Proceedings Paper:

Selivanov, A. orcid.org/0000-0001-5075-7229 and Fridman, E. (2019) Sampled-data H∞ filtering of a 2D heat equation under pointlike measurements. In: 2018 IEEE Conference on Decision and Control (CDC). 2018 IEEE Conference on Decision and Control (CDC), 17-19 Dec 2018, Miami Beach, FL, USA. IEEE , pp. 539-544. ISBN 9781538613962

https://doi.org/10.1109/cdc.2018.8619284

Sampled-data H_∞ filtering of a 2D heat equation under pointlike measurements

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Abstract— **The existing sampled-data observers for 2D heat equations use spatially averaged measurements, i.e., the state values averaged over subdomains covering the entire space domain. In this paper, we introduce an observer for a 2D heat equation that uses pointlike measurements, which are modeled as the state values averaged over small subsets that do not cover the space domain. The key result, allowing for an efficient analysis of such an observer, is a new inequality that bounds the L_2-norm of the difference between the state and its point value by the reciprocally convex combination of the L_2-norms of the first and second order space derivatives of the state. The convergence conditions are formulated in terms of linear matrix inequalities feasible for large enough observer gain and number of pointlike sensors. The results are extended to solve the H_∞ filtering problem under continuous and sampled in time pointlike measurements.**

I. Introduction

Partial differential equations model tremendous amount of processes: heat transfer, fluid dynamics, fusion reactions, wave propagation, etc. Such processes may require feedback control to remain stable, e.g., chemical reactors [1], oil drill strings [2], tokamaks [3], and rotating stall in axial compressors [4]. Here, we construct observers for 2D heat equations with continuous and sampled in time pointlike measurements. These observers can be combined with state-feedback controllers to stabilize 2D reaction-diffusion systems.

For 1D heat equations, *point* observers/controllers have been constructed and analyzed under continuous [5], [6], [7], [8], [9], [10] and sampled in time [7], [11] measurements. N-D diffusion equations with *averaged* measurements (i.e., the state values are averaged over subdomains covering the entire space domain) have been studied in [12], [13], [14]. Constructive analysis of 2D diffusion equations under pointlike measurements and actuators is a challenging problem, especially in the presence of time-delays and disturbances. In this paper, we develop a constructive method allowing to solve the H_∞ filtering problem in the case of a 2D reaction-diffusion system with continuous and sampled in time pointlike measurements.

Point measurements are often modeled using Dirac delta functions. This leads to certain difficulties in the well-posedness analysis and data sampling becomes hard to study due to the unboundedness of the corresponding input/output operators. We use a more convenient approach where pointlike measurements are presented as the average state value over a small enough domain subset. This allows to avoid technical difficulties related to the well-posedness and allows to study data sampling. Similarly to [15], which studied 1D domains, we use the mean value theorem to present such measurements as the state point values. Inspired by [16], we derive an inequality that bounds the L_2-norm of the difference between the state and its point value by the reciprocally convex combination of the L_2-norms of the first and second order space derivatives of the state (Lemma 5). Combining this inequality with a Lyapunov functional, we derive the observer convergence conditions in terms of linear matrix inequalities that are feasible for a high enough observer gain and large enough number of sensors (Section II). The results are extended to solve the H_∞ filtering problem under continuous (Section III) and sampled in time (Section IV) pointlike measurements.

Notations: $\|\cdot\|$ is the L^2-norm, $\operatorname{supp} f$ is the support of function f, $\operatorname{conv}(G)$ is the convex hull.

The following lemmas will be used in the analysis.

Lemma 1: Let $g_i \in L^2$, $i = 1, \ldots, n$. Then

$$\left\|\sum_{i=1}^{n} g_i\right\|^2 \le \sum_{i=1}^{n} \frac{1}{\alpha_i}\|g_i\|^2$$

for any $\alpha_i > 0$ such that $\sum_i \alpha_i = 1$.

Proof. By the convexity of $\|\cdot\|^2$,

$$\left\|\sum_{i=1}^n \alpha_i \tfrac{g_i}{\alpha_i}\right\|^2 \le \sum_{i=1}^n \alpha_i \left\|\tfrac{g_i}{\alpha_i}\right\|^2 = \sum_{i=1}^n \tfrac{1}{\alpha_i}\|g_i\|^2.$$

■

Lemma 2 (Wirtinger's inequality): For $f \in H^1(a,b)$,

$$\|f\| \le \frac{2(b-a)}{\pi}\|f'\| \quad \text{if } f(a)=0 \text{ or } f(b)=0,$$
$$\|f\| \le \frac{(b-a)}{\pi}\|f'\| \quad \text{if } f(a)=0 \text{ and } f(b)=0.$$

Proof. See [17, Chapter 7.7].

Lemma 3 (Exponential Wirtinger's inequality): If $\alpha \in \mathbb{R}$ and $f \in H^1(a,b)$ is such that $f(a)=0$ or $f(b)=0$, then

$$\int_a^b e^{2\alpha t} f^2(t)\,dt \le e^{2|\alpha|(b-a)}\frac{4(b-a)^2}{\pi^2}\int_a^b e^{2\alpha t}\dot{f}^2(t)\,dt.$$

Proof. See [18, Lemma A.18].

Lemma 4 (Jensen inequality): If $\rho\colon [a,b] \to [0,\infty)$ and $f\colon [a,b] \to \mathbb{R}$ are such that the integration concerned is well-defined, then

$$\left[\int_a^b \rho(s) f(s)\,ds\right]^2 \le \int_a^b \rho(s)\,ds \int_a^b \rho(s) f^2(s)\,ds.$$

Proof. See [19, Lemma 1].

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Supported by Israel Science Foundation (Grant No. 1128/14).

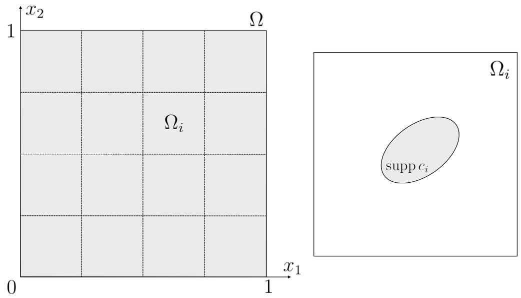

Fig. 1. Subdomains Ω_i and the subset $\operatorname{supp} c_i \subset \overline{\Omega}_i$

II. POINTLIKE OBSERVER FOR A 2D HEAT EQUATION

Consider the reaction-diffusion system

$$\begin{aligned} &z_t(x,t) = \Delta_D z(x,t) + az(x,t), \quad x \in \Omega,\ t>0, \\ &z|_{\partial\Omega} = 0, \quad z|_{t=0} = z_0 \end{aligned} \tag{1}$$

defined on $\Omega = (0,1)\times(0,1) \subset \mathbb{R}^2$ with the state $z\colon \overline{\Omega}\times[0,\infty) \to \mathbb{R}$, reaction coefficient a, and diffusion term

$$\Delta_D z = \operatorname{div}(D\nabla z), \quad 0 < D = \left[\begin{smallmatrix} d_1 & d_2 \\ d_2 & d_3 \end{smallmatrix}\right] \in \mathbb{R}^{2\times 2}. \tag{2}$$

Remark 1: Any open parallelogram $\widetilde{\Omega} \subset \mathbb{R}^2$ can be transformed to $\Omega = (0,1)\times(0,1)$ using a nonsingular change of variable $x = A\widetilde{x} + b$. In this case, $D = A\widetilde{D}A^T$.

Remark 2: For simplicity, we consider a *linear* system. The results can be extended to Lipschitz and sector-bounded nonlinearities in a manner similar to [7], [13].

Let Ω be divided into N square subdomains Ω_i (Fig. 1) with a sensor placed in each Ω_i providing the measurements

$$\begin{aligned} &y_i(t) = \int_{\Omega_i} c_i(\xi) z(\xi,t)\,d\xi, \\ &0 \le c_i \in L^2(\Omega_i), \quad \int_{\Omega_i} c_i = 1, \quad i = 1,\dots,N. \end{aligned} \tag{3}$$

For example,

$$c_i(\xi) = \begin{cases} \frac{1}{\varepsilon^2}, & |\xi - x_c^i|_\infty < \frac{\varepsilon}{2}, \\ 0, & |\xi - x_c^i|_\infty \ge \frac{\varepsilon}{2} \end{cases} \tag{4}$$

with a small $\varepsilon \in (0, 1/\sqrt{N}]$ model point measurements at $x_c^i \in \Omega_i$. The case of $\varepsilon = 1/\sqrt{N}$ was considered in [13].

We study the observer

$$\begin{aligned} &\hat{z}_t(x,t) = \Delta_D \hat{z}(x,t) + a\hat{z}(x,t) \\ &\qquad + L\textstyle\sum_{i=1}^N \chi_i(x)\left[y_i(t) - \int_{\Omega_i} c_i(\xi)\hat{z}(\xi,t)\,d\xi\right], \\ &\hat{z}|_{\partial\Omega} = 0, \quad \hat{z}|_{t=0} = 0 \end{aligned} \tag{5}$$

with the injection gain L and characteristic functions

$$\chi_i(x) = \begin{cases} 1, & x \in \Omega_i, \\ 0, & x \notin \Omega_i, \end{cases} \qquad i = 1,\dots,N. \tag{6}$$

The estimation error $\bar{z}(x,t) = z(x,t) - \hat{z}(x,t)$ satisfies

$$\begin{aligned} &\bar{z}_t = \Delta_D \bar{z} + a\bar{z} - L\textstyle\sum_{i=1}^N \chi_i(x)\int_{\Omega_i} c_i(\xi)\bar{z}(\xi,t)\,d\xi, \\ &\bar{z}|_{\partial\Omega} = 0, \quad \bar{z}|_{t=0} = z_0. \end{aligned} \tag{7}$$

Definition 1: A (classical) solution of (7) is a function $\bar{z} \in C^1([0,\infty); L^2(\Omega))$ such that $\bar{z}(\cdot,t) \in H^2(\Omega)\cap H_0^1(\Omega)$ for $t \ge 0$ and $\bar{z}$ satisfies (7).

Since $A\colon D(A) \subset L^2(\Omega) \to L^2(\Omega)$, $Aw = \Delta_D w$, with $D(A) = H^2(\Omega)\cap H_0^1(\Omega)$ generates a C_0-semigroup [20, Theorem 7.2.7] and $B\colon L^2(\Omega) \to L^2(\Omega)$, $Bw = aw - L\sum_{i=1}^N \chi_i \int_{\Omega_i} c_i w$, is bounded, the operator $A+B$ generates a C_0-semigroup [21, Theorem 3.2.1]. By [21, Theorem 3.1.3], (7) has a unique classical solution for

$$z_0 \in D(A) = H^2(\Omega)\cap H_0^1(\Omega).$$

By the mean value theorem (this idea comes from [15]),

$$\int_{\Omega_i} c_i(\xi)\bar{z}(\xi,t)\,d\xi = \bar{z}(x^i(t),t),$$

where $x^i(t) \in \operatorname{conv}(\operatorname{supp} c_i)$ for $t \ge 0$ and $i = 1,\dots,N$. Denoting

$$\sigma(x,t) = L\bar{z}(x,t) - L\textstyle\sum_{i=1}^N \chi_i(x)\bar{z}(x^i,t), \quad x \in \Omega,\ t \ge 0, \tag{8}$$

we present (7) as

$$\begin{aligned} &\bar{z}_t = \Delta_D \bar{z} + (a-L)\bar{z} + \sigma, \quad x \in \Omega,\ t>0, \\ &\bar{z}|_{\partial\Omega} = 0, \qquad \bar{z}|_{t=0} = z_0. \end{aligned} \tag{9}$$

If $\sigma \equiv 0$, then the system (9) is stable for a large enough injection gain L. If $\Omega = (0,1)$, the error $\sigma \not\equiv 0$ can be bounded using Wirtinger's inequality as $\|\sigma(\cdot,t)\| \le 2/(N\pi)\|\bar{z}_x(\cdot,t)\|$, which was used in [7] to prove the stability of (9) for large L and N. We prove the following lemma to bound the error σ in the case of $\Omega = (0,1)^2$. This lemma refines [16, Lemma 4.1].

Lemma 5: Let $f \in H^2((0,l)^2;\mathbb{R})$, $f(0,0) = 0$. Then

$$\begin{aligned} \|f\|^2 \le \frac{1}{\alpha_1}\left(\frac{2l}{\pi}\right)^2\left\|\frac{\partial f}{\partial x_1}\right\|^2 + \frac{1}{\alpha_2}\left(\frac{2l}{\pi}\right)^2\left\|\frac{\partial f}{\partial x_2}\right\|^2 \\ + \frac{1}{\alpha_3}\left(\frac{2l}{\pi}\right)^4\left\|\frac{\partial^2 f}{\partial x_1\partial x_2}\right\|^2 \end{aligned} \tag{10}$$

for any positive α_1, α_2, α_3 such that $\alpha_1+\alpha_2+\alpha_3 = 1$.

Proof. Since $\alpha_2 \in (0,1)$,

$$\begin{aligned} \|f\|^2 &= \|f(\cdot,0) + (f(\cdot,\cdot) - f(\cdot,0))\|^2 \\ &\overset{\text{Lem.1}}{\le} \tfrac{1}{1-\alpha_2}\|f(\cdot,0)\|^2 + \tfrac{1}{\alpha_2}\|f(\cdot,\cdot) - f(\cdot,0)\|^2 \\ &\overset{\text{Lem.2}}{\le} \tfrac{1}{1-\alpha_2}\left(\tfrac{2l}{\pi}\right)^2\|f_{x_1}(\cdot,0)\|^2 + \tfrac{1}{\alpha_2}\left(\tfrac{2l}{\pi}\right)^2\|f_{x_2}\|^2. \end{aligned}$$

Since $\frac{\alpha_1}{1-\alpha_2} + \frac{\alpha_3}{1-\alpha_2} = 1$,

$$\begin{aligned} \|f_{x_1}(\cdot,0)\|^2 &= \|f_{x_1}(\cdot,\cdot) + (f_{x_1}(\cdot,0) - f_{x_1}(\cdot,\cdot))\|^2 \\ &\overset{\text{Lem.1}}{\le} \tfrac{1-\alpha_2}{\alpha_1}\|f_{x_1}\|^2 + \tfrac{1-\alpha_2}{\alpha_3}\|f_{x_1}(\cdot,0) - f_{x_1}(\cdot,\cdot)\|^2 \\ &\overset{\text{Lem.2}}{\le} \tfrac{1-\alpha_2}{\alpha_1}\|f_{x_1}\|^2 + \tfrac{1-\alpha_2}{\alpha_3}\left(\tfrac{2l}{\pi}\right)^2\|f_{x_1x_2}\|^2. \end{aligned}$$

Combining these inequalities, we obtain (10). ∎

Corollary 1: Let $f \in H^2((0,l)^2;\mathbb{R})$, $f(0,0) = 0$, $\eta > 0$. Then

$$\begin{aligned} \eta\|f\|^2 \le \lambda_1\left(\frac{2l}{\pi}\right)^2\left\|\frac{\partial f}{\partial x_1}\right\|^2 + \lambda_2\left(\frac{2l}{\pi}\right)^2\left\|\frac{\partial f}{\partial x_2}\right\|^2 \\ + \lambda_3\left(\frac{2l}{\pi}\right)^4\left\|\frac{\partial^2 f}{\partial x_1\partial x_2}\right\|^2 \end{aligned} \tag{11}$$

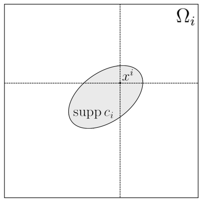

Fig. 2. Four rectangles cornered at $x^i \in \operatorname{supp} c_i$

for any λ_1, λ_2, λ_3 satisfying

$$\operatorname{diag}\{\lambda_1, \lambda_2, \lambda_3\} \geq \eta \left[\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix}\right]. \tag{12}$$

Proof. By the Schur complement, (12) is equivalent to

$$\begin{bmatrix} \eta^{-1} \operatorname{diag}\{\lambda_1, \lambda_2, \lambda_3\} & \mathbf{1}_3 \\ \mathbf{1}_3^T & 1 \end{bmatrix} \geq 0,$$

where $\mathbf{1}_3 = (1, 1, 1)^T$, which is equivalent to

$$0 < \operatorname{diag}\{\lambda_1, \lambda_2, \lambda_3\},$$
$$0 \leq 1 - \eta \mathbf{1}_3^T \operatorname{diag}\{\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}\} \mathbf{1}_3 = 1 - \eta \sum_{i=1}^{3} \lambda_i^{-1}.$$

Thus, for

$$\alpha_1 = \frac{\eta}{\lambda_1}, \quad \alpha_2 = \frac{\eta}{\lambda_2}, \quad \alpha_3 = \frac{\eta}{\lambda_3},$$

we have $\alpha_1 + \alpha_2 + \alpha_3 \leq 1$. Clearly, Lemma 5 remains true for $\alpha_1 + \alpha_2 + \alpha_3 \leq 1$ implying (11). ■

Each rectangle cornered at $x^i \in \operatorname{supp} c_i$ and lying in Ω_i (see Fig. 2) has sides smaller than

$$l = \max_{i=1,\dots,N} \max_{\substack{\omega \in \partial\Omega_i \\ d \in \operatorname{supp} c_i}} |\omega - d|_\infty. \tag{13}$$

Applying Corollary 1 to σ defined in (8) on each of such rectangles and summing over them, we obtain

$$0 \leq -\eta \frac{\|\sigma\|^2}{L^2} + \lambda_1 \left(\frac{2l}{\pi}\right)^2 \|\bar{z}_{x_1}\|^2 + \lambda_2 \left(\frac{2l}{\pi}\right)^2 \|\bar{z}_{x_2}\|^2 + \lambda_3 \left(\frac{2l}{\pi}\right)^4 \|\bar{z}_{x_1 x_2}\|^2 \tag{14}$$

with $\eta > 0$, λ_1, λ_2, λ_3 satisfying (12). The positive terms in (14) can be made arbitrarily small by reducing l, i.e., by increasing the number of sensors N.

Theorem 1: Consider the system (1) with the measurements (3). For a given injection gain L and decay rate $\alpha > 0$, let there exist[1]

$$P = \left[\begin{smallmatrix} p_1 & p_2 \\ p_2 & p_3 \end{smallmatrix}\right] > 0, \quad \eta > 0, \quad \lambda_i > 0, \quad i = 1, \dots, 6,$$

such that (12) is true, $\Phi \leq 0$, and $\Phi_\nabla \leq 0$, where

$$\Phi = \begin{bmatrix} \Phi_{11} & 0 & 1 \\ * & \Phi_{22} & -\bar{p} \\ * & * & -\eta/L^2 \end{bmatrix},$$
$$\Phi_{11} = 2(a - L + \alpha) - (\lambda_5 + \lambda_6)\pi^2,$$
$$\Phi_{22} = -\bar{p}\bar{d}^T - \bar{d}\bar{p}^T + \begin{bmatrix} 0 & 0 & \lambda_4 \\ 0 & \lambda_3 (2l/\pi)^4 - 2\lambda_4 & 0 \\ \lambda_4 & 0 & 0 \end{bmatrix},$$
$$\Phi_\nabla = -2D + 2(a - L + \alpha)P + \tfrac{(2l)^2}{\pi^2} \left[\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right] + \left[\begin{smallmatrix} \lambda_5 & 0 \\ 0 & \lambda_6 \end{smallmatrix}\right]$$

[1]MATLAB codes for solving the LMIs are available at https://github.com/AntonSelivanov/CDC18

with l defined in (13), $\bar{p} = (p_1, 2p_2, p_3)^T$, and $\bar{d} = (d_1, 2d_2, d_3)^T$. Then the state of the observer (5) exponentially converges to the state of the system (1) in the H_0^1-norm for any initial conditions $z_0 \in H^2(\Omega) \cap H_0^1(\Omega)$:

$$\exists\, C > 0: \quad \|z(\cdot, t) - \hat{z}(\cdot, t)\|_{H_0^1} \leq C e^{-\alpha t} \|z_0\|_{H_0^1}.$$

Proof. Differentiating $V_0 = \|\bar{z}\|^2$ along (9), we obtain

$$\dot{V}_0 = 2 \textstyle\int_\Omega \bar{z}\bar{z}_t = 2 \int_\Omega \bar{z} \left[\Delta_D \bar{z} + (a - L)\bar{z} + \sigma\right].$$

Since $\bar{z}|_{\partial\Omega} = 0$, by the divergence theorem,

$$2 \textstyle\int_\Omega \bar{z} \Delta_D \bar{z} = 2 \int_\Omega \bar{z} \operatorname{div}(D\nabla\bar{z}) = -2 \int_\Omega (\nabla\bar{z})^T D \nabla\bar{z}.$$

Therefore,

$$\dot{V}_0 + 2\alpha V_0 = -2 \textstyle\int_\Omega (\nabla\bar{z})^T D \nabla\bar{z} + 2(a - L + \alpha) \int_\Omega \bar{z}^2 + 2 \int_\Omega \bar{z}\sigma. \tag{15}$$

If $\sigma \equiv 0$, (15) is nonpositive for a large L implying the exponential stability of (9) in L^2. To compensate $\sigma \not\equiv 0$, we will use (14) that contains $\|\bar{z}_{x_1 x_2}\|^2$. For $\bar{z} \in C_0^\infty$ integration by parts yields

$$0 = -2\lambda_4 \int_\Omega \bar{z}_{x_1 x_2}^2 + 2\lambda_4 \int_\Omega \bar{z}_{x_1 x_1} \bar{z}_{x_2 x_2}. \tag{16}$$

Since C_0^∞ is dense in $H^2 \cap H_0^1$, the latter holds for $\bar{z} \in H^2 \cap H_0^1$. To compensate $\bar{z}_{x_1 x_1}$ and $\bar{z}_{x_2 x_2}$, we consider

$$V_1 = \textstyle\int_\Omega (\nabla\bar{z}(x, t))^T P \nabla\bar{z}(x, t)\, dx. \tag{17}$$

Since $\bar{z}|_{\partial\Omega} = 0$, by the divergence theorem,

$$\dot{V}_1 = 2 \int_\Omega (\nabla\bar{z})^T P \nabla\bar{z}_t = -2 \int_\Omega \operatorname{div}(P\nabla\bar{z})\, \bar{z}_t.$$

Substituting $\bar{z}_t$, we obtain

$$\dot{V}_1 + 2\alpha V_1 = -2 \textstyle\int_\Omega \operatorname{div}(P\nabla\bar{z}) \operatorname{div}(D\nabla\bar{z}) + 2(a - L + \alpha) \int_\Omega (\nabla\bar{z})^T P \nabla\bar{z} - 2 \int_\Omega \operatorname{div}(P\nabla\bar{z})\, \sigma, \tag{18}$$

where we used the divergence theorem to obtain the second summand. Wirtinger's inequality (Lemma 2) implies

$$0 \leq -(\lambda_5 + \lambda_6)\pi^2 \int_\Omega \bar{z}^2 + \int_\Omega (\nabla\bar{z})^T \left[\begin{smallmatrix} \lambda_5 & 0 \\ 0 & \lambda_6 \end{smallmatrix}\right] \nabla\bar{z}. \tag{19}$$

Summing up (14)–(19), for $V = V_0 + V_1$ we obtain

$$\dot{V} + 2\alpha V \leq \int_\Omega \varphi^T \Phi \varphi + \int_\Omega (\nabla\bar{z})^T \Phi_\nabla \nabla\bar{z} \leq 0,$$

where $\varphi = (\bar{z}, \bar{z}_{x_1 x_1}, \bar{z}_{x_1 x_2}, \bar{z}_{x_2 x_2}, \sigma)^T$. Thus, $\dot{V} \leq -2\alpha V$ implying the exponential stability of (9) in the H_0^1-norm. ■

Remark 3 (Feasibility of LMIs): The LMIs of Theorem 1 are always feasible for a large enough injection gain L and small enough l defined in (13). Indeed, $D > 0$ implies $d_1 d_3 - d_2^2/\nu > 0$ for a large enough $\nu < 1$. Since

$$2 \left[\begin{smallmatrix} 0 & -d_1 d_2 \\ -d_1 d_2 & 0 \end{smallmatrix}\right] \leq 2 \operatorname{diag}\{\nu d_1^2, d_2^2/\nu\},$$
$$2 \left[\begin{smallmatrix} 0 & -d_2 d_3 \\ -d_2 d_3 & 0 \end{smallmatrix}\right] \leq 2 \operatorname{diag}\{d_2^2/\nu, \nu d_3^2\},$$

for $l=0$, $p_1=d_3$, $p_2=0$, $p_3=d_1$, and $\lambda_4=d_1^2+d_3^2$,

$$\Phi_{22}\le\begin{bmatrix}-2(d_1d_3-\frac{d_2^2}{\nu}) & 0 & 0\\ 0 & -2(1-\nu)\lambda_4 & 0\\ 0 & 0 & -2(d_1d_3-\frac{d_2^2}{\nu})\end{bmatrix}<0.$$

Therefore, $\Phi<0$ for large enough L and η. Clearly, $\Phi_\nabla<0$ for a large enough L and (12) holds for large enough λ_1, λ_2, and λ_3. Thus, the LMIs of Theorem 1 are feasible for $l=0$. By continuity, they remain so for a small enough l.

Corollary 2: The observer (5) provides exponentially converging state estimate of the system (1), (3) if the injection gain L is large enough and l defined in (13) is small enough (i.e., the number of sensors N is large enough).

Remark 4 (Boundary conditions): The results can be extended to (1) with the boundary conditions

$$z|_{\Gamma_D}=0,\quad \tfrac{\partial z}{\partial \mathbf{n}}|_{\Gamma_N}=0,\quad \partial\Omega=\Gamma_D\cup\Gamma_N,\quad \Gamma_D\cap\Gamma_N=\varnothing,$$

where $\mathbf{n}$ denotes the normal to Γ_N. All the calculations of Theorem 1 remain valid except for (19), which according to the Wirtinger inequality (Lemma 2) should be replaced by

$$\begin{aligned}0&\le-\lambda_5c_1\pi^2\textstyle\int_\Omega\bar z^2+\lambda_5\int_\Omega\bar z_{x_1}^2,\\ 0&\le-\lambda_6c_2\pi^2\textstyle\int_\Omega\bar z^2+\lambda_6\int_\Omega\bar z_{x_2}^2,\end{aligned}$$

where

$$c_1=\begin{cases}1, & \text{if } z(0,x_2)=z(1,x_2)=0,\ \forall x_2\in(0,1)\\ \frac14, & \text{if } z(0,x_2)=0 \text{ or } z(1,x_2)=0,\ \forall x_2\in(0,1)\\ 0, & \text{otherwise,}\end{cases}$$

$$c_2=\begin{cases}1, & \text{if } z(x_1,0)=z(x_1,0)=0,\ \forall x_1\in(0,1)\\ \frac14, & \text{if } z(x_1,0)=0 \text{ or } z(x_1,0)=0,\ \forall x_1\in(0,1)\\ 0, & \text{otherwise.}\end{cases}$$

Remark 5 (3D domains): If $\Omega=(0,1)^3$, an upper bound for σ similar to (14) can be derived. This bound will involve the 3rd order space derivative, which we do not know how to compensate. Thus, it is not clear how to extend the proposed method to 3D domains.

III. H_∞ FILTERING OF A 2D HEAT EQUATION

Consider the reaction-diffusion system

$$\begin{aligned}&z_t(x,t)=\Delta_Dz(x,t)+az(x,t)+w(x,t),\quad t>0,\ x\in\Omega\\ &z|_{\partial\Omega}=0,\quad z|_{t=0}=z_0\end{aligned}\tag{20}$$

defined on $\Omega=(0,1)\times(0,1)\subset\mathbb{R}^2$ with the state $z\colon\overline{\Omega}\times[0,\infty)\to\mathbb{R}$, diffusion term (2), reaction coefficient a, and disturbance $w\in L^2(0,\infty;L^2(\Omega))$.

Similarly to the previous section, the domain Ω is divided into N square subdomains Ω_i (Fig. 1) with a sensor placed in each Ω_i providing the measurements

$$\begin{aligned}&y_i(t)=\int_{\Omega_i}c_i(\xi)z(\xi,t)\,d\xi+v_i(t),\\ &0\le c_i\in L^2(\Omega_i),\quad \int_{\Omega_i}c_i=1,\quad i=1,\dots,N,\end{aligned}\tag{21}$$

where $v_i\in L^2(0,\infty)$ is the measurement noise.

Consider the observer (5). The estimation error $\bar z(x,t)=z(x,t)-\hat z(x,t)$ satisfies (cf. (7))

$$\begin{aligned}&\bar z_t=\Delta_D\bar z+a\bar z-L\textstyle\sum_{i=1}^N\chi_i(x)\bar z(x^i,t)+w-v,\\ &\bar z|_{\partial\Omega}=0,\quad \bar z|_{t=0}=z_0,\end{aligned}\tag{22}$$

where

$$v(x,t)=L\textstyle\sum_{i=1}^N\chi_i(x)v_i(t).$$

We assume that w and v are such that (22) is well-posed. E.g., if $w,v\in C^1([0,\infty),L^2)$, the system (22) has a unique classical solution for any $z_0\in H^2(\Omega)\cap H_0^1(\Omega)$ [21, Theorem 3.1.3].

We say that (22) has an L^2-gain not greater than γ, if

$$\int_0^\infty\|\bar z(\cdot,t)\|^2\,dt\le\gamma^2\int_0^\infty\left[\|w(\cdot,t)\|^2+\|v(\cdot,t)\|^2\right]dt\tag{23}$$

for $z_0=0$ and any $w,v\in L^2(0,\infty;L^2(\Omega))$.

Theorem 2: Consider the system (20) with the measurements (21). For a given injection gain L and decay rate $\alpha>0$, let there exist[2]

$$P=\left[\begin{smallmatrix}p_1&p_2\\p_2&p_3\end{smallmatrix}\right]>0,\quad \eta,\gamma_1,\gamma_2>0,\quad \lambda_i>0,\quad i=1,\dots,6,$$

such that (12) is true, $\Psi\le0$, and $\Phi_\nabla\le0$, where

$$\Psi=\left[\begin{array}{ccc|cc} & & & 1 & 1\\ & \bar\Phi & & -\bar p & -\bar p\\ & & & 0 & 0\\ \hline * & * & * & -\gamma_2 & 0\\ * & * & * & * & -\gamma_2\end{array}\right],$$

$\bar\Phi$ coincides with Φ from Theorem 1 except for

$$\bar\Phi_{11}=2(a-L+\alpha)-(\lambda_5+\lambda_6)\pi^2+\gamma_1,$$

$\bar p=(p_1,2p_2,p_3)^T$, and Φ_∇ is given in Theorem 1. Then (22) has an L^2-gain not greater than $\gamma=\sqrt{\gamma_2/\gamma_1}$.

Proof. Using (8), we present (22) as (cf. (9))

$$\begin{aligned}&\bar z_t=\Delta_D\bar z+(a-L)\bar z+\sigma+w-v,\ \ x\in\Omega,\ t>0,\\ &\bar z|_{\partial\Omega}=0,\qquad \bar z|_{t=0}=z_0.\end{aligned}\tag{24}$$

Differentiating $V_0=\|\bar z\|^2$ and V_1 defined in (17) along (24) and using the divergence theorem, we obtain (cf. (15), (18))

$$\begin{aligned}\dot V_0+2\alpha V_0=&-2\textstyle\int_\Omega(\nabla\bar z)^TD\nabla\bar z+2(a-L+\alpha)\int_\Omega\bar z^2\\ &+2\textstyle\int_\Omega\bar z\sigma+2\int_\Omega\bar z[w-v],\end{aligned}\tag{25}$$

$$\begin{aligned}\dot V_1+2\alpha V_1=&-2\textstyle\int_\Omega\operatorname{div}(P\nabla\bar z)\operatorname{div}(D\nabla\bar z)\\ &+2(a-L+\alpha)\textstyle\int_\Omega(\nabla\bar z)^TP\nabla\bar z-2\int_\Omega\operatorname{div}(P\nabla\bar z)\,\sigma\\ &-2\textstyle\int_\Omega\operatorname{div}(P\nabla\bar z)\,[w-v].\end{aligned}\tag{26}$$

Summing up (14), (16), (19), (25), and (26), for $V=V_0+V_1$ we obtain

$$\begin{aligned}&\dot V+2\alpha V+\gamma_1\|\bar z(\cdot,t)\|^2-\gamma_2\left[\|w(\cdot,t)\|^2+\|v(\cdot,t)\|^2\right]\\ &\qquad\le\textstyle\int_\Omega\psi^T\Psi\psi+\int_\Omega(\nabla\bar z)^T\Phi_\nabla\nabla\bar z\le0,\end{aligned}\tag{27}$$

where $\psi=(\bar z,\bar z_{x_1x_1},\bar z_{x_1x_2},\bar z_{x_2x_2},\sigma,w,-v)^T$. Integrating (27) from 0 to ∞ with $\bar z(\cdot,0)=0$, we obtain (23) with $\gamma=\sqrt{\gamma_2/\gamma_1}$. ■

[2]MATLAB codes for solving the LMIs are available at https://github.com/AntonSelivanov/CDC18

IV. Sampled-data H_∞ filtering

Consider the reaction-diffusion system (20). Let the domain Ω be divided into N square subdomains Ω_i (Fig. 1) with a sensor placed in each Ω_i providing the sampled in time measurements (cf. (21))

$$\begin{aligned} &y_{i,k} = \int_{\Omega_i} c_i(\xi) z(\xi, t_k)\, d\xi + v_{i,k}, \\ &0 \le c_i \in L^\infty(\Omega_i), \quad \int_{\Omega_i} c_i = 1, \quad i = 1, \dots, N, \end{aligned} \tag{28}$$

where $v_{i,k}$ is the measurement noise and the sampling instants t_k with $k \in \mathbb{N}$ satisfy

$$0 = t_1 < t_2 < \cdots, \quad \lim t_k = \infty, \quad t_{k+1} - t_k \le h.$$

We study the sampled-data observer (cf. (5))

$$\begin{aligned} &\hat{z}_t(x,t) = \Delta_D \hat{z}(x,t) + a\hat{z}(x,t) + L\sum_{i=1}^N \chi_i(x) \times \\ &\qquad \left[y_{i,k} - \int_{\Omega_i} c_i(\xi)\hat{z}(\xi, t_k)\, d\xi\right], \quad t \in [t_k, t_{k+1}),\ k \in \mathbb{N}, \\ &\hat{z}|_{\partial\Omega} = 0, \quad \hat{z}|_{t=0} = 0 \end{aligned} \tag{29}$$

with the injection gain L and characteristic functions χ_i defined in (6). The estimation error $\bar{z}(x,t) = z(x,t) - \hat{z}(x,t)$ satisfies (cf. (7), (22))

$$\begin{aligned} &\bar{z}_t = \Delta_D \bar{z} + a\bar{z} - L\sum_{i=1}^N \chi_i(x) \int_{\Omega_i} c_i(\xi)\bar{z}(\xi, t_k)\, d\xi \\ &\qquad\qquad + w - v, \quad t \in [t_k, t_{k+1}), \\ &\bar{z}|_{\partial\Omega} = 0, \quad \bar{z}|_{t=0} = z_0, \end{aligned} \tag{30}$$

where $v(x,t) = L\sum_{i=1}^N \chi_i(x) v_{i,k}$ for $t \in [t_k, t_{k+1})$. The existence of a unique classical solution of (30) for $z_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ can be established using the step method.

Theorem 3: Consider the system (20) with the measurements (28). For a given injection gain L, decay rate $\alpha > 0$, and maximum sampling period h, let there exist[3]

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} > 0,\ \eta, \gamma_1, \gamma_2, \nu > 0,\ \lambda_i > 0,\ i = 1, \dots, 6,$$

such that (12) is true and $\Upsilon \le 0$, $\Phi_\nabla \le 0$, where

$$\Upsilon = \left[\begin{array}{ccccc|cc} & & & & & 1 & \nu h(a-L) \\ & & & & & -\bar{p} & \nu h\bar{d} \\ & & \Psi & & & 0 & \nu h \\ & & & & & 0 & \nu h \\ & & & & & 0 & \nu h \\ \hline * & * & * & * & * & -\nu & \nu h \\ * & * & * & * & * & * & \Upsilon_{77} \end{array}\right],$$

$$\Upsilon_{77} = -\frac{\pi^2 N \nu e^{-2\alpha h}}{4L^2 \max_i \|c_i\|_\infty},$$

Ψ is given in Theorem 2, $\bar{p} = (p_1, 2p_2, p_3)^T$, $\bar{d} = (d_1, 2d_2, d_3)^T$, and Φ_∇ is given in Theorem 1. Then (30) has an L^2-gain less or equal than $\gamma = \sqrt{\gamma_2/\gamma_1}$.

Proof. By the mean value theorem, for $t \in [t_k, t_{k+1})$

$$\begin{aligned} &L\sum_{i=1}^N \chi_i(x) \int_{\Omega_i} c_i(\xi)\bar{z}(\xi, t_k)\, d\xi \\ &\qquad = L\sum_{i=1}^N \chi_i(x) \int_{\Omega_i} c_i(\xi)\bar{z}(\xi, t)\, d\xi - \kappa(x,t) \\ &\qquad = L\sum_{i=1}^N \chi_i(x)\bar{z}(x^i(t), t) - \kappa(x,t) \\ &\qquad = L\bar{z}(x,t) - \sigma(x,t) - \kappa(x,t) \end{aligned}$$

with $x^i(t) \in \operatorname{conv}(\operatorname{supp} c_i)$, $\sigma(x,t)$ defined in (8), and

$$\begin{aligned} \kappa(x,t) = L\sum_{i=1}^N \chi_i(x) \int_{\Omega_i} c_i(\xi)\left[\bar{z}(\xi,t) - \bar{z}(\xi,t_k)\right] d\xi, \\ x \in \Omega, \quad t \in [t_k, t_{k+1}). \end{aligned}$$

Then the error system (30) takes the form (cf. (24))

$$\begin{aligned} &\bar{z}_t = \Delta_D \bar{z} + (a - L)\bar{z} + \sigma + \kappa + w - v, \\ &\bar{z}|_{\partial\Omega} = 0, \quad \bar{z}|_{t=0} = z_0. \end{aligned} \tag{31}$$

Let $V = V_0 + V_1 + V_\kappa$ with $V_0 = \|\bar{z}\|^2$, V_1 from (17), and

$$\begin{aligned} V_\kappa = &\tfrac{4\nu h^2}{\pi^2} e^{2\alpha h} \int_{t_k}^t e^{-2\alpha(t-s)} \int_\Omega \kappa_t^2(x,s)\, dx\, ds \\ &- \nu \int_{t_k}^t e^{-2\alpha(t-s)} \int_\Omega \kappa^2(x,s)\, dx\, ds, \quad t \in [t_k, t_{k+1}). \end{aligned}$$

Due to Lemma 3, $V_\kappa \ge 0$. Moreover, V_κ does not grow at the jumps t_k, since $V_\kappa(t_k) = 0$. Differentiating V along (31), we have (cf. (25), (26))

$$\begin{aligned} \dot{V}_0 + 2\alpha V_0 = &-2\int_\Omega (\nabla\bar{z})^T D\nabla\bar{z} + 2(a - L + \alpha)\int_\Omega \bar{z}^2 \\ &+ 2\int_\Omega \bar{z}\sigma + 2\int_\Omega \bar{z}\kappa + 2\int_\Omega \bar{z}[w - v], \end{aligned} \tag{32}$$

$$\begin{aligned} \dot{V}_1 + 2\alpha V_1 = &-2\int_\Omega \operatorname{div}(P\nabla\bar{z}) \operatorname{div}(D\nabla\bar{z}) \\ &+ 2(a - L + \alpha)\int_\Omega (\nabla\bar{z})^T P\nabla\bar{z} - 2\int_\Omega \operatorname{div}(P\nabla\bar{z})\,\sigma \\ &- 2\int_\Omega \operatorname{div}(P\nabla\bar{z})\,\kappa - 2\int_\Omega \operatorname{div}(P\nabla\bar{z})[w - v], \end{aligned} \tag{33}$$

$$\dot{V}_\kappa + 2\alpha V_\kappa = \tfrac{4\nu h^2}{\pi^2} e^{2\alpha h} \int_\Omega \kappa_t^2(x,t)\, dx - \nu \int_\Omega \kappa^2(x,t)\, dx. \tag{34}$$

The positive term in (34) can be bounded as

$$\begin{aligned} &\int_\Omega \kappa_t^2(x,t)\, dx \\ &\quad = \int_\Omega \left(L\sum_{i=1}^N \chi_i(x) \int_{\Omega_i} c_i(\xi)\bar{z}_t(\xi,t)\, d\xi\right)^2 dx \\ &\quad = L^2 \int_\Omega \sum_{i=1}^N \chi_i(x) \left(\int_{\Omega_i} c_i(\xi)\bar{z}_t(\xi,t)\, d\xi\right)^2 dx \\ &\quad \overset{\text{Lem.4}}{\le} L^2 \int_\Omega \sum_{i=1}^N \chi_i(x) \int_{\Omega_i} c_i(\xi)\bar{z}_t^2(\xi,t)\, d\xi\, dx \\ &\quad \le L^2 \max_i \|c_i\|_\infty \int_\Omega \sum_{i=1}^N \chi_i(x) \int_{\Omega_i} \bar{z}_t^2(\xi,t)\, d\xi\, dx \\ &\quad = \max_i \|c_i\|_\infty \tfrac{L^2}{N} \int_\Omega \bar{z}_t^2(\xi,t)\, d\xi. \end{aligned}$$

Summing up (14), (16), (19), (32)–(34), we obtain

$$\begin{aligned} &\dot{V} + 2\alpha V + \gamma_1\|\bar{z}(\cdot,t)\|^2 - \gamma_2\left[\|w(\cdot,t)\|^2 + \|v(\cdot,t)\|^2\right] \\ &\quad \le \int_\Omega \upsilon^T \bar{\Upsilon} \upsilon + \int_\Omega (\nabla\bar{z})^T \Phi_\nabla \nabla\bar{z} \\ &\qquad + \tfrac{4\nu h^2}{\pi^2} e^{2\alpha h} \max_i \|c_i\|_\infty \tfrac{L^2}{N} \int_\Omega \bar{z}_t^2(x,t)\, dx, \end{aligned}$$

where $\upsilon = (\bar{z}, \bar{z}_{x_1x_1}, \bar{z}_{x_1x_2}, \bar{z}_{x_2x_2}, \sigma, w, -v, \kappa)^T$ and $\bar{\Upsilon}$ is obtained from Υ by eliminating the last block-column and block-row. Substituting (31) for $\bar{z}_t$ and using the Schur complement, we obtain that $\Upsilon < 0$ and $\Phi_\nabla < 0$ guarantee

$$\dot{V} + 2\alpha V + \gamma_1\|\bar{z}(\cdot,t)\|^2 - \gamma_2\left[\|w(\cdot,t)\|^2 + \|v(\cdot,t)\|^2\right] \le 0.$$

Integrating it from 0 to ∞ with $\bar{z}(\cdot,0) = 0$, we obtain (23) with $\gamma = \sqrt{\gamma_2/\gamma_1}$. ■

[3]MATLAB codes for solving the LMIs are available at
https://github.com/AntonSelivanov/CDC18

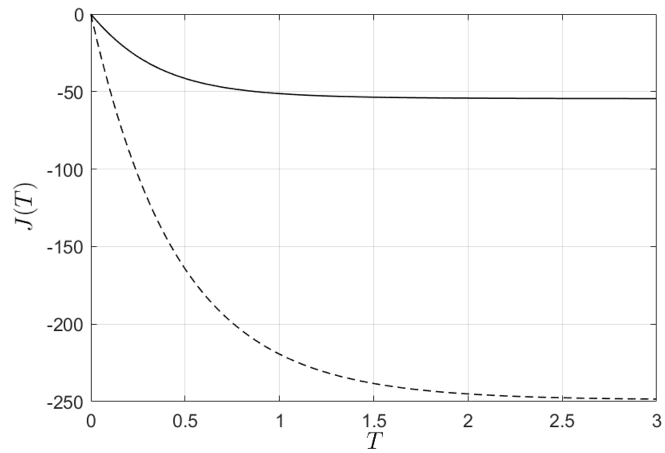

Fig. 3. Performance index $J(T)$ on $[0,3]$ for continuous-time measurements (21) with $\gamma = 2.4$ (solid line) and sampled-data measurements (28) with $h = 10^{-3}$ and $\gamma = 4.6$ (dashed line).

V. EXAMPLE

Consider the system (20) with $D = \operatorname{diag}\{1, 0.8\}$ and $a = 2\pi^2$. Let the domain $\Omega = (0,1)^2$ be divided into $N = 36$ squares of side length $1/\sqrt{N} = 1/6$. Let the measurements be given by (21) with c_i defined in (4), where x_c^i are the centers of Ω_i and $\varepsilon = 0.05$. Then $l = 1/(2\sqrt{N}) + \varepsilon/2 \approx 0.1$ according to (13). The LMIs of Theorem 2 are feasible for $L = 5$, $\gamma = 2.4$, $\alpha = 0.01$. Thus, the observer (5) provides H_∞ filtering of the system (20) with the L^2-gain not greater than $\gamma = 2.4$. Fig. 3 shows the evolution of

$$J(T) = \int_0^T \left[\|\bar{z}(\cdot,t)\|^2 - \gamma^2 \|w(\cdot,t)\|^2 - \gamma^2 \|v(\cdot,t)\|^2 \right] dt$$

for $z_0 \equiv 0$, $w(x,t) = e^{-t}\sin(\pi x)\sin(\pi y)$, $v_i(t) = e^{-t}$. It remains negative implying that (23) is satisfied. For this choice of w and v_i the smallest L_2-gain obtained from the numerical simulations is $\gamma = 1.2$.

The LMIs of Theorem 3 are feasible for $\gamma = 4.6$ and $h = 10^{-3}$ (other parameters are the same). Therefore, the sampled-data observer (29) provides H_∞ filtering of the system (20) with the L^2-gain not greater than $\gamma = 4.6$.

VI. CONCLUSIONS

Design of sampled-data observers for 2D parabolic systems with point measurements was an open problem [13]. This paper suggested a solution to this problem for linear 2D reaction-diffusion systems with the pointlike measurements modeled as the state values averaged over small subdomains. The solution is based on a novel bound on the L_2-norm of the difference between the state and its point value in terms of a reciprocally convex combination of the L_2-norms of the first and second order state derivatives. The results can be extended to semilinear systems as considered in [13]. Extension of the results to the controller design is a topic of the future research.

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