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Sampled-data relay control of diffusion PDEs

Anton Selivanov, Emilia Fridman

School of Electrical Engineering, Tel Aviv University, Israel

Abstract

We consider a vector reaction-advection-diffusion equation on a hypercube. The measurements are weighted averages of the state over different subdomains. These measurements are asynchronously sampled in time. Subject to matched disturbances, the discrete control signals are applied through shape functions and zero-order holds. The feature of this work is that we consider generalized relay control: the control signals take their values in a finite set. This allows for networked control through low capacity communication channels. First, we derive linear matrix inequalities (LMIs) whose feasibility guarantees the ultimate boundedness with a limit bound proportional to the sampling period. Then we construct a switching procedure for the controller parameters that ensures semi-global practical stability: for an arbitrarily large domain of initial conditions the trajectories converge to a set whose size does not depend on the domain size. For the disturbance-free system this procedure guarantees exponential convergence to the origin. The results are demonstrated by two examples: 2D catalytic slab and a chemical reactor.

Key words: Distributed parameter systems; Relay control; Networked control systems.

1 Introduction

Networked control systems (NCSs), which are comprised of spatially distributed sensors, actuators, and controllers connected via a communication network, have become widespread due to great advantages they bring: long distance control, low cost, ease of reconfiguration, reduced system wiring, etc. [1,2]. Networked control of distributed parameter systems may be applicable to long distance control of chemical reactors [3] or air polluted areas [4]. One of the main challenges in NCSs is a measurement sampling. A variety of methods has been developed to analyse PDEs in the presence of sampling: the discrete-time approach [5,6], the time-delay approach [7,8], the modal decomposition techniques [9,10], which were also used for sampled-data predictive control with state and control constraints [11,12]. To reduce the amount of transmitted signals, event-triggered approach has been developed for PDEs [13,14]. In this work we use the time-delay approach to develop sampled-data relay control for diffusion equation, where the control signals take their values in a finite set. This allows for networked control through low capacity communication channels.

Relay control is a well known approach in a wide range of technical domains [15]. It has undeniable advantages: simple implementation, control saturation/quantization, finite time convergence, full compensation of matched disturbances. However, the analysis of *sampled-data* relay control is not a trivial task even for linear finite-dimensional systems. In [16] it has been shown that relay control does not lead to the asymptotic stability of a finite-dimensional system in the presence of input delay. In this case ultimate boundedness is achieved with a limit bound proportional to the time-delay bound. In [17] a convex optimization approach has been used to study generalized relays for finite-dimensional systems. In that work sampled measurements were modeled as input delays and the size of the limit set was proportional to a sampling period.

In this work we consider sampled-data relay control of semilinear diffusion PDEs. We assume that the space domain is divided into several subdomains. In each subdomain, there is a sensor, which measures a weighted average of the state function, and a controller, which influences the dynamics through a shape function. The control signals are subject to unknown disturbances, take their values in a finite set, and remain constant within a sampling period. First, we derive linear matrix inequalities (LMIs) whose feasibility guarantees the ultimate boundedness with a limit bound proportional to the sampling period. Then we construct a switching procedure for the controller parameters that ensures semi-global practical stability: for an arbitrarily large domain of initial conditions the trajectories converge to a set whose size does not depend on the domain size. For the disturbance-free system this procedure guarantees expo-

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Email addresses: antonselivanov@gmail.com (Anton Selivanov), emilia@eng.tau.ac.il (Emilia Fridman).

nential convergence to the origin. The results are demonstrated by two examples: 2D catalytic slab and a chemical reactor. Preliminary results, presented in [18], are generalized here to a vector system with multidimensional domain, convection term, reaction term, and asynchronous sampling.

Notations: $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $1 : N_s = \{1, 2, \ldots, N_s\}$, $\mathcal{H}^1(\Omega)$ is the Sobolev space of absolutely continuous functions with square integrable first derivatives, $\operatorname{div} f$ is the divergence of a vector field f, $\nabla z(x,t)$ is the gradient with respect to x if z is scalar and $\nabla z(x,t) = \operatorname{col}\{\nabla z^1, \ldots, \nabla z^M\}$ if $z = (z^1, \ldots, z^M)^T$. Given a set $S \subset \mathbb{R}^N$, $l(S)$ is its diameter, $\lambda(S)$ is its volume, $\operatorname{Int}\{S\}$ is the interior, $\operatorname{conv}\{S\}$ is the closed convex hull. For a convex polytope $\mathcal{P}$, $\rho \in \mathbb{R}$, we denote $\rho\mathcal{P} = \{\rho v \,|\, v \in \mathcal{P}\}$. For a matrix $P \in \mathbb{R}^{N \times N}$, $P > 0$ denotes that it is symmetric and positive-definite, $\lambda_{\max}(P)$ is the maximum eigenvalue, $\otimes$ stands for the Kronecker product.

Lemma 1 (Exponential Wirtinger inequality [19]) *Let a, b, $\alpha \in \mathbb{R}$, $0 \le W \in \mathbb{R}^{n \times n}$, and $f\colon [a,b] \to \mathbb{R}^n$ be an $\mathcal{H}^1$ function such that $f(a) = 0$ or $f(b) = 0$. Then*

$$\int_a^b e^{2\alpha t} f^T(t) W f(t)\, dt \le e^{2|\alpha|(b-a)} \frac{4(b-a)^2}{\pi^2} \int_a^b e^{2\alpha t} \dot{f}^T(t) W \dot{f}(t)\, dt.$$

Lemma 2 (Wirtinger inequality on hypercube [8]) *Let $\Omega = [0,1]^N$ and $f \in \mathcal{H}^1(\Omega)$ be a scalar function such that $f|_{\partial\Omega} = 0$. Then*

$$N\pi^2 \int_\Omega f^2(x)\, dx \le \int_\Omega \|\nabla f(x)\|^2\, dx.$$

Lemma 3 (Poincaré inequality on rectangle [20]) *Let $\Omega \subset R^N$ be rectangular with a diameter $l(\Omega)$ and $f \in \mathcal{H}^1(\Omega)$ be a scalar function such that $\int_\Omega f(x)\, dx = 0$. Then*

$$\int_\Omega f^2(x)\, dx \le \frac{l^2(\Omega)}{\pi^2} \int_\Omega \|\nabla f(x)\|^2\, dx.$$

2 Preliminaries and problem formulation

2.1 Lyapunov-based relay control of ODEs

Before proceeding to PDEs, we explain the essential idea of the Lyapunov-based relay control for ODEs. Consider the plant

$$\dot{x} = Ax + B(u + w), \quad x \in \mathbb{R}^n,\ u, w \in \mathbb{R}$$

such that (A, B) is stabilizable. Then there exist $K \in \mathbb{R}^{1 \times n}$ and $0 < P \in \mathbb{R}^{n \times n}$ such that $P(A - BK) + (A - BK)^T P < 0$. For $V = \frac{1}{2} x^T P x$ one has

$$\begin{aligned}\dot{V} &= x^T P[Ax + B(u + w \pm Kx)] \\ &= x^T P[A - BK]x + x^T PB(u + w + Kx).\end{aligned}$$

If one requires $|w| \le \rho K_0$ and guarantees $|Kx| \le (1-\rho)K_0$ for some $\rho \in [0, 1)$, then $w + Kx \in [-K_0, K_0]$. Taking

$$u = -K_0 \operatorname{sign} x^T PB = \arg \min_{v \in [-K_0, K_0]} x^T PBv,$$

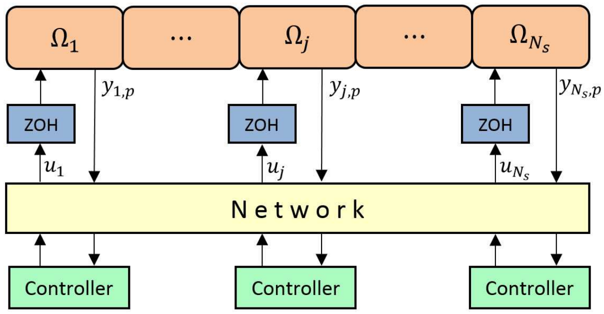

Fig. 1. The system representation

one gets

$$\begin{aligned}&x^T PBu \le x^T PB(-w - Kx) \\ &\text{for } -(w + Kx) \in [-K_0, K_0].\end{aligned} \tag{1}$$

Then $\dot{V} < 0$ for $x \neq 0$. To guarantee that $|Kx(t)| \le (1 - \rho)K_0$, note that it follows from

$$V(x(t)) < \min_{|Kx| \ge (1-\rho)K_0} V(x). \tag{2}$$

The minimum in (2) is positive, since the ellipsoid $V(x) = c$ with small enough $c > 0$ lies in the layer $|Kx| < (1 - \rho)K_0$. Since $V(x(t))$ cannot increase when $|Kx(t)| \le (1 - \rho)K_0$, if (2) holds for $t = 0$, it remains true for $t \ge 0$. For an arbitrary domain, (2) holds with $t = 0$ if the relay controller gain K_0 is large enough. This implies the semi-global stability.

Consider now sampled-data relay control with sampling $0 = t_0 < t_1 < t_2 < \ldots$ given by

$$u(t) = -K_0 \operatorname{sign} x^T(t_k) PB, \quad t \in [t_k, t_{k+1}).$$

For the same V one has

$$\begin{aligned}\dot{V} &= x^T P[A - BK]x + x^T PB(u + w + Kx) \\ &= x^T P[A - BK]x + x^T(t_k) PB(u + w + Kx) \\ &\quad + \int_{t_k}^t \dot{x}^T(s)\, ds PB(u + w + Kx).\end{aligned}$$

By a reasoning similar to the above, the term with $x^T(t_k)$ is nonpositive. If $\dot{x}$ is bounded, the integral term can be made arbitrarily small by reducing the maximum sampling, i.e. $\max_k\{t_{k+1} - t_k\}$. These allow to obtain ultimate boundedness proportional to the sampling and disturbance bounds.

In this paper we will extend these ideas to sampled-data relay control of a diffusion PDE.

2.2 Problem formulation

Consider a semilinear parabolic system

$$\begin{aligned}z_t(x,t) &= \Delta_D z(x,t) + \beta \nabla z(x,t) + Az(x,t) + f(x,t,z) \\ &\quad + B \textstyle\sum_{j=1}^{N_s} b_j(x)[u_j(t) + w_j(t)], \quad x \in \Omega,\end{aligned} \tag{3}$$

with the space domain $\Omega = [0,1]^N$, state $z\colon \Omega \times [t_0, \infty) \to \mathbb{R}^M$, matched disturbances $w_j(t)$, and matrices $\beta \in \mathbb{R}^{M \times MN}$, $A \in \mathbb{R}^{M \times M}$, $B \in \mathbb{R}^{M \times L}$. The diffusion term is defined as $\Delta_D z = (\Delta_D^1 z^1, \ldots, \Delta_D^M z^M)^T$,

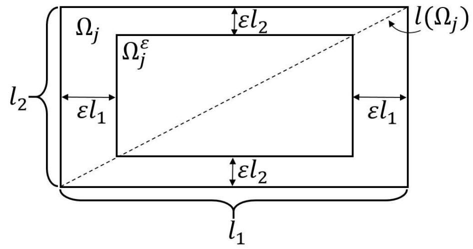

Fig. 2. Subdomain Ω_j and its subset Ω_j^ε

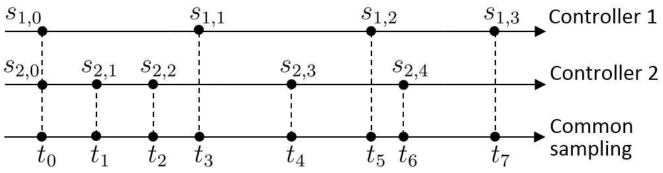

Fig. 3. Common sampling intervals

where $\Delta_D^m z^m(x,t) = \operatorname{div}(D^m(x)\nabla z^m(x,t))$ with $D^m(x) = (D^m(x))^T \in \mathcal{C}^1(\Omega, R^{N\times N})$ for $m \in 1:M$. The space domain Ω is divided into N_s rectangular subdomains Ω_j (Fig. 1), where the control signals are applied through shape functions $b_j(x) \in \mathcal{H}^1(0,1)$ such that

$$\begin{cases} b_j(x) = 0, & x \notin \Omega_j, \\ b_j(x) = 1, & x \in \Omega_j^\varepsilon, \\ b_j(x) \in [0,1], & x \in \Omega_j \backslash \Omega_j^\varepsilon \end{cases} \tag{4}$$

with Ω_j^ε being subsets of Ω_j depicted in Fig. 2.

Each control signal u_j is applied through zero-order hold changing its value at asynchronous sampling instants $t_0 = s_{j,0} < s_{j,1} < s_{j,2} < \ldots$ such that

$$s_{j,p+1} - s_{j,p} \le h, \quad \lim_p s_{j,p} = \infty, \quad \forall p \in \mathbb{N}_0,\ j \in 1:N_s.$$

By $[t_k, t_{k+1})$ we denote common sampling time intervals where all u_j are constant (see Fig. 3). We adopt the notation $t_{j,k} = \max_{p\in\mathbb{N}_0}\{s_{j,p} \mid s_{j,p} \le t_k\}$. For instance, in Fig. 3 $t_{1,0} = t_{1,1} = t_{1,2} = s_{1,0}$, $t_{1,3} = t_{1,4} = s_{1,1}$ and so on. Clearly, $t_{j,k+1} - t_{j,k} \le h$ and $[t_k, t_{k+1}) = \bigcap_{j=1}^{N_s}[t_{j,k}, t_{j,k+1})$.

We assume that the measurements of the system (3), (4) are given by

$$y_{j,p} = \int_{\Omega_j} b_j(x) z(x, s_{j,p})\, dx, \quad j \in 1:N_s,\ p \in \mathbb{N}_0.$$

Let $\mathcal{V} = \{v_1, v_2, \ldots, v_q\} \subset \mathbb{R}^L$ be a set of control values. Consider the generalized sampled-data relay control

$$\begin{aligned} u_j(t) &= \operatorname{argmin}_{v\in\mathcal{V}}\, y_{j,p}^T P_1 B v, \\ t &\in [s_{j,p}, s_{j,p+1}), \quad j \in 1:N_s, \quad p \in \mathbb{N}_0 \end{aligned} \tag{5}$$

with $P_1 \in \mathbb{R}^{M\times M}$ to be defined later. A concrete form of the set $\mathcal{V}$ is not important for our further analysis. For instance, if $\mathcal{V} = \{-v, v\}$ with $0 < v \in \mathbb{R}$, the minimum in (5) is delivered by $u_j(t) = -v \operatorname{sign}\{(P_1 B)^T y_{j,p}\}$, which coincides with the classical relay control.

Remark 1 *For the sake of simplicity, we consider the case of collocated sensors and actuators, i.e. the measurements $y_{j,p}$ depend on the controller shape functions $b_j(x)$. However, the results can be extended to the non-collocated case with the measurements $y_{j,p} = \int_{\Omega_j} \bar{b}_j(x) z(x, s_{j,p})\, dx$ provided $\|b_j(x) - \bar{b}_j(x)\|$ are small enough.*

We consider the system (3) under the Dirichlet boundary conditions

$$z(x,t)|_{x\in\partial\Omega} = 0 \tag{6}$$

and the Neumann boundary conditions

$$\langle z_x(x,t), \bar{n}\rangle|_{x\in\partial\Omega} = 0, \tag{7}$$

where $\bar{n}$ is a unit vector normal to the edge.

We adopt the following assumptions:

1) $\exists d_0^m : 0 < d_0^m I \le D^m(x), \forall x \in [0,1]$, $m \in 1:M$.
2) $\operatorname{conv}\{\mathcal{V}\} \ne \emptyset$ and $0 \in \operatorname{Int}\{\operatorname{conv}\{\mathcal{V}\}\}$.
3) $\forall j \in 1:N_s, w_j \in \mathcal{C}^1$ and $\exists \rho \in [0,1)$ such that
$$w_j(t) \in -\rho \operatorname{conv}\{\mathcal{V}\} \quad \forall t \ge t_0, \quad j \in 1:N_s.$$
4) $f = (f^1, \ldots, f^M)^T \in \mathcal{C}^1$ and $\forall m \in 1:M$, $z \in \mathbb{R}^N$, $x \in \Omega$, $t \in [t_0, \infty)$,
$$(\mu_T^m z^m - f^m(x,t,z))(f^m(x,t,z) - \mu_B^m z^m) \ge 0$$
for some $\mu_T^m \ge \mu_B^m$.
5) There exists $K \in \mathbb{R}^{L\times M}$ such that the system
$$z_t(x,t) = \Delta_D z(x,t) + Az(x,t) + Bu(x,t) \quad t \ge t_0 \tag{8}$$
is stable under the state-feedback control $u(x,t) = -Kz(x,t)$.

Assumption 1 determines a parabolic system with minimum diffusion rates d_0^m. Assumption 2 is a standard technical assumption. Assumption 3 allows to compensate the disturbances using the relay control $u \in \mathcal{V}$. Assumption 4 implies that the nonlinearity f^m belongs to the sector $[\mu_T^m, \mu_B^m]$. Assumption 5 guarantees that for a large enough number of subdomains N_s and small enough sampling h, the finite-dimensional controller $u_j(t) = -Ky_{j,p}$ (for $j \in 1:N_s$, $t \in [s_{j,p}, s_{j,p+1})$) stabilizes the system [8]. Similarly to Subsection 2.1, this controller can be replaced by relay control if the system state is bounded (see Remark 3).

Remark 2 *In order to verify Assumption 5, consider*

$$V_1(t) = \int_\Omega z^T(x,t) P_1 z(x,t)\, dx \tag{9}$$

with $P_1 = \operatorname{diag}\{p_1^1, \ldots, p_1^m\} > 0$. Then (8) implies

$$\dot{V}_1 = 2\int_\Omega z^T P_1[\Delta_D z + (A - BK)z]. \tag{10}$$

Using Green's formula and taking into account the boundary conditions (6) or (7), we obtain:

$$\begin{aligned} 2\textstyle\int_\Omega z^T P_1 \Delta_D z &= -2\textstyle\sum_{m=1}^M \int_\Omega (\nabla z^m)^T p_1^m D^m \nabla z^m \\ &\le -2\textstyle\int_\Omega (\nabla z)^T (P_1 D_0 \otimes I_N)\nabla z, \end{aligned}$$

where $D_0 = \operatorname{diag}\{d_0^1, \ldots, d_0^M\}$ *with* d_0^m *from Assumption 1. For the Dirichlet boundary conditions we can use the Wirtinger inequality (Lemma 2) to obtain*

$$-2\int_\Omega (\nabla z)^T (P_1 D_0 \otimes I_N)\nabla z \le -2N\pi^2 \int_\Omega z^T P_1 D_0 z.$$

Therefore, Assumption 5 is satisfied if

$$P_1[A - BK - \mu D_0] + [A - BK - \mu D_0]^T P_1 \le 0, \quad (11)$$

where $\mu = N\pi^2$ *for* (6) *and* $\mu = 0$ *for* (7). *Denoting* $P_1^{-1} = \bar{P}_1$, $Y = K\bar{P}_1$ *and multiplying* (11) *by* $\bar{P}_1$ *from both sides, we obtain that Assumption 5 is satisfied if there exist* $\bar{P}_1 = \operatorname{diag}\{\bar{p}_1^1, \ldots, \bar{p}_1^m\} > 0$ *and* $Y \in \mathbb{R}^{L\times M}$ *such that*

$$[A - \mu D_0]\bar{P}_1 + \bar{P}_1[A - \mu D_0]^T + BY + Y^T B^T \le 0$$

where $\mu = N\pi^2$ *for* (6) *and* $\mu = 0$ *for* (7). *The controller gain is given by* $K = -Y\bar{P}_1^{-1}$.

Remark 3 *Here we explain how Lyapunov-based relay control (Subsection 2.1) is extended to PDEs. Consider the system with continuous-time control*

$$z_t(x,t) = \Delta_D z(x,t) + Az(x,t) + B[u(t) + w(t)] \quad (12)$$

subject to boundary conditions (6) *or* (7). *Let the measurements be given by* $y(t) = \int_\Omega z(x,t)\,dx$. *For* V_1 *from* (9), *we have*

$$\dot{V}_1 = 2\int_\Omega z^T P_1[\Delta_D z + (A - BK)z] + 2\int_\Omega z^T P_1 BK[z - y] + 2\int_\Omega z^T(x,t)P_1 B[u(t) + w(t) + Ky(t)]\,dx.$$

The first integral term coincides with (10) *and is negative if* (11) *is true. The second term may be compensated using the Poincaré inequality (see* (A.8) *for details). The last term is equal to* $2y^T P_1 B[u + w + Ky]$. *If* $-w \in \rho\operatorname{conv}\{\mathcal{V}\}$ *and* $-Ky \in (1-\rho)\operatorname{conv}\{\mathcal{V}\}$, *then* $-w - Ky \in \operatorname{conv}\{\mathcal{V}\}$. *Taking*

$$u = \operatorname{argmin}_{v\in\mathcal{V}} y^T P_1 Bv = \operatorname{argmin}_{v\in\operatorname{conv}\{\mathcal{V}\}} y^T P_1 Bv,$$

one obtains

$$2y^T P_1 Bu \le 2y^T P_1 B[-w - Ky].$$

Thus, the last integral term of $\dot{V}_1$ *is nonpositive. In Theorem 1 this idea is extended to sampled-data control through shape functions on several subdomains.*

Remark 4 *Our main objective is to achieve ultimate bound for the trajectories that is proportional to a sampling period. In this remark we explain what prevents us from obtaining such results under point measurements. Consider the system* (12) *with point measurements* $\bar{y}(t) = z(\frac{1}{2}, t)$. *For* V_1 *from* (9), *we have*

$$\begin{aligned}\dot{V}_1 = {} & 2\int_\Omega z^T P_1[\Delta_D z + (A - BK)z] \\ & + 2\int_\Omega \bar{y}^T(t)P_1 B[u(t) + w(t) + Kz(x,t)]\,dx \\ & + 2\int_\Omega \delta^T(x,t)P_1 B[u(t) + w(t) + Kz(x,t)]\,dx,\end{aligned}$$

where $\delta(x,t) = z(x,t) - \bar{y}(t)$. *The first integral term coincides with* (10) *and is negative if* (11) *is true. If* $-w - K\int_\Omega z \in \operatorname{conv}\{\mathcal{V}\}$, *the second term is nonpositive for* $u = \operatorname{argmin}_{v\in\mathcal{V}} \bar{y}^T P_1 Bv$. *The difficulty arises when analysing the last term due to the presence of* u *and* w. *Using the boundedness of* u *and* w, *one can prove only ultimate boundedness of the sampling-free system. This eliminates the possibility of obtaining an ultimate bound proportional to a sampling period for the sampled-data system. The other types of functionals, like* $V_2 = \int_\Omega z_x^T P_2 z_x$, *seem to be inapplicable.*

Since $w, f \in \mathcal{C}^1$, by arguments of [21] we establish the existence of a unique strong solution of (3)–(5) initialized with $z(\cdot, t_0) \in \mathcal{H}^1(\Omega)$ subject to appropriate boundary conditions (6) or (7). Moreover, if $z(\cdot, t_0) \in \mathcal{H}^2$ subject to appropriate boundary conditions, then the solution $z(\cdot, t)$ is of class $\mathcal{C}^1$ in time as a function with values in $\mathcal{H}^1$ [22].

Remark 5 *For the proof of our main result (Theorem 1), we need the Lyapunov-Krasovskii functional* (A.1) *to be continuous on* (t_k, t_{k+1}). *To achieve this, it suffices to guarantee that the solution is continuous in* $\mathcal{H}^1$*-norm. This requires to take the shape functions* (4) *from* $\mathcal{H}^1$. *For smaller* ε *in* (4) *the stability conditions of Theorem 1 are less restrictive. Thus, if the system is stable for* $\varepsilon' > 0$, *it remains stable for all* $\varepsilon \in (0, \varepsilon')$. *For* $\varepsilon \to 0$ *the shape functions approach*

$$b_j(x) = \begin{cases} 1, & x \in \Omega_j, \\ 0, & x \notin \Omega_j, \end{cases} \quad j \in 1 : N_s,$$

which are not from $\mathcal{H}^1$. *However, after the stability is proved for all* $\varepsilon \in (0, \varepsilon')$, *one can prove the stability for* $\varepsilon = 0$ *using continuous dependence of the solutions on the parameters (see, e.g., [23, Theorem 3.4.4]).*

Our objective is to derive conditions for local practical stability of the closed-loop system (3)–(5) and to find a bound on the domain of attraction. Moreover, we construct a switching procedure that allows to obtain semi-global results, i.e. practical stability for an arbitrary set of initial conditions. For disturbance-free systems this procedure guarantees exponential convergence to the origin.

3 Regional stabilization

For convenience we define

$$\|z(\cdot,t)\|_V^2 = \int_\Omega z^T(x,t)P_1 z(x,t)\,dx + h\sum_{m=1}^M \int_\Omega p_3^m (\nabla z^m(x,t))^T D^m(x)\nabla z^m(x,t)\,dx,$$

where $P_1 = \operatorname{diag}\{p_1^1, \ldots, p_1^M\} \ge 0$, $P_3 = \operatorname{diag}\{p_3^1, \ldots, p_3^M\} \ge 0$, and $z(\cdot,t) \in \mathcal{H}^1(\Omega)$. The choice of such norm is motivated by the Lyapunov-Krasovskii functional (A.1). Similarly to [7,8], the terms with p_3^m appear due to sampling.

Denote by $a_i \in \mathbb{R}^L$, $i \in 1 : N_a$, the dual vectors of $\operatorname{conv}\{\mathcal{V}\}$:

$$\operatorname{conv}\{\mathcal{V}\} = \{v \in \mathbb{R}^L \mid a_i^T v \le 1, i \in 1 : N_a\}. \quad (13)$$

Such vectors always exist (see, e.g., [24, Theorem 1.1]).

The following theorem provides the ultimate boundedness conditions for the closed-loop system (3)–(5) under (6) or (7) with an ultimate bound C_∞ proportional to a product of the sampling period h and $\max_{v\in\mathcal{V}}\|v\|^2$.

Theorem 1 *Consider the system* (3), (4) *with control laws* (5) *and boundary conditions* (6) *or* (7) *under Assumptions 1–5. For given sampling period $h>0$, decay rate $\alpha>0$, and tuning parameter $\nu>0$ let there exist $P_2=\operatorname{diag}\{p_2^1,\dots,p_2^M\}$, $0\le W\in\mathbb{R}^{M\times M}$, $L\times L$ nonnegative matrices β_u,β_w, and $M\times M$ nonnegative diagonal matrices P_1, P_3, Λ_f, Λ_κ, Λ_D, where $\Lambda_D=0$ for the Neumann boundary conditions* (7)*, such that*[1] $\Phi\le 0$*, where $\Phi=\{\Phi_{ij}\}$ is a symmetric matrix composed from*

$$\begin{aligned}
\Phi_{11}&=P_1(A-BK)+(A-BK)^TP_1+2\alpha P_1-\mu_T\mu_B\Lambda_f\\
&\quad+2N\varepsilon(1+\nu^{-1})\Lambda_\kappa-N\pi^2\Lambda_D+h(P_2A+A^TP_2),\\
\Phi_{12}&=(P_1+hP_2)\beta,\quad \Phi_{13}=P_1+hP_2+\tfrac{1}{2}(\mu_T+\mu_B)\Lambda_f,\\
\Phi_{14}&=P_1BK,\quad \Phi_{15}=h(A^TP_3-P_2),\quad \Phi_{16}=h(P_1BK)^T,\\
\Phi_{17}&=\Phi_{18}=hP_2B,\quad \Phi_{22}=2(\alpha hP_3-P_1-hP_2)D_0\otimes I_N\\
&\quad+\Lambda_D\otimes I_N+(1+\nu)\tfrac{l^2}{\pi^2}(\Lambda_\kappa\otimes I_N),\quad \Phi_{25}=h(P_3\beta)^T,\\
\Phi_{33}&=-\Lambda_f,\ \Phi_{35}=hP_3,\ \Phi_{44}=-\Lambda_\kappa,\ \Phi_{46}=-h(P_1BK)^T,\\
\Phi_{55}&=h(e^{2\alpha h}W-2P_3),\ \Phi_{57}=\Phi_{58}=hP_3B,\ \Phi_{66}=-\tfrac{\pi^2h}{4}W,\\
\Phi_{67}&=\Phi_{68}=hP_1B,\quad \Phi_{77}=-h\beta_u,\quad \Phi_{88}=-h\beta_w,
\end{aligned}$$

$\mu_T=\operatorname{diag}\{\mu_T^1,\dots,\mu_T^M\}$, $\mu_B=\operatorname{diag}\{\mu_B^1,\dots,\mu_B^M\}$, $l=\max_j l(\Omega_j)$, $D_0=\operatorname{diag}\{d_0^1,\dots,d_0^M\}$. *Denote*

$$\begin{aligned}
C_0&=\min_{i\in 1:N_a}(a_i^TKP_1^{-1}K^Ta_i)^{-1}\min_{j=1:N_s}\lambda(\Omega_j),\\
C_\infty&=\tfrac{h}{2\alpha}(\lambda_{\max}(\beta_u)+\rho^2\lambda_{\max}(\beta_w))\max_{v\in\mathcal{V}}\|v\|^2.
\end{aligned}$$

If

$$C_\infty<(1-\rho)^2C_0 \tag{14}$$

then for initial conditions $z(\cdot,t_0)\in\mathcal{H}^1(\Omega)$ subject to appropriate boundary conditions (6) *or* (7)*, such that*

$$\|z(\cdot,t_0)\|_V^2<(1-\rho)^2C_0, \tag{15}$$

the strong solution of the system satisfies

$$\|z(\cdot,t)\|_V^2\le\|z(\cdot,t_0)\|_V^2e^{-2\alpha(t-t_0)}+C_\infty. \tag{16}$$

Proof is given in Appendix A.

Remark 6 *For zero values of ε, μ_T, μ_B, α, l, h, β the condition $\Phi\le0$ is reduced to*

$$\begin{aligned}
\operatorname{diag}\{&P_1(A-BK)+(A-BK)^TP_1-N\pi^2\Lambda_D,\\
&-2P_1D_0\otimes I_N+\Lambda_D\otimes I_N\}\le 0.
\end{aligned}$$

The latter inequality coincides with (11) *if one takes $\Lambda_D=-2P_1D_0$ for* (6) *or $\Lambda_D=0$ for* (7)*. Therefore, Assumption 5 guarantees $\Phi\le0$ for small enough ε, μ_T, μ_B, α, l, h, β and establishes a relation among the system parameters (such as sampling h, decay rate α, subdomains' maximum diameter l, etc.) that preserves the stability.*

Remark 7 *If the conditions of Theorem 1 are satisfied for $h=0$, they are also satisfied with the same decision variables for all $h\in[0,h^*]$, where h^* is sufficiently small (this can be verified using Schur complement formula). Since C_0 does not depend on h and C_∞ is linear in h, this implies that by decreasing the sampling period h one ensures exponential convergence of the solutions from the set* (15) *to an arbitrarily small vicinity of zero.*

Remark 8 *If K is unknown, the matrix inequalities of Theorem 1 are nonlinear. Similarly to [25], they can be linearized by seting $P_2=\mu_2P_1$, $P_3=\mu_3P_1$, $\bar P_1=P_1^{-1}$, multiplying Φ from both sides by $\operatorname{diag}\{\bar P_1\otimes I_{N+5},I_{2N}\}$ and denoting $Y=K\bar P_1$. The scalars μ_2 and μ_3 are tuning parameters.*

Remark 9 *Theorem 1 admits several straight-forward extensions. First, one may consider the boundary conditions*

$$z(x,t)|_{x\in\Gamma_1}=0,\quad \langle z_x(x,t),\bar n\rangle|_{x\in\Gamma_2}=0,$$

where $\Gamma_1\cup\Gamma_2=\partial\Omega$. Moreover, for constant diffusion coefficients $D^m(x)=D^m$ one may derive the stability conditions with non-diagonal matrices P_1, P_2, and P_3 (see [26]).

4 Semi-global stabilization by switching

The set of control values $\mathcal{V}$ has no impact on the feasibility of $\Phi\le0$ from Theorem 1. At the same time, $\mathcal{V}$ determines the sizes of the initial set $(1-\rho^2)C_0$ (through dual vectors a_i) and the limit set C_∞. Using this observation, we construct a switching procedure that ensures ultimate boundedness for an arbitrarily large domain with a limit bound independent of the domain size (Corollary 1). For disturbance-free systems this procedure guarantees exponential convergence to the origin.

Consider the system (3), (4) with boundary conditions (6) or (7) under Assumptions 1–5. Let us choose a "zooming" parameter $\sigma_k>0$ and switching period $T>0$. Assumption 3 can be rewritten as

$$w_j(t)\in-\rho\operatorname{conv}\{\mathcal{V}\}=-\frac{\rho}{\sigma_k}\operatorname{conv}\{\sigma_k\mathcal{V}\}.$$

Then the substitute $\mathcal{V}\to\sigma_k\mathcal{V}$ (with dual vectors $a_i\to\sigma_k^{-1}a_i$) in Theorem 1 leads to the following changes

$$C_0\to\sigma_k^2C_0,\quad C_\infty\to\sigma_k^2C_u+C_w,\quad \rho\to\frac{\rho}{\sigma_k},$$

where

$$\begin{aligned}
C_u&=\tfrac{h}{2\alpha}\lambda_{\max}(\beta_u)\max_{v\in\mathcal{V}}\|v\|^2,\\
C_w&=\tfrac{h}{2\alpha}\rho^2\lambda_{\max}(\beta_w)\max_{v\in\mathcal{V}}\|v\|^2.
\end{aligned}$$

In particular, the condition (14), which guarantees that the limit set is larger than the initial set, takes the form

$$\sigma_k^2C_u+C_w<\left(1-\frac{\rho}{\sigma_k}\right)^2\sigma_k^2C_0=U_k. \tag{17}$$

[1] MATLAB codes for solving the LMIs are available at `https://github.com/AntonSelivanov/Aut17`

The condition (15) was imposed to guarantee $V(t_0) < (1-\rho)^2 C_0$, which in our case can be written as

$$V(kT) < (\sigma_k - \rho)^2 C_0 = U_k. \tag{18}$$

If $\Phi \le 0$ and (17), (18) are true then, in a manner similar to the proof of Theorem 1, one obtains (cf. (A.3))

$$V(kT+T) \le (U_k - \sigma_k^2 C_u - C_w)e^{-2\alpha T} + \sigma_k^2 C_u + C_w. \tag{19}$$

Due to (17), this upper bound for $V(kT+T)$ is smaller than U_k, an upper bound for $V(kT)$. Thus, we can reduce the zooming parameter σ_{k+1} so that $U_{k+1} = (\sigma_{k+1} - \rho)^2 C_0$ satisfies

$$U_{k+1} = (U_k - \sigma_k^2 C_u - C_w)e^{-2\alpha T} + \sigma_k^2 C_u + C_w.$$

This leads to a switching control

$$\begin{aligned} &u_j(t) = \operatorname{argmin}_{v \in \sigma_k \mathcal{V}} \; y_{j,p}^T P_2 B v, \\ &t \in [s_{j,p}, s_{j,p+1}) \cap [kT, kT+T), \end{aligned} \tag{20}$$

where $j \in 1 : N_s$, $k, p \in \mathbb{N}_0$ and

$$\begin{aligned} &\sigma_k = \rho + \sqrt{U_k / C_0}, \\ &U_{k+1} = (U_k - \sigma_k^2 C_u - C_w)e^{-2\alpha T} + \sigma_k^2 C_u + C_w. \end{aligned} \tag{21}$$

To ensure the stability, it suffices to guarantee (17) and (18) for $k \in \mathbb{N}_0$. Let $C_u < C_0$. Then the parabola $\sigma^2 C_u + C_w - (\sigma - \rho)^2 C_0 = 0$ opens down with the largest (real) root

$$\sigma_\infty = \left(1 - \frac{C_u}{C_0}\right)^{-1} \left(\rho + \sqrt{\rho^2 \frac{C_u}{C_0} + \left(1 - \frac{C_u}{C_0}\right) \frac{C_w}{C_0}}\right). \tag{22}$$

Therefore, the relation (17) is satisfied for any $\sigma_k > \sigma_\infty$. By taking $\sigma_0 > \sigma_\infty$ such that $V(t_0) < U_0 = C_0(\sigma_0 - \rho)^2$, we guarantee (17) and (18) for $k = 0$. If (17) and (18) hold for some $k \in \mathbb{N}_0$ then (19) implies (18) for $k+1$. Moreover, (19) implies that $U_{k+1} < U_k$ and, consequently, $\sigma_{k+1} < \sigma_k$. Therefore,

$$U_{k+1} \overset{(21)}{>} \sigma_k^2 C_u + C_w > \sigma_{k+1}^2 C_u + C_w,$$

which guarantees (17) for $k+1$. By induction, (17) and (18) hold for $k \in \mathbb{N}_0$, therefore, $V(t) < U_k$ for $t \in [kT, kT+T)$, with U_k and σ_k being monotonically decreasing sequences of positive numbers. These sequences converge to a unique (real) positive root of (21) given by (22) and $U_\infty = C_0(\sigma_\infty - \rho)^2$. We have proved the following results.

Corollary 1 *Consider the system* (3), (4) *with boundary conditions* (6) *or* (7) *under Assumptions 1–5. Let* $\Phi \le 0$, *where* Φ *is given in Theorem 1, and* $C_u < C_0$. *Then, for an arbitrary set of initial conditions* $z(\cdot, t_0) \in \mathcal{H}^1(\Omega)$ *subject to appropriate boundary conditions, the switching controller* (20), (21) *with* $\sigma_0 > \sigma_\infty$ *such that*

$$\|z(\cdot, t_0)\|_V^2 < (\sigma_0 - \rho)^2 C_0 = U_0$$

guarantees

$$\|z(\cdot, t)\|_V^2 < U_k, \quad t \in [kT, kT+T),\; k \in \mathbb{N}_0. \tag{23}$$

Moreover, σ_k *and* U_k *monotonically decrease to* σ_∞ *and* $U_\infty = (\sigma_\infty - \rho)^2 C_0$.

	Example 1		Example 2	
k	σ_k	U_k	σ_k	U_k
0	1	0.1702	1	52.73
1	0.998	0.1694	0.69	24.97
2	0.996	0.1687	0.48	11.87
3	0.994	0.1680	0.33	5.68

Table 1
Parameters of switching

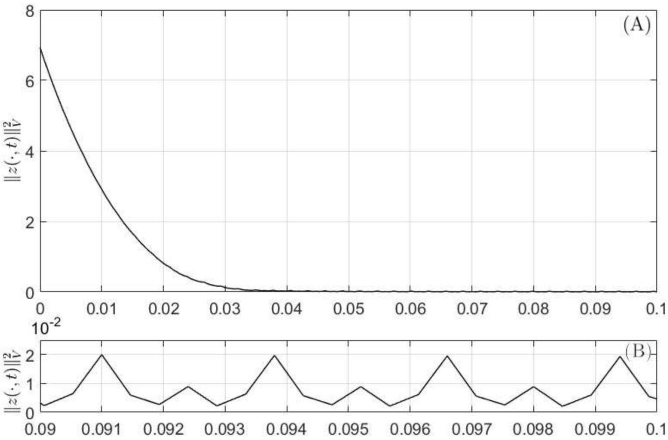

Fig. 4. Example 1: Evolution of $\|z(\cdot,t)\|_V^2$: (A) on $[0, 0.1]$; (B) on $[0.09, 0.1]$

Corollary 2 *Consider the disturbance-free system* (3), (4) *with* $w_j(t) \equiv 0$ *and boundary conditions* (6) *or* (7) *under Assumptions 1, 2, 4, 5. Let* $\Phi \le 0$, *where* Φ *is given in Theorem 1, and* $C_u < C_0$. *Then, for an arbitrary set of initial conditions* $z(\cdot, t_0) \in \mathcal{H}^1(\Omega)$ *subject to appropriate boundary conditions, the switching controller* (20) *with*

$$\sigma_k = \sqrt{U_k / C_0}, \quad U_{k+1} = \lambda U_k, \tag{24}$$

where

$$\lambda = \left(1 - \frac{C_u}{C_0}\right) e^{-2\alpha T} + \frac{C_u}{C_0}$$

and $\sigma_0 > 0$ *is such that*

$$\|z(\cdot, t_0)\|_V^2 < \sigma_0^2 C_0 = U_0,$$

guarantees the exponential stability with the decay rate

$$\delta = -\frac{\ln \lambda}{2T}.$$

For the disturbance-free case, switching algorithm (24) is obtained by substituting $\rho = 0$ (consequently, $C_w = 0$) in (21). The condition $C_u < C_0$ implies $\lambda < 1$, therefore, $U_k \to 0$ and $\sigma_k \to 0$ when $k \to \infty$. That is, the system is exponentially stable. Since U_k are upper bounds for the Lyapunov functional, the exponential decay rate δ is found from the equation $\lambda = e^{-2\delta T}$.

5 Examples

Example 1. Consider a 2D extension of the catalytic rod equation from [27]:

$$\frac{\partial z}{\partial t} = \frac{1}{\pi^2\sqrt{2}}\left[\frac{\partial^2 z}{\partial x_1^2} + \frac{\partial^2 z}{\partial x_2^2}\right] - \beta_U z + \beta_T\left(e^{-\frac{\gamma}{1+z}} - e^{-\gamma}\right) + \beta_U \sum_{j=1}^{N_s} \beta_j(x)[u_j(t_{j,k}) + w_j(t)], \quad t \in [t_k, t_{k+1}) \tag{25}$$

under the Dirichlet boundary conditions (6), where z is the temperature in the reactor, $\beta_T = 50$ is a heat of reaction, $\beta_U = 2$ is a heat transfer coefficient, $\gamma = 4$ is an activation energy, and the control u is the temperature of the cooling medium. For the above values the steady state $z(x,t) = 0$ is unstable.

To stabilize the system (25), we use the controllers (5). The nonlinearity $f(x,t,z) = \beta_T(e^{-\frac{\gamma}{1+z}} - e^{-\gamma})$ satisfies Assumption 4 with $\mu_T = 6.15$ and $\mu_B = 0$. The conditions of Theorem 1 are feasible with $\mathcal{V} = \{\pm 10\}$, $K = 4$, $N_s = 36$, $\alpha = 2.4$, $\rho = 0.01$, $\varepsilon = 10^{-9}$, $\nu = 10^{-5}$, $h = 1.4 \times 10^{-3}$. For such choice of $\mathcal{V}$ the dual vectors $a_{1,2} = \pm 0.1$ lead to $C_0 = 0.1736$, $C_\infty = 0.1696$. The initial conditions were chosen as

$$z(x,0) = 2\exp\left(\frac{-1}{1-(2x_1-1)^2-(2x_2-1)^2}\right)$$

if $(2x_1-1)^2 + (2x_2-1)^2 \leq 1$ and 0 otherwise. Note that $z(\cdot,0)$ satisfies (15). The disturbance $w_j(t)$ is piecewise linear function with $w_j(t_k) \in -\rho\,\mathrm{conv}\{\mathcal{V}\}$ being uniformly distributed random numbers. The evolution of $\|z(\cdot,t)\|_V^2$ is presented in Fig. 4. As one can see, the state $z(\cdot,t)$ converges to the vicinity of the origin.

Consider the switching controller (20). The values of the switching parameters (21) for $T = 1$ are given in Table 1. Note that the values of σ_k and U_k are decreasing. This indicates that the state, which gets smaller and smaller, requires smaller control effort after every switching time.

Note that by increasing the number of sensors N_s we reduce $l = \max_j l(\Omega_j)$ that appears in Φ_{22} of Theorem 1. Therefore, for larger N_s the LMI $\Phi \leq 0$ remains feasible. This corresponds to the general intuition, which says "the more sensors/actuators the better". On the other hand, larger N_s reduces the bound for initial conditions C_0. Thus, for large N_s, the condition $C_\infty^u < C_0$ may no longer hold. In the considered example, the LMIs are not feasible for $N_s \leq 25$ and the condition $C_\infty^u < C_0$ is violated for $N_s \geq 49$.

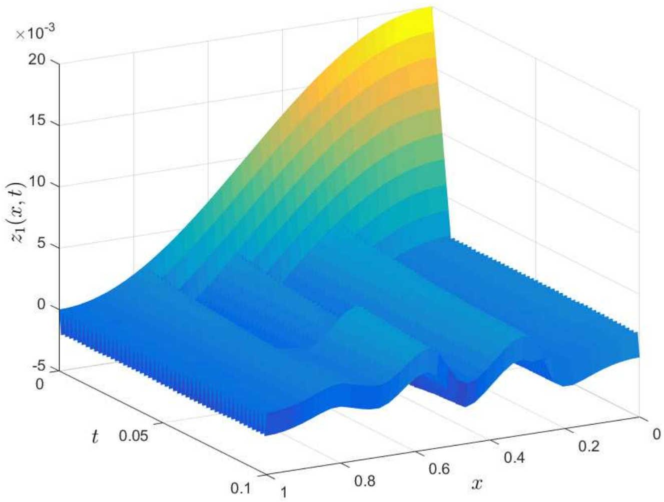

Fig. 5. Example 2: Evolution of $z_1(\cdot,t)$

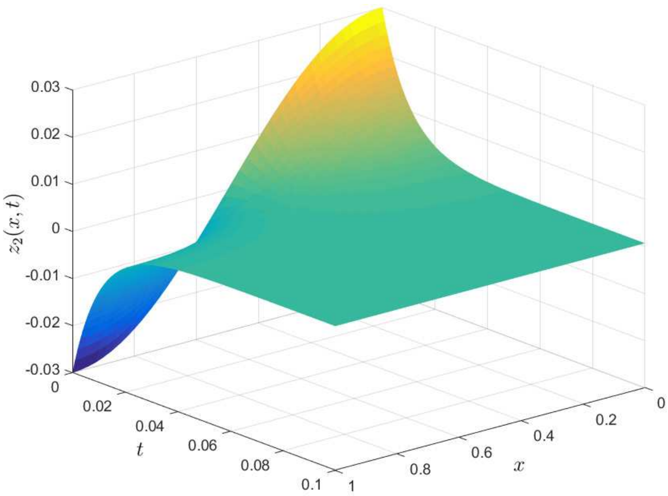

Fig. 6. Example 2: Evolution of $z_2(\cdot,t)$

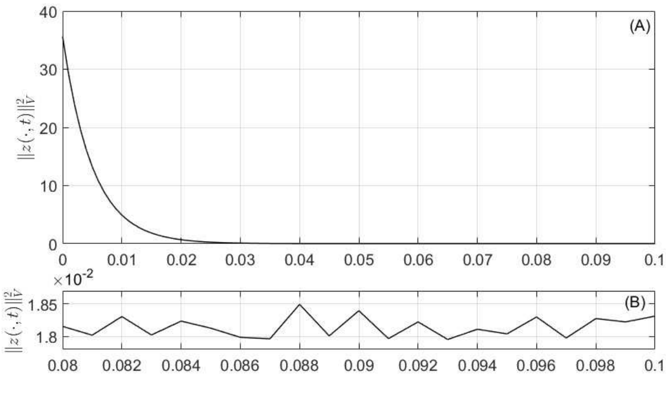

Fig. 7. Example 2: Evolution of $\|z(\cdot,t)\|_V^2$: (A) on $[0, 0.1]$; (B) on $[0.08, 0.1]$

Example 2. Consider the chemical reactor model from [3]

$$\begin{aligned} Le\frac{\partial z_1}{\partial t} + V\frac{\partial z_1}{\partial x} - \frac{\partial^2 z_1}{\partial x^2} =& f^*(z) \\ &+ \sum_{j=1}^{N_s} b_j(x)[u_j(t_{j,k}) + w_j(t)], \quad t \in [t_k, t_{k+1}), \\ \frac{\partial z_2}{\partial t} + V\frac{\partial z_2}{\partial x} - D\frac{\partial^2 z_2}{\partial x^2} =& g(z), \end{aligned}$$

under the Neumann boundary conditions (7), where $Le = 100$ is the Lewis number, $V = 1.1$ is convective velocity, $D = 10$ is diffusion coefficient. This model accounts for an activator z_1, which undergoes reaction (expressed as $f^*(z)$), advection and diffusion, and for a fast inhibitor z_2, which may be advected by the flow. The kinetics terms are given by

$$f^*(z) = z_1 \cos^2(z_1) + z_2, \quad g(z) = -\beta z_1 - dz_2,$$

where $\beta = 0.45$, $d = 0.2$. The conditions of Theorem 1 are feasible with $\mathcal{V} = \{\pm 2\}$, $K = [2, 0]$, $N_s = 4$, $\alpha = 0.14$, $\rho = 0.01$, $\varepsilon = 10^{-7}$, $\nu = 10^{-5}$, $h = 10^{-3}$. For such choice of $\mathcal{V}$ the dual vectors $a_{1,2} = \pm 0.5$ lead to $C_0 = 53.8$, $C_\infty = 15.9$. The results of numerical simulations on $[0, 0.1]$ for

$$z(x, 0) = \begin{bmatrix} \cos(\pi x)+1 \\ 3\cos(\pi x) \end{bmatrix} \times 10^{-2}$$

are presented in Figs. 5–7. As one can see, the state $z(\cdot, t)$ converges to the vicinity of the origin.

The switching parameters (21) of the controller (20) for $T = 5$ are given in Table 1. Similarly to Example 1, the values of σ_k and U_k are decreasing. That is, the state requires smaller control effort after every switching time.

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A Proof of Theorem 1

Throughout the proof we assume that the initial conditions are from $\mathcal{H}^2$. This guarantees that the solution $z(\cdot,t)$ is of class $\mathcal{C}^1$ in time as a function with values in $\mathcal{H}^1$. Then the Lyapunov-Krasovskii functional defined below is continuous on $[t_k,t_{k+1})$ and $V(t_k)\le V(t_k-0)$. After being proved for the initial conditions from $\mathcal{H}^2$, Theorem 1 for $z(\cdot,t_0)\in\mathcal{H}^1$ follows from continuous dependence of the solutions on the initial conditions (see, e.g., [28, Theorem 6.1.2]) and the density of $\mathcal{H}^2$ in $\mathcal{H}^1$.

Consider the functional $V=V_1+V_2+V_W$ with

$$
\begin{aligned}
V_1 &= \int_\Omega z^T(x,t)P_1z(x,t)\,dx,\\
V_2 &= h\sum_{m=1}^{M}\int_\Omega p_3^m(\nabla z^m(x,t))^T D^m(x)\nabla z^m(x,t)\,dx,\\
V_W &= he^{2\alpha h}\sum_{j=1}^{N_s}\int_{\Omega_j}\int_{t_{j,k}}^{t} e^{-2\alpha(t-s)}z_s^T(x,s)Wz_s(x,s)\,ds\,dx\\
&\quad -\frac{\pi^2 h}{4}\sum_{j=1}^{N_s}\int_{\Omega_j}\int_{t_{j,k}}^{t} e^{-2\alpha(t-s)}\eta^T(x,s)W\eta(x,s)\,ds\,dx,\\
&\qquad t\in[t_k,t_{k+1}),\quad k\in\mathbb{N}_0,
\end{aligned}
\tag{A.1}
$$

where $\eta(x,t)=\frac{1}{h}[z(x,t)-z(x,t_{j,k})]$ for $x\in\Omega_j$, $t\in[t_k,t_{k+1})$. Here V_1 and V_2 are chosen as in [8], V_W is an extension of the Wirtinger-based terms of [29] to the case of diffusion PDEs. The exponential Wirtinger inequality (Lemma 1) implies $V_W\ge 0$, therefore, $V\ge 0$.

We divide the proof into two parts. First, we assume that

$$
\frac{K}{\lambda(\Omega_j)}\int_{\Omega_j} z(x,t)\,dx\in -(1-\rho)\,\mathrm{conv}\{\mathcal{V}\},\quad \forall j\in 1:N_s \tag{A.2}
$$

and show that

$$
V(t)\le (V(t_0)-C_\infty)\,e^{-2\alpha(t-t_0)}+C_\infty,\quad t\ge t_0. \tag{A.3}
$$

Then we prove that the solutions of (3)–(5) satisfy (A.2) for $t\ge t_0$.

I. Proof of (A.3) *under the assumption* (A.2)

$$
\begin{aligned}
\dot V_1 &= 2\int_\Omega z^TP_1[\Delta_D z+\beta\nabla z+Az+f]\\
&\quad +2\sum_{j=1}^{N_s}\int_{\Omega_j} z^T(x,t)P_1Bb_j(x)[u_j(t_{j,k})+w_j(t)]\,dx.
\end{aligned}
\tag{A.4}
$$

The key idea is to transform the last term as follows:

$$
\begin{aligned}
&2\int_{\Omega_j} z^T(x,t)P_1Bb_j(x)[u_j(t_{j,k})+w_j(t)]\,dx\\
&\quad \pm 2\int_{\Omega_j} z^T(x,t)P_1BKz(x,t)\,dx\\
&\quad \pm 2\int_{\Omega_j} z^T(x,t)P_1BK\tfrac{b_j(x)}{\lambda(\Omega_j)}\int_{\Omega_j} z(y,t)\,dy\,dx\\
&= -2\int_{\Omega_j} z^T(x,t)P_1BKz(x,t)\,dx+2\int_{\Omega_j} z^T(x,t)P_1BK\times\\
&\quad \left[z(x,t)-\tfrac{b_j(x)}{\lambda(\Omega_j)}\int_{\Omega_j} z(y,t)\,dy\right]dx+2\int_{\Omega_j} z^T(x,t)P_1B\times\\
&\quad b_j(x)\left[\tfrac{K}{\lambda(\Omega_j)}\int_{\Omega_j} z(y,t)\,dy+u_j(t_{j,k})+w_j(t)\right]dx.
\end{aligned}
\tag{A.5}
$$

Denote

$$
\kappa(x,t)=z(x,t)-\tfrac{b_j(x)}{\lambda(\Omega_j)}\int_{\Omega_j} z(y,t)\,dy,\quad x\in\Omega_j.
$$

Now we derive the inequality

$$
\begin{aligned}
0\le -\sum_{j=1}^{N_s}\int_{\Omega_j}\kappa^T\Lambda_\kappa\kappa+2N\varepsilon(1+\nu^{-1})\sum_{j=1}^{N_s}\int_{\Omega_j} z^T\Lambda_\kappa z\\
+(1+\nu)\tfrac{l^2}{\pi^2}\sum_{j=1}^{N_s}\int_{\Omega_j}(\nabla z)^T(\Lambda_\kappa\otimes I_N)\nabla z,
\end{aligned}
\tag{A.6}
$$

which allows to bound $\kappa(x,t)$ and compensate the second term of (A.5). By Young's inequality,

$$
\begin{aligned}
&\int_{\Omega_j}(\kappa^m(x,t))^2\,dx=\int_{\Omega_j}\Big[z^m(x,t)-\tfrac{1}{\lambda(\Omega_j)}\int_{\Omega_j} z^m(y,t)\,dy\\
&\quad +\tfrac{1-b_j(x)}{\lambda(\Omega_j)}\int_{\Omega_j} z^m(y,t)\,dy\Big]^2dx\\
&\le (1+\nu)\int_{\Omega_j}\Big[z^m(x,t)-\tfrac{1}{\lambda(\Omega_j)}\int_{\Omega_j} z^m(y,t)\,dy\Big]^2dx\\
&\quad +(1+\nu^{-1})\int_{\Omega_j}\tfrac{(1-b_j(x))^2}{\lambda^2(\Omega_j)}\Big[\int_{\Omega_j} z^m(y,t)\,dy\Big]^2dx.
\end{aligned}
\tag{A.7}
$$

Since

$$
\int_{\Omega_j}\left[z^m(x,t)-\tfrac{1}{\lambda(\Omega_j)}\int_{\Omega_j} z^m(y,t)\,dy\right]dx=0,
$$

the Poincaré inequality (Lemma 3) allows to obtain

$$
\begin{aligned}
&(1+\nu)\int_{\Omega_j}\Big[z^m(x,t)-\tfrac{1}{\lambda(\Omega_j)}\int_{\Omega_j} z^m(y,t)\,dy\Big]^2dx\\
&\quad\le (1+\nu)\tfrac{l^2(\Omega_j)}{\pi^2}\int_{\Omega_j}(\nabla z^m(x,t))^T\nabla z^m(x,t)\,dx.
\end{aligned}
\tag{A.8}
$$

By Bernoulli's inequality,

$$
\int_{\Omega_j\setminus\Omega_j^\varepsilon} dx=\left[1-(1-2\varepsilon)^N\right]\lambda(\Omega_j)\le 2N\varepsilon\lambda(\Omega_j),
$$

which together with Jensen's inequality [30] implies

$$
\begin{aligned}
&(1+\nu^{-1})\int_{\Omega_j}\tfrac{(1-b_j(x))^2}{\lambda^2(\Omega_j)}\Big[\int_{\Omega_j} z^m(y,t)\,dy\Big]^2dx\\
&\quad\le (1+\nu^{-1})\tfrac{1}{\lambda^2(\Omega_j)}\int_{\Omega_j\setminus\Omega_j^\varepsilon} dx\,\lambda(\Omega_j)\int_{\Omega_j}(z^m(y,t))^2\,dy\\
&\quad\le 2N\varepsilon(1+\nu^{-1})\int_{\Omega_j}(z^m(y,t))^2dy.
\end{aligned}
\tag{A.9}
$$

Using the estimates (A.8) and (A.9) in (A.7), we obtain (A.6).

The last term of (A.5) can be presented in the form

$$
\begin{aligned}
&2\ \int_{\Omega_j} z^T(x,t)P_1 B b_j(x)\big[\tfrac{K}{\lambda(\Omega_j)}\int_{\Omega_j} z(y,t)\,dy \\
&\quad +u_j(t_{j,k})+w_j(t)\big]\,dx \\
&= 2\int_{\Omega_j} z^T(x,t_{j,k})P_1 B b_j(x)\,dx\big[\tfrac{K}{\lambda(\Omega_j)}\int_{\Omega_j} z(y,t)\,dy \\
&\quad +u_j(t_{j,k})+w_j(t)\big]+2\int_{\Omega_j} h\eta^T(x,t)P_1 B\times \\
&\quad \big[b_j(x)(u_j(t_{j,k})+w_j(t))+Kz(x,t)-K\kappa(x,t)\big]\,dx
\end{aligned}
\tag{A.10}
$$

where $\eta(x,t)=\frac{1}{h}[z(x,t)-z(x,t_{j,k})]$ for $x\in\Omega_j$, $t\in[t_k,t_{k+1})$. Due to Assumption 3 and (A.2),

$$
w_j(t)+\frac{K}{\lambda(\Omega_j)}\int_{\Omega_j} z(y,t)\,dy\in-\operatorname{conv}\{\mathcal{V}\},\quad \forall j\in 1:N_s.
$$

Then, (5) leads to (cf. (1))

$$
\begin{aligned}
&\int_{\Omega_j} z^T(x,t_{j,k})P_1Bb_j(x)\,dx\,u_j(t_{j,k}) \\
&=\min_{v\in\mathcal{V}}\int_{\Omega_j} z^T(x,t_{j,k})P_1Bb_j(x)\,dx\,v \\
&=\min_{v\in\operatorname{conv}\{\mathcal{V}\}}\int_{\Omega_j} z^T(x,t_{j,k})P_1Bb_j(x)\,dx\,v \\
&\le-\int_{\Omega_j} z^T(x,t_{j,k})P_1Bb_j(x)\,dx\big[w_j(t)+\tfrac{K}{\lambda(\Omega_j)}\int_{\Omega_j} z(y,t)\,dy\big].
\end{aligned}
\tag{A.11}
$$

Therefore, the first term in the right-hand side of (A.10) is nonpositive.

We use the following descriptor representation of (3) [8]:

$$
\begin{aligned}
0=\ &2h\sum_{j=1}^{N_s}\int_{\Omega_j}[z^T(x,t)P_2+z_t^T(x,t)P_3][-z_t(x,t) \\
&+\Delta_D z(x,t)+\beta\nabla z(x,t)+Az(x,t)+f(x,t,z) \\
&+Bb_j(x)(u_j(t_{j,k})+w_j(t))]\,dx,\qquad t\in[t_k,t_{k+1}).
\end{aligned}
\tag{A.12}
$$

Let us transform the terms of (A.4) and (A.12) that involve $\Delta_D z$. Using Green's formula and taking into account the boundary conditions (6) or (7), we obtain

$$
\begin{aligned}
&2\int_\Omega z^T[P_1+hP_2]\Delta_D z \\
&\quad=-2\sum_{m=1}^{M}\int_\Omega[p_1^m+hp_2^m](\nabla z^m)^T D^m\nabla z^m \\
&\quad\le-2\int_\Omega(\nabla z)^T([P_1+hP_2]D_0\otimes I_N)\nabla z, \\
&2h\int_\Omega z_t^T P_3\Delta_D z \\
&\quad=-2h\sum_{m=1}^{M}\int_\Omega p_3^m(\nabla z_t^m)^T D^m\nabla z^m=-\dot V_2.
\end{aligned}
\tag{A.13}
$$

Furthermore,

$$
\begin{aligned}
\dot V_W=-2\alpha V_W+he^{2\alpha h}\sum_{j=1}^{N_s}\int_{\Omega_j} z_t^T W z_t \\
-\tfrac{\pi^2 h}{4}\sum_{j=1}^{N_s}\int_{\Omega_j}\eta^T W\eta.
\end{aligned}
\tag{A.14}
$$

By multiplying the inequalities of Assumption 4 by $\lambda_f^m\ge 0$ and summing them up, we obtain

$$
0\le\sum_{j=1}^{N_s}\int_{\Omega_j}\begin{bmatrix} z\\ f\end{bmatrix}^T\begin{bmatrix}-\mu_T\mu_B\Lambda_f & \frac12(\mu_T+\mu_B)\Lambda_f\\ \frac12(\mu_T+\mu_B)\Lambda_f & -\Lambda_f\end{bmatrix}\begin{bmatrix} z\\ f\end{bmatrix}.
\tag{A.15}
$$

For the Dirichlet boundary conditions (6) we use the Wirtinger inequality (Lemma 2) to obtain

$$
0\le\int_\Omega(\nabla z)^T(\Lambda_D\otimes I_N)\nabla z-N\pi^2\int_\Omega z^T\Lambda_D z. \tag{A.16}
$$

By summing up (A.4), (A.14) with the right-hand sides of (A.6), (A.12), (A.15), (A.16) and taking into account (A.5), (A.10), (A.11), (A.13), we obtain

$$
\begin{aligned}
&\dot V+2\alpha V-\sum_{j=1}^{N_s}\int_{\Omega_j} hb_j^2(x)\big[u_j^T(t_{j,k})\beta_u u_j(t_{j,k}) \\
&+w_j^T(t)\beta_w w_j(t)\big]\,dx\le\sum_{j=1}^{N_s}\int_{\Omega_j}\varphi_j^T(x,t)\Phi\varphi_j(x,t)\,dx,
\end{aligned}
$$

where $\varphi_j=\operatorname{col}\{z,\nabla z,f,\kappa,z_t,\eta,b_j(x)u_j(t_{j,k}),b_j(x)w_j(t)\}$. Therefore, the condition $\Phi\le 0$ guarantees $\dot V\le-2\alpha V+2\alpha C_\infty$, which implies (A.3).

II. Proof of (A.2) *for* $t\ge t_0$

Due to (13), we need to prove

$$
-a_i^T K d_j\le(1-\rho),\quad i\in 1:N_a, \tag{A.17}
$$

where $d_j=\frac{1}{\lambda(\Omega_j)}\int_{\Omega_j} z$. Since for $i\in 1:N_a$,

$$
\min_{-a_i^T K d_j=(1-\rho)} d_j^T P_1 d_j=(1-\rho)^2(a_i^T K P_1^{-1}K^T a_i)^{-1},
$$

due to Assumption 2, it suffices to prove (cf. (2))

$$
d_j^T P_1 d_j<(1-\rho)^2\min_i(a_i^T K P_1^{-1}K^T a_i)^{-1}.
$$

Jensen's inequality implies

$$
\begin{aligned}
d_j^T P_1 d_j&=\lambda^{-2}(\Omega_j)\int_{\Omega_j} z^T P_1\int_{\Omega_j} z \\
&\le\lambda^{-1}(\Omega_j)\int_{\Omega_j} z^T P_1 z\le\tfrac{1}{\min_j\lambda(\Omega_j)}V_1.
\end{aligned}
$$

Therefore, it suffices to show

$$
\begin{aligned}
V_1(t)<\min_j\lambda(\Omega_j)(1-\rho)^2\min_i(a_i^T K P_1^{-1}K^T a_i)^{-1} \\
=(1-\rho)^2 C_0,\quad t\ge t_0.
\end{aligned}
\tag{A.18}
$$

Let (A.18) be false for some $t_1\ge t_0$. Then

$$
V_1(t_0)\le V(t_0)\overset{(15)}{<}(1-\rho)^2 C_0\le V_1(t_1).
$$

Since V_1 is continuous, there must exist $t_*\in(t_0,t_1)$ such that

$$
\begin{aligned}
&V_1(t)<(1-\rho)^2C_0,\quad t\in[t_0,t_*] \\
&V_1(t_*)>V(t_0).
\end{aligned}
\tag{A.19}
$$

The first relation of (A.19) guarantees (A.3) on $[t_0,t_*]$, which implies $V(t_*)\le V(t_0)$. This contradicts the second relation of (A.19), which implies $V(t_*)\ge V_1(t_*)>V(t_0)$. Thus, (A.18) and, consequently, (A.3) are true on $[t_0,\infty)$.

Remark 10 *Note that the error due to sampling* $\eta(x,t)=\frac{1}{h}[z(x,t)-z(x,t_{j,k})]$ *is compensated by the Wirtinger-based term* V_W. *Its derivative* (A.14) *contains* $he^{2\alpha h}\int_\Omega z_t^T W z_t$ *that we compensate using the descriptor representation* (A.12). *This allows to avoid the terms with* $\Delta_D z$ *that would arise if one substituted the expression for* z_t.