}$ with $a_1 < \cdots < a_r$, we put

$$\phi_A(X) = L_{K(a_1)} \cdots L_{K(a_r)} X.$$

Remark 1.3. Note that $A \angle B$ is vacuously satisfied if $A = \emptyset$ or $B = \emptyset$, and because of this, the relation is not transitive. (It is not reflexive or symmetric either.)

Definition 1.4. Let \mathbb{Q} be the set of all subsets of \mathbb{P} that are closed upwards, ordered by reverse inclusion. We define $u: \mathbb{P} \rightarrow \mathbb{Q}$ by $uA = \{B \mid A \subseteq B\}$, so u is a morphism of posets. We also define $v: \mathbb{P} \rightarrow \mathbb{Q}$ by $vA = \{B \mid B \cap A \neq \emptyset\}$, so v is order-reversing.

Remark 1.5. In \mathbb{P} , the smallest element is \emptyset and the largest element is N . In \mathbb{Q} , the smallest element is $u\emptyset = \mathbb{P}$ and the largest element is \emptyset . The element uN is second-largest in \mathbb{Q} , in the sense that every element $U \in \mathbb{Q}$ with $U \neq \emptyset$ satisfies $U \leq uN$.

Lemma 1.6. *There is a map $\mu: \mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{Q}$ of posets given by*

$$\mu(U, V) = U * V = \{A \cup B \mid A \in U, B \in V, A \angle B\}.$$

This operation is associative, with

$$U * V * W = \{A \cup B \cup C \mid A \in U, B \in V, C \in W, A \angle B, A \angle C, B \angle C\},$$

and $u\emptyset$ is a two-sided identity element. Moreover it is distributive on both sides with respect to the union.

Proof. Suppose that $A \in U$, $B \in V$, $A \angle B$ and $A \cup B \subseteq C$. We can then choose t such that $a \leq t$ for all $a \in A$, and $t \leq b$ for all $b \in B$. We put $A' = \{c \in C \mid c \leq t\}$ and $B' = \{c \in C \mid t \leq c\}$. Then $A \subseteq A'$ so $A' \in U$, and $B \subseteq B'$ so $B' \in V$. We also have $C = A' \cup B'$ with $A' \angle B'$, so $C \in U * V$. This proves that $U * V$ is closed upwards, so we have indeed defined a map $\mu: \mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{Q}$. It is clear that if $U \subseteq U'$ and $V \subseteq V'$ then $U * V \subseteq U' * V'$, so μ is a morphism of posets. All remaining claims are also easy. \square

Remark 1.7. We can define $\kappa: \mathbb{Q} \rightarrow \mathbb{P}$ by $\kappa(U) = \{n \mid \{n\} \in U\}$. This is order-reversing and satisfies $\kappa(U * V) = \kappa(U) \cap \kappa(V)$ and $\kappa(vA) = A$ and

$$\kappa(uA) = \begin{cases} N & \text{if } A = \emptyset \\ A & \text{if } |A| = 1 \\ \emptyset & \text{if } |A| > 1. \end{cases}$$

However, κ is very far from being injective, so this gives only crude insight into the monoid structure of \mathbb{Q} . We do not know any better example of a homomorphism to a more familiar monoid.

It is familiar that we can regard posets as categories with hom sets of size at most one. The above lemma then makes \mathbb{Q} into a monoidal category. We also have a monoidal category $\text{End}(\mathcal{B})$ of endofunctors of \mathcal{B} , with composition as the monoidal product.

We now give a preliminary statement of our main result. This will be imprecise in various ways, to be discussed below; the imprecision will be removed in the main body of the paper.

Theorem 1.8. *There is a strong monoidal functor $U \mapsto \theta_U$ from \mathbb{Q} to $\text{End}(\mathcal{B})$, with $\theta_{uA} = \phi_A$ and $\theta_{vA} = \lambda_A$. We therefore have $\theta_U \theta_V(X) \simeq \theta_{U*V}(X)$, and there are compatible natural maps $\theta_U(X) \rightarrow \theta_V(X)$ whenever $U \leq V$ in \mathbb{Q} , or equivalently $U \supseteq V$. The definition is that $\theta_U(X)$ is the homotopy inverse limit of the objects $\phi_A(X)$ for $A \in U$.*

As \mathbb{Q} is finite and the image of θ contains the generators of Λ , we see in particular that Λ is finite. We do not know whether θ is injective. We can spell out the relationship between \mathbb{Q} and Λ a little more explicitly as follows:

Definition 1.9. Let $\mathbb{A} = (A_1, \dots, A_r)$ be a list of subsets of N . A *thread* for \mathbb{A} is a list (a_1, \dots, a_r) with $a_i \in A_i$ for all i and $a_1 \leq a_2 \leq \dots \leq a_r$. A *thread set* for \mathbb{A} is a subset $A^* \subseteq N$ such that there exists a thread contained in A^* . We write $T(\mathbb{A})$ for the set of all thread sets. This is clearly closed upwards, so $T(\mathbb{A}) \in \mathbb{Q}$. We also write $\lambda_{\mathbb{A}}$ for the composite $\lambda_{A_1} \cdots \lambda_{A_r}$.

Proposition 1.10. *In \mathbb{Q} we have $T(\mathbb{A}) = vA_1 * \dots * vA_r$. Thus, Theorem 1.8 implies that $\lambda_{\mathbb{A}} = \theta_{T(\mathbb{A})}$.*

Proof. This follows from the definitions by a straightforward induction. \square

Example 1.11. We previously mentioned the functor $F = \lambda_{013} \lambda_{023}$. This is θ_U , where

$$\begin{aligned} U &= v\{0, 1, 3\} * v\{0, 2, 3\} = T(\{0, 1, 3\}, \{0, 2, 3\}) \\ &= \{A \mid 0 \in A \text{ or } 3 \in A \text{ or } \{1, 2\} \subseteq A\}. \end{aligned}$$

A check of cases shows that $U * U = v\{0, 3\}$ and then that $U^{*k} = v\{0, 3\}$ for $k \geq 2$. Thus, the monoid generated by F is $\{1, F, \lambda_{03}\}$.

Example 1.12. Take $N = \{0, 1, 2\}$ and $U = \{\{0, 1\}, \{1, 2\}, \{0, 1, 2\}\} \in \mathbb{Q}$. We claim that $U \neq T(\mathbb{A})$ for any \mathbb{A} , so \mathbb{Q} really is different from Λ . Indeed, suppose that $U = T(\mathbb{A})$ with $\mathbb{A} = (A_1, \dots, A_r)$. Then $\{1\} \notin T(\mathbb{A})$, so we can choose m with $1 \notin A_m$. On the other hand, we have $\{0, 1\} \in T(\mathbb{A})$, which means that there exists p with $1 < p \leq r$ and $0 \in A_i$ for $i < p$ and $1 \in A_i$ for $i \geq p$. From this it

is clear that $p > m$. Similarly, as $\{1, 2\} \in T(A)$ there must exist q with $1 \leq q < r$ and $1 \in A_i$ for $i \leq q$ and $2 \in A_i$ for $i > q$. From this we see that $q < m$ and so $q \leq p - 2$. It follows in turn that $\{0, 2\} \in T(\mathbb{A})$, contradicting our assumption that $T(\mathbb{A}) = U$.

One problem with our preliminary statement is that the formation of homotopy limits requires diagrams that commute in some underlying model category, whereas localisation functors are merely characterised by a homotopical universal property. To avoid these issues, we will need some foundational work with derivators and anafunctors. For any X , we would like to construct a coherent diagram containing all of the objects $\theta_U X$ and all the natural morphisms between them. Instead of constructing this directly, we will define a category of potential candidates (Definition 8.1), and prove that an appropriate forgetful functor to \mathcal{B} is an equivalence (Proposition 8.4). To make this work smoothly, we need a version that works uniformly when X is not just a single spectrum, but is itself a coherent diagram. This is precisely the kind of issue for which the theory of derivators is designed. However, it will still work out that θ_U is not really an honest functor, but is instead a kind of fraction in which we formally invert an equivalence of derivators. We will also need some foundational work to support this.

Remark 1.13. This paper contains a number of results about the combinatorial homotopy theory of \mathbb{P} , \mathbb{Q} and various other posets constructed from these. All of these results have been formalised in the Lean proof assistant. A snapshot of the code will be deposited on the arXiv. Active development will continue at <https://github.com/NeilStrickland>.

2. BASIC DEFINITIONS

Definition 2.1. We will fix a compactly generated triangulated category \mathcal{B} . We also fix an integer $n^* \geq 1$ and put $N = \{0, \dots, n^* - 1\}$ as before. We then fix a family of homology theories $K(n)_* : \mathcal{B} \rightarrow \text{Ab}_*$ for $n \in N$ and put $K(A)_* = \bigoplus_{a \in A} K(a)_*$ for any $A \subseteq N$. We let λ_A denote the localisation with respect to the localizing subcategory of $K(A)_*$ -acyclics. We assume the following condition, which we call the *fracture axiom*: if A is a nonempty subset of N , and $b \in N$ with $b > \max(A)$, then $K(b)_* \lambda_A(X) = 0$ for all X .

Remark 2.2. Since later we will employ the theory of stable derivators we need to assume that our triangulated category \mathcal{B} is the underlying category of a stable derivator, i.e. $\mathcal{B} \simeq \mathcal{C}(e)$ for some derivator \mathcal{C} . This condition is not too restrictive and it is verified if \mathcal{B} has a geometric model (see [2, Theorem 6.11] or the easier result [7, Proposition 1.36] for combinatorial model categories).

Lemma 2.3. *The fracture axiom implies the following statement (which we call the extended fracture axiom): if $A, B \subseteq N$ with $A \angle B$ and $K(B)_*(X) = 0$, then $K(B)_*(\lambda_A(X)) = 0$.*

Proof. If $A = \emptyset$ then $K(A)_* = 0$ and so $\lambda_A = 0$ and everything is trivial. We can thus assume that $A \neq \emptyset$, so $\max(A)$ is defined. The assumption $A \angle B$ then means that $b \geq \max(A)$ for all $b \in B$. We are given that $K(B)_*(X) = 0$, or in other words that $K(b)_*(X) = 0$ for all $b \in B$. We want to prove that $K(b)_*(\lambda_A(X)) = 0$. If $b > \max(A)$ then this is immediate from the fracture axiom. This just leaves the case where $b = \max(A)$, so $b \in A$. The map $X \rightarrow \lambda_A(X)$ is a $K(A)$ -equivalence,

so it is a $K(b)$ -equivalence (because $b \in A$), and $K(b)_*(X) = 0$ by assumption, so $K(b)_*(\lambda_A(X)) = 0$ as required. \square

All our examples will be verified using the following result:

Proposition 2.4. *Let \mathcal{B} be a stable homotopy category as in [12] (so it has a closed symmetric monoidal structure compatible with the triangulation). Suppose we have N as before and objects $K(n) \in \mathcal{B}$ representing the homology theories $K(n)_*(X) = \pi_*(K(n) \wedge X)$. Suppose we also have objects $T(n) \in \mathcal{B}$, and that the following axioms are satisfied:*

- (a) $T(n)$ is dualizable for all n .
- (b) For $m < n$ we have $K(m) \wedge T(n) = 0$.
- (c) For any object X and any n we have $K(n)_*(X) = 0$ iff $K(n) \wedge X = 0$ iff $K(n) \wedge T(n) \wedge X = 0$.

Then the fracture axiom is satisfied.

Proof. Suppose that $b > \max(A)$. We need to show that $K(b) \wedge \lambda_A(X) = 0$, and by axiom (c) it will suffice to show that $K(b) \wedge T(b) \wedge \lambda_A(X) = 0$. For this it will suffice to show that $T(b) \wedge \lambda_A(X) = 0$, or that the identity map of $T(b) \wedge \lambda_A(X)$ is zero, or that the adjoint map

$$DT(b) \wedge T(b) \wedge \lambda_A(X) \rightarrow \lambda_A(X)$$

is zero. Here $K(A) \wedge T(b) = 0$ by axiom (b), so the source of the above map is $K(A)$ -acyclic, whereas the target is $K(A)$ -local; this implies that the map is zero as required. \square

Example 2.5. For the simplest example, let \mathcal{B} be the derived category of modules over $\mathbb{Z}_{(p)}$, and put

$$\begin{aligned} K(0) &= \mathbb{Q} & K(1) &= \mathbb{Z}/p \\ T(0) &= \mathbb{Z}_{(p)} & T(1) &= \mathbb{Z}/p. \end{aligned}$$

It is then straightforward to check the hypotheses of Proposition 2.4, so the fracture axiom is satisfied.

Example 2.6. For the motivating example, fix a prime p . Let \mathcal{B}_0 denote the category of symmetric spectra of simplicial sets, equipped with the p -localisation of the usual model structure. Put $\mathcal{B} = \text{Ho}(\mathcal{B}_0)$ (so this is the usual stable homotopy category of p -local spectra). For any $n \in N = \{0, \dots, n^* - 1\}$, let $K(n)$ denote the Morava K -theory spectrum of height n at the prime p , and let $T(n)$ be any finite p -local spectrum of type n . It is again straightforward to check the hypotheses of Proposition 2.4, so the fracture axiom is satisfied.

Example 2.7. For another example, fix a cyclic group G of order p^d for some prime p and $d \geq 0$. Put $N = \{0, \dots, d\}$. For $n \in N$ let H_n be the unique subgroup of order p^{d-n} in G , and put $Q_n = G/H_n$, so that $|Q_n| = p^n$. In the G -equivariant stable homotopy category, put $T(n) = (Q_n)_+$ and

$$K(n) = (Q_n)_+ \wedge \tilde{\Sigma}EQ_{n+1},$$

where $\widetilde{\Sigma}$ denotes the unreduced suspension. For the case $n = d$, this should be interpreted as $K(d) = (Q_d)_+ = G_+$. We find that the geometric fixed points are

$$\phi^{H_m}T(n) = \begin{cases} 0 & \text{if } m < n \\ (Q_n)_+ & \text{if } m \geq n, \end{cases} \quad \phi^{H_m}K(n) = \begin{cases} 0 & \text{if } k \neq n \\ S^0 & \text{if } k = n. \end{cases}$$

Recall that ϕ^{H_m} preserves smash products, and that $X = 0$ iff $\phi^{H_m}(X) = 0$ (in the nonequivariant stable category) for all m . Using this, it is not hard to check the hypotheses of Proposition 2.4, and we again see that the fracture axiom is satisfied.

Remark 2.8. It would of course be interesting to see what one could say about more general finite groups, where the subgroup lattice is more complicated. We suspect that this will be substantially harder; we may return to the question in future work.

3. DERIVATORS AND HOMOTOPY (CO)LIMITS

We will use the theory of stable derivators, mostly following [7].

Definition 3.1. Let POSet denote the strict 2-category of finite posets, so the 0-cells are finite posets, and the 1-cells are nondecreasing maps. Given two nondecreasing maps $u, v: P \rightarrow Q$, there is one 2-cell from u to v if $u(p) \leq v(p)$ for all p , and no 2-cells otherwise. We write $[n]$ for the set $\{0, \dots, n\}$ with its usual order, so $[n] \in \text{POSet}$. Note that $[0]$ is the terminal poset, which will also be denoted by e .

Definition 3.2. For us, a *prederivator* is a strict 2-functor $\mathcal{C}: \text{POSet}^{\text{op}} \rightarrow \text{CAT}$. More explicitly, it consists of

- (a) For every finite poset P , a category $\mathcal{C}(P)$.
- (b) For every morphism $u: P \rightarrow Q$, a functor $u^*: \mathcal{C}(Q) \rightarrow \mathcal{C}(P)$, such that $1^* = 1$ and $(v \circ u)^* = u^* \circ v^*$ on the nose.
- (c) For every inequality $u \leq v$ between morphisms $P \rightarrow Q$, a natural map $u^* \rightarrow v^*$, satisfying some evident axioms.

(By restricting attention to finite posets rather than more general categories, we are following [7, Remark 1.8].)

Remark 3.3. For any prederivator \mathcal{C} we have a category $\mathcal{C}(e)$, which we call the *underlying category* of \mathcal{C} . We will often think of this as the key ingredient, with the other categories just adding extra structure to $\mathcal{C}(e)$ in some sense.

Definition 3.4. A *derivator* is a prederivator in which all the functors u^* have left and right adjoints with certain compatibility properties, as specified in [7, Definition 1.10]. These adjoints can be thought of as homotopy right and left Kan extensions. A *stable derivator* is a derivator \mathcal{C} subject to some additional conditions:

- (a) \mathcal{C} should be *strong*. To explain this, note that for any P there are evident inclusions $i_0, i_1: P \rightarrow [1] \times P$ with $i_0 \leq i_1$. We therefore have functors $i_0^*, i_1^*: \mathcal{C}([1] \times P) \rightarrow \mathcal{C}(P)$, together with a natural map between them. These can be combined in an obvious way to get a functor $\mathcal{C}([1] \times P) \rightarrow \mathcal{C}(P)^{[1]}$, and the strongness condition is that this functor should be full and essentially surjective (for all P). This is [7, Definition 1.13].

- (b) \mathcal{C} should be *pointed*, which means that $\mathcal{C}(e)$ should have an object that is both initial and terminal. Many consequences of this condition are investigated in [7, Section 3].
- (c) Homotopy pushouts squares in \mathcal{C} must coincide with homotopy pullback squares, in the sense spelled out in [7, Definition 4.1].

Remark 3.5. If \mathcal{C} is a derivator and P is a finite poset, then we can define a new derivator \mathcal{C}^P by $\mathcal{C}^P(Q) = \mathcal{C}(P \times Q)$. This is called a *shifted derivator* and it is often a useful device. In [7], Theorem 1.31 proves that \mathcal{C}^P is indeed a derivator, and Proposition 2.6 proves that for any $u: P \rightarrow Q$, the resulting morphism $u^*: \mathcal{C}^Q \rightarrow \mathcal{C}^P$ preserves left and right homotopy Kan extensions.

Remark 3.6. For every derivator \mathcal{C} there is a dual derivator $\mathcal{C}^{\text{op}}(P) = \mathcal{C}(P^{\text{op}})^{\text{op}}$, as in [7, Definition 1.15] and surrounding discussion.

Definition 3.7. Let \mathcal{C} be a derivator. Let P be a finite poset, and let c be the unique morphism $P \rightarrow e$, so we have functors $c_!, c_*: \mathcal{C}(P) \rightarrow \mathcal{C}(e)$. We define $\text{holim}_{\rightarrow P} X = c_!(X)$ and $\text{holim}_{\leftarrow P} (X) = c_*(X)$.

Definition 3.8. Let $f: P \rightarrow Q$ be a morphism of finite posets, and suppose that $q \in Q$. We use the following notation for comma posets:

$$\begin{aligned} f/q &= \{p \in P \mid f(p) \leq q\} \\ q/f &= \{p \in P \mid q \leq f(p)\}. \end{aligned}$$

Remark 3.9. We can now recall the key axiom from [7, Definition 1.10], which is known as the Kan formula. Suppose we have a morphism $f: P \rightarrow Q$, a derivator \mathcal{C} , an object $X \in \mathcal{C}(P)$, and an element $q \in Q$. The element q gives $i_q: e \rightarrow Q$, and we want to understand the object $i_q^* f_* X \in \mathcal{C}(e)$. There is an evident inclusion $j: q/f \rightarrow P$ so we have $j^* X \in \mathcal{C}(q/f)$ and $\text{holim}_{\leftarrow q/f} j^* X \in \mathcal{C}(e)$. Using various adjunctions one can write down a natural map $i_q^* f_*(X) \rightarrow \text{holim}_{\leftarrow q/f} j^* X$, and the axiom says that this should be an isomorphism. Dually, the natural map $\text{holim}_{\rightarrow f/q} j^* X \rightarrow i_q^* f_!(X)$ should also be an isomorphism.

This is a direct generalization of the Kan formula for Kan extensions of functors in the usual categorical setting. That is, for a type of derivators called *represented*, the above isomorphisms are exactly the expression of right or left Kan extensions via limits and colimits respectively. See the discussion following [7, Definition 1.9].

Theorem 3.10. *Let \mathcal{C} be a stable derivator. Then each category $\mathcal{C}(P)$ has a natural structure as a triangulated category. Moreover, for each $u: P \rightarrow Q$, the corresponding functor $u^*: \mathcal{C}(Q) \rightarrow \mathcal{C}(P)$ has a canonical natural isomorphism $u^* \Sigma \rightarrow \Sigma u^*$ with respect to which it is an exact functor, and the same applies to the left and right adjoint functors $u_!, u_*: \mathcal{C}(P) \rightarrow \mathcal{C}(Q)$.*

Proof. This is [7, Corollary 4.19]. □

Definition 3.11. Let \mathcal{C} be a stable derivator, and let P be a finite poset. Each $p \in P$ gives a morphism $i_p: e \rightarrow P$ and thus a functor $i_p^*: \mathcal{C}(P) \rightarrow \mathcal{C}(e)$. Given $X \in \mathcal{C}(P)$, we write X_p for $i_p^*(X) \in \mathcal{C}(e)$. We also put $\text{supp}(X) = \{p \mid X_p \neq 0\} \subseteq P$, and call this the *support* of X . Given $Q \subseteq P$, we say that X is *supported in Q* if $\text{supp}(X) \subseteq Q$. We write $\mathcal{C}_Q(P)$ for the full subcategory of $\mathcal{C}(P)$ consisting of objects supported in Q .

Remark 3.12. Derivators are defined in [7, Definition 1.10], and the second axiom says that a morphism $f: X \rightarrow Y$ in $\mathcal{C}(P)$ is an isomorphism iff $f_p: X_p \rightarrow Y_p$ is an isomorphism for all p . From this we see that $\text{supp}(X) = \emptyset$ iff $X = 0$. Similarly, for any $M \subseteq P$ we have an inclusion $j: M \rightarrow P$ and we write $X|_M$ for $j^*(Q)$. We then find that X is supported in Q iff $X|_{P \setminus Q} = 0$.

Definition 3.13. Let P be a finite poset, and let Q be a subset of P .

- (a) We say that Q is a *sieve* if it is closed downwards, so that whenever $p \leq q$ with $q \in Q$ we also have $p \in Q$.
- (b) We say that Q is a *cosieve* if it is closed upwards, so that whenever $q \leq p$ with $q \in Q$ we also have $p \in Q$.

If $Q \subseteq P$ is a (co)sieve, we also say that the inclusion morphism $Q \rightarrow P$ is a (co)sieve. More generally, if $i: Q \rightarrow P$ is an embedding (so $i(q) \leq i(q')$ iff $q \leq q'$) and $i(Q)$ is a (co)sieve, we say that i is a (co)sieve. This is the finite poset version of [7, Definition 1.28].

Lemma 3.14. *If \mathcal{C} is a derivator and $j: Q \rightarrow P$ is an embedding then the counit map $j^*j_*X \rightarrow X$ and the unit map $X \rightarrow j^*j_!X$ are both isomorphisms (for all $X \in \mathcal{C}(Q)$). Thus, the functors j_* and $j_!$ are both full and faithful embeddings. In particular, this holds if j is a sieve or a cosieve.*

Proof. We first consider the counit map $j^*j_*X \rightarrow X$. By the derivator axiom Der2, it will suffice to prove that the induced map $(j_*X)_{j(q)} = (j^*j_*X)_q \rightarrow X_q$ is an isomorphism in $\mathcal{C}(e)$ for all $q \in Q$. Put $j(q)/j = \{a \in Q \mid j(q) \leq j(a)\}$ and let $k: j(q)/j \rightarrow Q$ be the inclusion. The Kan formula identifies $(j_*X)_{j(q)}$ with the homotopy limit of $k^*(X) \in \mathcal{C}(j(q)/j)$. Note that q is initial in $j(q)/j$, so the inclusion $i_q: e \rightarrow j(q)/j$ is left adjoint to $c: j(q)/j \rightarrow e$, so the homotopy limit functor c_* is the same as i_q^* by [7, Lemma 1.23]. This gives $(j_*X)_{j(q)} = c_*k^*(X) = (ki_q)^*(X) = X_q$ as required. This proves that the counit map $j^*j_*X \rightarrow X$ is an isomorphism. To prove that the unit map $X \rightarrow j^*j_!X$ is also an isomorphism, we can either give a similar argument, or take adjoints, or appeal to a kind of self-duality of the theory of derivators. From the isomorphism $X \simeq j^*j_!X$ we obtain $[W, X] \simeq [W, j^*j_!X] = [j_!W, j_!X]$, so $j_!$ is full and faithful. A similar argument shows that j_* is full and faithful. \square

Proposition 3.15. *Let \mathcal{C} be a stable derivator, and let $j: Q \rightarrow P$ be a sieve. Then:*

- (a) *The functor $j_*: \mathcal{C}(Q) \rightarrow \mathcal{C}(P)$ has a right adjoint denoted by $j^!$, as well as the left adjoint j^* that exists by the definition of j_* .*
- (b) *The unit map $X \rightarrow j^!j_*(X)$ is an isomorphism (as are the unit map $X \rightarrow j^*j_!X$ and the counit map $j^*j_*X \rightarrow X$, as we saw in Lemma 3.14).*
- (c) *The functor j_* gives an equivalence from $\mathcal{C}(Q)$ to $\mathcal{C}_Q(P)$, with inverse given by j^* or $j^!$.*
- (d) *If $Y \in \mathcal{C}(Q)$ corresponds to $X \in \mathcal{C}_Q(P)$ then $\text{holim}_{\leftarrow Q} (Y) \simeq \text{holim}_{\leftarrow P} (X)$.*

Proof. Most of parts (a) to (c) can be obtained by combining Definition 3.4, Proposition 3.6 and Corollary 3.8 from [7]. More specifically, 3.6(ii) says that j_* is full and faithful with $\mathcal{C}_Q(P)$ as the essential image, and 3.8 gives us the functor $j^!$. From Lemma 3.14 we have $1 \simeq j^*j_*$, and by taking right adjoints we get $1 \simeq j^!j_*$. We leave it to the reader to check that this natural isomorphism is just the unit map. Given this, the claim (d) just reduces to $(c_P)_*j_* \simeq (c_Q)_*$, where c_P and c_Q are the unique morphisms $P \rightarrow e$ and $Q \rightarrow e$. But this is clear because $c_P j = c_Q$. \square

Proposition 3.16. *Let \mathcal{C} be a stable derivator, and let $i: R \rightarrow P$ be a cosieve. Then:*

- (a) *The functor $i_!: \mathcal{C}(R) \rightarrow \mathcal{C}(P)$ has a left adjoint denoted by $i^?$, as well as the right adjoint i^* that exists by the definition of $i_!$.*
- (b) *The counit map $i^?i_!(X) \rightarrow X$ is an isomorphism (as are the unit map $X \rightarrow i^*i_!X$ and the counit map $i^*i_*X \rightarrow X$, as we saw in Lemma 3.14).*
- (c) *The essential image of $i_!$ is precisely $\mathcal{C}_R(P)$, so in fact we have an equivalence $\mathcal{C}(R) \simeq \mathcal{C}_R(P)$.*
- (d) *If $Z \in \mathcal{C}(R)$ corresponds to $X \in \mathcal{C}_R(P)$ then $\text{holim}_{\rightarrow R} (Z) \simeq \text{holim}_{\rightarrow P} (X)$.*

Proof. This is dual to the previous proposition. \square

Proposition 3.17. *We now want to combine the above two propositions. Suppose that $Q \subseteq P$ is a sieve, and let $R = P \setminus Q$ be the complementary cosieve. Let $Q \xrightarrow{j} P \xleftarrow{i} R$ be the inclusions, so we have functors as follows, with each functor left adjoint to the one below it.*

$$\begin{array}{ccccc}
 & & j_! & & i^? \\
 & \curvearrowright & & \curvearrowleft & \\
 \mathcal{C}(Q) & \xleftarrow{j^*} & \mathcal{C}(P) & \xleftarrow{i_!} & \mathcal{C}(R) \\
 & \curvearrowleft & & \curvearrowright & \\
 & & j_* & & i^* \\
 & & j^! & & i_*
 \end{array}$$

- (a) *The composites $i^?j_!$, $j^*i_!$, i^*j_* and $j^!i_*$ (obtained by composing two functors at the same level in the diagram) are all zero. (We have nothing systematic to say about any other composite functors between $\mathcal{C}(Q)$ and $\mathcal{C}(R)$.)*
- (b) *The six adjunctions in the diagram involve six (co)unit maps to or from the identity functor of $\mathcal{C}(P)$. These fit into a diagram as follows, in which every straight line is part of a natural distinguished triangle:*

$$\begin{array}{ccc}
 i_!i^? & & j_*j^! \\
 & \swarrow & \searrow \\
 & & 1 \\
 i_!i^* & \xrightarrow{\quad} & j_*j^* \\
 & \swarrow & \searrow \\
 i_*i^* & & j_!j^*
 \end{array}$$

Proof. This is just a reorganisation of information extracted from [7, Example 4.25]. We will explain some parts of the argument. The Kan formula expresses the object $(j^*i_!Z)_q = (i_!Z)_{j(q)}$ as a homotopy colimit over a certain comma poset, but the (co)sieve properties of i and j ensure that this comma poset is empty, so $j^*i_! = 0$. By taking left and right adjoints repeatedly we deduce that the other composites in (a) are also zero. The horizontal composite $f: i_!i^* \rightarrow j_*j^*$ is adjoint to a map $i^* \rightarrow i^*j_*j^*$, and $i^*j_* = 0$, so it follows that $f = 0$. The same kind of argument shows that the two other straight line composites are also zero. For the remaining points we refer to [7], and to the paper [11] that is cited there. \square

4. HOMOTOPY THEORY OF PARTIALLY ORDERED SETS

Definition 4.1. Let $\text{POSet}(P, Q)$ be the set of monotone maps from P to Q . We give this the partial order such that $f \leq g$ iff $f(p) \leq g(p)$ in Q for all $p \in P$. It is easy to see that this makes the category of finite posets into a cartesian closed category.

Definition 4.2. Let P be a finite poset. Recall that a subset $\sigma \subseteq P$ is a *chain* if the induced order on σ is total. Given a map $x: P \rightarrow [0, 1]$, we define $\text{supp}(x) = \{p \mid x(p) > 0\}$. We define $|P|$ to be the set of maps $x: P \rightarrow [0, 1]$ such that $\text{supp}(x)$ is a chain and $\sum_{p \in P} x(p) = 1$. We call this the *geometric realisation* of P . A standard argument shows that this gives a functor $\text{POSet} \rightarrow \text{Top}$ that preserves finite coproducts and finite limits. In particular, we can apply geometric realisation to the evaluation map $\text{POSet}(P, Q) \times P \rightarrow Q$, and then take adjoints, to get a continuous map

$$|\text{POSet}(P, Q)| \rightarrow \text{Top}(|P|, |Q|).$$

Definition 4.3. For any finite poset P , we define $\pi_0(P)$ to be the quotient of P by the smallest equivalence relation such that $p \sim q$ whenever $p \leq q$. This is easily seen to be the same as the set of path components of $|P|$. It gives a functor from finite posets to finite sets, which preserves finite products and coproducts. It follows formally that we can construct a quotient category of POSet with morphism sets $\pi_0(\text{POSet}(P, Q))$. We call this the *strong homotopy category* of finite posets. It also follows that if f and g lie in the same equivalence class of $\pi_0(\text{POSet}(P, Q))$ then the resulting maps $|f|$ and $|g|$ are homotopic (by a straight-line homotopy, in the basic case where $f \leq g$ or $g \leq f$). Thus, geometric realisation gives a functor from the strong homotopy category of posets to the homotopy category of topological spaces.

Remark 4.4. Suppose we have morphisms $f: P \rightarrow Q$ and $g: Q \rightarrow P$ that are adjoint, in the sense that $f(p) \leq q$ iff $p \leq g(q)$. We then have (co)unit inequalities $1 \leq gf$ and $fg \leq 1$, showing that fg and gf give identities in the strong homotopy category, and thus that f and g are strong homotopy equivalences.

Definition 4.5. We say that P is *strongly contractible* if the map $c_P: P \rightarrow e$ is a strong homotopy equivalence.

We note that this holds if P has a smallest element or a largest element. We also note that if P is a strongly contractible poset, then $|P|$ is a contractible space.

Definition 4.6. Consider a morphism $f: P \rightarrow Q$ in POSet , and note that $c_Q f = c_P: P \rightarrow e$. For any derivator \mathcal{C} and any $X, Y \in \mathcal{C}(Q)$ we therefore get a map

$$f^*: \mathcal{C}(Q)(c_Q^* X, c_Q^* Y) \rightarrow \mathcal{C}(P)(c_P^* X, c_P^* Y).$$

We say that f is a *\mathcal{D} -equivalence* if this map is bijective for all \mathcal{C} , X and Y . We also say that P is *\mathcal{D} -contractible* if the map $c_P: P \rightarrow e$ is a \mathcal{D} -equivalence, or equivalently the functor

$$c_P^*: \mathcal{C}(e) \rightarrow \mathcal{C}(P)$$

is full and faithful.

Remark 4.7. Groth uses the term *homotopy contractible* rather than *\mathcal{D} -contractible*.

Proposition 4.8. *If $[f] = [g]$ in $\pi_0(\text{POSet}(P, Q))$ then*

$$f^* = g^* : \mathcal{C}(Q)(c_Q^* X, c_Q^* Y) \rightarrow \mathcal{C}(P)(c_P^* X, c_P^* Y).$$

Thus, f is a \mathcal{D} -equivalence iff g is a \mathcal{D} -equivalence.

Proof. We can reduce easily to the case where $f \leq g$. As $\mathcal{C} : \text{POSet}^{\text{op}} \rightarrow \text{CAT}$ is a strict 2-functor, the following diagram of categories and functors must commute on the nose:

$$\begin{array}{ccc} \text{POSet}(P, Q) \times \text{POSet}(Q, e) & \xrightarrow{\text{compose}} & \text{POSet}(P, e) \\ \downarrow & & \downarrow \\ [\mathcal{C}(Q), \mathcal{C}(P)] \times [\mathcal{C}(e), \mathcal{C}(Q)] & \xrightarrow{\text{compose}} & [\mathcal{C}(e), \mathcal{C}(P)]. \end{array}$$

The inequality $f \leq g$ gives a morphism $(f, c_Q) \rightarrow (g, c_Q)$ in the category $\text{POSet}(P, Q) \times \text{POSet}(Q, e)$, and this becomes the identity morphism of c_P in $\text{POSet}(P, e)$. The claim now follows by chasing the diagram. \square

Corollary 4.9. *If $f : P \rightarrow Q$ is a strong homotopy equivalence, then it is a \mathcal{D} -equivalence. In particular:*

- (a) *If f has a left or right adjoint, then it is a \mathcal{D} -equivalence.*
- (b) *If P is strongly contractible, then it is \mathcal{D} -contractible.* \square

The following definitions are taken from [10, Section 3]:

Definition 4.10.

- (a) We say that a map $f : Q \rightarrow P$ is *homotopy final* if the natural map

$$\text{holim}_{\overrightarrow{Q}} f^*(X) = (c_Q)_! f^*(X) = (c_P)_! f_! f^*(X) \rightarrow (c_P)_!(X) = \text{holim}_{\overrightarrow{P}}(X)$$

is an isomorphism for all derivators \mathcal{C} and all objects $X \in \mathcal{C}(P)$.

- (b) Dually, we say that a map $f : Q \rightarrow P$ is *homotopy cofinal* if the natural map

$$\text{holim}_{\overleftarrow{P}}(X) = (c_P)_*(X) \rightarrow (c_P)_* f_* f^*(X) = (c_Q)_* f^*(X) = \text{holim}_{\overleftarrow{Q}} f^*(X)$$

is an isomorphism for all derivators \mathcal{C} and all objects $X \in \mathcal{C}(P)$.

We do not really need the following result, but it helps to clarify the relationship between our definitions.

Proposition 4.11. *If $f : Q \rightarrow P$ is homotopy final or homotopy cofinal then it is a \mathcal{D} -equivalence.*

Proof. We will just treat the final case, as the other case is dual. Taking $X = c_P^*(U)$ in the definition, we see that the natural map $(c_Q)_! c_Q^*(U) \rightarrow (c_P)_! c_P^*(U)$ is an isomorphism. This gives an isomorphism

$$\mathcal{C}(e)((c_P)_! c_P^*(U), V) \simeq \mathcal{C}(e)((c_Q)_! c_Q^*(U), V)$$

for any V . By adjunction, we get an isomorphism

$$\mathcal{C}(P)(c_P^*(U), c_P^*(V)) \simeq \mathcal{C}(Q)(c_Q^*(U), c_Q^*(V)).$$

We leave it to the reader to check that this is just f^* , as required. \square

Proposition 4.12. *The map f is homotopy final iff p/f is strongly contractible for all p , and this holds if f has a left adjoint. Dually, f is homotopy cofinal iff f/p is strongly contractible for all p , and this holds if f has a right adjoint.*

Proof. See [10, Corollary 3.13] and surrounding discussion. \square

Proposition 4.13. *Consider a commutative square*

$$\begin{array}{ccc} P & \xrightarrow{t} & Q \\ u \downarrow & \alpha \nearrow & \downarrow v \\ R & \xrightarrow{w} & S, \end{array}$$

and the resulting Beck-Chevalley transform $\alpha_*: w^*v_* \rightarrow u_*t^*$ between morphisms $\mathcal{C}^Q \rightarrow \mathcal{C}^R$. For any $r \in R$ we have comma posets r/u and $w(r)/v$, and using t and α we can produce a morphism $t_r: r/u \rightarrow w(r)/v$. If this is homotopy cofinal for all r , then α is an isomorphism.

Proof. We must show that for any T and any $X \in \mathcal{C}^Q(T) = \mathcal{C}(Q \times T)$, the map $(\alpha_*)_X: w^*v_*X \rightarrow u_*t^*X$ is an isomorphism in $\mathcal{C}(R \times T)$. After replacing \mathcal{C} by \mathcal{C}^T we can assume that $T = e$, so $X \in \mathcal{C}(Q)$ and $(\alpha_*)_X$ is a morphism in $\mathcal{C}(R)$. It will suffice to show that $i_r^*(\alpha_*)_X$ is an isomorphism for all $r \in R$. The Kan formula expresses the source and target of $i_r^*(\alpha_*)_X$ as homotopy limits over comma posets. In more detail, there is a projection $\pi_r: w(r)/v \rightarrow Q$, and the source of $i_r^*(\alpha_*)_X$ is the homotopy limit of π_r^*X , whereas the target is the homotopy limit of $t_r^*\pi_r^*X$. The homotopy cofinality condition says that the natural map between these is an isomorphism. \square

Definition 4.14. Recall that a subset $\sigma \subseteq P$ is a *chain* if the induced order on σ is total. If σ is a chain, we write $\dim(\sigma) = |\sigma| - 1$. We put

$$\begin{aligned} s(P) &= \{ \text{nonempty chains } \sigma \subseteq P \} \\ s_d(P) &= \{ \sigma \in s(P) \mid \dim(\sigma) = d \} \\ s_{\leq d}(P) &= \{ \sigma \in s(P) \mid \dim(\sigma) \leq d \}. \end{aligned}$$

Note that every nonempty chain σ has a largest element, which we denote by $\max(\sigma)$. We order $s(P)$ by inclusion, which ensures that $\max: s(P) \rightarrow P$ is a morphism of posets.

This construction gives a functor $s: \text{POSet} \rightarrow \text{POSet}$, and $\max: s(P) \rightarrow P$ is natural. However, if $f \leq g$ then it is typically not the case that $s(f) \leq s(g)$. This makes the following proof more complex than one might expect.

Proposition 4.15. *If $[f] = [g]$ in $\pi_0(\text{POSet}(P, Q))$, then $[s(f)] = [s(g)]$. Thus, s induces an endofunctor of the strong homotopy category.*

Proof. We can easily reduce to the case where $f \leq g$. We then choose a minimal element p_1 in P , then a minimal element p_2 in $P \setminus \{p_1\}$ and so on, giving an enumeration $P = \{p_1, \dots, p_{m-1}\}$ say. We define $\phi: P \rightarrow [m] = \{0, \dots, m\}$ by $\phi(p_i) = i$ (so ϕ is injective and monotone, and 0 and m are not in the image). Then for $0 \leq k \leq m$ we define $u_k, v_k: s(P) \rightarrow s(Q)$ by

$$\begin{aligned} u_k(\sigma) &= \{f(p) \mid p \in \sigma, \phi(p) < k\} \cup \{g(p) \mid p \in \sigma, \phi(p) \geq k\} \\ v_k(\sigma) &= \{f(p) \mid p \in \sigma, \phi(p) \leq k\} \cup \{g(p) \mid p \in \sigma, \phi(p) \geq k\}. \end{aligned}$$

We find that $u_k(\sigma)$ and $v_k(\sigma)$ are nonempty chains in Q , so we really do have maps $u_k, v_k: s(P) \rightarrow s(Q)$ as advertised. It is also clear that when $\sigma \subseteq \tau$ we have $u_k(\sigma) \subseteq u_k(\tau)$ and $v_k(\sigma) \subseteq v_k(\tau)$, so u_k and v_k are maps of posets. Next, we find that $u_k, u_{k+1} \leq v_k$, which implies that all the maps u_k and v_k lie in the same path component. Finally, we have $u_0 = s(g)$ and $u_m = s(f)$, so $[s(f)] = [s(g)]$ as claimed. \square

Lemma 4.16. *The map $\max: s(P) \rightarrow P$ is homotopy cofinal (and so is a \mathcal{D} -equivalence).*

Proof. Fix $p \in P$; it will suffice to show that the poset

$$U = \max / p = \{\sigma \in sP \mid \max(\sigma) \leq p\}$$

is strongly contractible. Put

$$V = \{\sigma \in sP \mid \max(\sigma) = p\} = \{\sigma \in U \mid p \in \sigma\} \subseteq U.$$

As $\{p\}$ is smallest in V , we see that V is strongly contractible. We can define a poset map $t: U \rightarrow V$ by $t(\tau) = \tau \cup \{p\}$, and we find that this is left adjoint to the inclusion $V \rightarrow U$, so the inclusion is a strong homotopy equivalence by Remark 4.4. It follows that U is also strongly contractible. \square

Lemma 4.17. *The map $|\max|: |s(P)| \rightarrow |P|$ is a homotopy equivalence.*

Proof. We can define a barycentric subdivision map $\beta: |s(P)| \rightarrow \text{Map}(P, [0, 1])$ by

$$\beta(w)(p) = \sum_{p \in \sigma} |\sigma|^{-1} w(\sigma).$$

It is well-known that this gives a homeomorphism $|s(P)| \rightarrow |P|$. A typical simplex τ of $s(P)$ is a chain $\{\sigma_0, \dots, \sigma_d\}$ where each σ_i is a nonempty chain in P and $\sigma_0 \subset \dots \subset \sigma_d$. It is easy to see that both β and $|\max|$ send $|\tau|$ into $|\sigma_d|$, so the straight-line homotopy between β and $|\max|$ stays within $|P|$. This means that $|\max|$ is homotopic to a homeomorphism, and so is a homotopy equivalence. \square

Definition 4.18. The *weak homotopy category* of finite posets is obtained from the strong homotopy category by inverting the maps $\max: s(P) \rightarrow P$. Thus every morphism $P \rightarrow Q$ in the weak homotopy category can be represented as $f \circ \max^{-r}$ for some $f: s^r(P) \rightarrow Q$, with $f \circ \max^{-r} = g \circ \max^{-s}$ iff $[f \circ \max^{s+t}] = [g \circ \max^{r+t}]$ in $\pi_0(\text{POSet}(s^{r+s+t}(P), Q))$ for sufficiently large t .

Remark 4.19. Using Lemma 4.17 we see that geometric realisation gives a functor from the weak homotopy category to the homotopy category of finite simplicial complexes. A slight variant of the standard simplicial approximation theorem shows that this functor is an equivalence. However, we do not need this, so we will not spell out the details.

We will also use some theory of total fibres. We recall some basic ideas.

Definition 4.20. Let R be a finite set, and let PR be the poset of subsets of R . Let $j: e \rightarrow PR$ correspond to $\emptyset \in PR$, and put $P'R = PR \setminus \{\emptyset\}$, and let $i: P'R \rightarrow PR$ be the inclusion. As j is a sieve, Proposition 3.15 gives us a functor $j^!: \mathcal{C}(PR) \rightarrow \mathcal{C}(e)$ that is right adjoint to j_* . We also write $\text{tfib}(X)$ or $\text{tfib}_R(X)$ for $j^!(X)$, and call this the *total fibre* of an object $X \in \mathcal{C}(PR)$. We also say that X is *cartesian* if it is in the essential image of $i_*: \mathcal{C}(P'R) \rightarrow \mathcal{C}(R)$.

Lemma 4.21. X is cartesian iff $\mathrm{tfib}(X) = 0$.

Proof. We use Proposition 3.17. Part (a) of that result includes the relation $j^!i_* = 0$, which means that tfib is trivial on cartesian objects. For the converse, part (b) tells us that the fibre of the unit map $X \rightarrow i_*i^*(X)$ is $j_*j^!(X)$, so if $j^!(X) = 0$ we see that $X \simeq i_*i^*(X)$ and so X is cartesian. \square

Lemma 4.22. $\mathrm{tfib}(X)$ is the fibre of the natural map $X_\emptyset \rightarrow \mathrm{holim}_{\leftarrow P'R} j^*X$.

Proof. Let c and c' be the maps $PR \rightarrow e$ and $P'R \rightarrow e$, so $cj = 1$ and $ci = c'$. Proposition 3.17(b) gives us a distinguished triangle $j_*j^!X \rightarrow X \rightarrow i_*i^*X$. The functor $c_*: \mathcal{C}(PR) \rightarrow \mathcal{C}(e)$ is exact by [7, Corollary 4.19]. We can therefore apply it to get a distinguished triangle $c_*j_*j^!X \rightarrow c_*X \rightarrow c_*i_*i^*X$. As $cj = 1$, the first term is just $j^!X = \mathrm{tfib}(X)$. As j is left adjoint to c , we can identify c_* with j^* (as in [7, Lemma 1.23]), so the middle term is $j^*X = X_\emptyset$. As $ci = c'$, the last term is $c'_*i^*X = \mathrm{holim}_{\leftarrow P'R} i^*X$. \square

We now want to discuss another description of $\mathrm{tfib}(X)$. Suppose that $r \in R$, and put $R_0 = R \setminus \{r\}$. Define $k_-, k_+: PR_0 \rightarrow PR$ by $k_-(T) = T$ and $k_+(T) = T \cup \{r\}$. As $k_- \leq k_+$ we have a natural map $k_-^*X \rightarrow k_+^*X$ in $\mathcal{C}(R_0)$, and thus a map $\mathrm{tfib}(k_-^*X) \rightarrow \mathrm{tfib}(k_+^*X)$.

Proposition 4.23. *There is a natural distinguished triangle*

$$\mathrm{tfib}(X) \rightarrow \mathrm{tfib}(k_-^*X) \rightarrow \mathrm{tfib}(k_+^*X) \rightarrow \Sigma \mathrm{tfib}(X).$$

Proof. We can regard X as an object of $\mathcal{C}^{PR_0}(P\{r\})$. From this point of view k_-^*X is just X_\emptyset , and $P'\{r\} \simeq e$ so k_+^*X is just $\mathrm{holim}_{\leftarrow P'\{r\}} j^*X$. We also see that $k_-^!X = \mathrm{tfib}_{\{r\}}(X)$. Lemma 4.22 therefore gives us a distinguished triangle $k_-^!X \rightarrow k_-^*X \rightarrow k_+^*X$. Now let $i_0: e \rightarrow PR_0$ correspond to \emptyset , and apply the exact functor $i_0^!$ to the above triangle. As $k_-i_0 = i: e \rightarrow PR$, the first term is $i^!X = \mathrm{tfib}(X)$. The other two terms are $\mathrm{tfib}(k_-^*X)$ and $\mathrm{tfib}(k_+^*X)$, as required. \square

Proposition 4.24. *Let $t: e \rightarrow PR$ correspond to the element $R \in PR$. Then there is a natural isomorphism $\mathrm{tfib}(t_!X) \simeq \Omega^{|R|}(X)$.*

Proof. We can split R as $R_0 \amalg \{r\}$ as before, and let $t_0: e \rightarrow PR_0$ correspond to $R_0 \in PR_0$, and put $n_0 = |R_0|$. It is then easy to identify $k_+^*t_!(X)$ with $(t_0)_!(X)$, so by induction we have $\mathrm{tfib}(k_+^*(t_!(X))) = \Omega^{n_0}(X)$. On the other hand, we can check that $k_-^*(t_!(X)) = 0$, so $\mathrm{tfib}(k_-^*(t_!(X))) = 0$. Now Proposition 4.23 tells us that $\mathrm{tfib}(t_!(X))$ is the fibre of $0 \rightarrow \Omega^{n_0}(X)$, which is $\Omega^{n_0+1}(X)$ as required. \square

The following proposition is a derivator version of a standard result about homotopy limits in simplicial or topological categories.

Proposition 4.25. *Let n be the maximum length of any chain in P . Then for all stable derivators \mathcal{C} and objects $X \in \mathcal{C}(P)$ there is a natural tower*

$$\mathrm{holim}_{\leftarrow P} (X) = T^n(X) \rightarrow T^{n-1}(X) \rightarrow \cdots \rightarrow T^0(X) \rightarrow T^{-1}(X) = 0$$

and natural distinguished triangles

$$\bigoplus_{\sigma \in s_d(P)} \Omega^d X_{\max(\sigma)} \rightarrow T^d(X) \rightarrow T^{d-1}(X).$$

Proof. Put $Y = \max^*(X) \in \mathcal{C}(s(P))$. Lemma 4.16 identifies $\text{holim}_{\leftarrow P}(X)$ with $\text{holim}_{\leftarrow s(P)}(Y)$, so we will work with Y from now on.

Let $j_d: s_{\leq d}(P) \rightarrow s(P)$ be the inclusion, and put $T^d(X) = \text{holim}_{\leftarrow} j_d^*(Y)$. Note that $T^n(Y) = \text{holim}_{\leftarrow s(P)} Y = \text{holim}_{\leftarrow P} X$.

Now fix d and consider the object $Z = j_{\leq d}^*(Y)$ and the inclusions $j: s_{\leq (d-1)}(P) \rightarrow s_{\leq d}(P)$ and $i: s_d(P) \rightarrow s_{\leq d}(P)$. Proposition 3.17 gives a distinguished triangle $i_! i^*(Z) \rightarrow Z \rightarrow j_* j^*(Z)$. If we let c be the map $s_{\leq d}(P) \rightarrow e$ and apply c_* , we get a distinguished triangle $c_* i_! i^*(Z) \rightarrow T^d(X) \rightarrow T^{d-1}(X)$. Now note that the order on $s_d(P)$ is discrete: for $\sigma, \tau \in s_d(P)$ we can only have $\sigma \leq \tau$ if $\sigma = \tau$. Because of this, we see that $\mathcal{C}(s_d(P)) \simeq \prod_{\sigma \in s_d(P)} \mathcal{C}(e)$. Because of this, we can write $i^*(Z)$ as a coproduct of objects $W(\sigma)$, where $W(\sigma)_\tau = 0$ for $\tau \neq \sigma$, and $W(\sigma)_\sigma = Z_\sigma = Y_\sigma = X_{\max(\sigma)}$. It will now suffice to identify $c_* i_! W(\sigma)$ with $\Omega^d Y_\sigma$. Here i is a cosieve, and it follows that for $\tau \in s_{\leq d}(P)$ we have

$$(i_! W(\sigma))_\tau = \begin{cases} Y_\sigma & \text{if } \tau = \sigma \\ 0 & \text{otherwise} \end{cases}.$$

The poset $\{\tau \in s(P) \mid \tau \subseteq \sigma\}$ is naturally identified with $P'(\sigma)$. Let $k: P'(\sigma) \rightarrow s_{\leq d}(P)$ be the inclusion, which is a sieve. As the support of $i_! W(\sigma)$ is contained in the image of k , we see from Proposition 3.15 that the unit map $i_! W(\sigma) \rightarrow k_* k^* i_! W(\sigma)$ is an isomorphism. It follows that $c_* i_! W(\sigma)$ is the homotopy limit of $k^* i_! W(\sigma)$. Here it is easy to see that the object $k^* i_! W(\sigma) \in \mathcal{C}(P'(\sigma))$ is the restriction to $P'(\sigma)$ of the object $t_! W(\sigma)$ appearing in Proposition 4.24. That proposition tells us that the total fibre of $t_! W(\sigma)$ is $\Omega^{d+1} Y_\sigma$. On the other hand, Lemma 4.22 tells us that the total fibre is the same as the fibre of the natural map from $(t_! W(\sigma))_\emptyset = 0$ to $\text{holim}_{\leftarrow} k^* i_! W(\sigma)$, which is $\Omega \text{holim}_{\leftarrow} k^* i_! W(\sigma)$. As we are working with stable derivators we know that Ω is an equivalence of categories, so $\text{holim}_{\leftarrow} k^* i_! W(\sigma) = \Omega^d Y_\sigma$ as required. \square

Corollary 4.26. *Let n be the maximum length of any chain in P . Then for all stable derivators \mathcal{C} and objects $X \in \mathcal{C}(P)$ there is a natural diagram*

$$0 = T_{-1}(X) \rightarrow T_0(X) \rightarrow \cdots \rightarrow T_n(X) = \text{holim}_{\leftarrow P}(X)$$

and natural distinguished triangles

$$T_{d-1}(X) \rightarrow T_d(X) \rightarrow \bigoplus_{\sigma \in s_d(P)} \Sigma^d X_{\max(\sigma)}.$$

Proof. Apply the proposition to the dual derivator. \square

Proposition 4.27. *For any map $u: Q \rightarrow P$ of finite posets, the functors u^* , $u_!$ and u_* all preserve arbitrary products and coproducts.*

Proof. The functors u^* and $u_!$ have right adjoints, so they preserve coproducts. The functors u^* and u_* have left adjoints, so they preserve products. The key point is to prove that u_* preserves coproducts. Consider a family of objects $X_\alpha \in \mathcal{C}(Q)$, and the resulting map $f: \bigoplus_\alpha u_*(X_\alpha) \rightarrow u_*(\bigoplus_\alpha X_\alpha)$. We want to prove that f is an isomorphism, and it will suffice to show that $i_p^*(f)$ is an isomorphism for all $p \in P$. We have already observed that i_p^* preserves coproducts, and we have a Kan

formula expressing $i_p^* u_*$ as a homotopy limit over p/u . It will therefore suffice to show that all homotopy limit functors preserve coproducts. Note that the functor $\Omega: \mathcal{C}(e) \rightarrow \mathcal{C}(e)$ is an equivalence of categories, so it certainly preserves coproducts. It follows that all functors of the form $X \mapsto \Omega^d X_q$ also preserve coproducts. It then follows by induction that the functors T^d in Proposition 4.25 all preserve coproducts. By taking d sufficiently large, we see that homotopy limits preserve coproducts as required. This completes the proof that u_* preserves coproducts, and we can apply that to the dual derivator to see that $u_!$ preserves products. \square

5. THICK SUBDERIVATORS

Definition 5.1. Let \mathcal{C} be a stable derivator. By an *thick subderivator* $\mathcal{E} \subseteq \mathcal{C}$ we mean a system of full subcategories $\mathcal{E}(P) \subseteq \mathcal{C}(P)$ for all P , such that:

- (a) Each category $\mathcal{E}(P)$ is closed under finite coproducts (including the empty coproduct, so $0 \in \mathcal{E}(P)$).
- (b) Whenever $X \xrightarrow{j} Y \xrightarrow{q} X$ in $\mathcal{C}(P)$ with $qj = 1$, if $Y \in \mathcal{E}(P)$ then $X \in \mathcal{E}(P)$. In other words, \mathcal{E} is closed under retracts. In particular, if $X \simeq Y$ and $Y \in \mathcal{E}(P)$ then $X \in \mathcal{E}(P)$.
- (c) For any morphism $u: P \rightarrow Q$ of finite posets, we have $u^* \mathcal{E}(Q) \subseteq \mathcal{E}(P)$ and $u_! \mathcal{E}(P) \subseteq \mathcal{E}(Q)$ and $u_* \mathcal{E}(P) \subseteq \mathcal{E}(Q)$. More briefly, we say that the functors u^* , u_* and $u_!$ preserve \mathcal{E} .

Definition 5.2. Let \mathcal{C} be a stable derivator, and let \mathcal{E}_1 be a thick subcategory of $\mathcal{C}(e)$. For any finite poset P and any $p \in P$ we have a corresponding map $i_p: e \rightarrow P$.

- (a) We put

$$(\gamma_0 \mathcal{E}_1)(P) = \{X \in \mathcal{C}(P) \mid i_p^*(X) \in \mathcal{E}_1 \text{ for all } p \in P\}$$

- (b) We let $(\gamma_1 \mathcal{E}_1)(P)$ denote the smallest thick subcategory of $\mathcal{C}(P)$ containing $(i_p)_!(\mathcal{E}_1)$ for all p .
- (c) We let $(\gamma_2 \mathcal{E}_1)(P)$ denote the smallest thick subcategory of $\mathcal{C}(P)$ containing $(i_p)_*(\mathcal{E}_1)$ for all p .

Theorem 5.3. *The subcategories $(\gamma_i \mathcal{E}_1)(P)$ are the same for $i = 0, 1, 2$ (so we will just write $(\gamma \mathcal{E}_1)(P)$ in future). The map γ gives a bijection from thick subcategories of $\mathcal{C}(e)$ to thick subderivators of \mathcal{C} . Moreover, if \mathcal{E} is a thick subderivator of \mathcal{C} , then $\mathcal{E}(P)$ is a thick subcategory of $\mathcal{C}(P)$ for all P .*

The proof will be given after some lemmas.

Remark 5.4. It is important here that our derivators are indexed on finite posets rather than more general categories; the theorem would not be true without some restriction of this kind. In particular it would fail for derivators indexed by finite groups; this is related to the fact that classifying spaces of finite groups are infinite complexes.

Lemma 5.5. *If $\mathcal{E} \subseteq \mathcal{C}$ is a thick subderivator, then $\mathcal{E}(P)$ is a thick subcategory of $\mathcal{C}(P)$ for all P , and*

$$(\gamma_1 \mathcal{E}(e))(P) \cup (\gamma_2 \mathcal{E}(e))(P) \subseteq \mathcal{E}(P) \subseteq (\gamma_0 \mathcal{E}(e))(P).$$

Proof. The triangulation of $\mathcal{C}(P)$ is defined in terms of operations of the form u^* , $u_!$, u_* , $u^!$ and $u^?$. However, the operations $u^!$ and $u^?$ are themselves defined in terms of v^* , v_* and $v_!$ for various auxiliary morphisms v , as discussed in [7, Section

2]. From this it follows that $\mathcal{E}(P)$ is closed under the suspension functor and its inverse, and under cofibrations, so it is a thick subcategory. From the definitions we also know that the functors $(i_p)_!$ preserve \mathcal{E} , so $\mathcal{E}(P)$ contains the generators of $(\gamma_1\mathcal{E}(e))(P)$, so it contains the whole of $(\gamma_1\mathcal{E}(e))(P)$. It also contains $(\gamma_2\mathcal{E}(e))(P)$ by essentially the same argument. On the other hand, the functors i_p^* also preserve \mathcal{E} , and this means that $\mathcal{E}(P) \subseteq (\gamma_0\mathcal{E}(e))(P)$. \square

Lemma 5.6. *If \mathcal{E}_1 is a thick subcategory of $\mathcal{C}(e)$, then $\gamma_0\mathcal{E}_1$ is a thick subderivator of \mathcal{C} , with $(\gamma_0\mathcal{E}_1)(e) = \mathcal{E}_1$.*

Proof. Suppose we have a morphism $u: Q \rightarrow P$ of finite posets. It is clear from the definitions that $u^*((\gamma_0\mathcal{E}_1)(P)) \subseteq (\gamma_0\mathcal{E}_1)(Q)$, or more briefly that u^* preserves $\gamma_0\mathcal{E}_1$. We claim that the functors u_* and $u_!$ also preserve $\gamma_0\mathcal{E}_1$. In the case $Q = e$, this follows easily from Proposition 4.25 and Corollary 4.26. We can use the Kan formulae to deduce the general case from the case $Q = e$. It is also easy to check that $\gamma_0\mathcal{E}_1$ is closed under retracts, so $\gamma_0\mathcal{E}_1$ is a thick subderivator of \mathcal{C} . The relation $(\gamma_0\mathcal{E}_1)(e) = \mathcal{E}_1$ is clear. \square

Corollary 5.7. *If \mathcal{E}_1 is a thick subcategory of $\mathcal{C}(e)$, then $(\gamma_1\mathcal{E}_1)(P) \cup (\gamma_2\mathcal{E}_1)(P) \subseteq (\gamma_0\mathcal{E}_1)(P)$ for all P .*

Proof. Lemma 5.6 allows us to apply Lemma 5.5 to the case $\mathcal{E} = \gamma_0\mathcal{E}_1$. \square

Lemma 5.8. *If \mathcal{E}_1 is a thick subcategory of $\mathcal{C}(e)$, then all functors $u_!$ preserve $\gamma_1\mathcal{E}_1$, and all functors u_* preserve $\gamma_2\mathcal{E}_1$.*

Proof. Fix a map $u: P \rightarrow Q$, and put $\mathcal{U} = \{X \in \mathcal{C}(P) \mid u_*(X) \in (\gamma_1\mathcal{E}_1)(Q)\}$. It is easy to see that \mathcal{U} is thick. As $u_i^p = i_{u(p)}: e \rightarrow P$ we see that $u_!(i_p)_! = (i_{u(p)})_!$, and it follows that $(i_p)_!(\mathcal{E}_1) \subseteq \mathcal{U}$ for all $p \in P$. It follows that $(\gamma_1\mathcal{E}_1)(P) \subseteq \mathcal{U}$, which means that u_* preserves $\gamma_1\mathcal{E}_1$ as required. Dually, we see that u_* preserves $\gamma_2\mathcal{E}_1$. \square

Lemma 5.9. *If \mathcal{E}_1 is a thick subcategory of $\mathcal{C}(e)$, then $\gamma_0\mathcal{E}_1 = \gamma_1\mathcal{E}_1 = \gamma_2\mathcal{E}_1$.*

Proof. We will write $\Gamma_i = \gamma_i\mathcal{E}_1$ for brevity. It will be enough to prove that $\Gamma_0 = \Gamma_1$, as duality then gives $\Gamma_0 = \Gamma_2$. We already know from Corollary 5.7 that $\Gamma_1(P) \subseteq \Gamma_0(P)$ for all P , so it will suffice to prove that $\Gamma_0(P) \subseteq \Gamma_1(P)$. This is clear if P is empty. If P is nonempty, we can choose a minimal element $a \in P$ and put $Q = P \setminus \{a\}$. Let $j: e \rightarrow P$ correspond to a , and let $i: Q \rightarrow P$ be the inclusion. Consider an object $Y \in \Gamma_0(P)$; we must show that $Y \in \Gamma_1(P)$. Put $X = j_!j^*(Y)$, and let Z be the cofibre of the counit map $X \rightarrow Y$. From Lemma 3.14 we have $j^*j_! = 1$ and so $j^*Z = 0$, so the support of Z is contained in $i(Q)$. As i is a cosieve, we see that $Z \simeq i_!i^*(Z)$. Now $i^*(Z) \in \Gamma_0(Q)$, and we can assume by induction that $\Gamma_0(Q) \subseteq \Gamma_1(Q)$, so $Z \in i_!\Gamma_1(Q) \subseteq \Gamma_1(P)$. From the definitions we also have $j^*(X) \in \mathcal{E}_1$ and $X \in \Gamma_1(P)$. As $\Gamma_1(P)$ is thick and contains X and Z , it also contains Y as required. \square

Proof of Theorem 5.3. First suppose that we start with a thick subcategory $\mathcal{E}_1 \subseteq \mathcal{C}(e)$. Lemma 5.9 tells us that the $\gamma_i\mathcal{E}_1$ are all the same, so we can just write $\gamma\mathcal{E}_1$. Lemma 5.6 tells us that this is a thick subderivator, with $(\gamma\mathcal{E}_1)(e) = \mathcal{E}_1$.

Suppose instead that we start with a thick subderivator $\mathcal{E} \subseteq \mathcal{C}$, and we put $\mathcal{E}_1 = \mathcal{E}(e)$. Lemma 5.5 tells us that $\mathcal{E}(P)$ is a thick subcategory of $\mathcal{C}(P)$ for all P ,

and in particular that \mathcal{E}_1 is a thick subcategory of $\mathcal{C}(e)$. We can therefore apply Lemma 5.9 to \mathcal{E}_1 and combine this with Lemma 5.5 to see that $\mathcal{E} = \gamma\mathcal{E}_1$. \square

Corollary 5.10. *Let \mathcal{E} be a thick subderivator of \mathcal{C} , and let X be an object of $\mathcal{C}(P)$. Suppose that $P = \bigcup_i P_i$ for some family of subposets P_i . Then X lies in $\mathcal{E}(P)$ iff $X|_{P_i} \in \mathcal{E}(P_i)$ for all i .*

Proof. The identity $\mathcal{E} = \gamma\mathcal{E}(e)$ means that $X \in \mathcal{E}(P)$ iff $i_p^*(X) \in \mathcal{E}(e)$ for all $p \in P$. Similarly, $X|_{P_i} \in \mathcal{E}(P_i)$ iff $i_p^*(X) \in \mathcal{E}(e)$ for all $p \in P_i$. The claim is clear from this. \square

Definition 5.11. Let \mathcal{T} be a triangulated category with arbitrary coproducts. Recall that a *localising subcategory* of \mathcal{T} is a thick subcategory that is closed under arbitrary coproducts. Similarly, if \mathcal{C} is a stable derivator, a *localising subderivator* is a thick subderivator $\mathcal{E} \subseteq \mathcal{C}$ such that the subcategory $\mathcal{E}(P) \subseteq \mathcal{C}(P)$ is closed under arbitrary coproducts for all P .

Proposition 5.12. *The map γ gives a bijection between localising subcategories of $\mathcal{C}(e)$ and localising subderivators of \mathcal{C} , with inverse $\mathcal{E} \mapsto \mathcal{E}(e)$.*

Proof. Firstly, if \mathcal{E} is a localising subderivator of \mathcal{C} , then it is immediate from the definitions that $\mathcal{E}(e)$ is a localising subcategory of $\mathcal{C}(e)$.

In the opposite direction, suppose that \mathcal{E}_1 is a localising subcategory of $\mathcal{C}(e)$. Let (X_α) be a family of objects of $(\gamma_0\mathcal{E}_1)(P)$, so $i_p^*(X_\alpha) \in \mathcal{E}_1$ for all α and all $p \in P$. As i_p^* has a right adjoint we see that it preserves coproducts, so $i_p^*(\bigoplus_\alpha X_\alpha) = \bigoplus_\alpha i_p^*(X_\alpha) \in \mathcal{E}_1$. As this holds for all p we see that $\bigoplus_\alpha X_\alpha \in (\gamma_0\mathcal{E}_1)(P)$. This shows that $\gamma\mathcal{E}_1$ is a localising subderivator, as required. \square

Lemma 5.13. *Let \mathcal{C} be a stable derivator, and let $\mathcal{E}(P)$ be a thick subcategory of $\mathcal{C}(P)$ for all P . Suppose that for every $u: Q \rightarrow P$, the functors $u_!$ and u^* preserve \mathcal{E} . Then $\mathcal{E} = \gamma\mathcal{E}(e)$, so in particular \mathcal{E} is a thick subderivator.*

Proof. Put $\mathcal{E}' = \gamma\mathcal{E}(e) = \gamma_0\mathcal{E}(e) = \gamma_1\mathcal{E}(e)$. We now claim that $\mathcal{E}(P) \subseteq \mathcal{E}'(P)$ for all P . Using the description $\mathcal{E}' = \gamma_0\mathcal{E}(e)$, this reduces to the claim that $i_p^*\mathcal{E}(P) \subseteq \mathcal{E}(e)$ for all p , which is true because i_p^* preserves \mathcal{E} by assumption. In the opposite direction, we know that the functors $(i_p)_!$ preserve \mathcal{E} , which means that $\mathcal{E}(P)$ contains all the generators of $\mathcal{E}'(P) = (\gamma_1\mathcal{E}(e))(P)$. As $\mathcal{E}(P)$ is assumed to be thick, it follows that $\mathcal{E}'(P) \subseteq \mathcal{E}(P)$. \square

Definition 5.14. Let \mathcal{T} be a triangulated category with coproducts. We say that an object $X \in \mathcal{T}$ is *compact* if the natural map $\bigoplus_\alpha [X, Y_\alpha] \rightarrow [X, \bigoplus_\alpha Y_\alpha]$ is an isomorphism for every family of objects Y_α . We write \mathcal{T}_c for the full subcategory of compact objects (which is easily seen to be thick). If $\mathcal{T} = \mathcal{C}(P)$ for some derivator \mathcal{C} , then we will write $\mathcal{C}_c(P)$ rather than $\mathcal{C}(P)_c$. We say that \mathcal{T} is *compactly generated* if

- (a) The category \mathcal{T}_c is essentially small (so there is a skeleton that has a set of objects, rather than a proper class); and
- (b) \mathcal{T} is the only thick subcategory of \mathcal{T} that is closed under arbitrary coproducts and contains \mathcal{T}_c .

Lemma 5.15. *Let \mathcal{T} be a triangulated category, let \mathcal{G} be a set of objects of \mathcal{T} , and let \mathcal{U} be the smallest thick subcategory containing \mathcal{G} . Then \mathcal{U} is essentially small.*

Proof. Define full subcategories \mathcal{U}_n as follows. Start with $\mathcal{U}_0 = \mathcal{G} \cup \{0\}$. Let \mathcal{U}_{n+1} consist of $\bigcup_{k \in \mathbb{Z}} \Sigma^k \mathcal{U}_n$, together with a choice of cofibre for every morphism in \mathcal{U}_n , and a choice of splitting for every idempotent morphism in \mathcal{U}_n . Put $\mathcal{U}_\infty = \bigcup_n \mathcal{U}_n$. We then find that \mathcal{U}_∞ has only a set of objects, and contains a representative of every isomorphism class in \mathcal{U} . \square

Proposition 5.16. $\mathcal{C}_c(P)$ is a thick subderivator of \mathcal{C} (and so is the same as $\gamma\mathcal{C}_c(e)$).

Proof. Put $\mathcal{E} = \gamma\mathcal{C}_c(e)$, which is a thick subderivator; it will suffice to show that this is the same as \mathcal{C}_c .

If F is left adjoint to G and G preserves coproducts then for small X we have

$$[FX, \bigoplus_{\alpha} Y_{\alpha}] = [X, G \bigoplus_{\alpha} Y_{\alpha}] = [X, \bigoplus_{\alpha} GY_{\alpha}] = \bigoplus_{\alpha} [X, GY_{\alpha}] = \bigoplus_{\alpha} [FX, Y_{\alpha}],$$

so FX is small. Using this, we see that $u_!$ and u^* preserve \mathcal{C}_c . The claim now follows from Lemma 5.13. \square

Corollary 5.17. If $\mathcal{C}(e)$ is compactly generated, then $\mathcal{C}(P)$ is compactly generated for all P .

Proof. First, we can choose a small skeleton \mathcal{G} for $\mathcal{C}_c(e)$, and then put

$$\mathcal{G}(P) = \{(i_p)_!(X) \mid p \in P, X \in \mathcal{G}\} \subseteq \mathcal{C}_c(P).$$

From the description $\mathcal{C}_c(P) = (\gamma_1\mathcal{C}_c(e))(P)$ we see that $\mathcal{C}_c(P)$ is generated by $\mathcal{G}(P)$, and so is essentially small by Lemma 5.15.

Now let \mathcal{T} be a localising subcategory of $\mathcal{C}(P)$ that contains $\mathcal{C}_c(P)$. Put

$$\mathcal{U} = \{X \in \mathcal{C}(e) \mid (i_p)_!(X) \in \mathcal{T} \text{ for all } p \in P\}.$$

This is easily seen to be a localising subcategory of $\mathcal{C}(e)$ containing $\mathcal{C}_c(e)$, so $\mathcal{U} = \mathcal{C}(e)$. From Theorem 5.3 it follows that $\gamma\mathcal{U} = \mathcal{C}$, so in particular $\mathcal{C}(P) = (\gamma_1\mathcal{U})(P)$. However, from the definition of \mathcal{U} it is clear that $(\gamma_1\mathcal{U})(P) \subseteq \mathcal{T}$, so $\mathcal{T} = \mathcal{C}(P)$ as required. \square

Definition 5.18. We say that \mathcal{C} is *compactly generated* if it satisfies the equivalent conditions of Corollary 5.17.

Definition 5.19. Let \mathcal{U} be a thick subcategory of a triangulated category \mathcal{T} . We then write

$$\begin{aligned} \mathcal{U}^{\perp} &= \{Y \in \mathcal{T} \mid [U, Y] = 0 \text{ for all } U \in \mathcal{U}\} \\ {}^{\perp}\mathcal{U} &= \{X \in \mathcal{T} \mid [X, U] = 0 \text{ for all } U \in \mathcal{U}\}. \end{aligned}$$

Similarly, if \mathcal{E} is a thick subderivator of a stable derivator \mathcal{C} , we put $\mathcal{E}^{\perp}(P) = \mathcal{E}(P)^{\perp}$ and $({}^{\perp}\mathcal{E})(P) = {}^{\perp}(\mathcal{E}(P))$.

Proposition 5.20. \mathcal{E}^{\perp} and ${}^{\perp}\mathcal{E}$ are thick subderivators, so $\mathcal{E}^{\perp} = \gamma(\mathcal{E}(e)^{\perp})$ and ${}^{\perp}\mathcal{E} = \gamma({}^{\perp}\mathcal{E}(e))$.

Proof. Suppose that $X \in ({}^{\perp}\mathcal{E})(P)$ and $u: P \rightarrow Q$. For $V \in \mathcal{E}(Q)$ we have $u^*(V) \in \mathcal{E}(P)$ and so $[u_!(X), V] = [X, u^*(V)] = 0$. From this we see that $u_!$ preserves ${}^{\perp}\mathcal{E}$. As u^* is left adjoint to u_* and u_* preserves \mathcal{E} , we see in the same way that u^* preserves ${}^{\perp}\mathcal{E}$. It therefore follows from Lemma 5.13 that ${}^{\perp}\mathcal{E}$ is a thick subderivator. A dual argument shows that \mathcal{E}^{\perp} is also a thick subderivator. \square

6. ANAFUNCTORS FOR DERIVATORS

Suppose we have morphisms of derivators

$$\mathcal{C} \xleftarrow{F} \mathcal{X} \xrightarrow{G} \mathcal{D}$$

in which F is an equivalence (which means that $F: \mathcal{C}(P) \rightarrow \mathcal{X}(P)$ is full, faithful and essentially surjective for all P). We could choose an inverse for F and thus obtain a morphism $GF^{-1}: \mathcal{C} \rightarrow \mathcal{D}$. However, we prefer not to make arbitrary choices, so we will instead treat GF^{-1} as a formal fraction, or *anafunctor*. The bicategory DER' of derivators and anafunctors is thus obtained from the bicategory DER of derivators by inverting the equivalences. A formal framework for bicategories of fractions is given in [16, Section 2], and extended in [18]. To apply this framework to derivators, we need the following result:

Lemma 6.1. *The 2-category DER admits 2-pullbacks along equivalences.*

Proof. Consider a span $\mathcal{D} \xrightarrow{F} \mathcal{F} \xleftarrow{G} \mathcal{E}$, in which G is an equivalence of derivators. We will provide an explicit model for the 2-pullback. For each finite poset P we have a span of categories $\mathcal{D}(P) \xrightarrow{F_P} \mathcal{F} \xleftarrow{G_P} \mathcal{E}(P)$, in which G_P is an equivalence. We define $\mathcal{P}(P)$ to be the usual 2-pullback of this span. Explicitly, the objects are triples (X, Y, α) where $X \in \mathcal{D}(P)$, $Y \in \mathcal{E}(P)$ and $\alpha: F_P(X) \rightarrow G_P(Y)$ is an isomorphism in $\mathcal{F}(P)$. A morphism $(f, g): (X, Y, \alpha) \rightarrow (X', Y', \alpha')$ consists of a pair of morphisms $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ respectively in $\mathcal{D}(P)$ and $\mathcal{E}(P)$ such that the diagram

$$\begin{array}{ccc} F_P X & \xrightarrow{F_P f} & F_P X' \\ \alpha \downarrow & & \downarrow \alpha' \\ G_P Y & \xrightarrow{G_P g} & G_P Y' \end{array}$$

commutes. Now suppose we have a monotone map $u: P \rightarrow Q$. We define $u^*: \mathcal{P}(Q) \rightarrow \mathcal{P}(P)$ as follows. On objects we set $u^*(X, Y, \alpha) = (u^*X, u^*Y, \alpha_{u^*})$ where the isomorphism α_{u^*} is the unique one making the following square commute

$$\begin{array}{ccc} F_P u^* X & \xrightarrow{\alpha_{u^*}} & G_P u^* Y \\ \gamma_u^F \uparrow & & \uparrow \gamma_u^G \\ u^* F_Q X & \xrightarrow{u^* \alpha} & u^* G_Q Y. \end{array}$$

On morphisms we set $u^*(f, g) = (u^*f, u^*g)$. It is immediate that this is strictly functorial in u , so we have defined a prederivator. There are projections $\pi_1: \mathcal{P} \rightarrow \mathcal{D}$ and $\pi_2: \mathcal{P} \rightarrow \mathcal{E}$, and we can define an invertible modification $\varphi: F\pi_1 \rightarrow G\pi_2$ by $\varphi_{(X, Y, \alpha)} = \alpha$. We thus have a diagram as follows:

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{P_2} & \mathcal{E} \\ P_1 \downarrow & \nearrow \varphi & \downarrow G \\ \mathcal{D} & \xrightarrow{F} & \mathcal{F} \end{array}$$

That this square provides a model for the 2-pullback of prederivators is a bothersome computation that we leave to the reader.

From the fact that G is an equivalence, it follows in a standard way that P_1 is an equivalence. As \mathcal{D} is a derivator, it follows that \mathcal{P} is a derivator. Thus, we have a 2-pullback in the bicategory of derivators. \square

We can now argue as in [18, Prop. 2.8] to justify the existence of the right calculus of fractions with respect to the class of equivalences in DER. The preferred squares we use for the composition of fractions are the explicit 2-pullbacks constructed above.

Remark 6.2. We will need two key features of the resulting bicategory, as follows. Firstly, suppose we have a diagram of derivators as follows, which commutes on the nose:

$$\begin{array}{ccc} \mathcal{C} & \xleftarrow{F} & \mathcal{X} \\ H \uparrow \simeq & \swarrow J & \downarrow G \\ \mathcal{Y} & \xrightarrow{K} & \mathcal{D} \end{array}$$

Here it is assumed that F and H are equivalences, and it follows that J is also an equivalence. The diagram then gives rise to an isomorphism between the anafunctors GF^{-1} and KH^{-1} .

Next, suppose we fix an equivalence $F: \mathcal{X} \rightarrow \mathcal{C}$, and consider various different functors $G_i: \mathcal{X} \rightarrow \mathcal{D}$. Then any natural transformation $\alpha: G_0 \rightarrow G_1$ gives rise to a morphism $G_0F^{-1} \rightarrow G_1F^{-1}$ of anafunctors, and this is functorial in α .

7. THE ANAFUNCTORS ϕ_A

We now explain our preferred framework for Bousfield localisation in the context of derivators. This will rely on facts about Bousfield localisation in compactly generated triangulated categories. For the homotopy category of spectra, all statements are well-known with very classical proofs that rely on having an underlying geometric category of spectra. There are also proofs of similar results in more axiomatic frameworks, relying only on the theory of triangulated categories. These are typically formulated in the context of well-generated categories as defined by Neeman [14], and the proofs are somewhat complex. It is well-known to experts that everything becomes much simpler, and much closer to the original results for the category of spectra, if we restrict attention to compactly generated categories. However, it seems surprisingly hard to find an full account of this in the literature. We have therefore provided one in Appendix A.

Definition 7.1. Let \mathcal{C} be a compactly generated stable derivator, and let $K: \mathcal{C}(e) \rightarrow \text{Ab}$ be a homology theory. As usual, we extend this to a graded theory $K_*: \mathcal{C}(e) \rightarrow \text{Ab}_*$ by $K_n(X) = K(\Sigma^{-n}X)$. For $X \in \mathcal{C}(P)$ define $K^P(X) = \bigoplus_{p \in P} K(X_p)$, and note that this is again a homology theory. Using Theorem 5.3 we see that the subcategories $\ker(K_*^P) \subseteq \mathcal{C}(P)$ form a localising subderivator of \mathcal{C} , which we will just call $\ker(K_*)$.

Now suppose we have an object $X \in \mathcal{C}([1] \times P)$. This gives a morphism $u: X_0 \rightarrow X_1$ in $\mathcal{C}(P)$ in the usual way. We say that X is a *localisation object* if $\text{fib}(u) \in \ker(K_*^P)$ and $X_1 \in \ker(K_*^P)^\perp$. We write $\mathcal{L}(P)$ for the subcategory of localisation objects in $\mathcal{C}([1] \times P)$. This is clearly a subprederivator of $\mathcal{C}^{[1]}$, and we have a morphism $i_0^*: \mathcal{L} \rightarrow \mathcal{C}$ of prederivators.

Proposition 7.2. \mathcal{L} is a thick subderivator of $\mathcal{C}^{[1]}$, and $i_0^*: \mathcal{L} \rightarrow \mathcal{C}$ is an equivalence of derivators.

Proof. We know from Theorem 5.3 and Proposition 5.20 that $\ker(K_*)$ and $\ker(K_*)^\perp$ are thick subderivators. Together with the results that we recalled in Theorem 3.10, this implies that \mathcal{L} is a thick subderivator of $\mathcal{C}^{[1]}$. Now consider the functor $i_0^*: \mathcal{L}(P) \rightarrow \mathcal{C}(P)$. Given $X \in \mathcal{C}(P)$, we can find a distinguished triangle $CX \rightarrow X \rightarrow LX$ with $CX \in \ker(K_*^P)$ and $LX \in \ker(K_*^P)^\perp$, by Theorem A.4 and Proposition A.8. By the strongness condition (Definition 3.4(a)), we can find $Y \in \mathcal{C}([1] \times P)$ such that the resulting morphism $Y_0 \rightarrow Y_1$ is isomorphic to $X \rightarrow LX$. This proves that i_0^* is essentially surjective. Now suppose we have $X, Y \in \mathcal{L}(P)$. Proposition 3.17 gives us a distinguished triangle $i_{1!}i_1^*(Y) \rightarrow Y \rightarrow i_{0*}i_0^*(Y)$. By applying $[X, -]$ to this and using various adjunctions we obtain an exact sequence

$$[i_1^?X, i_1^*Y] \rightarrow [X, Y] \xrightarrow{i_0^*} [i_0^*X, i_0^*Y] \rightarrow [i_1^?X, i_1^*\Sigma Y].$$

Here $i_1^?X$ is just the cofibre of $i_0^*X \rightarrow i_1^*X$, which lies in $\ker(K_*^P)$ because $X \in \mathcal{L}(P)$. Also, because $Y \in \mathcal{L}(P)$ we have $i_1^*Y \in \ker(K_*^P)^\perp$, so $[i_1^?X, i_1^*Y] = [i_1^?X, i_1^*\Sigma Y] = 0$. It follows that the map $i_0^*: [X, Y] \rightarrow [i_0^*X, i_0^*Y]$ is an isomorphism as required. \square

We will sometimes need a slightly more general statement.

Lemma 7.3. *Suppose that $X, Y \in \mathcal{C}^{[1]}(P)$, giving maps $u: X_0 \rightarrow X_1$ and $v: Y_0 \rightarrow Y_1$ in $\mathcal{C}(P)$. Suppose that $\text{fib}(u) \in \ker(K_*^P)$ and $Y_1 \in \ker(K_*^P)^\perp$. Then the map*

$$i_0^*: \mathcal{C}([1] \times P)(X, Y) \rightarrow \mathcal{C}(P)(X_0, Y_0)$$

is bijective.

Proof. These weakened hypotheses are all that was used in the proof of Proposition 7.2 \square

Definition 7.4. We define L_K to be the anafunctor $i_1^*(i_0^*)^{-1}$, and call this *Bousfield localisation* with respect to K .

We now return to a framework similar to that of Section 2: we assume that we have a compactly generated stable derivator \mathcal{C} together with homology theories $K(i)$ on $\mathcal{C}(e)$ for $i \in N$, where N is a finite, totally ordered set. As before, we define \mathbb{P} to be the poset of subsets of N (ordered by inclusion). We also define \mathbb{Q} to be the poset of upper sets in \mathbb{P} (ordered by reverse inclusion), and define $u: \mathbb{P} \rightarrow \mathbb{Q}$ by $uT = \{U \mid T \subseteq U\}$.

Definition 7.5. Consider a finite poset R and an object $X \in \mathcal{C}(\mathbb{P} \times R)$. Suppose that $t \in N$ and $U \subseteq N$ with $t < u$ for all $u \in U$. We then write tU for $\{t\} \cup U$, so we have $U < tU$ in \mathbb{P} , giving maps $f_{t,U,r}: X_{U,r} \rightarrow X_{tU,r}$ in $\mathcal{C}(e)$. We say that X is (t, U) -localising if $f_{t,U}$ is a $K(t)$ -localisation. Equivalently, $X_{tU,r}$ should be $K(t)$ -local, and the fibre of $f_{t,U,r}$ should be $K(t)$ -acyclic. We also say that X is *fully localising* if it is (t, U) -localising for all t and U . We write $\mathcal{P}(R)$ for the full subcategory of fully localising objects in $\mathcal{C}(\mathbb{P} \times R)$. There is an evident inclusion $i_\emptyset: R \rightarrow \mathbb{P} \times R$, which gives a functor $i_\emptyset^*: \mathcal{P}(R) \rightarrow \mathcal{C}(R)$.

Example 7.6. Consider the case $N = \{0\}$, and suppose that our derivator \mathcal{C} arises from a stable model category \mathcal{C}_0 . An object of $\mathcal{P}(R)$ is then a diagram $X: [1] \times R \rightarrow \mathcal{C}_0$ such that the morphisms $X_{0,r} \rightarrow X_{1,r}$ are all localisations with respect to $K(0)$. Informally, we can therefore say that an object of \mathcal{P} is a diagram

of type $(X \rightarrow L_{K(0)}X)$. In the same sense, if $N = \{0, 1\}$ then an object of \mathcal{P} is essentially a diagram of the following type:

$$\begin{array}{ccc} X & \longrightarrow & L_{K(0)}X \\ \downarrow & & \downarrow \\ L_{K(1)}X & \longrightarrow & L_{K(0)}L_{K(1)}X \end{array} \qquad \begin{array}{ccc} X & \longrightarrow & \phi_0 X \\ \downarrow & & \downarrow \\ \phi_1 X & \longrightarrow & \phi_{01} X \end{array}$$

Here the right hand diagram is just alternate notation for the left hand one. For $N = \{0, 1, 2\}$, the diagram is as follows:

$$\begin{array}{ccccc} X & \longrightarrow & \phi_0 X & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & \phi_1 X & \longrightarrow & \phi_{01} X & \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ \phi_2 X & \longrightarrow & \phi_{02} X & & \\ & \downarrow & \downarrow & \downarrow & \\ & \phi_{12} X & \longrightarrow & \phi_{012} X & \end{array}$$

Proposition 7.7. \mathcal{P} is a thick subderivator of $\mathcal{C}^{\mathbb{P}}$, and $i_0^*: \mathcal{P} \rightarrow \mathcal{C}$ is an equivalence of derivators.

Proof. The claim is clear if $N = \emptyset$. If $N \neq \emptyset$, we let $n_0 \in N$ be the smallest element, so N can be decomposed as $\{n_0\} \amalg N_1$ say. This gives an obvious decomposition $\mathbb{P} = [1] \times \mathbb{P}_1$. We can define $\mathcal{P}_1 \subseteq \mathcal{C}^{\mathbb{P}_1}$ using N_1 , and by induction we can assume that this is a thick subderivator with $i_0^*: \mathcal{P}_1 \rightarrow \mathcal{C}$ being an equivalence.

Now define \mathcal{L} as in Definition 7.1, with respect to the homology theory $K(n_0)$ for the derivator \mathcal{P}_1 . We find that

$$\mathcal{P}(R) = \{X \in \mathcal{L}(\mathbb{P}_1 \times R) \mid i_0^*(X) \in \mathcal{P}_1(R)\},$$

and the claim now follows from Proposition 7.2 together with the induction hypothesis. \square

Again, we will sometimes need a slightly more general statement.

Lemma 7.8. *Suppose that $X, Y \in \mathcal{C}^{\mathbb{P}}(R)$. Suppose that for t, U and r as before, the map $f_{t,U,r}: X_U \rightarrow X_{tU}$ is a $K(t)$ -equivalence, and the object Y_{tU} is $K(t)$ -local. Then the map*

$$i_0^*: \mathcal{C}(\mathbb{P} \times R)(X, Y) \rightarrow \mathcal{C}(R)(X_\emptyset, Y_\emptyset)$$

is bijective.

Proof. As above taking n_0 the minimum of N we get a decomposition $\mathbb{P} = [1] \times \mathbb{P}_1$ and arguing by induction we can assume the map $\mathcal{C}(\mathbb{P}_1 \times R)(X_\emptyset, Y_\emptyset) \rightarrow \mathcal{C}(R)(X_\emptyset, Y_\emptyset)$ is a bijection. Now we have just to compose this with $\mathcal{C}(\mathbb{P} \times R)(X, Y) \rightarrow \mathcal{C}(\mathbb{P}_1 \times R)(X_\emptyset, Y_\emptyset)$ which is an isomorphism by Lemma 7.3. \square

Definition 7.9. For $A \subseteq N$ we consider the diagram

$$\mathcal{C} \xleftarrow{i_0^*} \mathcal{P} \xrightarrow{i_A^*} \mathcal{C}$$

and define an anafunctor $\phi_A: \mathcal{C} \rightarrow \mathcal{C}$ by $\phi_A = i_A^* \circ (i_0^*)^{-1}$.

Remark 7.10. Suppose that $A = \{a_1, \dots, a_r\}$ with $a_1 < \dots < a_r$. It is then not hard to see from the definitions that in some sense we have

$$\phi_A \simeq L_{K(a_1)} \cdots L_{K(a_r)},$$

so that Definition 7.9 is a more precise version of Definition 1.2 in the introduction. A rigorous formulation with anafunctors will be given in Corollary 7.18.

Proposition 7.11. *The anafunctor $\phi_{\{a\}}$ is equivalent to $L_{K(a)}$.*

Proof. Define $j: [1] \rightarrow \mathbb{P}$ by $j(0) = \emptyset$ and $j(1) = \{a\}$. This gives a morphism $j^*: \mathcal{C}^{\mathbb{P}} \rightarrow \mathcal{C}^{[1]}$. If we define \mathcal{L} as in Definition 7.1 with respect to $K(a)$, we find that j^* restricts to give a morphism $\mathcal{P} \rightarrow \mathcal{L}$. This fits into a diagram as follows, which commutes on the nose:

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{i_{\{a\}}^*} & \mathcal{C} \\ \downarrow i_{\emptyset}^* \simeq & \searrow j^* & \uparrow i_1^* \\ \mathcal{C} & \xleftarrow{i_{\emptyset}^* \simeq} & \mathcal{L} \end{array}$$

From this it is clear that the anafunctor $\phi_{\{a\}} = i_{\{a\}}^* (i_{\emptyset}^*)^{-1}$ is equivalent to $L_{K(a)} = i_1^* (i_{\emptyset}^*)^{-1}$. \square

Now let j be the inclusion of $\mathbb{P}' = \mathbb{P} \setminus \{\emptyset\}$ in \mathbb{P} , and consider the fibration

$$\mathrm{tfib}(X) \rightarrow i_{\emptyset}^*(X) \rightarrow \underset{\mathbb{P}'}{\mathrm{holim}} j^*(X)$$

as in Lemma 4.22.

We can now give a derivator formulation of the chromatic fracture argument.

Proposition 7.12. *For any $X \in \mathcal{P}(R)$, the above morphism $i_{\emptyset}^*(X) \rightarrow \underset{\mathbb{P}'}{\mathrm{holim}} j^*(X)$ is a localisation with respect to $K(N) = \bigoplus_{n \in N} K(n)$.*

Proof. From the definition of $\mathcal{P}(R)$ we see that for all $T \in \mathbb{P}'$, the object $j^*(X)_T$ is local with respect to $K(\min(T))$ and thus with respect to $K(N)$. Proposition 5.20 tells us that the $K(N)$ -local objects form a thick subderivator, so the object $LX = \underset{\mathbb{P}'}{\mathrm{holim}} j^*(X)$ is $K(N)$ -local. It will therefore suffice to show that the fibre $\mathrm{tfib}(X) = \mathrm{fib}(X_{\emptyset} \rightarrow LX)$ is $K(N)$ -acyclic, or equivalently, that it is $K(i)$ -acyclic for all i . Let Y_r be the total fibre of the subdiagram indexed by subsets of $\{r+1, \dots, n^*-1\}$, so $Y_{n^*-1} = 0$ and $Y_{-1} = \mathrm{tfib}(X)$. Let Z_r be the total fibre of the subdiagram indexed by subsets of $\{r, \dots, n^*-1\}$ containing r , so that $Y_r \rightarrow Z_r$ is a $K(r)$ -localisation, and the fibre is Y_{r-1} by Proposition 4.23. This shows that Y_{i-1} is $K(i)$ -acyclic. The fracture axiom then tells us that Z_{i-1} is also $K(i)$ -acyclic, and Y_{i-2} is the fibre of the map $Y_{i-1} \rightarrow Z_{i-1}$ so it is again $K(i)$ -acyclic. By iterating this, we find that Y_{-1} is $K(i)$ -acyclic as required. \square

Definition 7.13. We say that an object $X \in \mathcal{C}(\mathbb{P} \times \mathbb{P} \times R)$ is *doubly localising* if

- (a) X is fully localising relative to $\mathbb{P} \times R$, so it lies in $\mathcal{P}(\mathbb{P} \times R)$.
- (b) The restriction to $\{u\emptyset\} \times \mathbb{P} \times R \simeq \mathbb{P} \times R$ lies in $\mathcal{P}(R)$.

We write $\mathcal{P}_2(R)$ for the full subcategory of doubly localising objects in $\mathcal{C}(\mathbb{P} \times \mathbb{P} \times R)$.

Proposition 7.14. \mathcal{P}_2 is a thick subderivator of $\mathcal{C}^{\mathbb{P} \times \mathbb{P}}$, and $i_{(\emptyset, \emptyset)}^*: \mathcal{P}_2 \rightarrow \mathcal{C}$ is an equivalence.

Proof. The first claim follows from Theorem 5.3. The two properties of the previous definition easily imply that the equivalence $i_{\emptyset}^*: \mathcal{P} \rightarrow \mathcal{C}$ of Proposition 7.7 (shifted by a \mathbb{P} component) restricts to an equivalence $\mathcal{P}_2 \rightarrow \mathcal{P}$, compose this again with i_{\emptyset}^* and the second claim is verified. \square

It is easy to see that $X \in \mathcal{C}(\mathbb{P} \times \mathbb{P} \times R) = \mathcal{C}^R(\mathbb{P} \times \mathbb{P})$ is doubly localising iff the following conditions are satisfied:

- (a) For all a, A, B with $\{a\} \angle A$, the map $X_{A,B} \rightarrow X_{\{a\} \cup A, B}$ is a $K(a)$ -localisation.
- (b) For all b, B with $\{b\} \angle B$, the map $X_{\emptyset, B} \rightarrow X_{\emptyset, \{b\} \cup B}$ is a $K(b)$ -localisation.

This essentially means that if $A = \{a_1 < \dots < a_p\}$ and $B = \{b_1 < \dots < b_q\}$ we must have

$$X_{A,B} = L_{K(a_1)} \cdots L_{K(a_p)} L_{K(b_1)} \cdots L_{K(b_q)} X_{(\emptyset, \emptyset)}.$$

In particular, we see that $X_{(A,B)} = 0$ unless $A \angle B$. This motivates the following construction.

Definition 7.15. We put $\mathbb{M} = \{(A, B) \in \mathbb{P} \times \mathbb{P} \mid A \angle B\}$, and define $\sigma: \mathbb{M} \rightarrow \mathbb{P}$ by $\sigma(A, B) = A \cup B$. We say that an object $X \in \mathcal{C}(\mathbb{M} \times R) = \mathcal{C}^R(\mathbb{M})$ is *doubly localising* if

- (a) For all a, A, B with $\{a\} \angle A$ and $\{a\} \cup A \angle B$, the map $X_{A,B} \rightarrow X_{\{a\} \cup A, B}$ is a $K(a)$ -localisation.
- (b) For all b, B with $\{b\} \angle B$, the map $X_{\emptyset, B} \rightarrow X_{\emptyset, \{b\} \cup B}$ is a $K(b)$ -localisation.

We write $\mathcal{P}'_2(R)$ for the subcategory of doubly localising objects.

Proposition 7.16. \mathcal{P}'_2 is a thick subderivator of $\mathcal{C}^{\mathbb{M}}$, and the inclusion $\text{inc}: \mathbb{M} \rightarrow \mathbb{P} \times \mathbb{P}$ induces mutually inverse equivalences

$$\mathcal{P}'_2 \xrightarrow{\text{inc}_*} \mathcal{P}_2 \xrightarrow{\text{inc}^*} \mathcal{P}'_2,$$

and the morphism $i_{(\emptyset, \emptyset)}^*: \mathcal{P}'_2 \rightarrow \mathcal{C}$ is also an equivalence. Moreover, the map $\sigma: \mathbb{M} \rightarrow \mathbb{P}$ gives an equivalence $\sigma^*: \mathcal{P} \rightarrow \mathcal{P}'_2$ and thus an equivalence $\text{inc}_* \circ \sigma^*: \mathcal{P} \rightarrow \mathcal{P}_2$.

Proof. The subposet $\mathbb{M} \subseteq \mathbb{P} \times \mathbb{P}$ is a sieve, so Proposition 3.15 gives mutually inverse equivalences

$$\mathcal{C}(\mathbb{M} \times R) \xrightarrow{\text{inc}_*} \mathcal{C}_{\mathbb{M} \times R}(\mathbb{P} \times \mathbb{P} \times R) \xrightarrow{\text{inc}^*} \mathcal{C}(\mathbb{M} \times R).$$

We have observed that if X is doubly localising then $X_{A,B} = 0$ for $(A, B) \notin \mathbb{M}$, so $X \in \mathcal{C}_{\mathbb{M} \times R}(\mathbb{P} \times \mathbb{P} \times R)$. It follows that inc^* restricts to give an equivalence from $\mathcal{P}_2(R)$ to some subcategory of $\mathcal{C}(\mathbb{M} \times R)$, with inverse given by inc_* . It is easy to check that the relevant subcategory is $\mathcal{P}'_2(R)$. We have now seen that in the diagram

$$\mathcal{P}_2 \xrightarrow{\text{inc}^*} \mathcal{P}'_2 \xrightarrow{i_{(\emptyset, \emptyset)}^*} \mathcal{C},$$

the first map and the composite are both equivalences of (pre)derivators, so the second map is also an equivalence. From this it also follows that \mathcal{P}'_2 is a derivator. Finally, direct inspection of the definitions shows that $\sigma^*(\mathcal{P}) \subseteq \mathcal{P}'_2$, and $i_{(\emptyset, \emptyset)}^* \sigma^* = i_{\emptyset}^*$, which implies that $\sigma^*: \mathcal{P} \rightarrow \mathcal{P}'_2$ is an equivalence. \square

Proposition 7.17. If $A \angle B$ then there is an equivalence of anafunctors $\phi_A \phi_B \simeq \phi_{A \cup B}$.

Proof. Define $j_B: \mathbb{P} \rightarrow \mathbb{P} \times \mathbb{P}$ by $j_B(T) = (T, B)$. Consider the following diagram:

$$\begin{array}{ccccc}
 \mathcal{C} & \xleftarrow{\cong i_\emptyset^*} & \mathcal{P} & & \\
 & & \nearrow j_B^* & & \searrow i_A^* \\
 & & \mathcal{P}_2 & \xrightarrow{i_{(A,B)}^*} & \mathcal{C} \\
 & \swarrow \cong i_\emptyset^* & \searrow \cong \text{inc}^* & & \nearrow i_{(A,B)}^* \\
 \mathcal{P} & & \mathcal{P}'_2 & & \mathcal{C} \\
 & \searrow \cong i_\emptyset^* & \swarrow \cong i_{(\emptyset,\emptyset)}^* & & \nearrow i_{A \cup B}^* \\
 & & \mathcal{C} & \xleftarrow{\cong i_\emptyset^*} & \mathcal{P} \\
 & & & & \nwarrow \cong \sigma^*
 \end{array}$$

Given that \mathcal{C} is a strict 2-functor, we see that everything commutes on the nose. Several morphisms have been marked as equivalences; these are justified by Proposition 7.16. It follows that all routes from the middle bottom \mathcal{C} to the middle right \mathcal{C} give the same anafunctor up to equivalence. If we go clockwise around the edge of the diagram we get $\phi_A \phi_B$, and if we go anticlockwise we get $\phi_{A \cup B}$.

To be precise the composition of anafunctors $\phi_A \phi_B$ is given by the following composition of spans via pullback

$$\begin{array}{ccccc}
 & & \mathcal{P} \times_{\mathcal{C}} \mathcal{P} & & \\
 & \swarrow \pi_1 & & \searrow \pi_2 & \\
 \mathcal{P} & & & & \mathcal{P} \\
 \swarrow i_\emptyset^* & & & & \searrow i_A^* \\
 \mathcal{C} & & \mathcal{C} & & \mathcal{C} \\
 & \nearrow i_B^* & & \nwarrow i_\emptyset^* &
 \end{array}$$

The upper left part of the previous diagram means that we can easily produce an isomorphism of anafunctors

$$\begin{array}{ccccc}
 & & \mathcal{P} \times_{\mathcal{C}} \mathcal{P} & & \\
 & \swarrow i_\emptyset^* \pi_1 & \uparrow \psi & \searrow i_A^* \pi_2 & \\
 \mathcal{C} & & & & \mathcal{C} \\
 & \swarrow i_{(\emptyset,\emptyset)}^* & & \searrow i_{(A,B)}^* & \\
 & & \mathcal{P}_2 & &
 \end{array}$$

where ψ is obtained using i_\emptyset^* and j_B^* . □

Corollary 7.18. *Suppose that $A = \{a_1, \dots, a_r\}$ with $a_1 < \dots < a_r$. There is then an equivalence of anafunctors*

$$\phi_A \simeq L_{K(a_1)} \cdots L_{K(a_r)}.$$

Proof. This follows by induction using Propositions 7.11 and 7.17. (The base case $A = \emptyset$ says that ϕ_\emptyset is equivalent to the identity, which is clear because $\phi_\emptyset = i_\emptyset^*(i_\emptyset^*)^{-1}$ by definition.) \square

8. THE ANAFUNCTORS θ_U

We now start to define a more general class of iterated localisation functors.

Definition 8.1. Consider an object $X \in \mathcal{C}(\mathbb{Q} \times R)$, and the pullback $(u \times 1)^*(X) \in \mathcal{C}(\mathbb{P} \times R)$. We say that X is *u-cartesian* if the natural map

$$(u \times 1)^*: \mathcal{C}(\mathbb{Q} \times R)(W, X) \rightarrow \mathcal{C}(\mathbb{P} \times R)((u \times 1)^*(W), (u \times 1)^*(X))$$

is an isomorphism for all W , or equivalently, X is in the essential image of the functor

$$(u \times 1)_*: \mathcal{C}(\mathbb{P} \times R) \rightarrow \mathcal{C}(\mathbb{Q} \times R).$$

We say that X is a *fracture object* if it is *u-cartesian*, and $(u \times 1)^*(X)$ is fully localising. We write $\mathcal{F}(R)$ for the subcategory of fracture objects in $\mathcal{C}(\mathbb{Q} \times R)$. We also define $j: R \rightarrow \mathbb{Q} \times R$ by $j(r) = (u\emptyset, R)$.

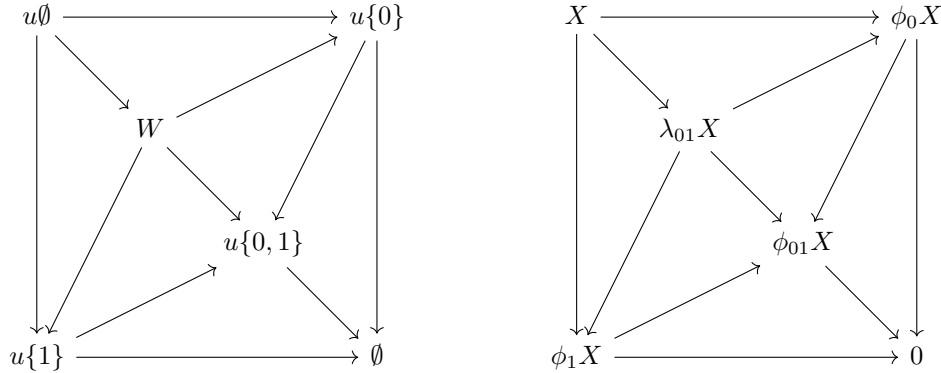
Remark 8.2. Because we have ordered \mathbb{Q} by reverse inclusion, we have $U \leq uA$ iff $uA \subseteq U$ iff $A \in U$. Using this together with the Kan formula, the *u-cartesian* condition becomes

$$X_U = \operatorname{holim}_{A \in U} X_{uA}.$$

Example 8.3. Consider the case $N = \{0, 1\}$, and put

$$W = vN = u\{0\} \cup u\{1\} = \{A \subseteq N \mid A \neq \emptyset\} \in \mathbb{Q}.$$

We then have $\mathbb{Q} = \{uA \mid A \in \mathbb{P}\} \cup \{W, \emptyset\}$, with partial order as shown on the left below. A fracture object is as shown on the right.

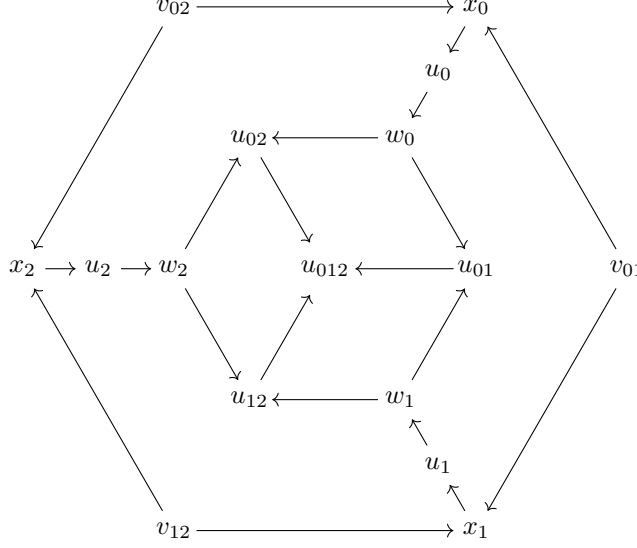


Now consider the case $N = \{0, 1, 2\}$, where $|\mathbb{P}| = 8$. It turns out that $|\mathbb{Q}| = 20$, with elements as follows:

- The smallest element is $u\emptyset = \mathbb{P}$.
- The next smallest element is $vN = \{A \in \mathbb{P} \mid A \neq \emptyset\}$.
- We write $v_{ij} = v\{i, j\} = \{A \mid i \in A \text{ or } j \in A\}$.
- We write $x_i = \{A \mid i \in A \text{ or } \{i\}^c \subseteq A\}$.
- We write $u_i = u\{i\} = v\{i\} = \{A \mid i \in A\}$.
- We write $w_i = \{A \mid A \supset \{i\}\}$ (strict inclusion).
- We write $u_{ij} = u\{i, j\} = \{A \mid \{i, j\} \subseteq A\} = \{\{i, j\}, N\}$.

- We write $u_{012} = uN = \{N\}$.
- We write $y = \{A \mid |A| \geq 2\}$.
- The largest element is \emptyset .

The Hasse diagram for $\mathbb{Q} \setminus \{u\emptyset, vN, y, \emptyset\}$ can be drawn in the plane as follows:



The remaining vertices fit in as follows. At the bottom we have $u\emptyset$, which is covered by vN , which is covered by the elements v_{ij} . At the top, u_{012} is covered by \emptyset . In the middle, y covers the elements x_i and is covered by the elements w_i . The u and v elements will correspond to functors ϕ and λ as we have discussed previously. One can check that the remaining elements other than w_1 can be factored as follows, and so will also correspond to iterated localisation functors:

$$\begin{array}{lll} x_0 = v_{01} * v_{02} & x_1 = v_{01} * v_{12} & x_2 = v_{02} * v_{12} \\ w_0 = u_0 * v_{12} & w_2 = v_{01} * u_2 & y = v_{01} * v_{02} * v_{12}. \end{array}$$

However, Example 1.12 shows that w_1 cannot be factored in this way.

Proposition 8.4. \mathcal{F} is a thick subderivator of $\mathcal{C}^{\mathbb{Q}}$, and $j^*: \mathcal{F} \rightarrow \mathcal{C}$ is an equivalence of derivators.

Proof. Let $\mathcal{E}(R)$ be the subcategory of u -cartesian objects in $\mathcal{C}(\mathbb{Q} \times R)$, so the functor $(u \times 1)^*: \mathcal{E}(R) \rightarrow \mathcal{C}(\mathbb{P} \times R)$ is an equivalence. An object is u -cartesian iff the unit map $X \rightarrow (u \times 1)_*(u \times 1)^*(X)$ is an isomorphism, and from this we see that $\mathcal{E}(R)$ is a thick subcategory of $\mathcal{C}(\mathbb{Q} \times R)$. (Here we have used Theorem 3.10, as we will do repeatedly without further comment.)

Now consider a morphism $f: R_0 \rightarrow R_1$ of finite posets. We know from [7, Proposition 2.6] that the resulting derivator morphism $f^*: \mathcal{C}^{R_1} \rightarrow \mathcal{C}^{R_0}$ preserves homotopy Kan extensions, so it commutes (in an evident sense) with the functors $(u \times 1)_*$, so it restricts to give a functor $f^*: \mathcal{E}(R_1) \rightarrow \mathcal{E}(R_0)$. It is even clearer that the functors $(1 \times f)_*$ commute with $(u \times 1)_*$, so they restrict to give $f_*: \mathcal{E}(R_0) \rightarrow \mathcal{E}(R_1)$. By the dual of Lemma 5.13, we deduce that \mathcal{E} is a thick subderivator of $\mathcal{E}^{\mathbb{Q}}$.

Next, consider the preimage under $u^*: \mathcal{C}^{\mathbb{Q}} \rightarrow \mathcal{C}^{\mathbb{P}}$ of the thick subderivator $\mathcal{P} \subseteq \mathcal{C}^{\mathbb{P}}$. Using [7, Proposition 2.6] again, we see that this preimage is again a thick

subderivator. By intersecting this with \mathcal{E} , we see that \mathcal{F} is a thick subderivator as claimed.

In conclusion we obtained the diagram of derivators

$$\begin{array}{ccc} \mathcal{C}^{\mathbb{P}} & \xrightarrow{u_*} & \mathcal{E} \\ \uparrow & & \uparrow \\ \mathcal{P} & \xrightarrow{u_*} & \mathcal{F} \end{array}$$

where the vertical arrows are inclusions and the horizontal ones equivalences. Thus composing the inverse equivalence $u^*: \mathcal{F} \rightarrow \mathcal{P}$ with i_{\emptyset}^* of Proposition 7.7 we get the second claim. \square

Definition 8.5. For $U \in \mathbb{Q}$, we define $\theta_U: \mathcal{C} \rightarrow \mathcal{C}$ to be the anafunctor

$$\mathcal{C} \xrightarrow{(j^*)^{-1}} \mathcal{F} \subseteq \mathcal{C}^{\mathbb{Q}} \xrightarrow{i_U^*} \mathcal{C}.$$

We also note that an inequality $U \leq V$ gives a natural transformation $i_U^* \rightarrow i_V^*$ and thus a morphism $\theta_U \rightarrow \theta_V$ of anafunctors, as discussed in Remark 6.2.

Remark 8.6. Consider an object $X \in \mathcal{C}(R)$, and a fracture object $Y \in \mathcal{F}(R)$ with $j^*Y \simeq X$. Then the object $Y_U = i_U^*Y \in \mathcal{C}(R)$ is a choice of $\theta_U(X)$. As Y is u -cartesian we have

$$Y_U = \operatorname{holim}_{U \leq uA} Y_{uA} = \operatorname{holim}_{A \in U} Y_{uA}.$$

Also, as $u^*Y \in \mathcal{P}(R)$ we know that Y_{uA} is a choice of $\phi_A(X)$. Thus, the basic idea is that $\theta_U(X) = \operatorname{holim}_{A \in U} \phi_A(X)$.

Remark 8.7. Consider the original chromatic context where the homology theory $K(n)$ is represented by a spectrum, so we can apply ϕ_A or θ_U to that spectrum. It is easy to see that $\phi_A(K(n))$ is $K(n)$ if $A \subseteq \{n\}$, and $\phi_A(K(n)) = 0$ in all other cases. From this we find that $\theta_U(K(n))$ is $K(n)$ if $\{n\} \in U$, and $\theta_U(K(n)) = 0$ in all other cases. In other words, with κ as in Remark 1.7 we have $\kappa(U) = \{n \mid \theta_U(K(n)) \neq 0\}$. In particular, if $\kappa(U) \neq \kappa(V)$ then $\theta_U \not\cong \theta_V$. However, it is common for $\kappa(U)$ to be empty, so this is not a very strong result.

Lemma 8.8. *If $A = \{a_1, \dots, a_r\}$ with $a_1 < \dots < a_r$ then there are equivalences of anafunctors*

$$\theta_{uA} \simeq \phi_A \simeq L_{K(a_1)} \cdots L_{K(a_r)}.$$

Proof. We have a diagram as follows, which commutes on the nose:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{i_{uA}^*} & \mathcal{C} \\ \downarrow j^* \simeq & \searrow u^* & \uparrow i_A^* \\ \mathcal{C} & \xleftarrow{i_{(\emptyset, \emptyset)}^*} & \mathcal{P} \end{array}$$

This gives an equivalence $i_{uA}^*(j^*)^{-1} \simeq i_A^*(i_{(\emptyset, \emptyset)}^*)^{-1}$ of anafunctors, or in other words $\theta_{uA} \simeq \phi_A$. Moreover, Corollary 7.18 gives $\phi_A \simeq L_{K(a_1)} \cdots L_{K(a_r)}$. \square

Lemma 8.9. *The functor θ_{\emptyset} is zero.*

Proof. Fix $X \in \mathcal{C}(R)$ and choose $Y \in \mathcal{F}(R)$ together with an isomorphism $X \simeq j^*(Y)$. It will suffice to prove that $Y_\emptyset = 0$. The u -cartesian property of Y allows us to write Y_\emptyset as a homotopy limit over the comma poset \emptyset/u . However, \emptyset is strictly larger than everything in the image of u (with respect to the reversed inclusion order that we are using on \mathbb{Q}). Thus, this comma poset is empty and the homotopy limit is zero as required. \square

Proposition 8.10. *For $vA = \{T \subseteq N \mid T \cap A \neq \emptyset\}$ we have $\theta_{vA} \simeq \lambda_A$.*

Proof. Let Y be any object of $\mathcal{F}(R)$. We claim that the morphism $Y_{u\emptyset} \rightarrow Y_{vA}$ is a localisation with respect to $K(A)$. In order to simplify notation, we replace \mathcal{C} by \mathcal{C}^R and thus reduce to the case $R = 1$. As Y is a u -cartesian object, we see that Y_{vA} is the homotopy inverse limit of $(u^*Y)|_{vA}$. Let P' be the poset of nonempty subsets of A . Note that the inclusion $i: P' \rightarrow vA$ is left adjoint to the map $r: vA \rightarrow P'$ given by $rT = T \cap A$. It follows from Proposition 4.12 that i is homotopy cofinal, so Y_{vA} is also the homotopy inverse limit of $(u^*Y)|_{P'}$. We can now apply Proposition 7.12 (with N replaced by A) to see that this homotopy limit is a $K(A)$ -localisation, as required.

Now define $k: [1] \rightarrow \mathbb{Q}$ by $k(0) = u\emptyset$ and $k(1) = vA$. This gives a morphism $k^*: \mathcal{C}^{\mathbb{Q}} \rightarrow \mathcal{C}^{[1]}$, and the previous paragraph shows that this restricts to give a morphism $\mathcal{F} \rightarrow \mathcal{L}$ (where \mathcal{L} is as in Definition 7.1, for localisation with respect to $K(A)$). We now have a diagram as follows, which commutes on the nose:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{i_{vA}^*} & \mathcal{C} \\ j^* \downarrow \simeq & \searrow k^* & \uparrow i_1^* \\ \mathcal{C} & \xleftarrow{i_0^*} & \mathcal{L} \end{array}$$

As i_0^* and j^* are equivalences, we see that k^* is also an equivalence. This gives an isomorphism $i_{vA}^*(j^*)^{-1} \simeq i_1^*(i_0^*)^{-1}$ of anafunctors, or in other words $\theta_{vA} \simeq \lambda_A$. \square

We now want to prove the following result:

Theorem 8.11. *The composite $\theta_U\theta_V$ is naturally isomorphic to θ_{U*V} .*

The proof will be given after some preliminaries.

We start with the following result, which will be needed in the proof of Theorem 8.11, and which also shows that Theorem 8.11 is consistent with Proposition 7.17.

Lemma 8.12. *For $A, B \in \mathcal{P}$ we have*

$$uA * uB = \begin{cases} u(A \cup B) & \text{if } A \angle B \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. By definition, we have

$$uA * uB = \{C \cup D \mid A \subseteq C, B \subseteq D, C \angle D\} \subseteq u(A \cup B).$$

If $A \angle B$ then we can choose k with $a \leq k$ for all $a \in A$, and $k \leq b$ for all $b \in B$. Then any $E \in u(A \cup B)$ can be written as $C \cup D$ with $C = \{j \in E \mid j \leq k\} \supseteq A$ and $D = \{j \in E \mid j \geq k\} \supseteq B$, so $E \in uA * uB$. We therefore have $uA * uB = u(A \cup B)$

in this case. On the other hand, if it is not true that $A \angle B$ then we can choose $a \in A$ and $b \in B$ with $a > b$. If C and D are as in the definition then $a \in C$ and $b \in D$ so it is not true that $C \angle D$. From this it follows that $uA * uB = \emptyset$. \square

Definition 8.13. We say that an object $X \in \mathcal{C}(\mathbb{Q} \times \mathbb{Q} \times R)$ is a *double fracture object* if

- (a) X is a fracture object relative to $\mathbb{Q} \times R$, so it lies in $\mathcal{F}(\mathbb{Q} \times R)$.
- (b) The restriction to $\{u\emptyset\} \times \mathbb{Q} \times R \simeq \mathbb{Q} \times R$ lies in $\mathcal{F}(R)$.

We write $\mathcal{F}_2(R)$ for the full subcategory of double fracture objects in $\mathcal{C}(\mathbb{Q} \times \mathbb{Q} \times R)$. We also define $k, l: \mathbb{Q} \rightarrow \mathbb{Q} \times \mathbb{Q}$ by $k(V) = (u\emptyset, V)$ and $l(U) = (U, u\emptyset)$. This gives functors $k^*, l^*: \mathcal{F}_2(R) \rightarrow \mathcal{C}(\mathbb{Q} \times R)$. Finally, we define $j_2: e \rightarrow \mathbb{Q} \times \mathbb{Q}$ to be the map with image $(u\emptyset, u\emptyset)$.

Proposition 8.14. \mathcal{F}_2 is a thick subderivator of $\mathcal{C}^{\mathbb{Q} \times \mathbb{Q}}$, and the maps k and l give equivalences as shown:

$$\begin{array}{ccc}
 \mathcal{F}_2 & \xrightarrow[k^*]{\simeq} & \mathcal{F} \\
 \downarrow l^* \simeq & \searrow j_2^* & \downarrow j^* \\
 \mathcal{F} & \xrightarrow[j^*]{\simeq} & \mathcal{C}
 \end{array}$$

Proof. Put $\mathcal{E}(R) = \mathcal{F}(\mathbb{Q} \times R) \subset \mathcal{C}(\mathbb{Q} \times \mathbb{Q} \times R)$ (so this is the subcategory of objects satisfying condition (a)). From Proposition 7.12 we see that \mathcal{E} is a thick subderivator of $\mathcal{C}^{\mathbb{Q} \times \mathbb{Q}}$ and that $k^*: \mathcal{E} \rightarrow \mathcal{C}^{\mathbb{Q}}$ is an equivalence of derivators. As \mathcal{F} is a thick subderivator of $\mathcal{C}^{\mathbb{Q}}$, it follows that the preimage under k^* of \mathcal{F} is a thick subderivator of \mathcal{E} . However, this preimage is just \mathcal{F}_2 , so \mathcal{F}_2 is a thick subderivator as claimed. It is also clear from this that $k^*: \mathcal{F}_2 \rightarrow \mathcal{F}$ is an equivalence. We have seen that $j^*: \mathcal{F} \rightarrow \mathcal{C}$ is also an equivalence.

Next, recall again that \mathcal{F} is a thick subderivator. Any monotone map $f: R \rightarrow R'$ gives a functor $(1 \times f)^*: \mathcal{C}(\mathbb{Q} \times R') \rightarrow \mathcal{C}(\mathbb{Q} \times R)$, and the subderivator property implies that $(1 \times f)^*(\mathcal{F}(R')) \subseteq \mathcal{F}(R)$. Take $R' = \mathbb{Q} \times R$ and $f(r) = (u\emptyset, r)$; the conclusion is then that $l^*(\mathcal{E}(R)) \subseteq \mathcal{F}(R)$, and so $l^*(\mathcal{F}_2(R)) \subseteq \mathcal{F}(R)$. This means that we have a diagram of functors as claimed, commuting up to natural isomorphism. As j^* and k^* are equivalences, we can chase the diagram to see that l^* and $i_{(u\emptyset, u\emptyset)}^*$ are equivalences as well. \square

Corollary 8.15. For any $U, V \in \mathbb{Q}$, the composite anafunctor $\theta_U \theta_V$ is isomorphic to the fraction

$$\mathcal{C} \xrightarrow{(j_2^*)^{-1}} \mathcal{F}_2 \xrightarrow{i_{(U, V)}^*} \mathcal{C}$$

Proof. Note that if $X \in \mathcal{F}_2(R)$ then $X \in \mathcal{F}(\mathbb{Q} \times R)$ and \mathcal{F} is a subderivator so we have $i_V^* X \in \mathcal{F}(R)$. We can thus interpret i_V^* as a morphism from \mathcal{F}_2 to \mathcal{F} . It fits into a diagram as follows, which commutes on the nose:

$$\begin{array}{ccccc}
& & & & \mathcal{C} \\
& & & & \uparrow i_U^* \\
& & & & \mathcal{F} \\
& & & \xrightarrow{i_V^*} & \\
& & \mathcal{F}_2 & & \\
& & \downarrow k^* & & \downarrow j^* \\
& & \mathcal{F} & & \mathcal{C} \\
& & \xrightarrow{i_V^*} & & \\
& & \mathcal{C} & & \\
& \swarrow j_2^* & \xrightarrow{j^*} & \swarrow i_V^* & \\
& \mathcal{C} & & &
\end{array}$$

The bottom edge represents the anafunctor θ_V , whereas the right hand edge represents θ_U . The claim is clear from this. \square

Proposition 8.16. *The morphism $u_*^2: \mathcal{C}^{\mathbb{P} \times \mathbb{P}} \rightarrow \mathcal{C}^{\mathbb{Q} \times \mathbb{Q}}$ restricts to give an equivalence $\mathcal{P}_2 \rightarrow \mathcal{F}_2$, with inverse $(u^2)^*$.*

Proof. Before starting we warn the reader that in this proof all the restrictions will be with respect the base derivator \mathcal{C} , even if we will apply them to elements we will prove are in the derivator of (doubly) localizing or fracture objects. This lets us avoid awkward notation and it is not restricting at all since the above derivators are subderivators of appropriate shifts of \mathcal{C} .

We must show that $\mathcal{P}_2(R) \simeq \mathcal{F}_2(R)$ for all R , but we can reduce to the case $R = e$ by replacing \mathcal{C} with \mathcal{C}^R .

We will factor the map $u^2: \mathbb{P} \times \mathbb{P} \rightarrow \mathbb{Q} \times \mathbb{Q}$ as $u^2 = u_1 \circ u_2$, where $u_1 = u \times 1: \mathbb{P} \times \mathbb{Q} \rightarrow \mathbb{Q} \times \mathbb{Q}$ and $u_2 = 1 \times u: \mathbb{P} \times \mathbb{P} \rightarrow \mathbb{P} \times \mathbb{Q}$. We also use the maps $i_\emptyset: \mathbb{P} \rightarrow \mathbb{P} \times \mathbb{P}$ and $i_{u\emptyset}: \mathbb{Q} \rightarrow \mathbb{Q} \times \mathbb{Q}$ given by $i_\emptyset(B) = (\emptyset, B)$ and $i_{u\emptyset}(V) = (u\emptyset, V)$. These fit in a commutative diagram

$$\begin{array}{ccc}
\mathbb{Q} \times \mathbb{Q} & \xleftarrow{i_{u\emptyset}} & \mathbb{Q} \\
\uparrow u^2 & & \uparrow u \\
\mathbb{P} \times \mathbb{P} & \xleftarrow{i_\emptyset} & \mathbb{P}
\end{array}$$

Note that an object $X \in \mathcal{C}(\mathbb{P} \times \mathbb{P})$ lies in $\mathcal{P}_2(e)$ iff it satisfies the following conditions:

- (a) $X \in \mathcal{P}(\mathbb{P})$
- (b) $i_\emptyset^* X \in \mathcal{P}(e)$.

Similarly, by unwinding the definitions a little we see that an object $Y \in \mathcal{C}(\mathbb{Q} \times \mathbb{Q})$ lies in $\mathcal{F}_2(e)$ iff the following hold:

- (c) $u_1^* Y \in \mathcal{P}(\mathbb{Q})$
- (d) $Y = (u_1)_*(u_1^* Y)$
- (e) $i_{u\emptyset}^* Y \in \mathcal{F}(e)$.

Suppose that $Y \in \mathcal{F}_2(e)$, so that (c), (d) and (e) are satisfied. Put $X = (u^2)^* Y \in \mathcal{C}(\mathbb{P} \times \mathbb{P})$; we must show that $X \in \mathcal{P}_2(e)$, or in other words that (a) and (b) are satisfied. Note that $X = u_2^*(u_1^* Y)$ and $u_1^* Y \in \mathcal{P}(\mathbb{Q})$ by (c) and \mathcal{P} is a subderivator so $u_2^*(u_1^* Y) \in \mathcal{P}(\mathbb{P})$ so (a) is satisfied. Moreover, the diagram shows that $i_\emptyset^* X = i_\emptyset^*(u_2^*(u_1^* Y)) = u^* i_{u\emptyset}^* Y$, and $i_{u\emptyset}^* Y \in \mathcal{F}(e)$ by (e), so $u^* i_{u\emptyset}^* Y \in \mathcal{P}(e)$, so (b) holds.

Suppose instead that we start with $X \in \mathcal{P}_2(e)$, so that (a) and (b) hold. Put $Y = u_*^2 X \in \mathcal{C}(\mathbb{Q} \times \mathbb{Q})$; we must then prove (c), (d) and (e). We first note that $Y = (u_1)_*(u_2)_* X$ and u_1 is an embedding so $u_1^*(u_1)_* \cong 1$ so $u_1^* Y \cong (u_2)_* X$. Moreover, we have $X \in \mathcal{P}(\mathbb{P})$ by (a) and \mathcal{P} is a subderivator so $(u_2)_* X \in \mathcal{P}(\mathbb{Q})$ and this proves (c). Condition (d) is also clear from this discussion. For condition (e), note that the diagram gives a Beck-Chevalley transformation

$$\alpha: i_{u\emptyset}^* Y = i_{u\emptyset}^* u_*^2 X \rightarrow u_* i_{\emptyset}^* X \in \mathcal{C}(\mathbb{Q}).$$

We know that $i_{\emptyset}^* X \in \mathcal{P}(e)$ by (b), and it follows that $u_* i_{\emptyset}^* X \in \mathcal{F}(e)$. For condition (e) it will therefore suffice to show that α is an isomorphism. For this it will in turn suffice to check that $i_V^* \alpha$ is an isomorphism in $\mathcal{C}(e)$ for all $V \in \mathbb{Q}$. Put

$$\begin{aligned} \mathbb{B} &= \{B \in \mathbb{P} \mid V \leq uB\} \\ \mathbb{C} &= \{(A, B) \in \mathbb{P} \times \mathbb{P} \mid (u\emptyset, V) \leq (uA, uB)\}. \end{aligned}$$

(These can in fact be simplified to $\mathbb{B} = V$ and $\mathbb{C} = \mathbb{P} \times V$.) The map i_{\emptyset} restricts to give a map $\mathbb{B} \rightarrow \mathbb{C}$. The Kan formula tells us that the domain of $i_V^* \alpha$ is $\text{holim}_{\leftarrow \mathbb{C}} X$, whereas the codomain is $\text{holim}_{\leftarrow \mathbb{B}} i_{\emptyset}^* X$. The evident projection $\mathbb{C} \rightarrow \mathbb{B}$ is right adjoint to i_{\emptyset} , so $i_{\emptyset}: \mathbb{B} \rightarrow \mathbb{C}$ is homotopy cofinal by Proposition 4.12, so α is an isomorphism as required.

We now have morphisms $u_*^2: \mathcal{P}_2 \rightarrow \mathcal{F}_2$ and $(u^2)^*: \mathcal{F}_2 \rightarrow \mathcal{P}_2$ with $(u^2)^* u_*^2 \simeq 1$ by Lemma 3.14. All that is left is to prove that when $Y \in \mathcal{F}_2(e)$, the unit map $Y \rightarrow u_*^2 (u^2)^* Y$ is an isomorphism. Put $Z = u_1^* Y$, so condition (d) gives $Y = (u_1)_* Z$. It will suffice to show that $Z = (u_2)_* u_2^* Z$. Note that $Z \in \mathcal{P}(\mathbb{Q})$ by condition (c), and \mathcal{P} is a subderivator, so $(u_2)_* u_2^* Z$ also lies in $\mathcal{P}(\mathbb{Q})$. As $i_{\emptyset}^*: \mathcal{P} \rightarrow \mathcal{C}$ is an equivalence, it will suffice to check that the map $i_{\emptyset}^* Z \rightarrow i_{\emptyset}^* (u_2)_* u_2^* Z$ is an isomorphism. For this, we claim that $i_{\emptyset}^* (u_2)_* u_2^* Z = u_* u^* i_{\emptyset}^* Z$. This can be checked using the Kan formula, or by recalling that $i_{\emptyset}^*: \mathcal{C}^{\mathbb{P}} \rightarrow \mathcal{C}$ is a morphism of derivators and so is compatible with u_* and u^* . We must therefore check that the map $i_{\emptyset}^* Z \rightarrow u_* u^* i_{\emptyset}^* Z$ is an isomorphism. Here $i_{\emptyset}^* Z = i_{\emptyset}^* u_1^* Y = i_{u\emptyset}^* Y$, and this lies in $\mathcal{F}(e)$ by condition (e), so the claim follows from the definition of \mathcal{F} . \square

Definition 8.17. Given $U, V \in \mathbb{Q}$ we put

$$U \boxtimes V = (U \times V) \cap \mathbb{M} = \{(A, B) \in \mathbb{P} \times \mathbb{P} \mid A \in U, B \in V, A \angle B\}.$$

The definition of $U * V$ can then be written as

$$U * V = \{A \cup B \mid (A, B) \in U \boxtimes V\}.$$

Note that $U \boxtimes V$ and $U * V$ can be seen as subposets of \mathbb{M} and \mathbb{P} respectively. We define $\sigma: U \boxtimes V \rightarrow U * V$ by $\sigma(A, B) = A \cup B$, and note that this is a morphism of posets.

Proposition 8.18. *The map $\sigma: U \boxtimes V \rightarrow U * V$ is homotopy cofinal.*

Proof. Consider an element $C \in U * V$ and the comma poset

$$\sigma/C = \{(A, B) \in U \boxtimes V \mid A \cup B \subseteq C\}.$$

By Proposition 4.12, it will be enough to show that this is strongly contractible. For $-1 \leq i \leq n^*$ we write $C_{\leq i} = \{c \in C \mid c \leq i\}$ and similarly for $C_{\geq i}$. As $C \in U * V$ we can write $C = A_0 \cup B_0$ for some $A_0 \in U$ and $B_0 \in V$ with $A_0 \angle B_0$. This means that we can choose k with $a \leq k$ for all $a \in A_0$, and $k \leq b$ for all $b \in B_0$. It follows

that $C_{\leq k} \in U$ and $C_{\geq k} \in V$. Let i be least such that $C_{\leq i} \in U$, and let j be largest such that $C_{\geq j} \in V$. Trivially $i \leq k \leq j$, thus $(C_{\leq i}, C_{\geq j}) \in \sigma/C$. Now consider an arbitrary element $(A, B) \in \sigma/C$. We define

$$\phi(A, B) = (C_{\leq \max(A)}, C_{\geq \min(B)}).$$

We use the conventions $\max(\emptyset) = -1$ and $\min(\emptyset) = n^*$ if necessary; this ensures that $\max(A) \leq \min(B)$ in all cases, so $C_{\leq \max(A)} \angle C_{\geq \min(B)}$. It is also clear that $A \subseteq C_{\leq \max(A)}$, so $C_{\leq \max(A)} \in U$, and similarly $C_{\geq \min(B)} \in V$. Thus, ϕ is a morphism of posets from σ/C to itself, with $\phi \geq 1$. On the other hand, if we define $\psi: \sigma/C \rightarrow \sigma/C$ to be the constant map with value $(C_{\leq i}, C_{\geq j})$, we find that $\psi \leq \phi$. This gives the required contraction of σ/C . \square

Proposition 8.19. *Consider the map $\mu: \mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{Q}$ (given by $(U, V) \mapsto U * V$) and the induced morphism $\mu^*: \mathcal{C}^{\mathbb{Q}} \rightarrow \mathcal{C}^{\mathbb{Q} \times \mathbb{Q}}$. This restricts to give an equivalence $\mu^*: \mathcal{F} \rightarrow \mathcal{F}_2$, making the following diagram commute up to natural isomorphism:*

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow[u_*]{\simeq} & \mathcal{F} \\ \text{inc}_* \sigma^* \downarrow \simeq & & \downarrow \simeq \mu^* \\ \mathcal{P}_2 & \xrightarrow[u_*^2]{\simeq} & \mathcal{F}_2 \end{array}$$

Proof. By the usual reduction, it will suffice to work with $\mathcal{P}(e)$, $\mathcal{F}(e)$ and so on. Proposition 7.16 gives the left hand equivalence, the top and bottom equivalences are given by Proposition 8.4 and Proposition 8.16 respectively. We claim that for $X \in \mathcal{P}(e)$, there is a natural isomorphism $\mu^* u_* X \simeq u_*^2 \text{inc}_* \sigma^* X$. Assuming this, everything else follows easily by chasing the diagram. To prove the claim, we apply Proposition 4.13 to the square

$$\begin{array}{ccc} \mathbb{M} & \xrightarrow{\sigma} & \mathbb{P} \\ u^2 \circ \text{inc} \downarrow & & \downarrow u \\ \mathbb{Q} \times \mathbb{Q} & \xrightarrow{\mu} & \mathbb{Q}, \end{array}$$

which commutes by Lemma 8.12. The square gives a Beck-Chevalley transform $\alpha: \mu^* u_* \rightarrow u_*^2 \text{inc}_* \sigma^*$, and the proposition tells us that this is an isomorphism provided that the map

$$\sigma_{(U, V)}: (U, V)/(u^2 \circ \text{inc}) \rightarrow \mu(U, V)/u$$

is homotopy cofinal for all $U, V \in \mathbb{Q}$. By unwinding the definitions, we see that this is just the map $U \boxtimes V \rightarrow U * V$ whose cofinality was proved in Proposition 8.18. \square

Proof of Theorem 8.11. Using Proposition 8.19 we obtain the following diagram, which commutes on the nose:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{i_{U * V}^*} & \mathcal{C} \\ \simeq \downarrow j^* & \searrow \mu^* & \uparrow i_{(U, V)}^* \\ \mathcal{C} & \xleftarrow[j_2^*]{\simeq} & \mathcal{F}_2 \end{array}$$

One route around the square gives θ_{U*V} by definition, and the other gives $\theta_U\theta_V$ by Corollary 8.15. \square

We conclude with an addendum to Proposition 8.18. We do not currently have any use for this, but the method of proof is interesting so we have included it.

Proposition 8.20. *The map $\sigma: U \boxtimes V \rightarrow U * V$ is also homotopy final.*

The proof will be given after some preliminaries.

Definition 8.21. We also define maps $\alpha_i, \beta_i: \mathbb{M} \rightarrow \mathbb{M}$ as follows:

$$\begin{aligned} \alpha_{2i}(A, B) &= (A_{<i}, A_{\geq i} \cup B) & \alpha_{2i+1}(A, B) &= (A_{\leq i}, A_{\geq i} \cup B) \\ \beta_{2i}(A, B) &= (A \cup B_{<i}, B_{\geq i}) & \beta_{2i+1}(A, B) &= (A \cup B_{\leq i}, B_{\geq i}) \end{aligned}$$

Here $A_{>i}$ means $\{a \in A \mid a > i\}$, and so on. We note that

$$\begin{aligned} \alpha_{2i+2}(A, B) &= (A_{\leq i}, A_{>i} \cup B) \\ \beta_{2i+2}(A, B) &= (A \cup B_{\leq i}, B_{>i}). \end{aligned}$$

Lemma 8.22. *All the above maps α_i and β_i are poset maps, with $\sigma\alpha_i = \sigma\beta_i = \sigma\gamma_i = \sigma\delta_i = \sigma$. The extreme cases are*

$$\begin{aligned} \alpha_0(A, B) &= (\emptyset, A \cup B) & \alpha_{2n^*}(A, B) &= (A, B) \\ \beta_0(A, B) &= (A, B) & \beta_{2n^*}(A, B) &= (A \cup B, \emptyset). \end{aligned}$$

There are inequalities $\alpha_{2i} \leq \alpha_{2i+1} \geq \alpha_{2i+2}$ and $\beta_{2i} \leq \beta_{2i+1} \geq \beta_{2i+2}$.

Proof. Straightforward from the definitions. \square

Proof of Proposition 8.20. Consider $C \in U * V$. By Proposition 4.12, it will suffice to prove that the poset

$$C/\sigma = \{(A, B) \in U \boxtimes V \mid C \subseteq A \cup B\}$$

is strongly contractible. As in the proof of Proposition 8.18, we can choose k between 0 and $n^* - 1$ such that $C_{\leq k} \in U$ and $C_{>k} \in V$. We claim that for all $i \geq 2k + 1$, the map $\alpha_i: \mathbb{M} \rightarrow \mathbb{M}$ preserves C/σ . To see this, suppose that $(A, B) \in C/\sigma$, so $A \in U$ and $B \in V$ and $A \angle B$ and $A \cup B \supseteq C$. We have $\alpha_i(A, B) = (A_{<u}, A_{\geq v} \cup B)$ for some u, v with $u > k$. We have seen that α_i preserves \mathbb{M} with $\sigma\alpha_i = \sigma$, so the only point to check is that $A_{\geq v} \cup B \in V$ and $A_{<u} \in U$. The first of these is clear because $B \in V$ and V is closed upwards. The second is also clear if $A_{<u} = A$. Suppose instead that $A_{<u} \neq A$, so there exists $a \in A$ with $a \geq u$. It follows that for $b \in B$ we have $b \geq a \geq u > k$, so $C_{\leq k} \cap B = \emptyset$. However, we have $C_{\leq k} \subseteq C \subseteq A \cup B$ by assumption, so $C_{\leq k} \subseteq A_{\leq k} \subseteq A_{<u}$. As $C_{\leq k} \in U$ and U is closed upwards, we see that $A_{<u} \in U$ as required. A symmetrical argument shows that β_i preserves C/σ for $i \leq 2k + 1$. As $\alpha_{2i} \leq \alpha_{2i+1} \geq \alpha_{2i+2}$ we see that $[\alpha_{2k+1}] = [\alpha_{2n^*}] = 1$ in the strong homotopy category. Similarly, we have $[\beta_{2k+1}] = [\beta_0] = 1$ and thus $[\alpha_{2k+1}\beta_{2k+1}] = 1$ in the strong homotopy category. However, it is not hard to see that

$$\alpha_{2k+1}\beta_{2k+1}(A, B) = ((A \cup B)_{\leq k}, (A \cup B)_{>k}) \geq (C_{\leq k}, C_{>k}),$$

so $\alpha_{2k+1}\beta_{2k+1}$ is equivalent to the constant map with value $(C_{\leq k}, C_{>k})$. \square

9. MONOIDAL STRUCTURES

Definition 9.1. A *symmetric monoidal derivator* \mathcal{C} is as specified in [8, Definition 2.4]. We will not give the full details, and we will use notation corresponding to stable homotopy theory rather than derived algebra. The key points are as follows:

- (a) Each category $\mathcal{C}(P)$ has a symmetric monoidal structure, with unit denoted by S and the monoidal product of X and Y called the *smash product* and denoted by $X \wedge Y$.
- (b) For each $u: P \rightarrow Q$, the pullback functor $u^*: \mathcal{C}(Q) \rightarrow \mathcal{C}(P)$ preserves smash products up to isomorphism.

Remark 9.2. It will not typically be true that the natural morphism $u_!(X \wedge u^*(Y)) \rightarrow u_!(X) \wedge Y$ is an isomorphism. Similarly, even if there exist function objects $F(X, Y)$ with $[W, F(X, Y)] \simeq [W \wedge X, Y]$, these will typically not satisfy $u^*F(X, Y) \simeq F(u^*X, u^*Y)$. These kinds of properties are valid for derivators indexed by groups or groupoids, but posets are the other extreme from that.

For the rest of this section, we will assume that \mathcal{C} is a symmetric monoidal derivator. We will also assume that the homology theories $K(n)$ have the property that $K(n)_*(X) = 0$ implies $K(n)_*(X \wedge Y) = 0$ for all Y . We will just give some simple results about how the monoidal structure interacts with chromatic fracture.

Proposition 9.3. *Suppose that $X, Y, Z \in \mathcal{F}(R)$. Then the natural map*

$$\mathcal{C}(\mathbb{Q} \times R)(X \wedge Y, Z) \rightarrow \mathcal{C}(R)(j^*X \wedge j^*Y, j^*Z)$$

is bijective.

Proof. As Z is u -cartesian, we have

$$\mathcal{C}(\mathbb{Q} \times R)(X \wedge Y, Z) = \mathcal{C}(\mathbb{P} \times R)(u^*(X \wedge Y), u^*Z).$$

Now just apply Lemma 7.8. □

We can use the above result to show that θ_U is lax monoidal, in the appropriate sense for anafunctors. In more detail, consider the following diagrams:

$$\begin{array}{ccc} \mathcal{C}(R) \times \mathcal{C}(R) & \xleftarrow{j^* \times j^*} & \mathcal{F}(R) \times \mathcal{F}(R) \xrightarrow{\wedge} \mathcal{C}(\mathbb{Q} \times R) \xrightarrow{i_U^*} \mathcal{C}(R) \\ & & \mathcal{C}(R) \times \mathcal{C}(R) \xrightarrow{\wedge} \mathcal{C}(R) \xleftarrow{j^*} \mathcal{F}(R) \xrightarrow{i_U^*} \mathcal{C}(R). \end{array}$$

The leftward-pointing maps are equivalences, so we can invert them to get two different anafunctors $\mathcal{C}(R) \times \mathcal{C}(R) \rightarrow \mathcal{C}(R)$. Informally, these are $(X, Y) \mapsto \theta_U(X) \wedge \theta_U(Y)$ and $(X, Y) \mapsto \theta_U(X \wedge Y)$. We need to provide a morphism between these anafunctors. For this, we introduce the category

$$\mathcal{P}(R) = \{(X, Y, Z, u) \mid X, Y, Z \in \mathcal{F}(R), u: j^*(X) \wedge j^*(Y) \xrightarrow{\simeq} j^*(Z)\},$$

and the diagram

$$\begin{array}{ccccc} \mathcal{P}(R) & \xrightarrow{r} & \mathcal{F}(R) & & \\ \downarrow l & & \downarrow \text{inc} & & \\ \mathcal{F}(R) \times \mathcal{F}(R) & \xrightarrow{\wedge} & \mathcal{C}(\mathbb{Q} \times R) & \xrightarrow{i_U^*} & \mathcal{C}(R) \\ \downarrow j^* \times j^* & & \downarrow j^* & & \\ \mathcal{C}(R) \times \mathcal{C}(R) & \xrightarrow{\wedge} & \mathcal{C}(R). & & \end{array}$$

Here $l(X, Y, Z, u) = (X, Y)$ and $r(X, Y, Z, u) = Z$. Using the fact that $j^*: \mathcal{F}(R) \rightarrow \mathcal{C}(R)$ is an equivalence, we find that l is also an equivalence and the rectangle formed of the two squares is a homotopy pullback. Because of this, our two anafunctors can be described as follows: we invert the equivalence $\mathcal{P}(R) \rightarrow \mathcal{C}(R) \times \mathcal{C}(R)$, then take one of the two routes around the top square, then apply i_U^* . Proposition 9.3 gives a natural map $X \wedge Y \rightarrow Z$ for all $(X, Y, Z, u) \in \mathcal{P}(R)$, or in other words, a natural map between the two composites around the top square. This gives our required morphism of anafunctors.

APPENDIX A. COMPACTLY GENERATED TRIANGULATED CATEGORIES

We next want some results about Brown representability and Bousfield localisation in triangulated categories and derivators. For the homotopy category of spectra, all statements are well-known with very classical proofs that rely on having an underlying geometric category of spectra [13, Chapter 7]. There are also proofs of similar results in more axiomatic frameworks, relying only on the theory of triangulated categories. These are typically formulated in the context of well-generated categories as defined by Neeman [14], and the proofs are somewhat complex. It is well-known to experts that everything becomes much simpler, and much closer to the original results for the category of spectra, if we restrict attention to compactly generated categories. However, it seems surprisingly hard to find an full account of this in the literature. We therefore provide one here.

Definition A.1. Until further notice, \mathcal{T} will be a compactly generated triangulated category with coproducts. We choose a small skeleton \mathcal{T}_0 of \mathcal{T}_c , and note that this is necessarily closed under suspensions and desuspensions and cofibres up to isomorphism. We also choose an infinite cardinal κ_0 such that the total number of morphisms in \mathcal{T}_0 is at most κ_0 .

Definition A.2. Let κ be a cardinal that is at least as large as κ_0 . We define subcategories $\mathcal{T}_n^\kappa \subseteq \mathcal{T}$ as follows. First, we let \mathcal{T}_0^κ be the subcategory of objects that can be expressed as a coproduct $\bigoplus_{i \in I} T_i$, where $|I| \leq \kappa$ and $T_i \in \mathcal{T}_c$ for all i . We then define \mathcal{T}_{n+1}^κ to be the subcategory of objects Z that can be expressed as the cofibre of a map from an object in \mathcal{T}_0^κ to an object in \mathcal{T}_n^κ . Finally, we let $\mathcal{T}_\infty^\kappa$ be the subcategory of objects X that can be expressed as the telescope of a sequence X_n with $X_n \in \mathcal{T}_n^\kappa$ for all n . Given that \mathcal{T}_c is essentially small, we find that \mathcal{T}_n^κ is essentially small for all $n \leq \infty$.

We now state a version of the Brown representability theorem:

Theorem A.3. *Let $K: \mathcal{T}^{\text{op}} \rightarrow \text{Ab}$ be a cohomology theory (so K converts all coproducts to products, and distinguished triangles to exact sequences). Then K is representable.*

The basic method of proof is due to Brown, and similar axiomatic versions have appeared with various different hypotheses in a number of places such as [13, Theorem 4.11] and [12, Theorem 2.3.2]. There are also various versions with weaker hypotheses and much more complicated proofs, such as [14, Chapter 8]. For completeness we will give a brief account here with our current hypotheses and notation.

Proof. We shall define recursively a sequence of objects

$$X(0) \xrightarrow{i_0} X(1) \xrightarrow{i_1} X(2) \xrightarrow{i_2} \dots$$

and elements $u(k) \in K(X(k))$ such that $i_k^* u(k+1) = u(k)$. We start with

$$X(0) = \bigoplus_{Z \in \mathcal{T}_0} \bigoplus_{v \in K(Z)} Z.$$

We take $u(0)$ to be the element of

$$K(X(0)) = \prod_{Z \in \mathcal{T}_0} \prod_{v \in K(Z)} K(Z)$$

whose (Z, v) th component is v . We then set

$$T(k) = \{(Z, f) \mid Z \in \mathcal{T}_0, f: Z \rightarrow X(k), f^* u(k) = 0\}.$$

We define $X(k+1)$ by the cofiber sequence

$$\bigoplus_{(Z, f) \in T(k)} Z \rightarrow X(k) \xrightarrow{i_k} X(k+1).$$

By applying K to this, we obtain a three-term exact sequence (with arrows reversed). It is clear by construction that $u(k)$ maps to zero in the left hand term, so that there exists $u(k+1) \in K(X(k+1))$ with $i_k^* u(k+1) = u(k)$ as required.

We now let X be the telescope of the objects $X(k)$. The cofibration defining this telescope gives rise to a short exact sequence

$$0 \rightarrow \lim_{\leftarrow k}^1 K(\Sigma X(k)) \rightarrow K(X) \rightarrow \lim_{\leftarrow k} K(X(k)) \rightarrow 0.$$

Using this, we find an element $u \in K(X)$ that maps to $u(k)$ in each $K(X(k))$. As in Yoneda's lemma, this induces a natural map $\tau_U: [U, X] \rightarrow K(U)$. It is easy to see that τ_Z is an isomorphism for each $Z \in \mathcal{T}_0$ (using the fact that these objects are small). It is also easy to see that

$$\{Z \mid \tau_{\Sigma^k Z} \text{ is an isomorphism for all } k\}$$

is a localizing category. It contains \mathcal{T}_0 , so it must be all of \mathcal{T} ; thus τ is an isomorphism. \square

Next, for any localising subcategory $\mathcal{U} \subseteq \mathcal{T}$, we can form the Verdier quotient category \mathcal{T}/\mathcal{U} ; see [14, Chapter 2] for a detailed treatment. However, we are implicitly assuming everywhere that the hom sets of our categories are small, or in other words that they are genuine sets rather than proper classes, and this can fail for \mathcal{T}/\mathcal{U} . If we can verify in a particular case that \mathcal{T}/\mathcal{U} has small hom sets, then we can use Brown representability to prove the existence of localisation functors. This general line of argument is well-known, going back to Adams and Bousfield. There are various known approaches to prove that \mathcal{T}/\mathcal{U} has small hom sets. One possibility is to assume that \mathcal{T} is the homotopy category of a Quillen model category or an infinity category in the sense of Lurie; but here we prefer to work solely with triangulated categories and derivators. In [13, Chapter 7], Margolis proves some results of this type for the category of spectra, and it is well-known to experts that his approach can be generalised to compactly generated triangulated categories. As with the representability theorem, there are also similar results with weaker hypotheses and much more complicated proofs, such as [14, Corollary 4.4.3]. However, we have not been able to find an explicit axiomatic version of the approach of Margolis in the literature, so we provide one here. We start by spelling out the argument that small hom sets give localisations.

Theorem A.4. *Suppose that \mathcal{T}/\mathcal{U} has small hom sets. Then for each $X \in \mathcal{T}$ there exists a distinguished triangle*

$$CX \xrightarrow{q} X \xrightarrow{j} LX \xrightarrow{d} \Sigma CX$$

with $CX \in \mathcal{U}$ and $LX \in \mathcal{U}^\perp$. In fact, q is terminal in \mathcal{U}/X and j is initial in X/\mathcal{U}^\perp .

Proof. From [14, Chapter 2] we have a triangulation of \mathcal{T}/\mathcal{U} such that the evident functor $\pi: \mathcal{C} \rightarrow \mathcal{C}/\mathcal{U}$ is exact. From [14, Corollary 3.2.11] we know that \mathcal{T}/\mathcal{U} has small coproducts and that π preserves coproducts. It follows that the functor $W \mapsto (\mathcal{T}/\mathcal{U})(\pi W, \pi X)$ is a cohomology theory on \mathcal{T} . By Theorem A.3, we can choose an object $LX \in \mathcal{T}$ and a natural isomorphism $\mathcal{T}(W, LX) \simeq (\mathcal{T}/\mathcal{U})(\pi W, \pi X)$ for all $W \in \mathcal{T}$. If $W \in \mathcal{U}$ then $\pi W = 0$ and so $\mathcal{T}(W, LX) = 0$; this proves that $LX \in {}^\perp\mathcal{U}$. The natural map

$$\mathcal{T}(W, X) \xrightarrow{\pi} (\mathcal{T}/\mathcal{U})(\pi W, \pi X) \simeq \mathcal{T}(W, LX)$$

corresponds (via the Yoneda Lemma) to a map $j: X \rightarrow LX$. We can then fit this into a distinguished triangle

$$CX \xrightarrow{q} X \xrightarrow{j} LX \xrightarrow{d} \Sigma CX.$$

Next, recall that every element of $(\mathcal{T}/\mathcal{U})(\pi W, \pi LX)$ can be represented as a fraction gf^{-1} for some diagram $(W \xleftarrow{f} V \xrightarrow{g} LX)$ with $\text{cof}(f) \in \mathcal{U}$. As $LX \in \mathcal{U}^\perp$ we see that $\mathcal{T}(\text{cof}(f), LX) = \mathcal{T}(\Sigma \text{cof}(f), LX) = 0$, and it follows that there is a unique $h: W \rightarrow LX$ with $hf = g$. Using this, we find that the map

$$\pi: \mathcal{T}(W, LX) \rightarrow (\mathcal{T}/\mathcal{U})(\pi W, \pi LX)$$

is an isomorphism for all W . By combining this with our isomorphism $\mathcal{T}(W, LX) \simeq (\mathcal{T}/\mathcal{U})(\pi W, \pi X)$, we see that $\pi(j): \pi X \rightarrow \pi LX$ is an isomorphism, so $\pi CX = 0$ and $CX \in \mathcal{U}$ as claimed. Now for $U \in \mathcal{U}$ we have $\mathcal{T}(U, LX)_* = 0$ hence the distinguished triangle gives $\mathcal{T}(U, CX) \simeq \mathcal{T}(U, X)$, which proves that $(CX \xrightarrow{q} X)$ is terminal in \mathcal{U}/X . Dually, if $Z \in \mathcal{U}^\perp$ then $\mathcal{T}(CX, Z)_* = 0$ so the distinguished triangle gives $\mathcal{T}(X, Z) \simeq \mathcal{T}(LX, Z)$, therefore $(X \xrightarrow{j} LX)$ is initial in X/\mathcal{U}^\perp . \square

Definition A.5. Let \mathcal{C} be a category, and let \mathcal{J} be a set of objects. We say that \mathcal{J} is *weakly initial* if for every $X \in \mathcal{C}$ there exists an object $T \in \mathcal{J}$ and a morphism $T \rightarrow X$.

Proposition A.6. *Suppose that \mathcal{U} is a localising subcategory of \mathcal{T} such that for each $X \in \mathcal{T}$, the comma category X/\mathcal{U} has a weakly initial set. Then the category \mathcal{T}/\mathcal{U} has small hom sets.*

Remark A.7. As we will explain in more detail below, this implies that there is a localisation functor L with kernel \mathcal{U} , and thus that the unit map $X \rightarrow LX$ is initial in the comma category X/\mathcal{U}^\perp (not X/\mathcal{U}). This interplay between X/\mathcal{U}^\perp and X/\mathcal{U} is a little surprising, but that is how the proof works.

Proof. Fix objects $X, Y \in \mathcal{T}$, and let $\{X \xrightarrow{e_i} U_i\}_{i \in I}$ be a weakly initial family in X/\mathcal{U} . Let $Z_i \xrightarrow{f_i} X$ be the fibre of e_i . As the cofiber of f_i is in \mathcal{U} , every map $g: Z_i \rightarrow Y$ gives a fraction $gf_i^{-1} \in (\mathcal{T}/\mathcal{U})(X, Y)$. It will suffice to show that every element of $(\mathcal{T}/\mathcal{U})(X, Y)$ is of this form. A general element of $(\mathcal{T}/\mathcal{U})(X, Y)$ can be

represented as qp^{-1} for some $X \xleftarrow{p} W \xrightarrow{q} Y$, where the cofibre of p is in \mathcal{U} . In more detail, we have a cofibration $W \xrightarrow{p} X \xrightarrow{m} V$ with $V \in \mathcal{U}$. By the weakly initial property, we can write m as ne_i for some i and some $n: U_i \rightarrow V$. This gives $mf_i = 0$, so f_i lifts to the fibre of m , so we can choose $k: Z_i \rightarrow W$ with $pk = f_i$. The cofibres of p and f_i are in \mathcal{U} , and \mathcal{U} is thick, so the octahedral axiom tells us that the cofibre of k is also in \mathcal{U} , so k becomes an isomorphism in \mathcal{T}/\mathcal{U} . We can now put $g = qk: Z_i \rightarrow Y$ and we have $qp^{-1} = (qk)(pk)^{-1} = gf_i^{-1}$ as required. \square

Proposition A.8. *Let $K: \mathcal{T} \rightarrow \text{Ab}$ be a homology theory (so K preserves all coproducts, and converts distinguished triangles to exact sequences). Put*

$$\mathcal{U} = \ker(K_*) = \{X \mid K(\Sigma^n X) = 0 \text{ for all } n \in \mathbb{Z}\}.$$

Then \mathcal{U} is a localising subcategory of \mathcal{T} such that X/\mathcal{U} has a weakly initial set for all X , so \mathcal{T}/\mathcal{U} has small hom sets.

The proof will be given after some preliminary definitions and lemmas.

Definition A.9. We define $K_*: \mathcal{T} \rightarrow \text{Ab}_*$ by $K_n(X) = K(\Sigma^{-n}X)$. We then define $\widehat{K}_*: \mathcal{T} \rightarrow \text{Ab}_*$ to be the left Kan extension of the restriction of K_* to $\mathcal{T}_0 \subset \mathcal{T}$. Explicitly, $\widehat{K}_*(X)$ is the colimit of the functor $\mathcal{T}_0/X \rightarrow \text{Ab}_*$ sending $(T \xrightarrow{t} X)$ to $K_*(T)$. There is an evident counit map $\phi_X: \widehat{K}_*(X) \rightarrow K_*(X)$.

Lemma A.10. *The counit map $\phi_X: \widehat{K}_*(X) \rightarrow K_*(X)$ is an isomorphism for all X .*

Proof. It is not hard to see that the category \mathcal{T}_0/X is filtered, and to deduce that \widehat{K} is again a homology theory. Details are given in [12, Section 2.3], for example. (In that reference there are officially some additional standing assumptions that we are not assuming here, such as that \mathcal{T} has a symmetric monoidal structure, but none of those assumptions are used in the relevant proofs.) It follows that the category $\{X \mid \phi_X \text{ is iso}\}$ is localising and contains \mathcal{T}_0 , so it must be all of \mathcal{T} , as required. \square

Definition A.11. We let κ_1 be a cardinal such that $\kappa_1 \geq \kappa_0$ and $\kappa_1 \geq |K_*(T)|$ for all $T \in \mathcal{T}_0$.

Corollary A.12. *Fix a cardinal κ such that $\kappa \geq \kappa_1$. Suppose that $X \in \mathcal{T}$ with $|K_*(X)| \leq \kappa$. Then one can choose a fibre sequence*

$$FX \xrightarrow{\alpha} X \xrightarrow{\beta} GX$$

such that

- (a) $FX \in \mathcal{T}_0^\kappa$.
- (b) $K_*(\alpha)$ is surjective, so $K_*(\beta) = 0$.
- (c) $|K_*(GX)| \leq \kappa$.

(We do not claim that F or G is a functor.)

Proof. Let $\{b_i \mid i \in I\}$ be a homogeneous generating set for $K_*(X)$ with $|I| \leq \kappa$. As $K_*(X) = \widehat{K}_*(X)$, we can choose objects $T_i \in \mathcal{T}_0$ and maps $t_i: T_i \rightarrow X$ and elements $a_i \in K_*(T_i)$ with $(t_i)_*(a_i) = b_i$. We then take $FX = \bigoplus_i T_i \in \mathcal{T}_0^\kappa$, and let $\alpha: FX \rightarrow X$ be the map given by t_i on the i 'th summand. Using $K_*(FX) = \bigoplus_i K_*(T_i)$ we find that $K_*(\alpha)$ is surjective as required. We then define $X \xrightarrow{\beta} GX$ to be the cofibre of α , so $K_*(\beta) = 0$ and we have a short exact sequence $K_{*+1}(GX) \rightarrow$

$K_*(FX) \rightarrow K_*(X)$. Standard cardinal arithmetic now gives $|K_*(FX)| \leq \kappa$ and then $|K_*(GX)| \leq \kappa$. \square

Construction A.13. We have a sequence $X \xrightarrow{\beta} GX \xrightarrow{\beta} G^2X \rightarrow \dots$, and we define $G^\infty X$ to be the telescope. For $0 \leq n \leq \infty$ we define $F_n X$ to be the fibre of the map $X \rightarrow G^n X$. The octahedral axiom then gives us a cofibration $\Sigma^{-1}FG^n X \rightarrow F_n X \rightarrow F_{n+1} X$, and one can also check that $F_\infty X$ is the telescope of the objects $F_n X$ for $n < \infty$.

Proposition A.14. *For X and κ as above, the fibre sequence*

$$F_\infty X \xrightarrow{\alpha_\infty} X \rightarrow G^\infty X$$

satisfies

- (a) $F_\infty X \in \mathcal{T}_\infty^\kappa$.
- (b) $K_*(\alpha_\infty)$ is an isomorphism.
- (c) $K_*(G^\infty X) = 0$, or in other words $G^\infty X \in \mathcal{U}$.

Proof. We see by induction that $|K_*(G^n X)| \leq \kappa$ for all n . It follows that $FG^n X \in \mathcal{T}_0^\kappa$, and thus that $F_n X \in \mathcal{T}_{n-1}^\kappa$, therefore $F_\infty X \in \mathcal{T}_\infty^\kappa$. Also, as the maps $K_*(\beta): K_*(G^n X) \rightarrow K_*(G^{n+1} X)$ are zero, we see that $K_*(G^\infty X) = 0$. It follows that the map $K_*(F_\infty X) \rightarrow K_*(X)$ is an isomorphism as desired. \square

Proof of Proposition A.8. It is straightforward to check that \mathcal{U} is a localising subcategory.

Now fix $X \in \mathcal{T}$, and choose $\kappa \geq \kappa_1$ such that $|K_*(X)| \leq \kappa$. Let \mathcal{A} be the subcategory of X/\mathcal{U} consisting of objects $(X \xrightarrow{f} U)$ such that the fibre of f lies in $\mathcal{T}_\infty^\kappa$. As $\mathcal{T}_\infty^\kappa$ is essentially small, we see that \mathcal{A} is also essentially small, so we can choose a small skeleton \mathcal{A}_0 .

Now let $(X \xrightarrow{g} V)$ is an arbitrary object of X/\mathcal{U} . Let $P \xrightarrow{j} X$ be the fibre of g , so $K_*(j)$ is an isomorphism, so $|K_*(P)| \leq \kappa$. Proposition A.14 therefore gives us a map $Q = F_\infty P \xrightarrow{q} P$ such that $Q \in \mathcal{T}_\infty^\kappa$ and $K_*(q)$ is an isomorphism. Let $X \xrightarrow{f} U$ be the cofibre of $jq: Q \rightarrow X$. As $K_*(jq)$ is an isomorphism, we see that $U \in \mathcal{U}$ and so $(X \xrightarrow{f} U) \in X/\mathcal{U}$. As $gj = 0$ we have $g(jq) = 0$ so the map $g: X \rightarrow V$ factors through f , so there is a morphism from $(X \xrightarrow{f} U)$ to $(X \xrightarrow{g} V)$ in X/\mathcal{U} . This proves that \mathcal{A} is weakly initial in X/\mathcal{U} , and it follows that the skeleton \mathcal{A}_0 has the same property. \square

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