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Observer-based input-to-state stabilization of networked control systems with large uncertain delays

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Abstract

We consider output-feedback predictor-based stabilization of networked control systems with large unknown time-varying communication delays. For systems with two networks (sensors-to-controller and controller-to-actuators), we design a sampled-data observer that gives an estimate of the system state. This estimate is used in a predictor that partially compensates unknown network delays. We emphasize the purely sampled-data nature of the measurement delays in the observer dynamics. This allows an efficient analysis via the Wirtinger inequality, which is extended here to obtain exponential stability. To reduce the number of sent control signals, we incorporate the event-triggering mechanism. For systems with only a controller-to-actuators network, we take advantage of continuously available measurements by using a continuous-time predictor and employing a recently proposed switching approach to event-triggered control. For systems with only a sensors-to-controller network, we construct a continuous observer that better estimates the system state and increases the maximum output sampling, therefore, reducing the number of required measurements. A numerical example illustrates that the predictor-based control allows one to significantly increase the network-induced delays, whereas the event-triggering mechanism significantly reduces the network workload.

Key words: Predictor-based control, Observer-based control, Networked control systems, Event-triggered control

1 Introduction

In networked control systems (NCSs), which are comprised of sensors, controllers, and actuators connected through a communication medium, transmitted signals are sampled in time and are subject to time-delays. Most existing papers on NCSs study robust stability with respect to small communication delays (see, e.g., [2,5,6,14]). To compensate large transport delays, a predictor-based approach can be employed. This was done in [10] for sampled-data state-feedback control of nonlinear systems and in [11] for an output-feedback control with approximate predictors. Sampled-data predictor-based state-feedback control of linear systems under continuous-time measurements has been considered in [18]. Nonlinear systems under sampled-data measurements and continuous output-feedback control have been studied in [1,12].

All the aforementioned works deal with *known constant* network-induced delays. Predictor-based networked control under *uncertain time-varying* delays has been considered in [22], where a *state*-feedback controller has been studied. In this paper, we propose a predictor-based dynamic *output*-feedback controller for NCSs with *uncertain time-varying* delays. We present a new model of a closed-loop observer-based NCS in the framework of the time-delay approach. In such a model, several delays appear due to sampling and network-induced delays. We emphasize the purely sampled-data nature of measurement delays in the observer dynamics. This allows an efficient analysis via the Wirtinger inequality, which is extended here to obtain exponential stability.

We start by considering the case of two networks: sensors-to-controller and controller-to-actuators (Section 2). Both networks introduce large time-varying delays. We assume that the messages sent from the sensors are time stamped [25]. This allows the controller to calculate the sensors-to-controller delay. The controller-to-actuators delay is assumed to be unknown but belongs to a known delay interval. We design an observer that is calculated on the controller side and gives an estimate of the system state. This estimate is used in a predictor, which partially compensates both delays. To reduce the workload of the controller-to-actuators network, we incorporate the event-triggering mechanism [23].

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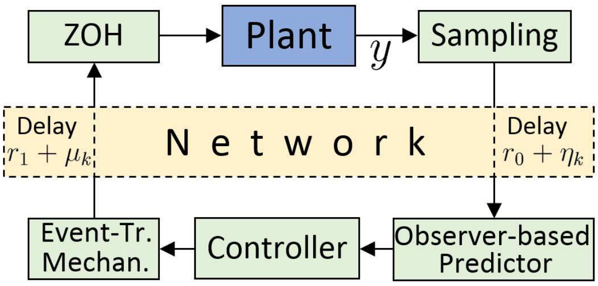

Fig. 1. NCS with two networks

In Section 3, we proceed to NCSs with continuous measurements and controller-to-actuators networks, where we demonstrate that a recently proposed switching approach to event-triggered control [21] takes advantage of continuously available measurements and further reduces the number of sent control signals. For the case of continuous control and sampled measurements, we construct a continuous observer that better estimates the system state and increases the maximum output sampling, therefore, reducing the number of required measurements (Section 4). All the results are demonstrated in Section 5 by an example borrowed from [25].

First, we present an extension of the Wirtinger inequality [17, Lemma 3.1].

Lemma 1 (Wirtinger inequality) *Let a, b, $\alpha \in \mathbb{R}$, $0 \le W \in \mathbb{R}^{n\times n}$, and $f\colon [a,b] \to \mathbb{R}^n$ be an absolutely continuous function with a square integrable first derivative such that $f(a) = 0$ or $f(b) = 0$. Then*

$$\int_a^b e^{2\alpha t} f^T(t)Wf(t)\,dt \le e^{2|\alpha|(b-a)}\frac{4(b-a)^2}{\pi^2}\int_a^b e^{2\alpha t}\dot f^T(t)W\dot f(t)\,dt.$$

Proof is based on an idea from [7, Lemma A.18]. If $\alpha \ge 0$, we have

$$\begin{aligned}\int_a^b e^{2\alpha t} f^T(t)Wf(t)\,dt &\le e^{2\alpha b}\int_a^b f^T(t)Wf(t)\,dt\\ &\le e^{2\alpha b}\frac{4(b-a)^2}{\pi^2}\int_a^b \dot f^T(t)W\dot f(t)\,dt\\ &\le e^{2|\alpha|(b-a)}\frac{4(b-a)^2}{\pi^2}\int_a^b e^{2\alpha t}\dot f^T(t)W\dot f(t)\,dt,\end{aligned} \tag{1}$$

where the second inequality follows from [17, Lemma 3.1]. If $\alpha < 0$, the proof is similar but $e^{2\alpha b}$ should be replaced by $e^{2\alpha a}$ after the first and second inequalities in (1). ■

If $\alpha = 0$, Lemma 1 coincides with [17, Lemma 3.1] that was used in [15] to construct a Lyapunov functional for stability analysis of a sampled-data system. Here we use the extended Wirtinger inequality of Lemma 1 for Lyapunov-based *exponential* stability analysis (see V_W term in (A.1)).

2 NCSs with two networks

Consider a linear system

$$\begin{aligned}\dot x(t) &= Ax(t) + Bu(t) + w_1(t),\\ y(t) &= Cx(t) + w_2(t),\end{aligned}\qquad t \ge 0 \tag{2}$$

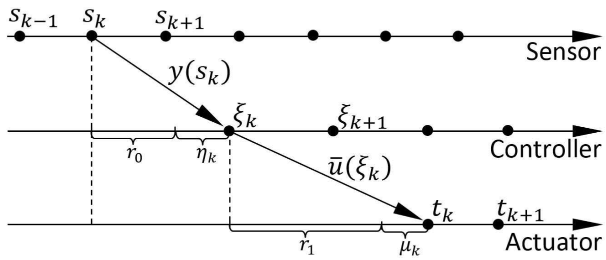

Fig. 2. Time-delays and updating times

with the state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^l$, exogenous disturbance $w_1 \in \mathbb{R}^n$, measurement noise $w_2 \in \mathbb{R}^l$, and constant matrices A, B, and C. We assume that (A, B) is stabilizable and (A, C) is detectable meaning that there exist constant matrices $K \in \mathbb{R}^{m\times n}$ and $L \in \mathbb{R}^{n\times l}$ such that $A + BK$ and $A + LC$ are Hurwitz. Let $\{s_k\}$ be sampling instants such that

$$0 = s_1 < s_2 < \ldots,\quad \lim_{k\to\infty} s_k = \infty,\quad s_{k+1} - s_k \le h.$$

In this section, we assume that at each sampling time s_k ($k \in \mathbb{N}$ throughout the paper) the output $y(s_k)$ is transmitted to a controller, which generates a control signal and transmits it to actuators, where it is applied through zero-order hold (see Fig. 1). The controller and actuators are event-driven with updating times (see Fig. 2)

$$\xi_k = s_k + r_0 + \eta_k,\qquad t_k = \xi_k + r_1 + \mu_k,$$

where r_0 and r_1 are known constant transport delays, η_k and μ_k are time-varying delays such that

$$0 \le \eta_k \le \eta_M,\; 0 \le \mu_k \le \mu_M,\; \xi_k \le \xi_{k+1},\; t_k \le t_{k+1}. \tag{3}$$

Note that the sequences $\{\xi_k\}$ and $\{t_k\}$ should be increasing, but we do not require $\eta_k + \mu_k$ to be less than a sampling interval. We assume that the sensors' and controller's clocks are synchronized [25] and together with $y(s_k)$ the time stamp s_k is transmitted so that $\eta_k = \xi_k - s_k - r_0$ can be calculated by the controller. The delay uncertainty μ_k is unknown.

To reduce the workload of a controller-to-actuators network, we incorporate the event-triggering mechanism [23]. The idea is to send only those control signals $u(\xi_k)$ which relative change is greater than some threshold. Namely, let the *nominal control* (without event-triggering) be

$$u(t) = \begin{cases} 0, & t < \xi_1,\\ u(\xi_k), & t \in [\xi_k, \xi_{k+1}),\end{cases}$$

where $u(\xi_k)$ will be constructed later. Then the *applied control* signal $\bar u(t)$ is 0 for $t < \xi_1$ and

$$\bar u(t) = \begin{cases} \bar u(\xi_{k-1}), & t \in [\xi_k, \xi_{k+1}),\quad (5) \text{ is true},\\ u(\xi_k), & t \in [\xi_k, \xi_{k+1}),\quad (5) \text{ is not true},\end{cases} \tag{4}$$

where the event-triggering rule is given by

$$[\bar u(\xi_{k-1}) - u(\xi_k)]^T\Omega\,[\bar u(\xi_{k-1}) - u(\xi_k)] \le \sigma u^T(\xi_k)\Omega u(\xi_k) \tag{5}$$

with event-triggering parameters $0 \le \Omega \in \mathbb{R}^{m\times m}$, $\sigma \in [0, 1)$, and initial value $\bar u(\xi_0) = 0$. Then the system (2) transforms into

$$\begin{aligned}
&\dot{x}(t) = Ax(t) + w_1(t), && t \in [0, t_1),\\
&\dot{x}(t) = Ax(t) + B\bar{u}(\xi_k) + w_1(t), && t \in [t_k, t_{k+1}),\\
&y(t) = Cx(s_k) + w_2(s_k), && t \in [s_k, s_{k+1}).
\end{aligned} \tag{6}$$

The purpose of this section is to construct a predictor-based controller that stabilizes (6). First, we construct the following observer for $x(t)$:

$$\begin{aligned}
&\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t - r_1) - L[y(t) - \hat{y}(t)], \quad t \geq 0,\\
&\hat{y}(t) = C\hat{x}(s_k), \quad t \in [s_k, s_{k+1})
\end{aligned} \tag{7}$$

with $\hat{x}(0) = 0$. The idea of this observer is the following. The system (6) suggests that one should use $\bar{u}(\xi_k)$ for $t \in [t_k, t_{k+1})$ instead of $u(t-r_1)$ in (7) to obtain a "control-free" system for the estimation error $x(t)-\hat{x}(t)$. However, this is not possible, since the value $t_k = \xi_k+r_1+\mu_k$ depends on the unknown μ_k. Then one may intend to use $\bar{u}(t-r_1) = \bar{u}(\xi_k)$ for $t \in [\xi_k + r_1, \xi_{k+1} + r_1)$ in (7). This, however, leads to additional event-triggering error, which can be avoided using $u(t - r_1)$.

Consider the change of variables [13,3]

$$\begin{aligned}
z(t) &= e^{A(r_0+r_1)}x(t) + \int_{t-r_1}^{t+r_0} e^{A(t+r_0-\theta)}Bu(\theta)\,d\theta,\\
\hat{z}(t) &= e^{A(r_0+r_1)}\hat{x}(t) + \int_{t-r_1}^{t+r_0} e^{A(t+r_0-\theta)}Bu(\theta)\,d\theta,
\end{aligned} \tag{8}$$

for $t \geq 0$ and $z(t) = \hat{z}(t) = 0$ for $t < 0$. Then we obtain

$$\begin{aligned}
\dot{z}(t) =& Az(t)+Bu(t+r_0)-e^{A(r_0+r_1)}Bu(t-r_1)\\
&+ e^{A(r_0+r_1)}w_1(t), \quad t \in [0, t_1),\\
\dot{z}(t) =& Az(t)+Bu(t+r_0)+e^{A(r_0+r_1)}B[\bar{u}(\xi_k)-u(t-r_1)],\\
&+ e^{A(r_0+r_1)}w_1(t), \quad t \in [t_k, t_{k+1}),\\
\dot{\hat{z}}(t) =& A\hat{z}(t)+Bu(t + r_0)-e^{A(r_0+r_1)}L\,[y(t)-\hat{y}(t)]\,, \ t \geq 0.
\end{aligned} \tag{9}$$

If $\mu_M = 0$, it is reasonable to take $\bar{u}(\theta)$ instead of $u(\theta)$ in (8) to obtain a more precise state prediction. For $\mu_M \neq 0$ we take $u(\theta)$ to avoid additional event-triggering errors that otherwise would appear in (9) (see [22] for details). As the nominal control law, we take $u(t) = 0$ for $t < \xi_1$ and

$$\begin{aligned}
u(t) = K\hat{z}(s_k) = K[&e^{A(r_0+r_1)}\hat{x}(s_k)\\
&+ \int_{s_k-r_1}^{s_k+r_0} e^{A(s_k+r_0-\theta)}Bu(\theta)\,d\theta], \quad t \in [\xi_k, \xi_{k+1}).
\end{aligned} \tag{10}$$

The value of $y(s_k)$ is available to the controller at time ξ_k, therefore, $\hat{x}(s_k)$ can be calculated by solving (7) on $[s_{k-1}, s_k]$. Since the time stamp s_k is sent together with $y(s_k)$, the control signal (10) can be calculated on the controller side. Moreover, no numerical difficulties arise while calculating the integral term in (10), since $u(\theta)$ is piecewise constant.

We analyse the system(9) under event-triggered control (4), (5), (10) using the time-delay approach to NCSs [5,6,4]. Define the following time-delays

$$\begin{aligned}
\tau_0(t) &= t - s_k, \quad t \in [\xi_k - r_0, \xi_{k+1} - r_0),\\
\tau_1(t) &= t - s_k, \quad t \in [\xi_k + r_1, \xi_{k+1} + r_1),\\
\tau_2(t) &= t - s_k, \quad t \in [t_k, t_{k+1}).
\end{aligned} \tag{11}$$

It is easy to check that for $t \geq t_1$ the following holds:

$$\begin{aligned}
&0 \leq \tau_0(t) \leq \bar{\tau} = h + \eta_M,\\
&r_0 + r_1 \leq \tau_1(t) \leq \tau_2(t) \leq \tau_M = r_0 + r_1 + h + \eta_M + \mu_M.
\end{aligned}$$

To avoid some technical complications, we assume that $\bar{\tau} \leq r_0+r_1$. The control law (10) implies $u(t+r_0) = K\hat{z}(s_k)$ for $t \in [\xi_k-r_0, \xi_{k+1}-r_0)$. Therefore, $u(t+r_0) = K\hat{z}(t-\tau_0(t))$. Similarly, $u(t - r_1) = K\hat{z}(t - \tau_1(t))$. Define the event-triggering error $e(t)$ by 0 for $t < t_1$ and

$$e(t) = \bar{u}(\xi_k) - u(\xi_k), \quad t \in [t_k, t_{k+1}).$$

Then for $t \in [t_k, t_{k+1})$ we have

$$\bar{u}(\xi_k) = u(\xi_k) + e(t) = K\hat{z}(t - \tau_2(t)) + e(t)$$

and the event-triggering (4), (5) for $t \geq t_1$ implies

$$0 \leq \sigma\hat{z}^T(t - \tau_2(t))K^T\Omega K\hat{z}(t - \tau_2(t)) - e^T(t)\Omega e(t). \tag{12}$$

For $t \in [s_k, s_{k+1})$ predictors (8) imply

$$\begin{aligned}
y(t) - \hat{y}(t) &= C[x(s_k) - \hat{x}(s_k)] + w_2(s_k)\\
&= Ce^{-A(r_0+r_1)}[z(s_k) - \hat{z}(s_k)] + w_2(s_k).
\end{aligned}$$

Using the notation $\delta_z(t) = z(t) - \hat{z}(t)$, we obtain

$$\begin{aligned}
\dot{\hat{z}}(t) =& A\hat{z}(t) + BK\hat{z}(t - \tau_0(t)) - D\delta_z(s_k)\\
&-e^{A(r_0+r_1)}Lw_2(s_k), \quad t \in [s_k, s_{k+1}) \cap [t_1, \infty)\\
\dot{\delta}_z(t) =& A\delta_z(t) + D\delta_z(s_k) + e^{A(r_0+r_1)}Be(t)\\
&+ e^{A(r_0+r_1)}BK[\hat{z}(t - \tau_2(t)) - \hat{z}(t - \tau_1(t))]\\
&+ e^{A(r_0+r_1)}[w_1(t) - Lw_2(s_k)],\\
&\qquad\qquad t \in [s_k, s_{k+1}) \cap [t_1, \infty),
\end{aligned} \tag{13}$$

where $D = e^{A(r_0+r_1)}LCe^{-A(r_0+r_1)}$.

Remark 1 *While the state estimate $\hat{z}$ enters* (13) *with different time-delays, the estimation error δ_z has a delay due to sampled-data only (does not undergo additional delays). This allows an efficient analysis via Wirtinger-based Lyapunov-Krasovskii functional (see V_W term in* (A.1)*).*

Theorem 1 *For given event-triggering parameter $\sigma \geq 0$ and decay rate $\alpha > 0$ let there exist $n \times n$ matrices $P_1 > 0$, $P_2 > 0$, $n \times n$ non-negative matrices S, S_0, S_1, R_0, R_1, W, an $m\times m$ matrix $\Omega \geq 0$, and $n\times n$ matrices G_i ($i = 0, \ldots, 3$) such that*

$$\Phi < 0, \quad \begin{bmatrix} R_0 & G_0 \\ * & R_0 \end{bmatrix} \geq 0, \quad \begin{bmatrix} R_1 & G_i \\ * & R_1 \end{bmatrix} \geq 0, \quad i = 1, 2, 3,$$

where $\Phi = \{\Phi_{ij}\}$ is a symmetric matrix composed from

$$\begin{aligned}
&\Phi_{11} = P_1A + A^TP_1 + 2\alpha P_1 + S_0 - \bar{\rho}R_0, \quad \Phi_{13} = \bar{\rho}G_0,\\
&\Phi_{12} = P_1BK + \bar{\rho}(R_0 - G_0), \Phi_{18} = \Phi_{19} = -P_1D, \Phi_{1,11} = A^TH,\\
&\Phi_{23} = \bar{\rho}(R_0 - G_0),\ \Phi_{22} = -\Phi_{23} - \Phi_{23}^T,\ \Phi_{2,11} = (BK)^TH,\\
&\Phi_{33} = \bar{\rho}(S - S_0 - R_0), \Phi_{44} = e^{-2\alpha(r_0+r_1)}(S_1 - S) - \rho_MR_1,\\
&\Phi_{45} = \rho_M(R_1 - G_1),\ \Phi_{46} = \rho_M(G_1 - G_2),\ \Phi_{47} = \rho_MG_2,\\
&\Phi_{55} = -\Phi_{45} - \Phi_{45}^T, \Phi_{57} = \rho_M(G_3 - G_2), \Phi_{56} = \Phi_{45} - \Phi_{57},\\
&\Phi_{67} = \rho_M(R_1 - G_3), \quad \Phi_{68} = -\Phi_{58} = (\rho_ABK)^TP_2,\\
&\Phi_{6,12} = -\Phi_{5,12} = he^{\alpha h}(\rho_ABK)^TW,\ \Phi_{77} = -\rho_M(S_1 + R_1),\\
&\Phi_{66} = -\Phi_{67} - \Phi_{67}^T + \sigma K^T\Omega K, \quad \Phi_{8,11} = \Phi_{9,11} = -D^TH,\\
&\Phi_{88} = P_2(A + D) + (A + D)^TP_2 + 2\alpha P_2, \quad \Phi_{89} = P_2D,
\end{aligned}$$

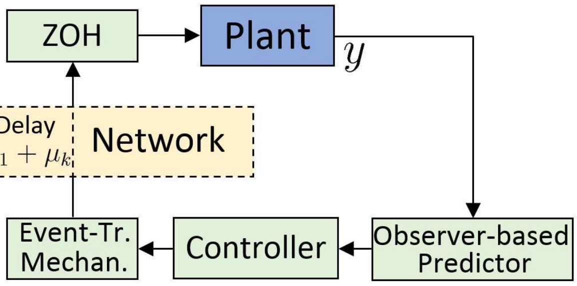

Fig. 3. NCS with a controller-to-actuators network

$$\Phi_{8,10}=P_2\rho_A B, \Phi_{8,12}=he^{\alpha h}(A+D)^T W, \Phi_{99}=-\pi^2 W/4,$$
$$\Phi_{9,12}=he^{\alpha h}D^T W, \Phi_{10,10}=-\Omega, \Phi_{10,12}=he^{\alpha h}(\rho_A B)^T W,$$
$$\Phi_{11,11}=-H, \qquad \Phi_{12,12}=-W$$

with $H = \bar\tau^2 R_0 + (\tau_M - r_0 - r_1)^2 R_1$, $\rho_A = e^{A(r_0+r_1)}$, $\bar\rho = e^{-2\alpha\bar\tau}$, $\rho_M = e^{-2\alpha\tau_M}$. *Then the system* (6), (7) *under the predictor-based event-triggered controller* (4), (5), (10) *is input-to-state stable with the decay rate* α, *i.e.*

$$\begin{aligned}\exists M\colon |x(t)| &\le Me^{-\alpha t}|x(0)| + M \sup_{s\in[0,t]} |w(t)|,\\ |\hat x(t)| &\le Me^{-\alpha t}|x(0)| + M \sup_{s\in[0,t]} |w(t)|,\end{aligned} \tag{14}$$

where $w(t) = \mathrm{col}\{w_1(t), w_2(s_k)\}$ *for* $t \in [s_k, s_{k+1})$.

Proof is given in Appendix A.

Remark 2 *The proposed approach can be easily extended to cope with packet dropouts with bounded number of consecutive packet losses. Consider an unreliable network with the maximum number of consecutive packet losses* d^{sc} *(from a sensor to a controller) and* d^{ca} *(from the controller to an actuator). To cope with this issue, we set the sensor to send the measurement* $y(s_k)$ $d^{sc}+1$ *times at time instants* $s_k + ih_d/d^{sc}$, *where* $i = 0,\dots,d^{sc}$, $h_d > 0$. *The same strategy is applied to the data sent from the controller. Denote by* r_k^{sc} *and* r_k^{ca} *network delays that correspond to the first successfully sent packets. Then the closed-loop system is given by* (13) *with*

$$\eta_k = (d_k^{sc}/d^{sc} + d_k^{ca}/d^{ca})h_d + r_k^{sc} + r_k^{ca} \le \eta_M,$$

where d_k^{sc} *and* d_k^{ca} *are the actual numbers of consecutive packets that were lost. If* $r_k^{sc} + r_k^{ca} < \eta_M$, *one can choose* $h_d > 0$ *such that* $\eta_k \le \eta_M$ *and apply the results of this section. This approach can be improved by introducing the acknowledgement signal of successful reception as suggested in [9].*

3 NCSs with a controller-to-actuators network

In this section, we consider a system with a controller-to-actuator network and continuously available measurements (see Fig. 3). Based on the available measurements, a controller continuously calculates a control signal and transmits it at the sampling times ξ_k. To obtain appropriate sequence of ξ_k, we use a switching approach to event-triggered control [21] that takes advantage of continuously available measurements. Namely, we take $\xi_1 = 0$ and

$$\begin{aligned}\xi_{k+1} = \min\{\xi \ge \xi_k + h \,|\, (u(\xi_k) - u(\xi))^T\Omega(u(\xi_k) - u(\xi))\\ \ge \sigma u^T(\xi)\Omega u(\xi)\},\end{aligned} \tag{15}$$

with event-triggering parameters $0 \le \Omega \in \mathbb{R}^{m\times m}$, $\sigma \in [0,1)$, $h > 0$. According to (15), after the controller sends out the control signal $u(\xi_k)$, it waits for at least the time period h. Then it starts to continuously check the event-triggering rule and sends the next control signal when the event-triggering condition is violated. The idea of a switching approach to event-triggered control is to present the closed-loop system as a switching between a system with sampling h and a system with continuous event-triggering mechanism. This ensures large inter-event times and reduces the number of sent signals [21].

The system takes the form

$$\begin{aligned}\dot x(t) &= Ax(t) + w_1(t), & t &\in [0, t_1),\\ \dot x(t) &= Ax(t) + Bu(\xi_k) + w_1(t), & t &\in [t_k, t_{k+1}),\\ y(t) &= Cx(t) + w_2(t).\end{aligned} \tag{16}$$

Recall that $t_k = \xi_k + r_1 + \mu_k$ are the actuators updating times. We take the observer (7) with continuously changing $u(t)$, $y(t)$, and $\hat y(t) = C\hat x(t)$. Performing the change of variable (8) (with $r_0 = 0$), we obtain the system (9) with $\bar u(\xi_k) = u(\xi_k)$. As the nominal control law we take

$$u(t) = K\hat z(t) = K[e^{Ar_1}\hat x(t) + \textstyle\int_{s_k-r_1}^{s_k} e^{A(s_k-\theta)}Bu(\theta)\,d\theta] \tag{17}$$

for $t \ge 0$ and $u(t) = 0$ for $t < 0$. Since $u(\theta)$ enters the integral term in (17), one needs to continuously calculate $u(\theta)$ and $\hat x(t)$. Therefore, the implementation of (17) is more complicated than that of (10) with a piecewise constant $u(\theta)$ [19]. On the other hand, (7) with continuously changing $u(t)$, $y(t)$, $\hat y(t)$ gives a better estimate of the state $x(t)$ and, as a results, allows to transmit less control signals (see Section 5 for details). Further analysis is based on a switching approach to event-triggered control [21]. Define

$$\begin{aligned} t_k^* &= \min\{t_k + h, t_{k+1}\},\\ \tau_3(t) &= t - \xi_k - r_1, \quad t \in [t_k, t_k^*),\\ \mu(t) &= \mu_k + (t - t_k^*)\frac{\mu_{k+1} - \mu_k}{t_{k+1} - t_k^*}, \quad t \in [t_k^*, t_{k+1}),\\ e_1(t) &= u(\xi_k) - u(t - r_1 - \mu(t)), \quad t \in [t_k^*, t_{k+1}),\end{aligned}$$

where $0 \le \tau_3(t) \le \bar\tau_3 = h + \mu_M$. The function $\mu(t)$ is chosen so that $t - r_1 - \mu(t) \in [\xi_k + h, \xi_{k+1})$ for $t \in [t_k^*, t_{k+1})$, therefore, (15) implies

$$0 \le \sigma\hat z^T(t - r_1 - \mu(t))K^T\Omega K\hat z(t - r_1 - \mu(t)) - e_1^T(t)\Omega e_1(t) \tag{18}$$

for $t \in [t_k^*, t_{k+1})$. Using the time-delay approach described in the previous section and denoting $\delta_z(t) = z(t) - \hat z(t)$, we obtain

$$\begin{aligned}\dot{\hat z}(t) =&(A+BK)\hat z(t)-D\delta(t)-e^{Ar_1}Lw_2(t),\ t\ge 0,\\ \dot{\delta_z}(t) =&(A+D)\delta_z(t)-e^{Ar_1}BK\hat z(t-r_1)\\ &+e^{Ar_1}[w_1(t)-Lw_2(t)],\ t\in[0,t_1),\\ \dot{\delta_z}(t) =&(A+D)\delta_z(t)+e^{Ar_1}[w_1(t)-Lw_2(t)]\\ &+e^{Ar_1}BK[\hat z(t-r_1-\tau_3(t))-\hat z(t-r_1)], t\in[t_k,t_k^*),\\ \dot{\delta_z}(t) =&(A+D)\delta_z(t)+e^{Ar_1}Be_1(t)\\ &+e^{Ar_1}BK[\hat z(t-r_1-\mu(t))-\hat z(t-r_1)]\\ &+e^{Ar_1}[w_1(t)-Lw_2(t)],\ t\in[t_k^*,t_{k+1}),\end{aligned} \tag{19}$$

with $D=e^{Ar_1}LCe^{-Ar_1}$.

Theorem 2 *For given event-triggering parameter $\sigma\ge 0$ and decay rate $\alpha>0$ let there exist $n\times n$ matrices $P_1>0$, $P_2>0$, $n\times n$ non-negative matrices S, S_0, S_1, R_0, R_1, an $m\times m$ matrix $\Omega\ge 0$, and $n\times n$ matrices G_0, G_1 such that*

$$\Xi\le 0,\quad \Psi\le 0,\quad \begin{bmatrix} R_0 & G_0\\ * & R_0\end{bmatrix}\ge 0,\quad \begin{bmatrix} R_1 & G_1\\ * & R_1\end{bmatrix}\ge 0,$$

where $\Xi=\{\Xi_{ij}\}$, $\Psi=\{\Psi_{ij}\}$ are symmetric matrices composed from

$$\begin{aligned}&\Xi_{11}=\Psi_{11}=P_2(A+D)+(A+D)^TP_2+2\alpha P_2,\\ &\Xi_{12}=\Psi_{12}=-D^TP_1, \Xi_{15}=-\Xi_{13}=\Psi_{14}=-\Psi_{13}=P_2\rho_ABK,\\ &\Xi_{22}=\Psi_{22}=P_1(A+BK)+(A+BK)^TP_1+2\alpha P_1+S,\\ &\Xi_{27}=\Psi_{28}=(A+BK)^TH, \Xi_{17}=\Psi_{18}=-D^TH, \Xi_{34}=\rho_MR_0,\\ &\Xi_{33}=\Psi_{33}=e^{-2\alpha r_1}(S_0-S)-\rho_MR_0, \Xi_{45}=\Xi_{56}=\bar\rho(R_1-G_1),\\ &\Xi_{44}=\Psi_{55}=\rho_M(S_1-S_0-R_0)-\bar\rho R_1,\ \Xi_{55}=-\Xi_{45}-\Xi_{45}^T,\\ &\Xi_{46}=\bar\rho G_1,\quad \Xi_{66}=-\bar\rho(S_1+R_1),\quad \Xi_{77}=\Psi_{88}=-H,\\ &\Psi_{17}=P_2\rho_AB,\ \Psi_{35}=\rho_MG_0,\ \Psi_{34}=\Psi_{45}=\rho_M(R_0-G_0),\\ &\Psi_{44}=-\Psi_{34}-\Psi_{34}^T+\sigma K^T\Omega K,\quad \Psi_{56}=\bar\rho R_1,\\ &\Psi_{66}=-\bar\rho(S_1+R_1),\quad \Psi_{77}=-\Omega\end{aligned}$$

with $H=\mu_M^2R_0+h^2R_1$, $\rho_A=e^{Ar_1}$, $\rho_M=e^{-2\alpha(r_1+\mu_M)}$, $\bar\rho=e^{-2\alpha(r_1+\bar\tau_3)}$. Then the system (7), (15)–(17) with $\hat y(t)=C\hat x(t)$ is input-to-state stable with the decay rate α in the sense of (14).

Proof is given in Appendix B.

Remark 3 *Note that the event-triggering rules (5) and (15) guarantee the Zeno behaviour avoidance (that is $\lim_k\xi_k=\infty$). For (5), this follows from the condition $\lim_k s_k=\infty$ and the definition of ξ_k. The event-triggering rule (15) explicitly guarantees that $\xi_{k+1}-\xi_k\ge h>0$.*

4 NCSs with a sensors-to-controller network

In this section, we consider a systems with a continuous control and a sensor-to-controller network (see Fig. 4) with sampling instants $s_k=kh$, $k\in\mathbb{N}$:

$$\begin{aligned}&\dot x(t)=Ax(t)+Bu(t)+w_1(t),\\ &y(t)=Cx(s_k)+w_2(s_k),\quad t\in[s_k,s_{k+1}).\end{aligned} \tag{20}$$

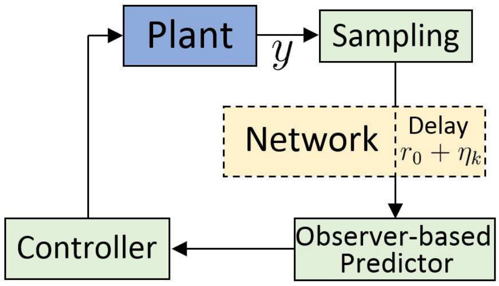

Fig. 4. NCS with a sensors-to-controller network

We use the observer (7) with $r_1=0$. The change of variable (8) for $t\ge 0$ leads to

$$\begin{aligned}&\dot z(t)=Az(t)+Bu(t+r_0)+e^{A(r_0+r_1)}w_1(t),\\ &\dot{\hat z}(t)=A\hat z(t)+Bu(t+r_0)-e^{A(r_0+r_1)}L(y(t)-\hat y(t)).\end{aligned} \tag{21}$$

As one can see from (21), the time delay r_0 is compensated by the predictors (8), therefore, one could intend to use $u(t)=K\hat z(t-r_0)$. However, the value of $\hat z(t-r_0)$ cannot be calculated by the controller for $t\in[\xi_k^*,\xi_{k+1})$, where $\xi_k^*=\min\{\xi_k,s_{k+1}+r_0\}$, since it depends on $y(s_{k+1})$ that is not available to the controller. The latest value of $\hat z$ that can be calculated by the controller for $t\in[\xi_k^*,\xi_{k+1})$ is $\hat z(s_k+h)=\hat z(s_{k+1})$. Therefore, we take $u(t)=\hat z(t-r_0)$ for $t\in[\xi_k,\xi_k^*)$ and $u(t)=\hat z(s_{k+1})$ for $t\in[\xi_k^*,\xi_{k+1})$ or, equivalently, $u(t)=0$ for $t<\xi_1$ and

$$u(t)=K\hat z(t-r_0-\eta(t)), \tag{22}$$

where

$$\eta(t)=\begin{cases}0, & t\in[\xi_k,\xi_k^*),\\ t-s_{k+1}-r_0, & t\in[\xi_k^*,\xi_{k+1}).\end{cases}$$

Note that $\eta(t)\le\eta_M$. Using the time-delay approach described in Section 2 and denoting $\delta_z(t)=z(t)-\hat z(t)$ for $t\in[s_k,s_{k+1})$, we obtain

$$\begin{aligned}\dot{\hat z}(t)&=A\hat z(t)+BK\hat z(t-\eta(t))-D\delta_z(s_k)\\ &\qquad -e^{A(r_0+r_1)}Lw_2(s_k),\\ \dot{\delta_z}(t)&=A\delta_z(t)+D\delta_z(s_k)+e^{A(r_0+r_1)}[w_1(t)-Lw_2(s_k)],\end{aligned}$$

where $D=e^{A(r_0+r_1)}LCe^{-A(r_0+r_1)}$.

Theorem 3 *For a given decay rate $\alpha>0$ let there exist $n\times n$ matrices $P_1>0$, $P_2>0$, $n\times n$ non-negative matrices S, R, W, and $n\times n$ matrix G such that*

$$N\le 0,\quad \begin{bmatrix} R & G\\ * & R\end{bmatrix}\ge 0,$$

where $N=\{N_{ij}\}$ is a symmetric matrix composed from

$$\begin{aligned}&N_{11}=P_1A+A^TP_1+2\alpha P_1+S-\rho_MR, N_{12}=N_{15}=-P_1D,\\ &N_{13}=P_1BK+\rho_M(R-G), N_{14}=\rho_MG, N_{16}=\eta_MA^TR,\\ &N_{22}=P_2(A+D)+(A+D)^TP_2+2\alpha P_2,\quad N_{25}=P_2D,\\ &N_{26}=N_{56}=-\eta_MD^TR,\quad N_{27}=he^{\alpha h}(A+D)^TW,\\ &N_{34}=\rho_M(R-G), N_{33}=-N_{34}-N_{34}^T, N_{36}=\eta_M(BK)^TR,\\ &N_{44}=-\rho_M(S+R), N_{55}=-\pi^2W/4, N_{56}=-\eta_MD^TR,\\ &N_{57}=he^{\alpha h}D^TW,\quad N_{66}=-R,\quad N_{77}=-W\end{aligned}$$

	$r_0 = 0.1$, $\eta_M = 0.005$			$r_0 = \eta_M = 0$		
	σ	h	SCS	σ	h	SCS
Sampled predictor (10)	0	0.044	228	0	0.057	176
Sampled event-triggering (4), (5), (10)	0.01	0.039	147.7	0.01	0.052	124.6
Continuous predictor (17)	—	—	—	0	0.088	114
Switching event-triggering (15), (17)	—	—	—	0.01	0.088	97.6

Table 1
Average numbers of sent control signals (SCS) for different control strategies ($r_1 = 0.1$, $\mu_M = 0.005$, $\alpha = 0.001$, $\|w_1(t)\| \le 10^{-3}$, $\|w_2(t)\| \le 10^{-3}$)

with $\rho_M = e^{-2\alpha\eta_M}$. *Then the system* (7), (8), (20), (22) *is input-to-state stable with the decay rate* α *in the sense of* (14).

Proof is similar to that of Theorem 1 and, therefore, is omitted here.

Remark 4 *MATLAB codes for solving the LMIs of Theorems 1–3 are available at* ***https://github.com/AntonSelivanov/Aut16***

5 Example: an inverted pendulum on a cart

Consider an inverted pendulum on a moving cart [24] described by (2) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -mg/M & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & g/l & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1/M \\ 0 \\ -1/(Ml) \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and $x(0) = [0.98, 0, 0.2, 0]^T$, where x_1 is the cart's position, x_2 is the cart's speed, x_3 is the pendulum's bob angle with respect to the vertical, x_4 is its speed, $M = 10$ is the cart mass, $m = 1$ is the pendulum mass, $l = 3$ is the length of the pendulum arm, and $g = 10$ is the gravitational acceleration. For

$$K = \begin{bmatrix} 2 & 12 & 378 & 210 \end{bmatrix}, \quad L = \begin{bmatrix} -11.7 & 1.2 \\ -37 & 8.9 \\ 1.2 & -11 \\ 7.9 & -36 \end{bmatrix}$$

the matrices $A + BK$ and $A + LC$ are Hurwitz. Below we compare different control strategies proposed in this paper.

First, consider a system with two networks (sensors-to-controller and controller-to-actuators). According to numerical simulations, the system (6), (7) under the control input $\bar{u}(\xi_k) = K\hat{x}(s_k)$ (without a predictor and event-triggering) is not stable for $r_0 = r_1 = 0.1$, $h = 0.035$, and $\eta_M = \mu_M = 0$. If $\sigma = 0$ (no event-triggering), the conditions of Theorem 1 are satisfied for the same h and larger $r_0 = r_1 = 0.17$, $\eta_M = \mu_M = 0.005$. That is, the predictor-based controller (10) admits larger network delays.

For $r_0 = r_1 = 0.1$, $\eta_M = \mu_M = 0.005$, $\alpha = 0.001$, $\sigma = 0$ Theorem 1 gives the maximum sampling period $h = 0.044$. This implies that, without event-triggering, within 10 seconds of simulations $\lfloor 10/h \rfloor + 1 = 228$ control signals are sent in the system (6), (7) under the predictor-based controller (10) ($\lfloor \cdot \rfloor$ stands for the integer part). For the event-triggered controller (4), (5), (10) with $\sigma = 0.01$ Theorem 1 gives $h = 0.039$. To obtain the number of sent control signals under the event-triggering, we perform 10 numerical simulations with random i.i.d. η_k and μ_k satisfying (3) and $w_1(t)$, $w_2(t)$ satisfying $\|w_1(t)\| \le 10^{-3}$, $\|w_2(t)\| \le 10^{-3}$. The results are given in Table 1. As one can see, event-triggering allows to reduce the workload of the controller-to-actuators network by more than 35%. Note that for event-triggered control ($\sigma > 0$) the sampling period h is smaller than for periodic control. That is, by introducing the event-triggering mechanism, we reduce the number of sent control signals but increase the number of sent measurements. However, the total number of signals sent through both sensors-to-controller and controller-to-actuators networks is reduced by more than 10%.

Now we consider a system with a controller-to-actuators network and continuous measurements ($r_0 = \eta_M = 0$). For this case, one can apply the sampled predictor-based controller (10) or the sampled event-triggered controller (4), (5), (10) (with $s_k = \xi_k$). The sampled approach simplifies the calculation of the integral term in (8) but does not take advantage of continuously available measurements. Indeed, as one can see from Table 1, the continuous predictor (17) without event-triggering ($\xi_k = kh$) reduces the network workload compared to the sampled predictor (10) by more than 35%.

To compare the sampled event-triggering mechanism (4), (5), (10) and the switching event-triggering (15), (17) for $\alpha = 0.001$ and $\sigma = 0.01$ we apply Theorems 1 and 2 to find the maximum allowable h. Then we perform 10 numerical simulations with random i.i.d. μ_k subject to (3) ($r_1 = 0.1$, $\mu_M = 0.005$). In Table 1 one can see that the switching event-triggering reduces the number of sent control signals by more than 20% compared to the sampled event-triggering and by almost 15% compared to the continuous predictor without event-triggering. The total numbers of sent measurements are reduced by 33% and 7%, respectively.

Finally, consider the system (20) with only sensors-to-controller network ($s_k = kh$). For the continuous controller (22) with the observer (7) Theorem 3 gives $h = 0.124$. For the sampled controller (10) with the observer (7) Theorem 1 gives $h = 0.056$. That is, by using the continuous controller, one can significantly reduce the number of required measurements $y(kh)$.

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A Proof of Theorem 1

For $t \geq t_1$ consider the functional

$$V = V_1 + V_2 + V_{S_0} + V_{R_0} + V_S + V_{S_1} + V_{R_1} + V_W,$$

where

$$\begin{aligned}
&V_1 = \hat{z}^T(t) P_1 \hat{z}(t), \quad V_2 = \delta_z^T(t) P_2 \delta_z(t),\\
&V_{S_0} = \int_{t-\bar{\tau}}^{t} e^{2\alpha(s-t)} \hat{z}^T(s) S_0 \hat{z}(s)\, ds,\\
&V_{R_0} = \bar{\tau} \int_{-\bar{\tau}}^{0} \int_{t+\theta}^{t} e^{2\alpha(s-t)} \dot{\hat{z}}^T(s) R_0 \dot{\hat{z}}(s)\, ds\, d\theta,\\
&V_S = \int_{t-r_0-r_1}^{t-\bar{\tau}} e^{2\alpha(s-t)} \hat{z}^T(s) S \hat{z}(s)\, ds,\\
&V_{S_1} = \int_{t-\tau_M}^{t-r_0-r_1} e^{2\alpha(s-t)} \hat{z}^T(s) S_1 \hat{z}(s)\, ds,\\
&V_{R_1} = (\tau_M - r_0 - r_1)\times\\
&\qquad \int_{-\tau_M}^{-r_0-r_1} \int_{t+\theta}^{t} e^{2\alpha(s-t)} \dot{\hat{z}}^T(s) R_1 \dot{\hat{z}}(s)\, ds\, d\theta,\\
&V_W = h^2 e^{2\alpha h} \int_{s_k}^{t} e^{2\alpha(s-t)} \dot{\delta}_z^T(s) W \dot{\delta}_z(s)\, ds\\
&\qquad -\tfrac{\pi^2}{4} \int_{s_k}^{t} e^{2\alpha(s-t)} v^T(s) W v(s)\, ds,\ t \in [s_k, s_{k+1})
\end{aligned} \tag{A.1}$$

with $v(t) = \delta_z(s_k) - \delta_z(t)$ for $t \in [s_k, s_{k+1})$. Wirtinger-based term V_W is non-negative due to Lemma 1, therefore, V is positive-definite. Due to V_W, the functional V has finite jumps at $t = s_k$, but since $V_W = 0$ for $t = s_k$, $V(s_k - 0) \geq V(s_k)$.

Jensen's inequality [8] and Park's theorem [20] lead to

$$\begin{aligned}
&\dot{V}_{R_0} + 2\alpha V_{R_0} = \bar{\tau}^2 \dot{\hat{z}}^T(t) R_0 \dot{\hat{z}}(t)\\
&-\bar{\tau} \int_{t-\bar{\tau}}^{t} e^{2\alpha(s-t)} \dot{\hat{z}}^T(s) R_0 \dot{\hat{z}}(s)\, ds \leq \bar{\tau}^2 \dot{\hat{z}}^T(t) R_0 \dot{\hat{z}}(t)\\
&-e^{-2\alpha\bar{\tau}} \begin{bmatrix} \hat{z}(t)-\hat{z}(t-\tau_0(t)) \\ \hat{z}(t-\tau_0(t))-\hat{z}(t-\bar{\tau}) \end{bmatrix}^T \begin{bmatrix} R_0 & G_0 \\ G_0^T & R_0 \end{bmatrix} \begin{bmatrix} \hat{z}(t)-\hat{z}(t-\tau_0(t)) \\ \hat{z}(t-\tau_0(t))-\hat{z}(t-\bar{\tau}) \end{bmatrix},
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
&\dot{V}_{R_1} + 2\alpha V_{R_1} = (\tau_M - r_0 - r_1)^2 \dot{\hat{z}}^T(t) R_1 \dot{\hat{z}}(t)\\
&-(\tau_M - r_0 - r_1) \int_{t-\tau_M}^{t-r_0-r_1} e^{2\alpha(s-t)} \dot{\hat{z}}^T(s) R_1 \dot{\hat{z}}(s)\, ds\\
&\leq (\tau_M - r_0 - r_1)^2 \dot{\hat{z}}^T(t) R_1 \dot{\hat{z}}(t) - e^{-2\alpha\tau_M} \times\\
&\begin{bmatrix} \hat{z}(t-r_0-r_1)-\hat{z}(t-\tau_1(t)) \\ \hat{z}(t-\tau_1(t))-\hat{z}(t-\tau_2(t)) \\ \hat{z}(t-\tau_2(t))-\hat{z}(t-\tau_M) \end{bmatrix}^T \begin{bmatrix} R_1 & G_1 & G_2 \\ * & R_1 & G_3 \\ * & * & R_1 \end{bmatrix} \begin{bmatrix} \hat{z}(t-r_0-r_1)-\hat{z}(t-\tau_1(t)) \\ \hat{z}(t-\tau_1(t))-\hat{z}(t-\tau_2(t)) \\ \hat{z}(t-\tau_2(t))-\hat{z}(t-\tau_M) \end{bmatrix}.
\end{aligned} \tag{A.3}$$

By calculating $\dot{V}$ and adding (12), in view of (A.2), (A.3), we obtain

$$\begin{aligned}
\dot{V} + 2\alpha V - \beta \leq{}& \varphi^T(t) \overline{\Phi} \varphi(t) + \varphi^T(t) \Psi \psi(t)\\
&+ \dot{\hat{z}}^T(t) H \dot{\hat{z}} + e^{2\alpha h} h^2 \dot{\delta}_z^T(t) W \dot{\delta}_z(t) - \beta,
\end{aligned}$$

where $\varphi(t) = \mathrm{col}\{\hat{z}(t), \hat{z}(t-\tau_0(t)), \hat{z}(t-\bar{\tau}), \hat{z}(t-r_0-r_1), \hat{z}(t-\tau_1(t)), \hat{z}(t-\tau_2(t)), \hat{z}(t-\tau_M), \delta_z(t), v(t), e(t)\}$, $\psi(t) = \mathrm{col}\{e^{A(r_0+r_1)}w_1(t), e^{A(r_0+r_1)}Lw_2(s_k)\}$, $\overline{\Phi}$ is obtained from Φ by taking away the last two block-columns and block-rows, and Ψ is $(9n+m)\times 2n$ matrix. By taking

$$\beta = \beta_w \sup_{s\in[0,t]} |e^{A(r_0+r_1)}w_1(t)|^2 + \beta_w \sup_{s\in[0,t]} |e^{A(r_0+r_1)}Lw_2(t)|^2,$$

substituting $\dot{\hat{z}}(t)$, $\dot{\delta}_z(t)$ and applying Schur's complement formula, we obtain that if

$$\begin{bmatrix} \Phi & \Psi' \\ * & -\beta_w I_{2n} \end{bmatrix} \le 0, \tag{A.4}$$

where Ψ' is $(11n+m)\times 2n$ matrix, then $\dot{V}(t) \le -2\alpha V(t) + \beta$. Since $\Phi < 0$, the relation (A.4) is true for large enough β_w. Therefore,

$$V(t) \le e^{-2\alpha(t-t_1)}V(t_1) + \frac{\beta}{2\alpha}, \quad t \ge t_1.$$

Since $z(t) = \hat{z}(t) + \delta_z(t)$ and the initial time interval does not influence exponential decay rate analysis [16], the latter implies

$$\begin{aligned} |\hat{z}(t)| &\le C_1(e^{-\alpha t}|z(0)| + \sup_{s\in[0,t]} |w(t)|), \\ |z(t)| &\le C_1(e^{-\alpha t}|z(0)| + \sup_{s\in[0,t]} |w(t)|), \end{aligned} \quad t \ge 0$$

for some $C_1 > 0$. From (8), we have

$$\begin{aligned} &x(t) = e^{-A(r_0+r_1)}z(t) \\ &- \int_{t-r_1}^{t+r_0} e^{A(t-r_1-\theta)}BK\hat{z}(\theta - r_0 - \tau_0(\theta - r_0))\,d\theta, \quad t \ge 0, \end{aligned}$$

therefore, there exists $C_2 > 0$ such that

$$\begin{aligned} |x(t)| &\le C_2(e^{-\alpha t}|z(0)| + \sup_{s\in[0,t]} |w(t)|) \\ &\le C_2 e^{-\alpha t}\left\|e^{A(r_0+r_1)}\right\| |x(0)| + C_2 \sup_{s\in[0,t]} |w(t)|. \end{aligned}$$

Similarly, $|\hat{x}(t)| \le Me^{-\alpha t}|x(0)| + M\sup_{s\in[0,t]} |w(t)|$.

B Proof of Theorem 2

For $t \ge \xi_1$ consider the functional

$$V = V_1 + V_2 + V_S + V_{S_0} + V_{R_0} + V_{S_1} + V_{R_1},$$

where V_1, V_2 are given in (A.1) and

$$\begin{aligned} V_S &= \int_{t-r_1}^{t} e^{2\alpha(s-t)}\hat{z}^T(s)S\hat{z}(s)\,ds, \\ V_{S_0} &= \int_{t-r_1-\mu_M}^{t-r_1} e^{2\alpha(s-t)}\hat{z}^T(s)S_0\hat{z}(s)\,ds, \\ V_{R_0} &= \mu_M \int_{-r_1-\mu_M}^{-r_1}\int_{t+\theta}^{t} e^{2\alpha(s-t)}\dot{\hat{z}}^T(s)R_0\dot{\hat{z}}(s)\,ds\,d\theta, \\ V_{S_1} &= \int_{t-r_1-\bar{\tau}_3}^{t-r_1-\mu_M} e^{2\alpha(s-t)}\hat{z}^T(s)S_1\hat{z}(s)\,ds, \\ V_{R_1} &= (\bar{\tau}_3 - \mu_M)\times \\ &\quad \int_{-r_1-\bar{\tau}_3}^{-r_1-\mu_M}\int_{t+\theta}^{t} e^{2\alpha(s-t)}\dot{\hat{z}}^T(s)R_1\dot{\hat{z}}(s)\,ds\,d\theta. \end{aligned}$$

We have

$$\begin{aligned} \dot{V}_{R_0} + 2\alpha V_{R_0} &= \mu_M^2\dot{\hat{z}}^T(t)R_0\dot{\hat{z}}(t) \\ &\quad -\mu_M\int_{t-r_1-\mu_M}^{t-r_1} e^{2\alpha(s-t)}\dot{\hat{z}}^T(s)R_0\dot{\hat{z}}(s)\,ds, \\ \dot{V}_{R_1} + 2\alpha V_{R_1} &= h^2\dot{\hat{z}}^T(t)R_1\dot{\hat{z}}(t) \\ &\quad -h\int_{t-r_1-\bar{\tau}_3}^{t-r_1-\mu_M} e^{2\alpha(s-t)}\dot{\hat{z}}^T(s)R_1\dot{\hat{z}}(s)\,ds. \end{aligned}$$

For $t \in [t_k, t_k^*)$, $\tau_3(t) \in [\mu_M, \bar{\tau}_3]$ Jensen's inequality and Park's theorem imply

$$\begin{aligned} &-\mu_M\int_{t-r_1-\mu_M}^{t-r_1} e^{2\alpha(s-t)}\dot{\hat{z}}^T(s)R_0\dot{\hat{z}}(s)\,ds \le -e^{-2\alpha(r_1+\mu_M)}\times \\ &[\hat{z}(t-r_1)-\hat{z}(t-r_1-\mu_M)]^T R_0[\hat{z}(t-r_1)-\hat{z}(t-r_1-\mu_M)], \end{aligned} \tag{B.1}$$

$$\begin{aligned} &-h\int_{t-r_1-\bar{\tau}_3}^{t-r_1-\mu_M} e^{2\alpha(s-t)}\dot{\hat{z}}^T(s)R_1\dot{\hat{z}}(s)\,ds \le -e^{-2\alpha(r_1+\bar{\tau}_3)}\times \\ &\begin{bmatrix} \hat{z}(t-r_1-\mu_M)-\hat{z}(t-r_1-\tau_3(t)) \\ \hat{z}(t-r_1-\tau_3(t))-\hat{z}(t-r_1-\bar{\tau}_3) \end{bmatrix}^T \begin{bmatrix} R_1 & G_1 \\ G_1^T & R_1 \end{bmatrix} \begin{bmatrix} \hat{z}(t-r_1-\mu_M)-\hat{z}(t-r_1-\tau_3(t)) \\ \hat{z}(t-r_1-\tau_3(t))-\hat{z}(t-r_1-\bar{\tau}_3) \end{bmatrix}. \end{aligned} \tag{B.2}$$

Calculating $\dot{V}$ for $t \in [t_k, t_k^*)$, $\tau_3(t) \in [\mu_M, \bar{\tau}_3]$ in view of (B.1), (B.2), we obtain

$$\dot{V} + 2\alpha V - \beta \le \xi^T(t)\overline{\Xi}\xi(t) + \xi^T(t)\Phi\phi^T(t) + \dot{\hat{z}}^T(t)H\dot{\hat{z}}(t) - \beta,$$

where $\xi(t) = \mathrm{col}\{\delta_z(t), \hat{z}(t), \hat{z}(t-r_1), \hat{z}(t-r_1-\mu_M), \hat{z}(t-r_1-\tau_3(t)), \hat{z}(t-r_1-\bar{\tau}_3)\}$, $\phi(t) = \mathrm{col}\{e^{A(r_0+r_1)}w_1(t), e^{A(r_0+r_1)}Lw_2(s_k)\}$, $\overline{\Xi}$ is obtained from Ξ by taking away the last block-column and block-row, and Φ is $6n \times 2n$ matrix. By taking

$$\beta = \beta_w \sup_{s\in[0,t]} |e^{A(r_0+r_1)}w_1(t)|^2 + \beta_w \sup_{s\in[0,t]} |e^{A(r_0+r_1)}Lw_2(t)|^2,$$

substituting $\dot{\hat{z}}(t)$ and applying Schur's complement formula, we obtain that if

$$\begin{bmatrix} \Xi & \Phi' \\ * & -\beta_w I_{2n} \end{bmatrix} \le 0, \tag{B.3}$$

where Φ' is $7n \times 2n$ matrix, then $\dot{V}(t) \le -2\alpha V(t) + \beta$. Since $\Xi < 0$, the relation (B.3) is true for large enough β_w. Therefore, $\dot{V}(t) \le -2\alpha V(t) + \beta$ for $t \in [t_k, t_k^*)$, $\tau_3(t) \in [\mu_M, \bar{\tau}]$.

For $t \in [t_k^*, t_{k+1})$ Jensen's inequality and Park's theorem imply

$$\begin{aligned} &-\mu_M\int_{t-r_1-\mu_M}^{t-r_1} e^{2\alpha(s-t)}\dot{\hat{z}}^T(s)R_0\dot{\hat{z}}(s)\,ds \le -e^{-2\alpha(r_1+\mu_M)}\times \\ &\begin{bmatrix} \hat{z}(t-r_1)-\hat{z}(t-r_1-\mu(t)) \\ \hat{z}(t-r_1-\mu(t))-\hat{z}(t-r_1-\mu_M) \end{bmatrix}^T \begin{bmatrix} R_0 & G_0 \\ G_0^T & R_0 \end{bmatrix} \begin{bmatrix} \hat{z}(t-r_1)-\hat{z}(t-r_1-\mu(t)) \\ \hat{z}(t-r_1-\mu(t))-\hat{z}(t-r_1-\mu_M) \end{bmatrix}, \end{aligned} \tag{B.4}$$

$$\begin{aligned} &-h\int_{t-r_1-\bar{\tau}_3}^{t-r_1-\mu_M} e^{2\alpha(s-t)}\dot{\hat{z}}^T(s)R_1\dot{\hat{z}}(s)\,ds \le -e^{-2\alpha(r_1+\bar{\tau}_3)}\times \\ &[\hat{z}(t-r_1-\mu_M)-\hat{z}(t-r_1-\bar{\tau}_3)]^T R_1[\hat{z}(t-r_1-\mu_M)-\hat{z}(t-r_1-\bar{\tau}_3)]. \end{aligned} \tag{B.5}$$

Calculating $\dot{V}$ for $t \in [t_k^*, t_{k+1})$ in view of (B.4), (B.5) and adding (18), we obtain

$$\dot{V} + 2\alpha V - \beta \le \psi^T(t)\overline{\Psi}\psi(t) + \psi^T(t)\overline{\Phi}\phi(t) + \dot{\hat{z}}^T(t)H\dot{\hat{z}}(t) - \beta,$$

where $\psi(t) = \mathrm{col}\{\delta_z(t), \hat{z}(t), \hat{z}(t-r_1), \hat{z}(t-r_1-\mu(t)), \hat{z}(t-r_1-\mu_M), \hat{z}(t-r_1-\bar{\tau}), e_1(t)\}$, $\overline{\Psi}$ is obtained from Ψ by taking away the last block-column and block-row, and $\overline{\Phi}$ is $(6n+m)\times 2n$ matrix. Similarly to the previous case, we obtain $\dot{V}(t) \le -2\alpha V(t) + \beta$ for $t \in [t_k^*, t_{k+1})$.

For $t \in [t_k, t_k^*)$, $\tau_3(t) \in [0, \mu_M)$ the system (19) is described by the last line of (19) with $e_1(t) = 0$ satisfying (18), therefore, $\dot{V} \leq -2\alpha V + \beta$ for $t \geq \xi_1$. The end of the proof is similar to that of Theorem 1.