Article:

Selivanov, A. orcid.org/0000-0001-5075-7229 and Fridman, E. (2019) Delayed H∞ control of 2D diffusion systems under delayed pointlike measurements. Automatica, 109. 108541. ISSN 0005-1098

https://doi.org/10.1016/j.automatica.2019.108541

Delayed H_∞ control of 2D diffusion systems under delayed pointlike measurements

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Abstract

Up to now, robust control of multi-dimensional diffusion systems was confined to *averaged* measurements. In this paper, we consider 2D diffusion systems with delayed *pointlike* measurements. A pointlike measurement is the state value averaged over a small subdomain that approximates its point value. The main novelty enabling the study of such measurements is a new inequality, which we call the reciprocally convex variation of Friedrich's inequality. It bounds the difference between a function and its point values in the L^2-norm using the function's derivatives. Combining this result with a new Lyapunov–Krasovskii functional, which has a spatially-varying kernel, we solve the H_∞ control and filtering problems in the presence of time-varying input and output delays. We show that any 2D semilinear diffusion system with pointlike measurements can be stabilized by static output feedback applied through characteristic functions if the controller gain and number of sensors/actuators are large enough while the input and output delays are sufficiently small. The results are demonstrated on a 2D catalytic slab model.

Key words: Distributed parameter systems, time-delays, Lyapunov–Krasovskii functionals, LMIs

1 Introduction

Partial differential equations model tremendous amount of processes: heat transfer, fluid dynamics, fusion reactions, wave propagation, etc. Many of these processes require feedback control to remain stable, e.g., chemical reactors [1], oil drill strings [2], tokamaks [3], and axial compressors [4]. In this paper, we study robust stabilization of 2D semilinear diffusion systems (i.e., those composed of a linear diffusion and a nonlinearity) under delayed pointlike measurements.

A *pointlike measurement* is the state value average over a small subdomain, which approximates its point value [5]. Point measurements are usually modeled by the Dirac delta function. Such an approach is quite theoretical since a physical device occupies a certain region and cannot operate in one point. Moreover, it leads to considerable difficulties in the stability and performance analysis, especially in the presence of time-delays.

For 1D heat equations, *point* observers/controllers have been constructed and analyzed under continuous [6–11] and sampled in time [8,12,13] measurements. N-D diffusion equations with *averaged* measurements (i.e., the state values are averaged over subdomains covering the entire space domain) were studied in [14–16]. Robust stabilization of N-D diffusion systems under *pointlike* measurements is an open challenging problem. In this paper, we resolve this problem for 2D domains and provide robust stability conditions in terms of linear matrix inequalities (LMIs). The key steps that allowed us to do so are the following:

(1) We derive a new inequality, which is a reciprocally convex variation of Friedrich's inequality. It bounds the difference between a function and its point value in the L^2-norm using the reciprocally convex combination of its derivatives (Section 2). This inequality refines and generalizes Lemma 4.1 of [17].
(2) We reduce the pointlike measurements to the point values of the state using the mean value theorem (Section 3). This idea comes from [18], where 1D domains were considered.
(3) In the presence of time-delays, we first isolate the delay-induced error (similarly to [14,15]) and only then apply the mean value theorem to the non-delayed measurements. Then, the delay-induced error enters the systems through a bounded operator. This enables the introduction of a new Lyapunov–Krasovskii

⋆ Supported by Israel Science Foundation (grant No. 1128/14) and C. and H. Manderman Chair at Tel Aviv University.

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term with a spatially-varying kernel that compensates the delay-induced error. Subsequently, this aids in solving the H_∞ control problem (Section 4).

We show that any 2D semilinear diffusion system with pointlike measurements can be stabilized by static output feedback applied through characteristic functions (or shape functions close to them) if the controller gain and number of sensors/actuators are large enough while the input/output delays are sufficiently small. The results are demonstrated on a model of a 2D catalytic slab (Section 6). Preliminary results on H_∞ filtering under pointlike measurements are presented in [19].

Notations: For $\Omega \subset \mathbb{R}^n$, $\overline{\Omega}$ denotes its closure, $\partial\Omega$ is its boundary, $|\Omega|$ is its volume, and $\operatorname{Conv}(\Omega)$ is its convex hull. If $z\colon \overline{\Omega}\times[0,\infty)\to\mathbb{R}$, then z_{x_i} and z_t are the partial derivatives and $\nabla z = (z_{x_1}, z_{x_2})^T$ is the spatial gradient. The divergence of a vector field f is denoted by $\operatorname{div}(f)$. For a matrix P, the notation $P > 0$ implies that P is square, symmetric, and positive definite with the symmetric elements sometimes marked as $*$. We denote $\mathbf{1}_n = (1,\dots,1)^T \in \mathbb{R}^n$, $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$, $\mathbb{R}_{>0} = \{x\in\mathbb{R} \mid x>0\}$. The symbols H^p and H^1_0 correspond to the Sobolev spaces, while $\|\cdot\|$ always stands for the L^2-norm. The support of a function f is denoted by $\operatorname{supp} f$.

2 Reciprocally convex variation of Friedrich's inequality

In this section, we present a new inequality (Theorem 1), which enables studying pointlike measurements on 2D domains (see (23)). This inequality bounds the L^2-norm of the difference between a function and its point value by the reciprocally convex combination of the L^2-norms of its derivatives. Theorem 1 refines and generalizes [17, Lemma 4.1].

For any n-dimensional multi-index $\alpha = (\alpha_1,\dots,\alpha_n)\in\mathbb{N}_0^n$ and sufficiently differentiable function $f\colon\mathbb{R}^n\to\mathbb{R}$, we denote

$$\partial^\alpha f = \frac{\partial^{\alpha_1+\cdots+\alpha_n} f}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}.$$

For example, $\partial^{(3,0,2)} f = \frac{\partial^5 f}{\partial x_1^3 \partial x_3^2}$.

Theorem 1 *Let $f\in H^n((0,l_1)\times\cdots\times(0,l_n))$. Then*

$$\|f(\cdot)-f(0)\|^2 \le \sum_{\alpha\in\mathcal{I}_n} \frac{c_\alpha}{\lambda_\alpha}\|\partial^\alpha f\|^2 \tag{1}$$

for any $\lambda_\alpha\in\mathbb{R}_{>0}$ such that $\sum_\alpha \lambda_\alpha = 1$, where

$\mathcal{I}_n = \{(\alpha_1,\dots,\alpha_n) \mid \alpha_i\in\{0,1\},\ 1\le i\le n,\ \sum_i \alpha_i > 0\}$

are the binary multi-indices with nonzero lengths and

$$c_\alpha = \left(\frac{2l_1}{\pi}\right)^{2\alpha_1}\cdots\left(\frac{2l_n}{\pi}\right)^{2\alpha_n},\quad \alpha=(\alpha_1,\dots,\alpha_n)\in\mathcal{I}_n.$$

PROOF. See Appendix A.

For $n=1$, Theorem 1 coincides with Wirtinger's inequality (Lemma 4). For $f\in H^2((0,l_1)\times(0,l_2))$, Theorem 1 implies

$$\|f(\cdot)-f(0)\|^2 \le \frac{1}{\lambda_{(1,0)}}\left(\frac{2l_1}{\pi}\right)^2\|f_{x_1}\|^2 + \frac{1}{\lambda_{(0,1)}}\left(\frac{2l_2}{\pi}\right)^2\|f_{x_2}\|^2 + \frac{1}{\lambda_{(1,1)}}\left(\frac{2l_1}{\pi}\right)^2\left(\frac{2l_2}{\pi}\right)^2\|f_{x_1x_2}\|^2 \tag{2}$$

with any $\lambda_{(1,0)},\lambda_{(0,1)},\lambda_{(1,1)}\in\mathbb{R}_{>0}$ such that $\lambda_{(1,0)}+\lambda_{(0,1)}+\lambda_{(1,1)}=1$.

Remark 1 *If $l_1=\cdots=l_n=\pi/2$, then $c_\alpha=1$ for all $\alpha\in\mathcal{I}_n$ and the right-hand side of (1) is a reciprocally convex combination of $\|\partial^\alpha f\|^2$.*

Remark 2 *Theorem 1 remains valid for $f\in H^n(\Omega)$ with a non-rectangular $\Omega\subset\mathbb{R}^n$ such that, for all $(x_1,\dots,x_n)\in\Omega$ and $k\in\{1,\dots,n\}$, the vector $(x_1,\dots,x_{k-1},0,\dots,0)$ belongs to Ω.*

The following lemma allows the conditions on λ_α from Theorem 1 to be reformulated as an LMI.

Lemma 1 *The conditions*

$$\mu_i > 0\quad \forall i\in\{1,\dots,n\},\quad \sum_{i=1}^n \mu_i^{-1}\le 1 \tag{3}$$

are equivalent to

$$\operatorname{diag}\{\mu_1,\dots,\mu_n\}\ge \mathbf{1}_n\mathbf{1}_n^T. \tag{4}$$

PROOF. By Schur's complement lemma, (4) is equivalent to

$$\begin{bmatrix}\operatorname{diag}\{\mu_1,\dots,\mu_n\} & \mathbf{1}_n\\ \mathbf{1}_n^T & 1\end{bmatrix}\ge 0,$$

which is equivalent to

$$0<\operatorname{diag}\{\mu_1,\dots,\mu_n\},$$

$$0\le 1-\mathbf{1}_n^T\operatorname{diag}\{\mu_1^{-1},\dots,\mu_n^{-1}\}\mathbf{1}_n = 1-\sum_{i=1}^n\mu_i^{-1}.$$

The latter coincides with (3).

Corollary 1 *For $f\in H^2((0,l_1)\times(0,l_2))$,*

$$\mu_0\|f(\cdot)-f(0)\|^2\le\mu_1\left(\frac{2l}{\pi}\right)^2\|f_{x_1}\|^2+\mu_2\left(\frac{2l}{\pi}\right)^2\|f_{x_2}\|^2 + \mu_3\left(\frac{2l}{\pi}\right)^4\|f_{x_1x_2}\|^2 \tag{5}$$

with $l=\max\{l_1,l_2\}$ and any $\mu_0,\mu_1,\mu_2,\mu_3\in\mathbb{R}_{>0}$ such that

$$\operatorname{diag}\{\mu_1,\mu_2,\mu_3\}\ge\mu_0\mathbf{1}_3\mathbf{1}_3^T. \tag{6}$$

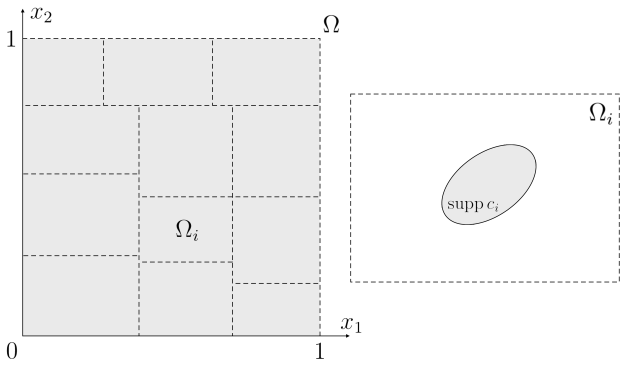

Fig. 1. Subdomains Ω_i and the subset $\operatorname{supp} c_i \subset \overline{\Omega}_i$

PROOF. Let $\lambda_{(1,0)} = \mu_0/\mu_1$, $\lambda_{(0,1)} = \mu_0/\mu_2$, and $\lambda_{(1,1)} = \mu_0/\mu_3$. By Lemma 1, the condition (6) guarantees that $\sum_\alpha \lambda_\alpha = \mu_0 \sum_{i=1}^3 \mu_i^{-1} \le 1$. Clearly, Theorem 1 remains valid if $\sum_\alpha \lambda_\alpha \le 1$. Thus, (2) implies (5).

3 Stabilization under pointlike measurements

Consider the semilinear diffusion system

$$
\begin{aligned}
&z_t(x,t) = \Delta_D z(x,t) + f(x,t,z(\cdot,t)) + \sum_{i=1}^N b_i(x)u_i(t),\\
&z|_{\partial\Omega} = 0, \quad z|_{t=0} = z_0 \in H_0^1(\Omega)
\end{aligned}
\tag{7}
$$

with the domain $\Omega = (0,1)\times(0,1) \subset \mathbb{R}^2$, state $z\colon \overline{\Omega}\times[0,\infty)\to\mathbb{R}$, diffusion operator

$$
\Delta_D z(x,t) = \operatorname{div}(D\nabla z(x,t)), \quad D = \left[\begin{smallmatrix} d_1 & d_2 \\ d_2 & d_3 \end{smallmatrix}\right] > 0, \tag{8}
$$

and nonlinearity $f\colon \Omega\times(0,\infty)\times H_0^1(\Omega)\to\mathbb{R}$ such that $f(\cdot,t,z)\in L^2(\Omega)$ and

$$
\|f(\cdot,t,z)\|^2 \le c_f\|z\|^2 + \int_\Omega (\nabla z)^T F(\nabla z) \tag{9}
$$

for all $t>0$ and $z \in H_0^1(\Omega)$, where $c_f > 0$ and $0 < F \in \mathbb{R}^{2\times 2}$. This system models numerous physical phenomena, such as air pollution [20], rotating stalls in axial compressors [4], and heat transfer in catalytic slabs (see Section 6).

Remark 3 (Nonsquare domains) *We consider $\Omega = (0,1)\times(0,1)$ for simplicity. The results are applicable to any open parallelogram $\widetilde{\Omega}\subset\mathbb{R}^2$, which can be transformed to Ω using a nonsingular change of variables $x = A\widetilde{x} + b$. In this case, $D = A\widetilde{D}A^T$ and $F = A\widetilde{F}A^T$, where $\widetilde{D}$ and $\widetilde{F}$ are the matrices from* (8) *and* (9) *for the domain $\widetilde{\Omega}$.*

We assume that Ω is divided into N rectangular subdomains Ω_i (Fig. 1) with an actuator and a sensor placed in each Ω_i. The actuators are modeled by

$$
b_i \in L^2(\Omega)\colon \quad \operatorname{supp} b_i \subset \overline{\Omega}_i, \quad i\in\{1,\dots,N\}. \tag{10}
$$

We assume that b_i approximate the characteristic functions

$$
\chi_i(x) = \begin{cases} 1, & x\in\Omega_i,\\ 0, & x\notin\Omega_i,\end{cases} \qquad i\in\{1,\dots,N\}, \tag{11}
$$

so that the quantity

$$
c_b = \max_{1\le i\le N} \frac{\|b_i-\chi_i\|^2}{|\Omega_i|} \tag{12}
$$

is small enough. Examples of such actuators are air injectors in axial compressors or cooling medium in catalytic slabs. The sensors provide the measurements

$$
\begin{aligned}
&y_i(t) = \int_{\Omega_i} c_i(\xi)z(\xi,t)\,d\xi,\\
&0\le c_i\in L^2(\Omega_i), \quad \int_{\Omega_i} c_i = 1, \quad i\in\{1,\dots,N\}.
\end{aligned}
\tag{13}
$$

The averaged measurements correspond to $c_i = \chi_i/|\Omega_i|$, which were considered in [15]. Here, we do not demand $\operatorname{supp} c_i$ to cover Ω_i. This allows the consideration of

$$
c_i(\xi) = \begin{cases} \dfrac{1}{\varepsilon^2}, & |\xi - x_c^i|_\infty < \dfrac{\varepsilon}{2},\\ 0, & |\xi - x_c^i|_\infty \ge \dfrac{\varepsilon}{2}\end{cases} \tag{14}
$$

with $x_c^i\in\Omega_i$ and small $\varepsilon>0$ (such that $\operatorname{supp} c_i\subset\overline{\Omega}_i$). Such c_i approximate the Dirac delta functions $\delta(\xi - x_c^i)$ corresponding to the point measurements at x_c^i. Thus, we call (13), (14) *pointlike* measurements. An example of such measurements is the average temperature in the vicinity of a given point.

We study (7) under the static output feedback

$$
u_i(t) = -Ky_i(t), \quad i\in\{1,\dots,N\}, \tag{15}
$$

which leads to the closed-loop system

$$
\begin{aligned}
&z_t = \Delta_D z + f - K\sum_{i=1}^N b_i(x)y_i(t),\\
&z|_{\partial\Omega} = 0, \quad z|_{t=0} = z_0\in H_0^1(\Omega).
\end{aligned}
\tag{16}
$$

A classical solution of (13), (16) is a function

$$
\begin{aligned}
&z\in C^1((0,\infty),L^2)\cap C([0,\infty),L^2),\\
&z(t)\in H_0^1(\Omega)\cap H^2(\Omega) \quad \forall t>0
\end{aligned}
\tag{17}
$$

that satisfies (13), (16). The existence of a unique classical solution to (13), (16) follows from [21, Theorem 6.3.3].

The last term of (16) approximates the stabilizing feedback $-Kz$:

$$
\begin{aligned}
-K\sum_{i=1}^N b_iy_i &= -K\sum_{i=1}^N (b_i-\chi_i)y_i - K\sum_{i=1}^N \chi_iy_i\\
&= \left[-K\sum_{i=1}^N (b_i-\chi_i)y_i\right] + \left[Kz - K\sum_{i=1}^N\chi_iy_i\right] - Kz\\
&= \epsilon + \sigma - Kz,
\end{aligned}
\tag{18}
$$

where

$$
\begin{aligned}
&\epsilon(x,t) = -K\sum_{i=1}^N (b_i(x)-\chi_i(x))y_i(t),\\
&\sigma(x,t) = Kz(x,t) - K\sum_{i=1}^N \chi_i(x)y_i(t).
\end{aligned}
\tag{19}
$$

Since $\operatorname{supp} b_i \subset \overline{\Omega}_i$, $\operatorname{supp} \chi_i = \overline{\Omega}_i$, and Ω_i are disjoint, the error ϵ can be bounded as

$$
\begin{aligned}
\|\epsilon(\cdot,t)\|^2 &= \int_\Omega \left[-K\sum_{i=1}^N (b_i(x)-\chi_i(x))y_i(t)\right]^2 dx \\
&= \int_\Omega \left[\sum_{i=1}^N (b_i(x)-\chi_i(x))^2(Ky_i(t))^2\right] dx \\
&= \sum_{i=1}^N \|b_i-\chi_i\|^2(Ky_i(t))^2 \\
&= \sum_{i=1}^N \frac{\|b_i-\chi_i\|^2}{|\Omega_i|}\int_\Omega (\chi_i(x)Ky_i(t))^2\,dx \\
&\overset{(12)}{\le} c_b\sum_{i=1}^N \|\chi_i K y_i\|^2 = c_b\left\|\sum_{i=1}^N \chi_i K y_i\right\|^2 \\
&\overset{(19)}{=} c_b\|Kz-\sigma\|^2.
\end{aligned}
\tag{20}
$$

Thus, for any $\mu_4 > 0$,

$$
0 \le -\mu_4\|\epsilon(\cdot,t)\|^2 + \mu_4 c_b\|Kz(\cdot,t)-\sigma(\cdot,t)\|^2 \quad \forall t \ge 0. \tag{21}
$$

The error σ appears because the state z is approximated using the measurements y_i. Below, we explain the main idea that allows us to bound σ. By the mean value theorem [1], for every $t \ge 0$ and $i \in \{1,\dots,N\}$,

$$
\exists\, x^i(t) \in \operatorname{Conv}(\operatorname{supp} c_i)\colon \quad \int_{\Omega_i} c_i(\xi)z(\xi,t)\,d\xi = z(x^i(t),t).
$$

(The convex hull appears because we do not require $\operatorname{supp} c_i$ to be path-connected.) Thus, $\sigma(x^i(t),t)=0$. Each rectangle cornered at x^i and lying in Ω_i (see Fig. 2) has sides smaller than

$$
l = \max_{1\le i\le N}\ \max_{\substack{\omega\in\partial\Omega_i\\ d\in\operatorname{supp} c_i}} |\omega-d|_\infty. \tag{22}
$$

Applying Corollary 1 on each of such rectangles and summing over them, we obtain

$$
\begin{aligned}
0 \le -\mu_0\frac{\|\sigma\|^2}{K^2} + \mu_1\left(\frac{2l}{\pi}\right)^2\|z_{x_1}\|^2 + \mu_2\left(\frac{2l}{\pi}\right)^2\|z_{x_2}\|^2 \\
+ \mu_3\left(\frac{2l}{\pi}\right)^4\|z_{x_1x_2}\|^2
\end{aligned}
\tag{23}
$$

for any $\mu_0,\mu_1,\mu_2,\mu_3 \in \mathbb{R}_{>0}$ satisfying (6). The positive terms in (23) can be made arbitrarily small by reducing l, which always can be achieved by increasing the number of sensors N. This corresponds to the general intuition that a larger amount of sensors allows for better estimation of the state.

Using (18), we present the closed-loop system (16) as

$$
\begin{aligned}
&z_t = \Delta_D z + f - Kz + \epsilon + \sigma,\\
&z|_{\partial\Omega} = 0, \quad z|_{t=0} = z_0 \in H_0^1(\Omega).
\end{aligned}
\tag{24}
$$

[1] The idea to use the mean value theorem comes from [18], where a scalar domain $\Omega \subset \mathbb{R}$ was considered

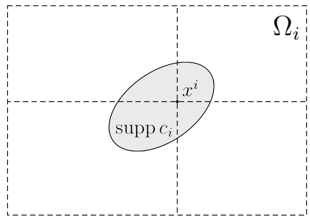

Fig. 2. Four rectangles cornered at $x^i \in \operatorname{Conv}(\operatorname{supp} c_i)$

To study its stability, consider $V_0 = \|z\|^2$. We have

$$
\dot V_0 = 2\int_\Omega z z_t \overset{(24)}{=} 2\int_\Omega z\left[\Delta_D z + f - Kz + \epsilon + \sigma\right].
$$

Since $z|_{\partial\Omega}=0$, by the divergence theorem,

$$
2\int_\Omega z\Delta_D z = 2\int_\Omega z\operatorname{div}(D\nabla z) = -2\int_\Omega (\nabla z)^T D\nabla z.
$$

Therefore,

$$
\dot V_0 = -2\int_\Omega (\nabla z)^T D\nabla z - 2K\int_\Omega z^2 + 2\int_\Omega z\,(f+\epsilon+\sigma). \tag{25}
$$

Clearly, $\Delta_D z$ and $-Kz$ from (24) give "stabilizing" negative summands in (25). To compensate the cross term with f, we add to $\dot V_0$ the right-hand side of

$$
0 \le -\mu_5\|f(\cdot,t,z)\|^2 + \mu_5 c_f\|z\|^2 + \mu_5\textstyle\int_\Omega (\nabla z)^T F(\nabla z), \tag{26}
$$

which follows from (9) for any $\mu_5 \ge 0$. To compensate the errors ϵ and σ, we will add the right-hand sides of (21) and (23), respectively. The first order derivatives from (23) are balanced by the first term of (25). To compensate the second order derivative from (23), we introduce

$$
V_1 = \int_\Omega (\nabla z(x,t))^T P\nabla z(x,t)\,dx, \quad P = \left[\begin{smallmatrix} p_1 & p_2\\ p_2 & p_3\end{smallmatrix}\right] > 0. \tag{27}
$$

Since $z|_{\partial\Omega}=0$ and $z_t|_{\partial\Omega}=0$, applying the divergence theorem twice, we obtain

$$
\begin{aligned}
\dot V_1 &= 2\textstyle\int_\Omega (\nabla z)^T P\nabla z_t = -2\int_\Omega \operatorname{div}(P\nabla z)\, z_t \\
&\overset{(16)}{=} -2\textstyle\int_\Omega \operatorname{div}(P\nabla z)\left[\Delta_D z + f - Kz + \epsilon + \sigma\right] \\
&= -2\textstyle\int_\Omega \operatorname{div}(P\nabla z)\,\Delta_D z - 2K\int_\Omega (\nabla z)^T P\nabla z \\
&\qquad - 2\textstyle\int_\Omega \operatorname{div}(P\nabla z)\left[f+\epsilon+\sigma\right].
\end{aligned}
\tag{28}
$$

If $P = p_0 D$, then the first term is $-2p_0\|\Delta_D z\|^2$, which compensates $\|z_{x_1x_2}\|^2$ from (23). Such P can be used to study a spatially varying diffusion matrix $D(x)$ in (8) (as considered in [15] for the case of averaged measurements). Here, we consider P of a more general form but for a constant D. This leads to less restrictive convergence conditions.

Theorem 2 *Consider the system* (7) *subject to* (9) *and* (10) *with the measurements* (13). *For given controller gain* K *and decay rate* $\alpha > 0$, *let there exist*

$$
P = \left[\begin{smallmatrix} p_1 & p_2\\ p_2 & p_3\end{smallmatrix}\right] > 0, \quad \mu_i > 0 \quad \forall i \in \{0,\dots,8\}
$$

such that[2] (6) *is true*, $\Phi \le 0$, *and* $\Phi_\triangledown \le 0$, *where*

$$\Phi = \begin{bmatrix} \Phi_{11} & 0 & 1 & 1-\mu_4 c_b K & 1 \\ * & \Phi_{22} & -\bar p & -\bar p & -\bar p \\ * & * & -\mu_5 & 0 & 0 \\ * & * & * & -\mu_0/K^2+\mu_4 c_b & 0 \\ * & * & * & * & -\mu_4 \end{bmatrix},$$

$$\Phi_{11} = -2(K-\alpha) - (\mu_7+\mu_8)\pi^2 + \mu_5 c_f + \mu_4 c_b K^2,$$

$$\Phi_{22} = -\bar p \bar d^T - \bar d \bar p^T + \left[\begin{smallmatrix} 0 & 0 & \mu_6 \\ 0 & \mu_3(2l/\pi)^4 - 2\mu_6 & 0 \\ \mu_6 & 0 & 0 \end{smallmatrix}\right],$$

$$\Phi_\triangledown = -2D - 2(K-\alpha)P + \mu_5 F + \left(\frac{2l}{\pi}\right)^2 \left[\begin{smallmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{smallmatrix}\right] + \left[\begin{smallmatrix} \mu_7 & 0 \\ 0 & \mu_8 \end{smallmatrix}\right],$$

c_b *is given in* (12), l *is defined in* (22), $\bar p = (p_1, 2p_2, p_3)^T$, *and* $\bar d = (d_1, 2d_2, d_3)^T$. *Then the static output feedback* (15) *exponentially stabilizes the system* (7) *in the* H_0^1*-norm with the decay rate* α, *i.e.,*

$$\exists C\colon \quad \|z(\cdot,t)\|_{H_0^1} \le C e^{-\alpha t} \|z_0\|_{H_0^1} \quad \forall t \ge 0.$$

PROOF. See Appendix B.

Remark 4 (Feasibility of LMIs) *The LMIs of Theorem 2 are always feasible for a large enough controller gain* K, *small enough* c_b *given in* (12), *and small enough* l *defined in* (22). *Indeed,* $D > 0$ *implies* $d_1 d_3 - d_2^2/q > 0$ *for a large enough* $q < 1$. *By Young's inequality,*

$$2\left[\begin{smallmatrix} 0 & -d_1 d_2 \\ -d_1 d_2 & 0 \end{smallmatrix}\right] \le 2\,\mathrm{diag}\{q d_1^2, d_2^2/q\},$$
$$2\left[\begin{smallmatrix} 0 & -d_2 d_3 \\ -d_2 d_3 & 0 \end{smallmatrix}\right] \le 2\,\mathrm{diag}\{d_2^2/q, q d_3^2\}.$$

Then, for $l = 0$, $\bar p = (d_3, 0, d_1)^T$, *and* $\mu_6 = d_1^2 + d_3^2$, *we obtain*

$$\Phi_{22} \le \begin{bmatrix} -2(d_1 d_3 - \frac{d_2^2}{q}) & 0 & 0 \\ 0 & -2(1-q)\mu_6 & 0 \\ 0 & 0 & -2(d_1 d_3 - \frac{d_2^2}{q}) \end{bmatrix} < 0.$$

Therefore, $\Phi < 0$ *for* $c_b = 0$ *and large enough* μ_4, μ_5, K, *and* μ_0. *Clearly,* $\Phi_\triangledown < 0$ *for a large enough* K *and* (6) *holds for large enough* μ_1, μ_2, *and* μ_3. *Thus, the LMIs of Theorem 2 are feasible for* $c_b = 0$ *and* $l = 0$. *By continuity, they remain feasible for small enough* c_b *and* l.

Corollary 2 *The semilinear diffusion system* (7) *with the measurements* (13) *is exponentially stable under the static output feedback* (15) *with a large enough controller gain* K *if* c_b *given in* (12) *and* l *defined in* (22) *are small enough (i.e., the shape functions* b_i *are close to* χ_i *and the number of sensors* N *is large enough).*

Remark 5 (Different boundary conditions) *The results can be extended to* (7) *with the boundary conditions*

$$z|_{\Gamma_D} = 0, \qquad \frac{\partial z}{\partial \mathbf{n}}|_{\Gamma_N} = 0, \tag{29}$$

where $\Gamma_D \cup \Gamma_N = \partial\Omega$, $\Gamma_D \cap \Gamma_N = \varnothing$, *and* $\mathbf{n}$ *is the normal to* Γ_N. *All calculations in the proof of Theorem 2 remain valid except for* (B.2), *which, according to Lemma 4, should be replaced by*

$$0 \le -\mu_7 q_1 \pi^2 \int_\Omega z^2 + \mu_7 \int_\Omega z_{x_1}^2,$$
$$0 \le -\mu_8 q_2 \pi^2 \int_\Omega z^2 + \mu_8 \int_\Omega z_{x_2}^2,$$

where

$$q_1 = \begin{cases} 1 & \text{if } z|_{x_1=0} = z|_{x_1=1} = 0, \\ \frac{1}{4} & \text{if } z|_{x_1=0} \text{ or } z|_{x_1=1} = 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$q_2 = \begin{cases} 1 & \text{if } z|_{x_2=0} = z|_{x_2=1} = 0, \\ \frac{1}{4} & \text{if } z|_{x_2=0} = 0 \text{ or } z|_{x_2=1} = 0, \\ 0 & \text{otherwise}. \end{cases}$$

Remark 6 (Point measurements) *Theorem 2 remains valid if* $c_i(x) = \delta(x - x_c^i)$, *which correspond to the point measurements* $y_i(t) = z(x_c^i, t)$. *In this case,* $l = \max_{1\le i\le N} \max_{\omega\in\partial\Omega_i} \|\omega - x_c^i\|_\infty$.

Remark 7 (3D domains) *If* $\Omega = (0,1)^3 \subset \mathbb{R}^3$, *then one can use Theorem 1 to bound the approximation error* σ *in a manner similar to* (23). *This bound involves the* 3*rd order space derivative, which we do not know how to compensate. Thus, it is not clear how to extend the proposed method to 3D domains.*

4 Delayed H_∞ control under delayed pointlike measurements

Consider the perturbed semilinear diffusion system

$$\begin{aligned} z_t(x,t) &= \Delta_D z(x,t) + f(x,t,z(\cdot,t)) \\ &\quad + \sum_{i=1}^N b_i(x) u_i(t - \tau_i^u(t)) + w(x,t), \end{aligned} \tag{30}$$

$$z|_{\partial\Omega} = 0, \quad z|_{t=0} = z_0 \in H_0^1(\Omega),$$

where $w\colon \Omega \times (0,\infty) \to \mathbb{R}$ is a disturbance and $\tau_i^u(t)$ are unknown time-varying input delays satisfying

$$0 \le \tau_i^u(t) \le \tau_M^u \quad \forall t \ge 0 \tag{31}$$

with a known bound τ_M^u. The other terms are as in (7). We assume that Ω is divided into N rectangular subdomains Ω_i (Fig. 1) with an actuator and a sensor placed in each Ω_i. The actuators are modeled by b_i subject to (10). The

[2] MATLAB code for solving the LMIs is available at
`https://github.com/AntonSelivanov/Aut19`

sensors provide the time-delayed noisy measurements

$$\tilde{y}_i(t) = \begin{cases} \int_{\Omega_i} c_i(\xi) z(\xi, t - \tau_i^y(t))\, d\xi + v_i(t), & t \geq \tau_i^y(t), \\ 0, & t < \tau_i^y(t), \end{cases}$$
$$0 \leq c_i \in L^\infty(\Omega_i), \quad \int_{\Omega_i} c_i = 1, \quad i \in \{1, \dots, N\}, \tag{32}$$

with the measurement noise $v_i\colon [0, \infty) \to \mathbb{R}$ and unknown time-varying output delays $\tau_i^y(t)$ satisfying

$$0 < \tau_m^y \leq \tau_i^y(t) \leq \tau_M^y \quad \forall t \geq 0. \tag{33}$$

We study (30) under the static output feedback

$$u_i(t) = -K\tilde{y}_i(t), \quad i \in \{1, \dots, N\}, \tag{34}$$

which leads to the closed-loop system

$$z_t = \Delta_D z + f - K \sum_{i=1}^N b_i(x) \tilde{y}_i(t - \tau_i^u(t)) + w, \tag{35}$$
$$z|_{\partial\Omega} = 0, \quad z|_{t=0} = z_0 \in H_0^1(\Omega).$$

The disturbance $w(x, t)$ and measurement noise $v_i(t)$ are called *admissible* if there exists a unique classical (in the sense of (17)) solution to (32), (35). This can be established for $w(\cdot, t)$ and $v_i(t)$ that are Hölder continuous in t: one needs to apply [21, Theorem 6.3.3] consecutively on each interval $[0, \tau_m^y]$, $[\tau_m^y, 2\tau_m^y]$, ... treating the delayed terms as inhomogeneities. Less restrictive conditions on w and v_i can be imposed by considering *strong* solutions.

Clearly,

$$\tilde{y}_i(t - \tau_i^u(t)) = y_i(t - \tau_i(t)) + v_i(t - \tau_i^u(t)), \tag{36}$$

where y_i are from (13) and

$$\tau_i = \tau_i^u(t) + \tau_i^y(t - \tau_i^u(t)).$$

In view of (31) and (33),

$$0 \leq \tau_m^y \leq \tau_i(t) \leq \tau_M = \tau_M^u + \tau_M^y. \tag{37}$$

We denote

$$\kappa(x, t) = K \sum_{i=1}^N \chi_i(x)[y_i(t) - y_i(t - \tau_i(t))],$$
$$v(x, t) = \sum_{i=1}^N b_i(x) v_i(t - \tau_i^u(t)).$$

The function κ represents the delay-induced error, while v is the distributed effect of the measurement noise. Similarly to (18), we have

$$\begin{aligned} &-K \sum_{i=1}^N b_i(x) \tilde{y}_i(t - \tau_i^u(t)) \\ &\overset{(36)}{=} -K \sum_{i=1}^N b_i(x) y_i(t - \tau_i(t)) - K \sum_{i=1}^N b_i(x) v_i(t - \tau_i^u(t)) \\ &\overset{(19)}{=} \epsilon(x, t - \tau_i(t)) + \sigma(x, t) + \kappa(x, t) - Kz(x, t) - Kv(x, t). \end{aligned} \tag{38}$$

Thus, the closed-loop system (32), (35) takes the form

$$z_t = \Delta_D z + f - Kz + \epsilon(x, t - \tau_i(t)) + \sigma + \kappa - Kv + w$$
$$z|_{\partial\Omega} = 0, \quad z|_{t=0} = z_0 \in H_0^1(\Omega). \tag{39}$$

Similarly to (20),

$$\begin{aligned} \|\epsilon(\cdot, t - \tau_i(t))\|^2 &\leq c_b \left\| \sum_{i=1}^N \chi_i(\cdot) K y_i(t - \tau_i(t)) \right\|^2 \\ &= c_b \|Kz - \sigma - \kappa\|^2, \end{aligned}$$

which implies

$$0 \leq -\mu_4 \|\epsilon(\cdot, t - \tau_i(t))\|^2 + \mu_4 c_b \|Kz - \sigma - \kappa\|^2 \quad \forall t \geq 0 \tag{40}$$

for any $\mu_4 > 0$. The approximation error σ and nonlinearity f will be compensated using (23) and (26). To compensate the delay-induced error κ, we introduce the Lyapunov–Krasovskii term

$$V_r = c_r r \int_{-\tau_M}^0 \int_{t+\theta}^t \sum_{i=1}^N \int_{\Omega_i} e^{-2\alpha(t-s)} c_i(\xi) z_s^2(\xi, s)\, d\xi\, ds\, d\theta \tag{41}$$

with the constant $c_r > 0$ to be defined hereafter.

Remark 8 (New Lyapunov–Krasovskii term) *The term* (41) *originates from [22,23], where a double integral term was used to study fast-varying delays in finite dimensional systems. The operator form of the double integral term was considered in [24] for infinite dimensional time-delay systems. In [8], it was transformed to a triple integral term to study parabolic PDEs with time-delays. Here, we make one further step by introducing the spatially-varying kernel $c_i(\xi)$, which compensates for the output delays in* (32)*. More sophisticated functionals used to study finite dimensional systems (see, e.g., [25]) might be generalized in a similar way. This should lead to less conservative but more complicated convergence conditions.*

Making the change of variable $\varsigma = t + \theta$, we get

$$\begin{aligned} \dot{V}_r + 2\alpha V_r = c_r r \int_{-\tau_M}^0 \sum_{i=1}^N \int_{\Omega_i} c_i(\xi) z_t^2(\xi, t)\, d\xi\, d\theta \\ - c_r r \int_{t-\tau_M}^t \sum_{i=1}^N \int_{\Omega_i} e^{-2\alpha(t-\varsigma)} c_i(\xi) z_\varsigma^2(\xi, \varsigma)\, d\xi\, d\varsigma. \end{aligned}$$

Now we show that the negative term is upper bounded by $-r\|\kappa(\cdot, t)\|^2$ for the appropriate c_r. To do so, we use the following version of Jensen's inequality.

Lemma 2 (Jensen's inequality [26]) *For Lebesgue-integrable $f\colon [a, b] \to \mathbb{R}$ and $\rho\colon [a, b] \to [0, \infty)$,*

$$\left[\int_a^b \rho(s) f(s)\, ds\right]^2 \leq \int_a^b \rho(s)\, ds \int_a^b \rho(s) f^2(s)\, ds.$$

Using this lemma with $\rho = c_i$ (recall that $\int_{\Omega_i} c_i = 1$) and $\rho \equiv 1$, we obtain

$$\begin{aligned}
& - c_r r \int_{t-\tau_M}^{t} \sum_{i=1}^{N} \int_{\Omega_i} e^{-2\alpha(t-\varsigma)} c_i(\xi) z_\varsigma^2(\xi,\varsigma)\, d\xi\, d\varsigma \\
& \quad \leq \; -c_r r e^{-2\alpha\tau_M} \sum_{i=1}^{N} \int_{t-\tau_i}^{t} \int_{\Omega_i} c_i(\xi) z_\varsigma^2(\xi,\varsigma)\, d\xi\, d\varsigma \\
& \quad \overset{\text{Lem.2}}{\leq} \; -c_r r e^{-2\alpha\tau_M} \sum_{i=1}^{N} \int_{t-\tau_i}^{t} \left[y_i'(\varsigma)\right]^2 d\varsigma \\
& \quad \overset{\text{Lem.2}}{\leq} \; -c_r r e^{-2\alpha\tau_M} \sum_{i=1}^{N} \frac{1}{\tau_i} \left[\int_{t-\tau_i}^{t} y_i'(\varsigma)\, d\varsigma\right]^2 \\
& \quad = \; -c_r r e^{-2\alpha\tau_M} \sum_{i=1}^{N} \frac{1}{\tau_i |\Omega_i|} \int_\Omega \chi_i(x) \left[\int_{t-\tau_i}^{t} y_i'(\varsigma)\, d\varsigma\right]^2 dx \\
& \quad \overset{(11)}{\leq} \; -\frac{c_r r e^{-2\alpha\tau_M}}{\tau_M \max_i |\Omega_i|} \int_\Omega \left[\sum_{i=1}^{N} \chi_i(x) \int_{t-\tau_i}^{t} y_i'(\varsigma)\, d\varsigma\right]^2 dx \\
& \quad = \; -\frac{c_r r e^{-2\alpha\tau_M}}{\tau_M \max_i |\Omega_i| K^2} \int_\Omega \kappa^2(x,t) dx = -r\|\kappa(\cdot,t)\|^2
\end{aligned}$$

if

$$c_r = e^{2\alpha\tau_M} \tau_M \max_{1\leq i\leq N} |\Omega_i| K^2. \tag{42}$$

That is,

$$\dot V_r + 2\alpha V_r \leq c_r r \tau_M \max_{1\leq i\leq N} \|c_i\|_\infty \int_\Omega z_t^2(x,t)\, dx - r\|\kappa(\cdot,t)\|^2. \tag{43}$$

The negative term $-r\|\kappa(\cdot,t)\|^2$ will compensate the cross terms with $\kappa(x,t)$ in the derivative of the Lyapunov–Krasovskii functional.

The system (32), (35) is *internally exponentially stable* if it is exponentially stable for $w = v \equiv 0$. For given $d_u > 0$ and $\gamma > 0$, consider the cost functional

$$\begin{aligned} J(u) = \int_0^\infty \int_\Omega [z^2(x,t) + d_u^2 u^2(x,t) \\ - \gamma^2 w^2(x,t) - \gamma^2 v^2(x,t)]\, dx\, dt, \end{aligned} \tag{44}$$

where $u(x,t) = \sum_{i=1}^N b_i(x) u_i(t - \tau_i^u(t))$. We say that the output feedback (34) solves the H_∞ control problem for the system (30), (32) if it leads to an internally exponentially stable system (32), (35) and guarantees that $J(u) \leq 0$ for any solution of (30) with $z|_{t=0} = 0$ and admissible $w, v \in L^2((0,\infty), L^2(\Omega))$. We solve the H_∞ control problem using the method described in [25, Section 4.3].

Theorem 3 *Consider the system* (30) *subject to* (9), (10), *and* (31) *with the measurements* (32) *subject to* (33). *For given controller gain K and decay rate $\alpha > 0$, let there exist*

$$P = \left[\begin{smallmatrix} p_1 & p_2 \\ p_2 & p_3 \end{smallmatrix}\right] > 0, \quad \mu_i > 0 \quad \forall i \in \{0,\ldots,8\},$$

$$r > 0, \quad \gamma_1 > 0, \quad \gamma_2 > 0, \quad \gamma_3 > 0$$

such that[3] (6) *is true,* $\widetilde\Phi \leq 0$, *and* $\Phi_\nabla \leq 0$, *where*

$$\widetilde\Phi = \left[\begin{array}{ccccc:cccc}
 & & & & & \widetilde\Phi_{16} & 1-\gamma_3 K & 1 & -\tau_M r K \\
 & & & & & -\bar p & -\bar p & -\bar p & \tau_M r \bar d \\
 & & \Phi' & & & 0 & 0 & 0 & \tau_M r \\
 & & & & & \mu_4 c_b + \gamma_3 & \gamma_3 & 0 & \tau_M r \\
 & & & & & \gamma_3 & \gamma_3 & 0 & \tau_M r \\ \hdashline
 * & * & * & * & * & \widetilde\Phi_{66} & \gamma_3 & 0 & \tau_M r \\
 * & * & * & * & * & * & \widetilde\Phi_{77} & 0 & \tau_M r \\
 * & * & * & * & * & * & * & -\gamma_2 & \tau_M r \\
 * & * & * & * & * & * & * & * & \widetilde\Phi_{99}
\end{array}\right],$$

$\widetilde\Phi_{16} = 1 - \mu_4 K c_b - \gamma_3 K,$
$\widetilde\Phi_{66} = -r + \mu_4 c_b + \gamma_3,$
$\widetilde\Phi_{77} = -\gamma_2/K^2 + \gamma_3,$
$\widetilde\Phi_{99} = -re^{-2\alpha\tau_M}/(K^2 \max_i |\Omega_i| \max_i \|c_i\|_\infty),$

$$\begin{aligned}\Phi' = \Phi + \gamma_1 \left[\begin{smallmatrix} 1 & 0 & 0 & 0 & 0 \end{smallmatrix}\right]^T \left[\begin{smallmatrix} 1 & 0 & 0 & 0 & 0 \end{smallmatrix}\right] \\ + \gamma_3 \left[\begin{smallmatrix} -K & 0 & 0 & 1 & 1 \end{smallmatrix}\right]^T \left[\begin{smallmatrix} -K & 0 & 0 & 1 & 1 \end{smallmatrix}\right],\end{aligned}$$

Φ and Φ_∇ are from Theorem 2, $\bar p = (p_1, 2p_2, p_3)^T$, and $\bar d = (d_1, 2d_2, d_3)^T$. Then the static output feedback (15) *solves the H_∞ control problem with $J(u)$ given in* (44), *$\gamma = \sqrt{\gamma_2/\gamma_1}$, and $d_u = \sqrt{\gamma_3/\gamma_1}$.*

PROOF. See Appendix C.

Remark 9 (Feasibility of LMIs) *The LMIs of Theorem 3 are always feasible for large enough K and γ_2 and small enough c_b, l, τ_M, γ_1, and γ_3. This follows from Remark 4 for $\tau_M = 0$ and remains so for a small enough τ_M by continuity.*

Corollary 3 *The static output feedback* (34) *solves the H_∞ control problem for the semilinear diffusion system* (30) *under the delayed noisy measurements* (32) *if the controller gain K and the desired L^2-gain γ are large enough while c_b from* (12), *l from* (22), *and τ_M from* (37) *are small enough.*

Remark 10 (Point measurements) *In the absence of delays ($\tau_M = 0$), the conditions of Theorem 3 can be simplified by eliminating the last column and row from $\widetilde\Phi$. Modified in this way, Theorem 3 with l given in Remark 6 provides conditions guaranteeing that the output feedback* (34) *solves the H_∞ control problem under the noisy point measurements $\tilde y_i(t) = z(x_c^i, t) + v_i(t)$. In the presence of delays ($\tau_M \neq 0$), Theorem 3 cannot be applied with $c_i(x) = \delta(x - x_c^i)$ since it includes $\max_i \|c_i\|_\infty$. This happens because the delay-induced error $\kappa(x,t)$ containing an unbounded operator is hard to compensate using Lyapunov–Krasovskii*

[3] MATLAB code for solving the LMIs is available at `https://github.com/AntonSelivanov/Aut19`

terms. (For instance, it can be compensated in the 1D case using Halanay's inequality [8], but this approach does not work in 2D due to the presence of $z_{x_1x_2}$ in (23)*.) If $\delta(x-x_c^i)$ are approximated by c_i from* (14)*, then $\max_i \|c_i\|_\infty$ is increasing while $\varepsilon \to 0$ leading to a smaller bound on the admissible delays τ_M that vanishes at the limit.*

Remark 11 (Different boundary conditions) *The results of this section can be extended to the boundary conditions* (29) *with the same adjustments as in Remark 5.*

5 H_∞ filtering under delayed pointlike measurements

Consider the semilinear diffusion system

$$\begin{aligned} &z_t(x,t) = \Delta_D z(x,t) + f(x,t,z(\cdot,t)) + w(x,t), \\ &z|_{\partial\Omega} = 0, \quad z|_{t=0} = z_0 \in H_0^1(\Omega) \end{aligned} \tag{45}$$

with the nonlinearity $f\colon \Omega \times (0,\infty) \times H_0^1(\Omega) \to \mathbb{R}$ such that $f(\cdot,t,z) \in L^2(\Omega)$ and

$$\begin{aligned} \|f(\cdot,t,z_1) - f(\cdot,t,z_2)\|^2 &\le c_f \|z_1 - z_2\|^2 \\ &+ \int_\Omega (\nabla z_1 - \nabla z_2)^T F (\nabla z_1 - \nabla z_2) \end{aligned} \tag{46}$$

for all $t > 0$ and $z_1, z_2 \in H_0^1(\Omega)$, where $c_f > 0$ and $0 < F \in \mathbb{R}^{2\times 2}$. The other terms are as in (7) and (30). Let the measurements be given by (32) with *known* time-varying delays $\tau_i^y(t)$ satisfying (33). To estimate the state of (45), we construct the observer

$$\begin{aligned} &\hat z_t(x,t) = \Delta_D \hat z(x,t) + f(x,t,\hat z(\cdot,t)) \\ &- L \sum_{i=1}^N \chi_i(x) \left(\int_{\Omega_i} c_i(\xi) \hat z(\xi, t - \tau_i^y(t))\, d\xi - \tilde y_i(t) \right), \\ &\hat z|_{\partial\Omega} = 0, \quad \hat z|_{t=0} = \hat z_0 \in H_0^1(\Omega), \quad \hat z|_{t<0} = 0 \end{aligned} \tag{47}$$

with the injection gain L and characteristic functions χ_i defined in (11). The estimation error $\bar z(x,t) = z(x,t) - \hat z(x,t)$ satisfies

$$\begin{aligned} \bar z_t = \Delta_D \bar z + \bar f - L \sum_{i=1}^N \chi_i \int_{\Omega_i} c_i(\xi) \bar z(\xi, t - \tau_i^y(t))\, d\xi \\ - Lv + w, \\ \bar z|_{\partial\Omega} = 0, \quad \bar z|_{t=0} = z_0 - \hat z_0 \in H_0^1(\Omega), \quad \bar z|_{t<0} = 0 \end{aligned} \tag{48}$$

with $\bar f(t,z,\hat z) = f(t,z) - f(t,\hat z)$ and the distributed disturbance $v(x,t) = \sum_{i=1}^N \chi_i(x) v_i(t)$.

Remark 12 (Unknown delays) *We assume that the delays $\tau_i^y(t)$ are known to guarantee that the observer* (47) *is implementable. If $\tau_i^y(t)$ are not known and replaced by* 0 *in* (47)*, then the error system* (48) *depends on the plant state z. This requires more sophisticated analysis (see, e.g., [27]).*

The system (48) is *internally exponentially stable* if it is exponentially stable for $w = v \equiv 0$. The system (48) has the L^2-gain not greater than $\gamma > 0$ if

$$\int_0^\infty \int_\Omega [\bar z^2(x,t) - \gamma^2 w^2(x,t) - \gamma^2 v^2(x,t)]\, dx\, dt \le 0 \tag{49}$$

for any solution of (48) with $\bar z|_{t=0} = 0$ and admissible $w, v \in L^2((0,\infty), L^2(\Omega))$.

The error system (48) coincides with (32), (35) if $b_i = \chi_i$, $K = L$, $\tau_i^u \equiv 0$, z is replaced by $\bar z$, and f is replaced by $\bar f$. Thus, Theorem 3 implies the following result.

Theorem 4 *Consider the system* (45) *subject to* (46) *with the measurements* (32) *subject to* (33)*. Let the conditions of Theorem 3 be feasible with $c_b = 0$, $\tau_M^u = 0$, and $K = L$. Then the observer* (47) *estimates the state of the system* (45) *with the L^2-gain not greater than $\gamma = \sqrt{\gamma_2/\gamma_1}$.*

6 Example: catalytic slab

Consider the catalytic slab model

$$\begin{aligned} &z_t = \frac{1}{2\pi^2} \Delta z + f(z) + \sum_{i=1}^N b_i(x) u_i(t - \tau_u^i(t)) + w, \\ &z|_{\partial\Omega} = 0, \quad z|_{t=0} = z_0 \end{aligned} \tag{50}$$

with the domain $\Omega = (0,1) \times (0,1)$, state z representing the temperature, disturbance w, and

$$f(z) = -\beta_U z + \beta_T (e^{-\gamma_a/(1+z)} - e^{-\gamma_a}),$$

where $\beta_T = 50$ is the heat of the reaction, $\beta_U = 2$ is the heat transfer coefficient, and $\gamma_a = 4$ is the activation energy. The controls u_i represent the temperature of the cooling medium, b_i subject to (10) model the actuators, and the unknown time-varying input delays $\tau_u^i(t)$ satisfy (31). The model (50) is a 2D extension of the catalytic rod model from [6, Section 4.3.1]. Clearly, $z \ge 0$ if $z_0 \ge 0$ and $w \equiv 0$. We assume that w is such that this property is preserved. Then f satisfies (9) with $c_f = \max_{z\ge 0} |f'|^2 \approx 22.72$ and $F = 0$. We assume that Ω is divided into N square subdomains (this implies $\sqrt{N} \in \mathbb{N}$) with a sensor and an actuator placed in the center of each subdomain.

First, we consider the system (50) without disturbances ($w \equiv 0$) under the output feedback (15) with the pointlike measurements (13), (14). In this case, (22) implies

$$l = (1/\sqrt{N} + \varepsilon)/2. \tag{51}$$

The LMIs of Theorem 2 are feasible for

$$K = 10, \quad \alpha = 0.01, \quad l = l_M = 0.0785, \quad c_b = 0.01.$$

The value of l given in (51) is not greater than $l_M = 0.0785$ if $N = 49$ and $\varepsilon \le 0.014$ or $N = 64$ and $\varepsilon \le 0.032$. Fig. 3 (blue dashed line) shows $\|z(\cdot,t)\|_{H_0^1} = \|\nabla z(\cdot,t)\|$ for $N = 64$, $\varepsilon = 0.0125$, $b_i = \chi_i$, and the initial conditions

$$z_0(x_1,x_2) = \sin(\pi x_1)\sin(\pi x_2), \quad x_1, x_2 \in [0,1]. \tag{52}$$

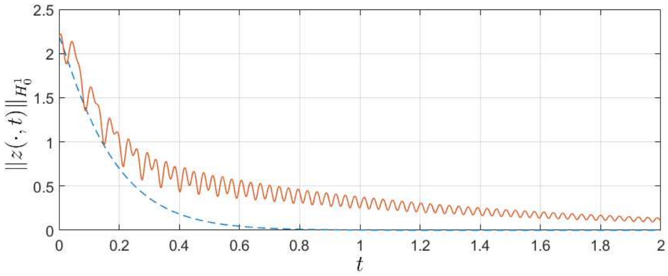

Fig. 3. Blue dashed line – the state of (50) with $w \equiv 0$ under (13)–(15); red solid line – the state of (50) with $w \not\equiv 0$ under (14), (32), (34)

Now, consider the system (50) with disturbances ($w \not\equiv 0$) under the delayed output feedback (34) with the delayed pointlike measurements (14), (32) subject to (33). Clearly, $\max_i |\Omega_i| = \frac{1}{N}, \quad \max_i \|c_i\|_\infty = \varepsilon^{-2}$.

The LMIs of Theorem 3 are feasible for

$$K = 10, \quad \alpha = 0.01, \quad N = 64, \quad \varepsilon = 0.0125,$$
$$c_b = 0.01, \quad \tau_M = 10^{-3}, \quad \gamma = 100, \quad d_u = 0.1,$$

and l given in (51). Fig. 3 (red solid line) shows $\|z(\cdot,t)\|_{H_0^1}$ for the initial conditions (52), $\tau_i^y(t) = \tau_i^u(t) \equiv \tau_M/2$, and

$$w(x,t) = \sin(10x_1 + t)\sin(10x_2 + t)e^{-t},$$
$$v_i(t) = \cos(100t)e^{-t} \quad \forall i \in \{1,\ldots,N\}.$$

Thus, we constructed an output-feedback ensuring the desired temperature of the catalytic slab.

7 Conclusions

Robust control of multi-dimensional diffusion systems was confined to averaged measurements. In this paper, we solve the H_∞ control problem for 2D semilinear diffusion systems with delayed pointlike measurements. The results are based on a new inequality, which is a reciprocally convex variation of Friedrich's inequality, and a new Lyapunov-Krasovskii term.

The presented approach can be extended to other types of multi-dimensional PDEs, including Kuramoto–Sivashinsky and 2D Navier–Stokes equations.

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A Proof of Theorem 1

The proof is based on the following two lemmas.

Lemma 3 *For any $v_1, \dots, v_n$ from a normed space X and any $\lambda_1, \dots, \lambda_n \in \mathbb{R}_{>0}$ such that $\lambda_1 + \dots + \lambda_n = 1$,*

$$\left\| \sum_{i=1}^n v_i \right\|_X^2 \le \sum_{i=1}^n \lambda_i^{-1} \|v_i\|_X^2.$$

PROOF. By the convexity of $\| \cdot \|_X^2$,

$$\left\| \sum_{i=1}^n \lambda_i \lambda_i^{-1} v_i \right\|_X^2 \le \sum_{i=1}^n \lambda_i \left\| \lambda_i^{-1} v_i \right\|_X^2 = \sum_{i=1}^n \lambda_i^{-1} \|v_i\|_X^2.$$

Lemma 4 (Wirtinger's inequality [28]) *For $f \in H^1(a,b)$,*

$$\|f\| \le \frac{2(b-a)}{\pi} \|f'\| \qquad \text{if } f(a) = 0 \text{ or } f(b) = 0,$$
$$\|f\| \le \frac{(b-a)}{\pi} \|f'\| \qquad \text{if } f(a) = f(b) = 0.$$

For $f \in H^2((0,l_1) \times (0,l_2))$ and any $\beta \in (0,1)$,

$$\begin{aligned}
\|f(\cdot) - f(0)\|^2 &= \|(f(\cdot,\cdot) - f(\cdot,0)) + (f(\cdot,0) - f(0,0))\|^2 \\
&\overset{\text{Lem.3}}{\le} \frac{1}{\beta} \|f(\cdot,\cdot) - f(\cdot,0)\|^2 + \frac{1}{1-\beta} \|f(\cdot,0) - f(0,0)\|^2 \\
&\overset{\text{Lem.4}}{\le} \frac{1}{\beta} \left(\frac{2l_2}{\pi} \right)^2 \|f_{x_2}\|^2 + \frac{1}{1-\beta} \left(\frac{2l_1}{\pi} \right)^2 \|f_{x_1}(\cdot,0)\|^2.
\end{aligned}$$

For any $\gamma \in (0,1)$, we have

$$\begin{aligned}
\|f_{x_1}(\cdot,0)\|^2 &= \|(f_{x_1}(\cdot,0) - f_{x_1}(\cdot,\cdot)) + f_{x_1}(\cdot,\cdot)\|^2 \\
&\overset{\text{Lem.3}}{\le} \frac{1}{\gamma} \|f_{x_1}(\cdot,0) - f_{x_1}(\cdot,\cdot)\|^2 + \frac{1}{1-\gamma} \|f_{x_1}\|^2 \\
&\overset{\text{Lem.4}}{\le} \frac{1}{\gamma} \left(\frac{2l_2}{\pi} \right)^2 \|f_{x_1 x_2}\|^2 + \frac{1}{1-\gamma} \|f_{x_1}\|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|f(\cdot) - f(0)\|^2 &\le \frac{1}{(1-\beta)(1-\gamma)} \left(\frac{2l_1}{\pi} \right)^2 \|f_{x_1}\|^2 \\
&+ \frac{1}{\beta} \left(\frac{2l_2}{\pi} \right)^2 \|f_{x_2}\|^2 + \frac{1}{(1-\beta)\gamma} \left(\frac{2l_1}{\pi} \right)^2 \left(\frac{2l_2}{\pi} \right)^2 \|f_{x_1 x_2}\|^2,
\end{aligned}$$

which coincides with (2) if $\beta = \lambda_{(0,1)}$ and $\gamma = \lambda_{(1,1)}/(1-\beta)$.

To prove the inductive step, let (1) be true for all $g \in H^{n-1}(\Omega)$ with $\Omega = (0,l_1) \times \dots \times (0,l_{n-1})$. Taking $g(x) = f(x,0)$, where $f \in H^n(\Omega \times (0,l_n))$, we obtain

$$\int_\Omega [f(x,0) - f(0,0)]^2 dx \le \sum_{\alpha \in \mathcal{I}_{n-1}} \frac{c_\alpha}{\lambda_\alpha} \int_\Omega [\partial^\alpha f(x,0)]^2 dx$$

for any $\lambda_\alpha \in \mathbb{R}_{>0}$ such that $\sum_\alpha \lambda_\alpha = 1$. Thus, for any $\beta \in (0,1)$,

$$\begin{aligned}
\|f(\cdot) - f(0)\|^2 &= \int_0^{l_n} \int_\Omega [(f(x,x_n) - f(x,0)) \\
&\quad + (f(x,0) - f(0,0))]^2 \, dx \, dx_n \\
&\overset{\text{Lem.3}}{\le} \frac{1}{\beta} \int_0^{l_n} \int_\Omega [f(x,x_n) - f(x,0)]^2 dx \, dx_n \\
&\quad + \frac{1}{1-\beta} \int_0^{l_n} \int_\Omega [f(x,0) - f(0,0)]^2 \, dx \, dx_n \\
&\overset{\text{Lem.4}}{\le} \frac{1}{\beta} \left(\frac{2l_n}{\pi} \right)^2 \|f_{x_n}\|^2 \\
&\quad + \sum_{\alpha \in \mathcal{I}_{n-1}} \frac{c_\alpha}{(1-\beta)\lambda_\alpha} \int_0^{l_n} \int_\Omega [\partial^\alpha f(x,0)]^2 dx \, dx_n.
\end{aligned}$$

For any $\gamma \in (0,1)$ and $\alpha \in \mathcal{I}_{n-1}$, we have

$$\begin{aligned}
&\int_0^{l_n} \int_\Omega (\partial^\alpha f(x,0))^2 \, dx \, dx_n \\
&= \int_0^{l_n} \int_\Omega [(\partial^\alpha f(x,0) - \partial^\alpha f(x,x_n)) + \partial^\alpha f(x,x_n)]^2 dx \, dx_n \\
&\overset{\text{Lem.3}}{\le} \frac{1}{\gamma} \int_0^{l_n} \int_\Omega [\partial^\alpha f(x,0) - \partial^\alpha f(x,x_n)]^2 \, dx \, dx_n \\
&\quad + \frac{1}{1-\gamma} \|\partial^\alpha f\|^2 \\
&\overset{\text{Lem.4}}{\le} \frac{1}{\gamma} \left(\frac{2l_n}{\pi} \right)^2 \left\| \frac{\partial}{\partial x_n} \partial^\alpha f \right\|^2 + \frac{1}{1-\gamma} \|\partial^\alpha f\|^2.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\|f(\cdot)-f(0)\|^2 &\le \frac{1}{\beta}\left(\frac{2l_n}{\pi}\right)^2\|f_{x_n}\|^2 \\
&+\sum_{\alpha\in\mathcal{I}_{n-1}}\left(\frac{2l_n}{\pi}\right)^2\frac{c_\alpha}{(1-\beta)\lambda_\alpha\gamma}\left\|\frac{\partial}{\partial x_n}\partial^\alpha f\right\|^2 \\
&+\sum_{\alpha\in\mathcal{I}_{n-1}}\frac{c_\alpha}{(1-\beta)\lambda_\alpha(1-\gamma)}\|\partial^\alpha f\|^2=\sum_{\alpha\in\mathcal{I}_n}\frac{c_\alpha}{\lambda_\alpha}\|\partial^\alpha f\|^2,
\end{aligned}
$$

where

$\lambda_{(0,\dots,0,1)}=\beta,$

$\lambda_{(\alpha_1,\dots,\alpha_{n-1},1)}=(1-\beta)\lambda_{(\alpha_1,\dots,\alpha_{n-1})}\gamma,$

$\lambda_{(\alpha_1,\dots,\alpha_{n-1},0)}=(1-\beta)\lambda_{(\alpha_1,\dots,\alpha_{n-1})}(1-\gamma).$

Note that c_α with $\alpha\in\mathcal{I}_{n-1}$ differ from c_α with $\alpha\in\mathcal{I}_n$. Clearly, the condition $\sum_{\alpha\in\mathcal{I}_{n-1}}\lambda_\alpha=1$ is equivalent to $\sum_{\alpha\in\mathcal{I}_n}\lambda_\alpha=1$. By induction, (1) holds for any $n\in\mathbb{N}$.

B Proof of Theorem 2

For $z\in C_0^\infty$, integration by parts yields

$$
0=-2\mu_6\int_\Omega z_{x_1x_2}^2+2\mu_6\int_\Omega z_{x_1x_1}z_{x_2x_2}. \tag{B.1}
$$

Since C_0^∞ is dense in H_0^1, the latter holds for $z\in H_0^1\cap H^2$. Since $z|_{\partial\Omega}=0$, Lemma 4 with $a=0$ and $b=\frac{1}{2}$ implies

$$
0\le-(\mu_7+\mu_8)\pi^2\int_\Omega z^2+\int_\Omega(\nabla z)^T\left[\begin{smallmatrix}\mu_7&0\\0&\mu_8\end{smallmatrix}\right]\nabla z. \tag{B.2}
$$

Consider $V=V_0+V_1$ with $V_0=\|z\|^2$ and V_1 from (27). Calculating its derivative (see (25) and (28)) and adding the right-hand sides of (21), (23), (26), (B.1), and (B.2), we obtain

$$
\dot V+2\alpha V\le\int_\Omega\varphi^T\Phi\varphi+\int_\Omega(\nabla z)^T\Phi_\nabla\nabla z\le0,
$$

where $\varphi=(z,z_{x_1x_1},z_{x_1x_2},z_{x_2x_2},f,\sigma,\epsilon)^T$. Thus, $\dot V\le-2\alpha V$, which implies the exponential stability of (16) in the H_0^1-norm with the decay rate α.

C Proof of Theorem 3

For $t\ge\tau_M$, consider $V=V_0+V_1+V_r$, where $V_0=\|z\|^2$, V_1 is defined in (27), and V_r is given by (41) with c_r from (42). Similarly to (25), we have

$$
\begin{aligned}
\dot V_0+2\alpha V_0 &\overset{(39)}{=}-2\int_\Omega(\nabla z)^TD\nabla z-2(K-\alpha)\int_\Omega z^2 \\
&+2\int_\Omega z[f+\epsilon(x,t-\tau_i)+\sigma+\kappa-Kv+w].
\end{aligned}
$$

Similarly to (28), we have

$$
\begin{aligned}
\dot V_1+2\alpha V_1 &=-2\int_\Omega\operatorname{div}(P\nabla z)\,\Delta_Dz-2(K-\alpha)\int_\Omega(\nabla z)^TP\nabla z \\
&-2\int_\Omega\operatorname{div}(P\nabla z)\left[f+\epsilon(x,t-\tau_i)+\sigma+\kappa-Kv+w\right].
\end{aligned}
$$

Summing up (23), (26), (40), (43), (B.1), (B.2), and the above expressions, we obtain

$$
\begin{aligned}
&\dot V+2\alpha V+\gamma_1\|z(\cdot,t)\|^2+\gamma_3\|u(\cdot,t)\|^2 \\
&\qquad-\gamma_2\|w(\cdot,t)\|^2-\gamma_2\|v(\cdot,t)\|^2 \\
&\le\int_\Omega\tilde\varphi^T\widetilde\Phi_s\tilde\varphi+\int_\Omega(\nabla z)^T\Phi_\nabla\nabla z \\
&\qquad+c_rr\tau_M\max_i\|c_i\|_\infty\|z_t(\cdot,t)\|^2,
\end{aligned}
$$

where $u(x,t)\overset{(38)}{=}\epsilon(x,t-\tau_i(t))+\sigma+\kappa-Kz-Kv$,

$\tilde\varphi=(z,z_{x_1x_1},z_{x_1x_2},z_{x_2x_2},f,\sigma,\epsilon(x,t-\tau_i(t)),\kappa,-Kv,w)^T$,

and $\widetilde\Phi_s$ is obtained from $\widetilde\Phi$ by eliminating the last column and row. Substituting (39) for z_t and using the Schur's complement lemma, we deduce that the conditions $\widetilde\Phi\le0$ and $\Phi_\nabla\le0$ guarantee

$$
\begin{aligned}
&\dot V+2\alpha V+\gamma_1\|z(\cdot,t)\|^2+\gamma_3\|u(\cdot,t)\|^2 \\
&\qquad-\gamma_2\|w(\cdot,t)\|^2-\gamma_2\|v(\cdot,t)\|^2\le0.
\end{aligned}
$$

Since the initial time interval $[0,\tau_M)$ does not influence the decay-rate analysis [29], the latter implies the internal exponential stability in the H_0^1-norm with the decay rate α. If $z|_{t\le0}=0$, then the functional V is well-defined for $t\ge0$ and $V|_{t=0}=0$. Thus, integrating the previous inequality from 0 to ∞, we prove that $J(u)\le0$ with J given in (44), $\gamma=\sqrt{\gamma_2/\gamma_1}$, and $d_u=\sqrt{\gamma_3/\gamma_1}$.