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Sampled-data relay control of semilinear diffusion PDEs

Anton Selivanov and Emilia Fridman

Abstract— **We consider sampled-data relay control of semilinear diffusion PDEs. Several control signals, subject to unknown bounded disturbances, enter the system through shape functions. The only information required for calculating the control signal is the sign of a weighted average of the state. First, for a nonlinearity from an arbitrary sector, we derive LMI-based conditions that determine how many controllers one should use to ensure local convergence to a bounded set. For a fixed domain of initial conditions the size of a limit set is proportional to a sampling period. Then we propose a switching procedure for controllers' gains that ensures convergence from an arbitrary domain to the same limit set.**

I. INTRODUCTION

Networked control systems (NCSs), which are comprised of spatially distributed sensors, actuators, and controllers connected via a communication network, have become widespread due to great advantages they bring: long distance control, low cost, ease of reconfiguration, reduced system wiring, etc [1], [2]. Networked control of distributed parameter systems is applicable to a long distance control of chemical reactors [3] or air polluted areas [4]. Since it is usually problematic to transmit continuous signals through communication networks, measurements sampling is one of the main challenges in NCSs. A variety of methods has been developed to analyze PDEs under a sampled-data control: the discrete-time approach has been studied in [5], [6], the model decomposition techniques have been applied in [7], [8], the time-delay approach has been proposed in [9], [10]. To save computational and communicational resources, event-triggered approach can be used [11]. In [12] event-triggered control of parabolic PDEs with quantized measurements has been considered. In this work we develop a control strategy that is even less demanding to system resources, namely, relay control.

Relay control has undeniable advantages: simple implementation, control saturation/quantization, finite time convergence, full compensation of matched disturbances [13]. However, analysis of a relay control is not a trivial task even for linear systems. In [14] it has been shown that a relay control does not lead to the asymptotic stability of a finite-dimensional linear system in the presence of input delay. In this case ultimate boundedness is achieved with a limit set whose size is proportional to the time-delay bound. In [15] a convex optimization approach has been used to study generalized relays for finite-dimensional systems. In that work sampled measurements were modeled as input delays and the size of the limit set was proportional to a sampling period.

In this work we consider sampled-data relay control of semilinear diffusion PDEs. The control signals are subject to unknown disturbances, enter the system through shape functions, and remain constant within a sampling period. The only information required for calculating the control signal is the sign of a weighted average of the state. First, for a nonlinearity from an arbitrary sector, we derive LMI-based conditions that determine how many controllers one should use to ensure local convergence to a bounded set. For a fixed domain of initial conditions the size of the limit set is proportional to a sampling period. Then we propose a switching procedure for controllers' gains that ensures convergence from an arbitrary domain to the same limit set. The results are demonstrated by an example.

Notations and preliminaries

The partial (weak) derivatives of function $z(x,t)$ are denoted by z_t, z_x, z_{xx}. The symbol $\mathbb{N}_0$ stands for nonnegative integers, $\mathcal{H}^1(0,1)$ is the Sobolev space of absolutely continuous functions with square integrable first derivatives, $\mathcal{H}_0^1(0,1)=\{f\in\mathcal{H}^1(0,1)\colon f(0)=f(1)=0\}$. For a square matrix P the notation $P>0$ indicates that P is symmetric and positive-definite, the symbol $*$ denotes its symmetric elements.

Lemma 1 (Wirtinger inequality [16]): For $a<b$ let $f\in\mathcal{H}^1(a,b)$ be a scalar function such that $f(a)=0$ or $f(b)=0$. Then, for any $\alpha\geq 0$,

$$\int_a^b e^{2\alpha t}f^2(t)\,dt\leq e^{2\alpha(b-a)}\frac{4(b-a)^2}{\pi^2}\int_a^b e^{2\alpha t}\dot f^2(t)\,dt.$$

Moreover, if $z(0)=z(1)=0$ then

$$\int_a^b f^2(t)\,dt\leq\frac{(b-a)^2}{\pi^2}\int_a^b \dot f^2(t)\,dt.$$

Lemma 2 (Poincaré inequality [17]): For $a<b$ let $f\in\mathcal{H}^1(a,b)$ be a scalar function with $\int_a^b f(x)\,dx=0$. Then

$$\int_a^b f^2(x)\,dx\leq\frac{(b-a)^2}{\pi^2}\int_a^b\left[\frac{df}{dx}(x)\right]^2dx.$$

II. SYSTEM DESCRIPTION

Consider a semilinear diffusion PDE

$$z_t(x,t)=z_{xx}(x,t)+\varphi(x,t,z)+\sum_{j=1}^N b_j(x)(u_j(t_k)+w_j(t)),$$
$$x\in[0,1],\quad t\in[t_k,t_{k+1}),\quad k\in\mathbb{N}_0\quad(1)$$

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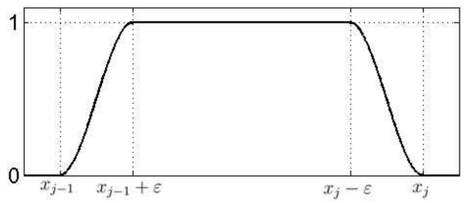

Fig. 1. Example of the shape function $b_j(x)$

with sampling instants $t_0 < t_1 < t_2 < \dots$ and a scalar state $z\colon [0,1]\times[t_0,\infty)\to\mathbb{R}$.

Assumption 1: For all $x \in [0,1]$, $t \ge t_0$, $z \in \mathbb{R}$ the nonlinearity φ satisfies the sector condition

$$(\varphi(x,t,z)-\varphi_m z)(\varphi_M z-\varphi(x,t,z)) \ge 0$$

with some $\varphi_m, \varphi_M \in \mathbb{R}$.

If $\varphi_m < \varphi_M$, Assumption 1 indicates that φ lies in the sector $[\varphi_m, \varphi_M]$ [18].

Assumption 2: The sampling instants satisfy

$$\lim_{k\to\infty} t_k = \infty, \quad 0 < t_{k+1}-t_k \le h, \quad k\in\mathbb{N}_0.$$

Consider the points $x_j = j/N$, $j=0,\dots,N$ that divide $[0,1]$ into N subintervals. The control inputs $u_j(t_k)\in\mathbb{R}$ and the matched disturbances $w_j(t)\in\mathbb{R}$ enter (1) through the shape functions $b_j(x)\in\mathcal{H}^1(0,1)$ such that (see Fig. 1)

$$\begin{cases} b_j(x)=1, & x\in[x_{j-1}+\varepsilon, x_j-\varepsilon],\\ b_j(x)=0, & x\notin[x_{j-1},x_j],\\ b_j(x)\in[0,1], & x\in(x_{j-1},x_{j-1}+\varepsilon)\cup(x_j-\varepsilon,x_j), \end{cases} \tag{2}$$

where $j=1,\dots,N$ and $\varepsilon\in(0,\frac{1}{2N})$ is a parameter. Similar shape functions appear, e.g., in the problem of compressor rotating stall with air injection actuator [19], where $z(x,t)$ denotes the axial flow through the compressor.

Remark 1: The shape functions (2) are chosen to be from $\mathcal{H}^1$ to guarantee well-posedness of the resulting closed-loop system. For $\varepsilon \to 0$ these functions approach piecewise constant functions

$$b_j(x)=\begin{cases}1, & x\in[x_{j-1},x_j),\\ 0, & x\notin[x_{j-1},x_j),\end{cases} \quad j=1,\dots,N.$$

Note that for $\varepsilon=0$, the practical stability conditions of Theorem 1 below are less restrictive.

We consider (1) under the Dirichlet boundary conditions, Neumann boundary conditions, or both:

$$\begin{aligned} &z(0,t)=0 \text{ or } z_x(0,t)=0,\\ &z(1,t)=0 \text{ or } z_x(1,t)=0. \end{aligned} \tag{3}$$

The system (1) may be unstable for large φ_m or φ_M (see [20] for $\varphi(z,x,t)=\varphi_M z$). We study (1) under the control laws

$$u_j(t_k) = -K \operatorname{sign} \int_{x_{j-1}}^{x_j} b_j(x) z(x,t_k)\,dx, \tag{4}$$

where $j=1,\dots,N$, $k\in\mathbb{N}_0$, $K>0$. The implementation of the control laws (4) is very simple. It requires to transmit through a communication network only the signs of different weighted averages of the state $z(x,t_k)$.

Assumption 3: There exists $\rho\in[0,1)$ such that

$$|w_j(t)| \le \rho K, \quad \forall t\ge t_0, \quad j=1,\dots,N.$$

This assumption guarantees that, in the absence of time-sampling, continuous versions of the controllers (4) can compensate the matched disturbances $w_j(t)$. The disturbance-free case corresponds to $\rho=0$.

A. *Well-posedness of* (1)–(4)

For the boundary conditions $z(0,t)=0$, $z_x(1,t)=0$ consider a Hilbert space $X=\{g\in\mathcal{H}^1(0,1)\colon g(0)=0\}$ with a norm $\|\cdot\|_X = \|\cdot\|_{\mathcal{H}^1}$. Denoting $\zeta(t)=z(\cdot,t)\in X$, we rewrite the system (1)–(4) in the form

$$\frac{d}{dt}\zeta(t) = \mathcal{A}\zeta(t) + f(t,\zeta(t)), \quad t\in[t_k,t_{k+1}),\ k\in\mathbb{N}_0, \tag{5}$$

where $\mathcal{A}=\frac{\partial^2}{\partial x^2}$ has a dense in X domain

$$D(\mathcal{A}) = \{g\in\mathcal{H}^2(0,1)\colon g(0)=0, g'(1)=0\}$$

and $f\colon [t_0,\infty)\times X\to X$ is given by

$$f(t,\zeta) = \psi(t,\zeta) + \sum_{j=1}^N b_j(\cdot)(u_j(t_k)+w_j(t)),$$

where $\psi\colon [t_0,\infty)\times X\to X$,

$$(\psi(t,\zeta))(x) = \varphi(x,t,(\zeta(t))(x)).$$

Assumption 4: $\psi\in\mathcal{C}^1([t_k,t_{k+1})\times X\to X)$ and $w_j\in\mathcal{C}^1[t_k,t_{k+1})$ for all $k\in\mathbb{N}_0$, $j=1,\dots,N$.

This assumption guarantees that f is continuously differentiable from any $[t_k,t_{k+1})\times X$ to X. Then, since $\mathcal{A}$ is the infinitesimal generator of a $\mathcal{C}_0$ semigroup, Theorem 1.5 from [21, p.187] guarantees that for $\zeta(t_0)\in D(A)$ the system (5) has a classical solution ζ on $[t_0,\infty)$.

The existence of a classical solution for other boundary conditions (3) can be established in a similar manner with

$$\begin{aligned} &X=\mathcal{H}^1_0(0,1) && \text{if } z(0,t)=z(1,t)=0,\\ &X=\{g\in\mathcal{H}^1(0,1)\colon g(1)=0\} && \text{if } z_x(0,t)=0, z(1,t)=0,\\ &X=\mathcal{H}^1(0,1) && \text{if } z_x(0,t)=z_x(1,t)=0. \end{aligned}$$

Note that the controllers (4) are discontinuous in time. However, the motion along the discontinuity surface is not possible due to sampling. Therefore, there is no need to consider Filippov solutions [22].

III. Stability conditions

For $h>0$, $q\ge 0$, $z(\cdot,t)\in\mathcal{H}^1(0,1)$ define

$$\|z(\cdot,t)\|_q^2 = \int_0^1 z^2(x,t)\,dx + qh\int_0^1 z_x^2(x,t)\,dx.$$

The choice of such a norm is motivated by the Lyapunov-Krasovskii functional (10) used in the proof of the stability conditions.

$$
\Xi = \begin{bmatrix}
\Xi_1 & 1+\lambda_\varphi(\varphi_m+\varphi_M)/2 & M & 0 & hM & 0 & 0 & 0 \\
* & -\lambda_\varphi & 0 & -qh & 0 & 0 & 0 & phe^{\alpha h} \\
* & * & -\lambda_\kappa N^2\pi^2/(1+\nu) & 0 & -hM & 0 & 0 & 0 \\
* & * & * & -2qh & 0 & -qh & -qh & phe^{\alpha h} \\
* & * & * & * & -ph\pi^2/4 & h & h & 0 \\
* & * & * & * & * & -\beta_u h & 0 & phe^{\alpha h} \\
* & * & * & * & * & * & -\beta_w h & phe^{\alpha h} \\
* & * & * & * & * & * & * & -ph
\end{bmatrix}, \tag{6}
$$
$$
\Xi_1 = -2M + 2\lambda_\kappa \varepsilon N^3\pi^2/\nu + 2\alpha - \lambda_\varphi\varphi_m\varphi_M - d\lambda\pi^2/4
$$

The following theorem provides the ultimate boundedness conditions with an ultimate bound C_∞ proportional to the sampling intervals bound h.

Theorem 1: For a given controller gain $K > 0$ consider the system (1), (2) with the boundary conditions (3) and the control laws (4) subject to Assumptions 1–4. For given decay rate $\alpha > 0$ and tuning parameter $\nu > 0$, let there exist nonnegative scalars p, q, M, λ_φ, λ_κ, λ, β_u, and β_w such that

$$
\begin{gathered}
\Xi \le 0, \\
2\alpha qh + \lambda_\kappa + \lambda \le 2, \\
\lambda = 0 \text{ if } z_x(0,t) = z_x(1,t) = 0, \\
d = \begin{cases} 4 \text{ if } z(0,t) = z(1,t) = 0, \\ 1 \text{ otherwise} \end{cases}
\end{gathered} \tag{7}
$$

whith Ξ given in (6). Denote

$$
\begin{aligned}
C_0 &= (1-\rho)^2 \frac{K^2}{NM^2}, \\
C_\infty &= (\beta_u + \beta_w \rho^2)\frac{K^2 h}{2\alpha}.
\end{aligned}
$$

If $C_\infty < C_0$ then for initial conditions $z(\cdot, t_0) \in \mathcal{H}^2(0,1)$ subject to (3), such that

$$
\|z(\cdot,t_0)\|_q^2 < C_0 \tag{8}
$$

a unique classical solution of the system satisfies

$$
\|z(\cdot,t)\|_q^2 \le \|z(\cdot,t_0)\|_q^2 e^{-2\alpha(t-t_0)} + C_\infty. \tag{9}
$$

Remark 2: A MATLAB code for solving the LMI of Theorem 1 is available at `https://github.com/AntonSelivanov/CDC16b`

Proof: Consider the Lyapunov function

$$
V(t) = V_1(t) + V_2(t) + V_W(t), \tag{10}
$$

where

$$
\begin{aligned}
V_1(t) &= \int_0^1 z^2(x,t)\,dx, \\
V_2(t) &= qh \int_0^1 z_x^2(x,t)\,dx, \\
V_W(t) &= phe^{2\alpha h}\int_0^1 \int_{t_k}^t e^{2\alpha(s-t)} z_s^2(x,s)\,ds\,dx \\
&\quad - \frac{\pi^2 ph}{4}\int_0^1\int_{t_k}^t e^{2\alpha(s-t)}\eta^2(x,s)\,ds\,dx, \quad t \in [t_k, t_{k+1}).
\end{aligned}
$$

The functional V_W is nonnegative due to Lemma 1.

We divide the proof into two parts. First, we assume that

$$
MN\left|\int_{x_{j-1}}^{x_j} z(x,t)\,dx\right| \le (1-\rho)K, \quad j = 1,\dots,N \tag{11}
$$

and show that

$$
V(t) \le (V(t_0) - C_\infty)\,e^{-2\alpha(t-t_0)} + C_\infty. \tag{12}
$$

Then we prove (11) for $t \ge t_0$.

I. Proof of (12) *under the assumption* (11)

Denoting $\eta(x,t) = [z(x,t) - z(x,t_k)]/h$ and integrating by parts, for $t \in [t_k, t_{k+1})$ we obtain

$$
\begin{aligned}
\dot V_1 &= 2\int_0^1 z(x,t)z_{xx}(x,t)\,dx + 2\int_0^1 z(x,t)\varphi(x,t,z)\,dx \\
&\quad + 2\sum_{j=1}^N \int_{x_{j-1}}^{x_j} z(x,t)b_j(x)(u_j(t_k) + w_j(t))\,dx \\
&= -2\int_0^1 z_x^2(x,t)\,dx + 2\int_0^1 z(x,t)\varphi(x,t,z)\,dx \\
&\quad + 2\sum_{j=1}^N \int_{x_{j-1}}^{x_j} h\eta(x,t)b_j(x)(\rho u_j(t_k) + w_j(t))\,dx \\
&\quad + 2\sum_{j=1}^N \int_{x_{j-1}}^{x_j} z(x,t_k)b_j(x)(\rho u_j(t_k) + w_j(t))\,dx \\
&\quad + 2\sum_{j=1}^N (1-\rho)\int_{x_{j-1}}^{x_j} z(x,t)b_j(x)u_j(t_k)\,dx.
\end{aligned} \tag{13}
$$

The penultimate term is not positive. Indeed, since

$$
\begin{aligned}
\rho u_j(t_k) &= -\rho K \operatorname{sign}\int_{x_{j-1}}^{x_j} b_j(x)z(x,t_k)\,dx \\
&= \arg\min_{v\in[-\rho K, \rho K]} v\int_{x_{j-1}}^{x_j} b_j(x)z(x,t_k)\,dx,
\end{aligned}
$$

for any $w_j(t)$ satisfying Assumption 3, we have

$$
\begin{aligned}
&\rho u_j(t_k)\int_{x_{j-1}}^{x_j} b_j(x)z(x,t_k)\,dx \\
&\qquad\qquad \le -w_j(t)\int_{x_{j-1}}^{x_j} b_j(x)z(x,t_k)\,dx.
\end{aligned} \tag{14}
$$

Now consider the last term of $\dot V_1$. Denoting

$$
\kappa(x,t) = z(x,t) - Nb_j(x)\int_{x_{j-1}}^{x_j} z(y,t)\,dy, \quad x \in [x_{j-1}, x_j),
$$

for $j=1,\dots,N$ we obtain

$$\begin{aligned}
&2(1-\rho)\int_{x_{j-1}}^{x_j} z(x,t)b_j(x)u_j(t_k)\,dx\\
&=2(1-\rho)\int_{x_{j-1}}^{x_j} z(x,t)b_j(x)u_j(t_k)\,dx\\
&\quad\pm 2M\int_{x_{j-1}}^{x_j} z^2(x,t)\,dx\\
&\quad\pm 2MN\int_{x_{j-1}}^{x_j} b_j(x)z(x,t)\,dx\int_{x_{j-1}}^{x_j} z(y,t)\,dy\\
&=-2M\int_{x_{j-1}}^{x_j} z^2(x,t)\,dx+2M\int_{x_{j-1}}^{x_j} z(x,t)\kappa(x,t)\,dx\\
&\quad+2\left[(1-\rho)u_j(t_k)+MN\int_{x_{j-1}}^{x_j} z(y,t)\,dy\right]\times\\
&\quad\int_{x_{j-1}}^{x_j} z(x,t)b_j(x)\,dx\\
&=-2M\int_{x_{j-1}}^{x_j} z^2(x,t)\,dx+2M\int_{x_{j-1}}^{x_j} z(x,t)\kappa(x,t)\,dx\\
&\quad+2\left[(1-\rho)u_j(t_k)+MN\int_{x_{j-1}}^{x_j} z(y,t)\,dy\right]\times\\
&\quad\int_{x_{j-1}}^{x_j} h\eta(x,t)b_j(x)\,dx\\
&\quad+2\left[(1-\rho)u_j(t_k)+MN\int_{x_{j-1}}^{x_j} z(y,t)\,dy\right]\times\\
&\quad\int_{x_{j-1}}^{x_j} z(x,t_k)b_j(x)\,dx.
\end{aligned}\tag{15}$$

Similarly to (14), the last term of (15) is not positive. Indeed, since

$$(1-\rho)u_j(t_k)=\operatorname*{arg\,min}_{v\in[-(1-\rho)K,(1-\rho)K]} v\int_{x_{j-1}}^{x_j} b_j(x)z(x,t_k)\,dx,$$

the condition (11) implies

$$\begin{aligned}
&(1-\rho)u_j(t_k)\int_{x_{j-1}}^{x_j} z(x,t_k)b_j(x)\,dx\\
&\quad\le -MN\int_{x_{j-1}}^{x_j} z(y,t)\,dy\int_{x_{j-1}}^{x_j} z(x,t_k)b_j(x)\,dx.
\end{aligned}\tag{16}$$

In view of (13)–(16), we obtain

$$\begin{aligned}
\dot V_1\le \sum_{j=1}^N\int_{x_{j-1}}^{x_j}\big\{&-2Mz^2(x,t)-2z_x^2(x,t)\\
&+2z(x,t)\varphi(x,t,z)+2Mz(x,t)\kappa(x,t)+2h\eta(x,t)\times\\
&[b_j(x)u_j(t_k)+Mz(x,t)-M\kappa(x,t)+b_j(x)w_j(t)]\big\}dx.
\end{aligned}\tag{17}$$

To compensate the cross terms with φ, we use S-procedure [23] by adding

$$\lambda_\varphi\sum_{j=1}^N\int_{x_{j-1}}^{x_j}(\varphi(x,t,z)-\varphi_m z(x,t))(\varphi_M z(x,t)-\varphi(x,t,z))\,dx,\tag{18}$$

which is nonnegative due to Assumption 1. The terms with $\kappa(x,t)$ will be compensated in a manner similar to [12], [24]. Namely, Young's inequality implies

$$\begin{aligned}
&\int_{x_{j-1}}^{x_j}\kappa^2(x,t)\,dx=\int_{x_{j-1}}^{x_j}\Big[z(x,t)-N\int_{x_{j-1}}^{x_j} z(y,t)\,dy\\
&\qquad\qquad+(1-b_j(x))N\int_{x_{j-1}}^{x_j} z(y,t)\,dy\Big]^2\,dx\\
&\le(1+\nu)\int_{x_{j-1}}^{x_j}\left(z(x,t)-N\int_{x_{j-1}}^{x_j} z(y,t)\,dy\right)^2\,dx\\
&\quad+(1+\nu^{-1})\int_{x_{j-1}}^{x_j}(1-b_j(x))^2N^2\left(\int_{x_{j-1}}^{x_j} z(y,t)\,dy\right)^2\,dx.
\end{aligned}$$

Since $\int_{x_{j-1}}^{x_j}\left(z(x,t)-N\int_{x_{j-1}}^{x_j} z(y,t)dy\right)dx=0$, Lemma 2 implies

$$\begin{aligned}
(1+\nu)\int_{x_{j-1}}^{x_j}&\left(z(x,t)-N\int_{x_{j-1}}^{x_j} z(y,t)\,dy\right)^2\,dx\\
&\le\tfrac{1+\nu}{(N\pi)^2}\int_{x_{j-1}}^{x_j} z_x^2(x,t)\,dx.
\end{aligned}$$

Furthermore, the definition of $b_j(x)$ and Jensen's inequality [25] imply

$$\begin{aligned}
(1+\nu^{-1})\int_{x_{j-1}}^{x_j}(1-b_j(x))^2N^2&\left(\int_{x_{j-1}}^{x_j} z(y,t)\,dy\right)^2\,dx\\
&\le(1+\nu^{-1})2\varepsilon N\int_{x_{j-1}}^{x_j} z^2(x,t)\,dx
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lambda_\kappa\sum_{j=1}^N\Big[\int_{x_{j-1}}^{x_j} z_x^2(x,t)\,dx-\tfrac{(N\pi)^2}{1+\nu}\int_{x_{j-1}}^{x_j}\kappa^2(x,t)\,dx\\
+\tfrac{2\varepsilon N^3\pi^2}{\nu}\int_{x_{j-1}}^{x_j} z^2(x,t)\,dx\Big]\ge 0.
\end{aligned}\tag{19}$$

The terms with $\eta(x,t)$ are compensated by

$$\begin{aligned}
\dot V_W=-2\alpha V_W-\tfrac{\pi^2ph}{4}\sum_{j=1}^N\int_{x_{j-1}}^{x_j}\eta^2(x,t)\,dx\\
+phe^{2\alpha h}\sum_{j=1}^N\int_{x_{j-1}}^{x_j}(z_{xx}(x,t)+\varphi(x,t,z)\\
+b_j(x)u_j(t_k)+b_j(x)w_j(t))^2\,dx.
\end{aligned}\tag{20}$$

The term z_{xx} that appeared in $\dot V_W$ is compensated by

$$\begin{aligned}
\dot V_2(t)&=2qh\int_0^1 z_xz_{xt}=-2qh\int_0^1 z_{xx}z_t\\
&=-2qh\int_0^1 z_{xx}^2-2qh\int_0^1 z_{xx}\varphi\\
&\quad-2qh\sum_{j=1}^N\int_{x_{j-1}}^{x_j} z_{xx}(x,t)b_j(x)(u_j(t_k)+w_j(t))dx.
\end{aligned}\tag{21}$$

If $z(0,t)=0$ or $z(1,t)=0$, Lemma 1 (with $\alpha=0$) implies

$$\lambda\left[\int_0^1 z_x^2(x,t)\,dx-\frac{d\pi^2}{4}\int_0^1 z^2(x,t)\,dx\right]\ge 0.\tag{22}$$

By summing up (17)–(22), for $t\in[t_k,t_{k+1})$ we obtain

$$\begin{aligned}
\dot V+2\alpha V-&\sum_{j=1}^N\int_{x_{j-1}}^{x_j} hb_j^2(x)(\beta_u u_j^2(t_k)+\beta_w w_j^2(t))\,dx\\
&\le\sum_{j=1}^N\int_{x_{j-1}}^{x_j}\xi_j^T(x,t)\Xi'\xi_j(x,t)\,dx\\
&\quad+(2\alpha qh+\lambda_\kappa+\lambda-2)\int_0^1 z_x^2\\
&\quad+phe^{2\alpha h}\sum_{j=1}^N\int_{x_{j-1}}^{x_j}(z_{xx}(x,t)+\varphi(x,t,z)\\
&\quad+b_j(x)u_j(t_k)+b_j(x)w_j(t))^2\,dx,
\end{aligned}\tag{23}$$

where $\xi_j=\operatorname{col}\{z,\varphi,\kappa,z_{xx},\eta,b_ju_j(t_k),b_jw_j\}$ and Ξ' is obtained from Ξ by deleting the last column and the last row. By applying Schur complement formula [25] to the last term, we obtain that relations (7) of the theorem guarantee that the right-hand side of (23) is not positive. Therefore,

$$\dot V\le -2\alpha V+2\alpha C_\infty,$$

which implies (12).

II. Proof of (11) *for* $t\ge t_0$

Using Jensen inequality and Lemma 1 we obtain

$$\left(\int_{x_{j-1}}^{x_j} z(x,t)\,dx\right)^2\le\frac{1}{N}\int_{x_{j-1}}^{x_j} z^2(x,t)\,dx\le\frac{V(t)}{N}.\tag{24}$$

Therefore, if $V(t)\le C_0$ then (11) is true. Initial conditions (8) imply $V(t_0)<C_0$ (note that $V_W(t_0)=0$). Let $t_*\in(t_0,\infty)$ be the smallest time instance such that $V(t_*)\ge C_0$. Since V is continuous on $[t_k,t_{k+1})$ and $V(t_k)\le V(t_k-0)$, we have $V(t_*)=C_0$ and $V(t)<C_0$ for $t\in[t_0,t_*)$. Together with (24) this implies (11) and, therefore, (12) is true for $t\in[t_0,t_*]$. Since $C_\infty<C_0$ and $V(t_0)<C_0$,

(12) guarantees that $V(t_*) < C_0$, what contradicts to the definition of t_*. Thus, for $t \geq t_0$ we have

$$V(t) < C_0 \Rightarrow (11) \Rightarrow (12) \Rightarrow (9).$$

■

Remark 3: If the conditions of Theorem 1 are satisfied for $h > 0$ then they are satisfied for all $h' \in [0, h]$ with the same decision variables (this can be verified using Schur complement formula). Since C_0 does not depend on h and C_∞ is linear in h, this implies that by decreasing h one ensures exponential convergence of the solutions from a fixed set (8) to an arbitrary small vicinity of zero. For $h \to 0$ one obtains exponential convergence to zero.

Remark 4: Consider (6) with $h = \varepsilon = \nu = 0$. Then for any φ_m and φ_M from Assumption 1 one can always ensure the feasibility of (7) by increasing N. Then the conditions of Theorem 1 will be feasible for small enough h, ε, and ν. That is, for a nonlinearity from an arbitrary sector $[\varphi_m, \varphi_M]$, the relations (7) determine how many controllers (i.e., what N) one should take to ensure the ultimate boundedness of the system.

Remark 5: The presented sampled-data control may be efficiently used for network-based control of diffusion PDEs. The control laws (4) allow to use the event-triggering mechanism that sends the messages only when the sign of the state weighted average changes its value. This allows to significantly reduce the network workload during the transient period. When the norm of the state starts to oscillate in the vicinity of zero, the sign has to be sent almost every sampling period.

Remark 6: Consider the system (1) with local disturbances (i.e., $w_j(t) \not\equiv 0$ for some j and $w_l(t) \equiv 0$ for $l \neq j$) and $\varphi \equiv 0$. By using collocated controllers (4) (assuming $b_l(t) \equiv 0$ for $l \neq j$) and slightly modifying the proof of Theorem 1, one can achieve ultimate boundedness with an arbitrary small limit bound for small enough sampling period h whereas the open-loop system is input-to-state stable with an ultimate bound proportional to the disturbance bound.

IV. Switching control

The relations (7) do not depend on the controller gain K. The feasibility of the relation $C_\infty < C_0$ also does not depend on K. Therefore, if the conditions of Theorem 1 are satisfied for some K, they remain true for any K such that $\sup_{j,t} |w_j(t)| \leq \rho K$. This observation allows to construct a switching controller that ensures convergence of the system trajectories from an arbitrary set to a fixed vicinity of zero.

Corollary 1: Consider the system (1)–(3) subject to Assumptions 1–4. Let the relations (7) be satisfied and assume that $C_\infty + \delta < C_0$ for some $\delta > 0$. For initial conditions from an arbitrary subset of X (defined in Subsection II-A) choose some $\mu_0 > 1$ such that

$$\|z(\cdot, t_0)\|_q^2 < \mu_0^2 C_0. \tag{25}$$

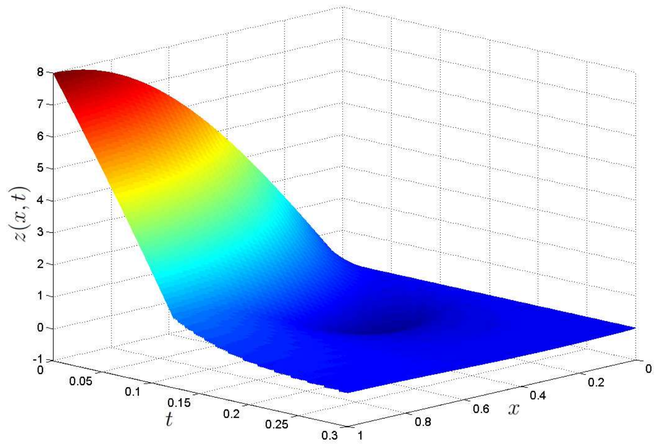

Fig. 2. Solution of the system

Consider the controllers

$$u_j(t_k) = -\mu_i K \operatorname{sign} \int_{x_{j-1}}^{x_j} b_j(x) z(x, t_k)\, dx,$$
$$t \in [t^i, t^{i+1}), \quad i \in \mathbb{N}_0, \quad j = 1, \ldots, N, \tag{26}$$

where

$$t^i = t_0 + \frac{i}{2\alpha} \ln \frac{C_0}{\delta}, \quad i \in \mathbb{N}_0,$$
$$\mu_{i+1} = \max\left\{1, \mu_i \sqrt{(C_\infty + \delta)/C_0}\right\}, \quad i \in \mathbb{N}_0.$$

Then the system trajectories converge to the set

$$\|z(\cdot, t)\|_q^2 \leq C_\infty + \delta.$$

Proof: Since $C_\infty + \delta < C_0$, the sequence t^i monotonically increases to infinity and the controllers (26) are well-defined for all $t \geq t_0$. Theorem 1 implies

$$\|z(\cdot, t)\|_q^2 \leq \mu_0^2 (C_0 e^{-2\alpha(t - t_0)} + C_\infty), \quad t \in [t^0, t^1].$$

If $\mu_1 > 1$, this implies

$$\|z(\cdot, t^1)\|_q^2 \leq \mu_0^2 (C_\infty + \delta) = \mu_1^2 C_0.$$

By induction we obtain

$$\|z(\cdot, t^{i+1})\|_q^2 \leq \mu_i^2 (C_\infty + \delta), \quad i \in \mathbb{N}_0.$$

The assertion of the corollary follows from the fact that μ_i monotonically decrease to 1. ■

V. Numerical example

Consider the system (1) with $N = 1$, boundary conditions

$$z(0, t) = 0, \quad z_x(1, t) = 0,$$

and the controllers (4) with $K = 50$. Let Assumptions 1–4 be satisfied with $[\varphi_m, \varphi_M] = [-5, 3]$, $h = 10^{-3}$, and $\rho = 0.1$. For $\varphi(x, t, z) = \varphi_M z \cos(0.1z)$ the system is unstable. For the decay rate $\alpha = 1$ the conditions of Theorem 1 are satisfied with $C_0 = 36$, $C_\infty = 1.93$. In Fig. 2 one can see the solution of the system with the above nonlinearity and

$$z(x, 0) = 8 \sin \frac{\pi x}{2}, \quad x \in [0, 1].$$

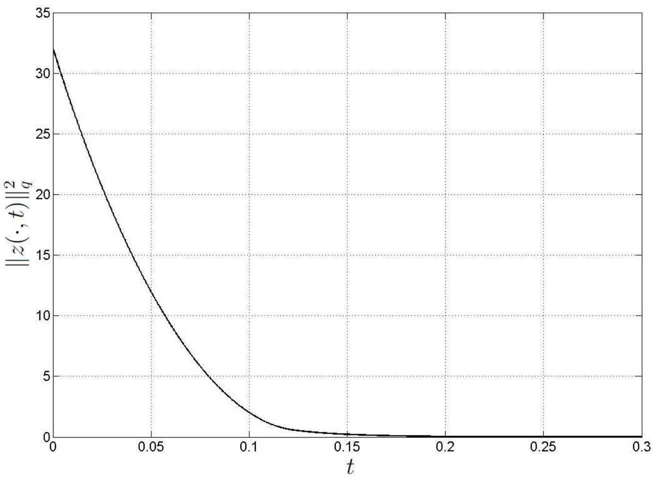

Fig. 3. Evolution of $\|z(\cdot,t)\|_q^2$ for $t \in [0, 0.3]$

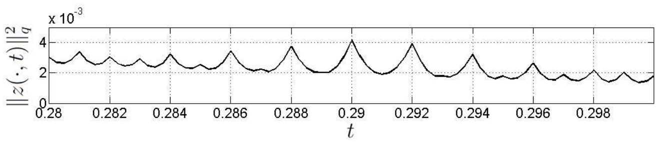

Fig. 4. Evolution of $\|z(\cdot,t)\|_q^2$ for $t \in [0.28, 0.3]$

In Figs. 3, 4 one can see that the norm $\|z(\cdot,t)\|_q^2$ converges to a small vicinity of zero and starts to shake. After the sign of the state weighted average is sent at $t = 0$, it does not change till $t = 0.123$. This time corresponds to a transient period. When the norm of the state starts to shake in a vicinity of zero, the state weighted average is sent almost every sampling period. Note that unknown disturbances subject to $\sup_{j,t} |w_j(t)| \leq \rho K = 5$ are compensated by the controllers (4).

VI. Conclusions

In this work we studied sampled-data relay control of semilinear diffusion PDEs. We derived LMI-based conditions ensuring that the system state locally converges to a vicinity of zero. Then we propose a switching procedure for controllers gains that ensures convergence from an arbitrary domain to the same limit set. The future work will be devoted to the extension of these results to vector N-D parabolic systems with nonuniform diffusion and asynchronous sampling.

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