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A COMPARISON PRINCIPLE FOR STOCHASTIC INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. A comparison principle for stochastic integro-differential equations driven by Lévy processes is proved. This result is obtained via an extension of an Itô formula, proved by N.V. Krylov, for the square of the norm of the positive part of L_2-valued, continuous semimartingales, to the case of discontinuous semimartingales. Comparison principle and Itô's formula and SPDE and Lévy processes

1. Introduction

Our goal is to prove a comparison principle for stochastic integro-differential equations (SIDEs) driven by Lévy processes. For this, first we present an Itô's formula for the square of the L_2-norm of the positive part of (possibly) discontinuous semimartingales with values in L_2-spaces. Our formula extends an Itô formula from [16] proved for continuous semimartingales. In [16] Itô's formulas for the square of L_2-norm of certain convex functions $r(u)$ of continuous semimartingales $u = u_t$ with values in L_2-spaces are obtained, and the important special case, $r(u) = (u)^+ = \max(u, 0)$, is then applied to prove a maximum principle for linear stochastic partial differential equations (SPDEs). The present paper is organized as follows. In Section 2 we formulate and prove our Itô formula. The main results concerning comparison theorems are stated in Section 3. We also give an existence and uniqueness result as a simple consequence of a theorem on stochastic evolution equations from [11]. For recent results concerning the solvability of SPDEs driven by Lévy processes we refer to [2]. In Section 4 we give some tools that will be needed in order to prove the main theorems in Section 5. For notions and results in SPDEs we refer to [19].

Comparison principles are powerful tools and play important role in PDE theory. Comparison theorems for SPDEs are known in various generalities in the literature. To the best of our knowledge, the first results on comparison of solutions of SPDEs appear in [14] and [6]. Recent results appear in [16], [4], [3] and [5]. In [3] and [4] quasi linear SPDEs, and in [5] quasi-linear SPDEs with obstacle are considered, and the p-th moments of the positive part of the supremum norm of the solutions are also estimated. In the above publications, SPDEs driven by Wiener processes, or cylindrical Wiener processes are considered. Our main result, Theorems 3.2 and 3.4, are comparison theorems for two classes of quasilinear SIDEs, linear versions of which, arise in non-linear filtering. For the theory of non-linear filtering of processes with jumps we refer to [8] and [9]. We will apply our result to investigate the solvability of a class of SPDEs driven by Lévy processes in another paper.

In conclusion we introduce some basic notation of the paper. Let $(\Omega, \mathscr{F}, P)$ be a probability space equipped with a right-continuous filtration $(\mathscr{F}_t)_{t\geq 0}$, such that $\mathscr{F}_0$ contains all P-zero sets. We consider a σ-finite measure space, $(Z, \mathcal{Z}, \nu)$ and a quasi left-continuous, adapted point process $(p_t)_{t\in[0,T]}$ in Z, for a finite $T > 0$. Let $N(dt, dz)$ be the random measure on $[0, T] \times Z$, corresponding to the point processes p. We assume that its compensator is $dt\nu(dz)$ and we use the notation

$$\tilde{N}(dt, dz) = N(dt, dz) - dt\nu(dz).$$

We also consider a sequence of independent real valued $\mathscr{F}_t$-Wiener processes $\{w_t^k\}_{k=1}^{\infty}$.

If X is a topological space then $\mathscr{B}(X)$ is the Borel σ-algebra on X. The notation $\mathscr{P}$ is used for the predictable σ-algebra on $\Omega \times [0, T]$. If X is a normed linear space then $|x|_X$ denotes the norm of $x \in X$, X^* is the dual of X, and $\langle x, x^*\rangle$ denotes the action of $x^* \in X^*$ on $x \in X$. The notation Q stands for the whole space $\mathbb{R}^d$ or for a bounded Lipschitz domain in $\mathbb{R}^d$. We write

$$D_i u := \frac{\partial u}{\partial x_i},\ D_{ij} u := \frac{\partial^2 u}{\partial x_i \partial x_j},\ \text{for } i, j = 1, ..., d,$$

for the first and second order partial derivatives of a function u defined on Q. As usual we denote by $W_p^k(Q)$ the space of functions $u \in L_p(Q)$, whose generalized derivatives up to order k lie in $L_p(Q)$. We set $H^1(Q) := W_2^1(Q)$ and we write $H_0^1(Q)$ for the closure of $C_c^\infty(Q)$ in $H^1(Q)$ under the norm

$$|u|_{H^1} = \Big(\sum_{i=1}^{d} |D_i u|_{L_2} + |u|_{L_2} \Big)^{1/2}.$$

We will use the notation $H^{-1}(Q)$ for the dual of $H_0^1(Q)$. Finally, we note that unless otherwise indicated, the summation convention is used with respect to repeated integer-valued indices throughout the paper.

2. Itô's formula for the square of the norm of the positive part

We are interested in an Itô's formula for $|u_t^+|^2_{L_2(Q)}$, where u_t is an $H^{-1}(Q)$-valued semimartingale taking values in $H_0^1(Q)$ for $dP \times dt$ almost every $(\omega, t) \in \Omega \times [0, T]$. Our approach to obtain it is similar to that in [16]. To state the formula we set

$$V := H_0^1(Q), \quad H := L_2(Q), \quad V^* := H^{-1}(Q),$$

and we consider the following processes

$$v : \Omega \times [0, T] \to V,\ v^* : \Omega \times [0, T] \to V^*,\ h^k : \Omega \times [0, T] \to H,$$

$$K : \Omega \times [0,T] \times Z \to H,$$

for integers $k \geq 1$, where v is progressively measurable, v^* and h^k are $\mathscr{F}_t$-adapted, measurable in (ω, t), and K is $\mathscr{P} \times \mathcal{Z}$ measurable. We consider also ψ, an $\mathscr{F}_0$-measurable random variable in H.

It is easy to see that $V = H_0^1(Q)$ is continuously and densely embedded into $L_2(Q)$. Hence, by identifying $H = L_2(Q)$ with its dual H^* by the help of the inner product $(,)$ in $L_2(Q)$, we get the *normal triple of spaces*

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*,$$

where $H^* \hookrightarrow V^*$ is the adjoint embedding of $V \hookrightarrow H$. We use the notation $\langle v^*, v\rangle$ for the duality product of $v^* \in V^*$ and $v \in V$. Notice that $\langle v^*, v\rangle = (v^*, v)$ when $v^* \in H$.

A stochastic process $v = (v_t)_{t\in[0,T]}$, taking values in a Banach space $\mathbb{B}$, is called a $\mathbb{B}$-valued strongly cádlág process if with probability one the trajectories of v are continuous from the right in $t \in [0, T)$ and have limits from the left at every $t \in (0, T]$ in the strong topology of $\mathbb{B}$, i.e., in the topology given by the norm in $\mathbb{B}$.

We make the following assumption.

Assumption 2.1.

(i) Almost surely

$$\int_{(0,T]} \Big(|v_t|_V^2 + |v_t^*|_{V^*}^2 + \sum_k |h_t^k|_H^2 + \int_Z |K_t(z)|_H^2 \nu(dz)\Big) dt < \infty,$$

(ii) for each $\phi \in V$ and for $dP \times dt$-almost every (ω, t), we have

$$(v_t, \phi) = (\psi, \phi) + \int_{(0,t]} \langle v_s^*, \phi\rangle ds + \int_{(0,t]} (h_s^k, \phi) dw_s^k$$
$$+ \int_{(0,t]} \int_Z (K_s(z), \phi) \tilde{N}(ds, dz).$$

Theorem 2.1. *Suppose that Assumption 2.1 is satisfied. Then there exists a set $\tilde{\Omega} \subset \Omega$ of probability one, and an H-valued strongly cádlág adapted process u_t such that $u_t = v_t$ for $dP \times dt$-almost every (ω, t). Moreover for $\omega \in \tilde{\Omega}$, $t \in [0, T]$ we have*

$$i)\ u_t = \psi + \int_{(0,t]} v_s^* ds + \int_{(0,t]} h_s^k dw_s^k + \int_{(0,t]} \int_Z K_s(z) \tilde{N}(ds, dz), \tag{2.1}$$

$$ii)\ |u_t^+|_H^2 = |\psi^+|_H^2 + 2\int_{(0,t]} \langle v_s^*, u_s^+\rangle ds + 2\int_{(0,t]} (h_s^k, u_s^+) dw_s^k$$
$$+2\int_{(0,t]} \int_Z (K_s(z), u_{s-}^+) \tilde{N}(dz, ds) + \int_{(0,t]} \sum_k |I_{u_s>0} h_s^k|_H^2 ds$$
$$+ \int_{(0,t]} \int_Z |(u_{s-} + K_s(z))^+|_H^2 - |u_{s-}^+|_H^2 - 2(K_s(z), u_{s-}^+)_H \ N(dz, ds).$$

To prove Theorem 2.1 we need two lemmas.

Lemma 2.2. *Let (X, Σ, μ) be a measure space, and let $u_n, u \in L_1(X)$ such that $u_n \to u$ in $L_1(X)$ as $n \to \infty$. Then there exists a subsequence $\{u_{n(k)}\}_{k=1}^{\infty}$ and a function $v \in L_1(X)$ such that for all $k \geq 1$ we have $|u_{n(k)}(x)| \leq v(x)$ for all $x \in X$, and $u_{n(k)}(x) \to u(x)$ for μ-almost every x as $k \to \infty$.*

Proof. There exists a subsequence $u_{n(k)}$ such that

$$|u_{n(k)} - u|_{L_1(X)} \leq 1/2^k \quad \text{for } k \geq 1.$$

Set $v(x) = |u(x)| + \sum_k |u_{n(k)}(x) - u(x)|$. Then v has the desired properties. Moreover, $\sum_k |u_{n(k+1)} - u_{n(k)}|_{L_1(X)} < \infty$, which implies that $u_{n(k)}$ converges μ-almost everywhere. □

The next lemma is from [4].

Lemma 2.3. *Let $\mathcal{Q}$ be a bounded Lipschitz domain in $\mathbb{R}^d$. Take $\phi_n \in C_c^{\infty}(\mathcal{Q})$, $n \in \mathbb{N}$, with*

i) $0 \leq \phi_n \leq 1$
ii) $\phi_n = 1$ *on* $\{x \in \mathcal{Q}, r(x) \geq 1/n\}$
iii) $\phi_n = 0$ *on* $\{x \in \mathcal{Q}, r(x) \leq 1/2n\}$
iv) $|(\phi_n)_{x_i}| \leq Cn$,

where C is a constant and $r(x) = dist(x, \partial\mathcal{Q})$. Then $\phi_n v \to v$ in $H_0^1(\mathcal{Q})$ for all $v \in H_0^1(\mathcal{Q})$, and for some constant C we have

$$\sup_n |\phi_n v|_{H_0^1} \leq C|v|_{H_0^1}, \qquad \forall v \in H_0^1(\mathcal{Q}).$$

Remark 2.1. One can easily see the existence of a sequence $(\phi_n)_{n\in\mathbb{N}}$ satisfying the conditions of the previous lemma. Then note that ϕ_n^2 also satisfies i)-iv). Hence, $\phi_n^2 v \to v$ in $H_0^1(\mathcal{Q})$, for all $v \in H_0^1(\mathcal{Q})$, and for some constant C we have

$$\sup_n |\phi_n^2 v|_{H_0^1} \leq C|v|_{H_0^1}, \qquad \forall v \in H_0^1(\mathcal{Q}).$$

We introduce now the functions $\alpha_\delta(r)$, $\beta_\delta(r)$ and $\gamma_\delta(r)$ on $\mathbb{R}$, for $\delta > 0$, given by

$$a_\delta(r) = \begin{cases} 1 & \text{if } r > \delta \\ \frac{r}{\delta} & \text{if } 0 \leq r \leq \delta \\ 0 & \text{if } r < 0, \end{cases}$$

$$\beta_\delta(r) = \int_0^r a_\delta(s)ds, \qquad \gamma_\delta(r) = \int_0^r \beta_\delta(s)ds.$$

For all $r \in \mathbb{R}$ we have $\alpha_\delta(r) \to I_{r>0}$, $\beta_\delta(r) \to r^+$ and $\gamma_\delta(r) \to (r^+)^2/2$ as $\delta \to 0$. Also, for all r, r_1, r_2 and δ, the following inequalities hold

$$|\alpha_\delta(r)| \leq 1, \ |\beta_\delta(r)| \leq |r|, \ |\gamma_\delta(r)| \leq \frac{r^2}{2},$$

$$|\gamma_\delta(r_1 + r_2) - \gamma_\delta(r_1) - \beta_\delta(r_1)r_2| \leq |r_2|^2. \tag{2.2}$$

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. We only prove ii) since the rest of the assertions are proved in [10], in greater generality. First we prove the statement when $Q=\mathbb{R}^d$. We have that equality (2.1) is satisfied if and only if, almost surely, for all $\varphi \in V$ and t we have

$$(u_t,\varphi)=(\psi,\varphi)+\int_{(0,t]}\langle v^*_s,\varphi\rangle ds+\int_{(0,t]}(h^k_s,\varphi)dw^k_s + \int_{(0,t]}\int_Z (K_s(z),\varphi)\tilde N(ds,dz). \tag{2.3}$$

Let ϕ be a mollifier with compact support and set $\phi_\epsilon(x):=\epsilon^{-d}\phi(x/\epsilon)$. For fixed x, the function $\phi_\epsilon(x-\cdot)$ is in V, so we can plug it in (2.3) instead of φ, to get that almost surely, for all $t\in[0,T]$

$$u^\epsilon_t(x)=u^\epsilon_0(x)+\int_{(0,t]}v^{*\epsilon}_s(x)ds+\int_{(0,t]}h^{k\epsilon}_s(x)dw^k_s + \int_{(0,t]}\int_Z K^\epsilon_s(z,x)\tilde N(ds,dz),$$

where for $g\in V^*$ we use the notation $g^\epsilon(x):=\langle g,\phi_\epsilon(x-\cdot)\rangle$. Note that u^ϵ_0 is $\mathscr{F}_0\times\mathscr{B}(\mathbb{R}^d)$ measurable. Also $u^\epsilon, v^{*\epsilon}$ and $h^{k\epsilon}$ are jointly measurable in (t,ω,x), $\mathscr{F}_t\times\mathscr{B}(\mathbb{R}^d)$ measurable for each t, and K^ϵ is $\mathscr{P}\times\mathcal{Z}\times\mathscr{B}(\mathbb{R}^d)$ measurable. It is also easy to see that there exists a constant C_ϵ, depending on ϵ, such that for all t,ω,x,z

$$|u^\epsilon_t(x)|\le C_\epsilon|u_t|_H,\ |u^\epsilon_0(x)|\le C_\epsilon|u_0|_H, |v^{*\epsilon}_t|_H\le C_\epsilon|v^*_t|_{V^*}$$
$$|v^{*\epsilon}_t(x)|\le C_\epsilon|v^*_t|_{V^*},\ |h^{k\epsilon}_t(x)|\le C_\epsilon|h^k_s|_H,$$
$$|K^\epsilon_t(x,z)|\le C_\epsilon|K_t(z)|_H.$$

One can also check that for a constant C, for all ϵ

$$|u^\epsilon_t|_H\le C|u_t|_H,\ |u^\epsilon_0|_H\le C|u_0|_H,\ |K^\epsilon_t(z)|_H\le C|K_t(z)|_H$$
$$|h^{k\epsilon}_t|_H\le C|h^k_s|_H, |v^{*\epsilon}_t|_{V^*}\le C|v^*_t|_{V^*},\ |u^\epsilon_t|_V\le C|u_t|_V.$$

Now let $\alpha_\delta,\beta_\delta,\gamma_\delta$ be as before, and fix x. By Itô's formula (see for example [13] or [21]), for each x there exists a set Ω_x of full probability, such that for all $\omega\in\Omega_x$ and $t\in[0,T]$ we have

$$\begin{aligned}\gamma_\delta(u^\epsilon_t(x))&=\gamma_\delta(u^\epsilon_0(x))+\int_{(0,t]}\beta_\delta(u^\epsilon_s(x))v^{*\epsilon}_s(x)ds\\ &+\sum_k\int_{(0,t]}\beta_\delta(u^\epsilon_s(x))h^{k\epsilon}_s(x)dw^k_s+\frac12\sum_k\int_{(0,t]}\alpha_\delta(u^\epsilon_s(x))|h^{k\epsilon}_s(x)|^2ds\\ &+\int_{(0,t]}\int_Z\beta_\delta(u^\epsilon_{s-}(x))K^\epsilon_s(z,x)\tilde N(ds,dz)\\ &+\int_{(0,t]}\int_Z\gamma_\delta(u^\epsilon_s(x)+K^\epsilon_s(z,x))\\ &-\gamma_\delta(u^\epsilon_{s-}(x))-\beta_\delta(u^\epsilon_{s-}(x))K^\epsilon_s(z,x)N(dz,ds).\end{aligned} \tag{2.4}$$

One can redefine the stochastic integrals such that (2.4) holds for all (ω, t, x). Integrating (2.4) over $\mathbb{R}^d$, taking appropriate versions of the stochastic integrals and using the Fubini and the stochastic Fubini theorems we get for each $t \in [0, T]$,

$$
\begin{aligned}
\int_{\mathbb{R}^d} \gamma_\delta(u^\epsilon_t(x))\,dx &= \int_{\mathbb{R}^d} \gamma_\delta(u^\epsilon_0(x))\,dx + \int_{(0,t]} \int_{\mathbb{R}^d} \beta_\delta(u^\epsilon_s(x)) v^{*\epsilon}_s(x)\,dx\,ds \\
\int_{(0,t]} \int_{\mathbb{R}^d} \beta_\delta(u^\epsilon_s(x)) h^{k\epsilon}_s(x)\,dx\,dw^k_s &+ \frac{1}{2}\sum_k \int_{(0,t]} \int_{\mathbb{R}^d} \alpha_\delta(u^\epsilon_s(x)) |h^{k\epsilon}_t(x)|^2\,dx\,ds \\
&+ \int_{(0,t]} \int_Z \int_{\mathbb{R}^d} \beta_\delta(u^\epsilon_{s-}(x)) K^\epsilon_s(z,x) dx \tilde{N}(ds, dz) \\
&+ \int_{(0,t]} \int_Z \int_{\mathbb{R}^d} \gamma_\delta(u^\epsilon_{s-}(x) + K^\epsilon_{s-}(z,x)) \\
&- \gamma_\delta(u^\epsilon_{s-}(x)) - \beta_\delta(u^\epsilon_{s-}(x)) K^\epsilon_s(z,x) dx N(dz, ds) \ (a.s.). \qquad (2.5)
\end{aligned}
$$

For a stochastic Fubini theorem we refer to [17], noting that the Fubini theorem there can be extended easily, by obvious changes in its proof, to our situation. Since each term in the above equation is a cádlág process in t, we see that (2.5) holds almost surely, for all $t \in [0, T]$. We claim that for each $t \in [0, T]$ both sides of (2.5) converges in probability as $\epsilon \to 0$ to give that

$$
\begin{aligned}
\int_{\mathbb{R}^d} \gamma_\delta(u_t(x)) dx &= \int_{\mathbb{R}^d} \gamma_\delta(u_0(x)) dx + \int_{(0,t]} \langle v^*_s, \beta_\delta(u_s) \rangle ds \\
\int_{(0,t]} \int_{\mathbb{R}^d} \beta_\delta(u_s(x)) h^k_s(x) dx dw^k_s &+ \frac{1}{2}\sum_k \int_{(0,t]} \int_{\mathbb{R}^d} \alpha_\delta(u_s(x)) |h^k_s(x)|^2 dx ds \\
&+ \int_{(0,t]} \int_Z \int_{\mathbb{R}^d} \beta_\delta(u_{s-}(x)) K_s(z,x) dx \tilde{N}(ds, dz) \\
&+ \int_{(0,t]} \int_Z \int_{\mathbb{R}^d} \gamma_\delta(u_{s-}(x) + K_s(z,x)) \\
&- \gamma_\delta(u_{s-}(x)) - \beta_\delta(u_{s-}(x)) K_s(z,x) dx N(dz, ds). \qquad (2.6)
\end{aligned}
$$

holds almost surely for each $t \in [0, T]$. We are going to show that each term in (2.5) converges in probability to the corresponding one in (2.6). Since for any sequence $\epsilon_k \downarrow 0$ we have $u^{\epsilon_k}_t \to u_t$ in $L_2(\mathbb{R}^d)$ for every $\omega \in \Omega$, by the equality $a^2 - b^2 = (a-b)(a+b)$ we have $(u^{\epsilon_k}_t)^2 \to u^2_t$ in $L_1(\mathbb{R}^d)$. Thus for every $\omega \in \Omega$ by Lemma 2.2 there exist $g \in L_1(\mathbb{R}^d)$ and a subsequence, denoted again by ϵ_k, such that for all $k \geq 1$

$$
|\gamma_\delta(u^{\epsilon_k}_t(x))| \leq \frac{(u^{\epsilon_k}_t(x))^2}{2} \leq \frac{g(x)}{2} \text{ for all } x.
$$

Since $\gamma_\delta(u^{\epsilon_k}_t(x)) \to \gamma_\delta(u_t(x))$ for almost every x, as $k \to \infty$, by Lebesgue's theorem on dominated convergence we obtain

$$
\int_{\mathbb{R}^d} \gamma_\delta(u^{\epsilon_k}_t(x)) dx \to \int_{\mathbb{R}^d} \gamma_\delta(u_t(x)) dx \quad \text{as } k \to \infty.
$$

Thus, for $\epsilon \downarrow 0$ the left-hand side of (2.5) converges to the left-hand side of (2.6) for every $\omega \in \Omega$, and hence also in probability, for each $t \in [0,T]$. To see the convergence of the second term in the right-hand side of (2.5) we fix (s,ω) such that $u_s \in V$. Then it is straightforward to check that

$$|\beta_\delta(u_s^\epsilon) - \beta_\delta(u_s)|_V \to 0, \text{ as } \epsilon \to 0.$$

Taking into account the well-known fact that there exist f_s^0 and $f_s^i \in L^2(\mathbb{R}^d)$ for $i = 1, ..., d$ such that

$$v_s^* = f_s^0 + D_i f_s^i,$$

we have

$$v_s^{*\epsilon} = f_s^{0\epsilon} + D_i f_s^{i\epsilon},$$

which gives

$$|v_s^* - v_s^{*\epsilon}|_{V^*} \le \sum_{i=0}^{d} |f_s^{i\epsilon} - f_s^i|_H \to 0, \text{ as } \epsilon \to 0.$$

Hence we conclude

$$\int_{\mathbb{R}^d} \beta_\delta(u_s^\epsilon(x)) v_s^{*\epsilon}(x)dx = \langle v_s^{*\epsilon}, \beta_\delta(u_s^\epsilon)\rangle \to \langle v_s^*, \beta_\delta(u_s)\rangle.$$

Notice that there is a constant C such that

$$\Big|\int_{\mathbb{R}^n} \beta_\delta(u_s^\epsilon(x)) v_s^{*\epsilon}(x)dx\Big| \le C(|u_s|_V^2 + |v_s^*|_{V^*}^2)$$

for all $\epsilon > 0$, $\omega \in \Omega$ and $s \in [0,T]$. Therefore, almost surely

$$\int_{(0,t]}\int_{\mathbb{R}^d} \beta_\delta(u_s^\epsilon(x)) v_s^{*\epsilon}(x)dxds \to \int_{(0,t]} \langle v_s^*, \beta_\delta(u_s)\rangle ds \quad \text{for all } t.$$

For the sum of the stochastic integrals against the Wiener processes we just note that almost surely for all $s \in [0,T]$

$$\sum_k \Big|\int_{\mathbb{R}^d} \beta_\delta(u_s^\epsilon(x)) h_s^{k\epsilon}(x)dx - \int_{\mathbb{R}^d} \beta_\delta(u_s(x)) h_s^k(x)dx\Big|^2 \to 0 \quad \text{as } \epsilon \downarrow 0,$$

and

$$\sum_k \Big|\int_{\mathbb{R}^d} \beta_\delta(u_s^\epsilon(x)) h_s^{k\epsilon}(x)dx - \int_{\mathbb{R}^d} \beta_\delta(u_s(x)) h_s^k(x)dx\Big|^2$$
$$\le 4 \sup_{t\le T} |u_t|_{L^2}^2 \sum_k |h_s^k|_{L^2}^2 \quad \text{for all } \epsilon > 0.$$

Hence almost surely

$$\int_{(0,T]} \sum_k \Big|\int_{\mathbb{R}^d} \beta_\delta(u_s^\epsilon(x)) h_s^{k\epsilon}(x)dx - \int_{\mathbb{R}^d} \beta_\delta(u_s(x)) h_s^k(x)dx\Big|^2 ds \to 0,$$

which implies that for $\epsilon \downarrow 0$

$$\int_{(0,t]}\int_{\mathbb{R}^d} \beta_\delta(u_s^\epsilon(x)) h_s^{k\epsilon}(x)\,dx\,dw_s^k \to \int_{(0,t]}\int_{\mathbb{R}^d} \beta_\delta(u_s(x)) h_s^k(x)\,dx\,dw_s^k$$

in probability, uniformly in $t \in [0,T]$. Note that for each k we have

$$\Big| \int_{\mathbb{R}^d} \alpha_\delta(u^\epsilon_s(x))|h^{k\epsilon}_s(x)|^2 - \alpha_\delta(u_s(x))|h^k_s(x)|^2 dx \Big|$$

$$\leq \int_{\mathbb{R}^d} |(h^{k\epsilon}_s(x))^2 - (h^k_s(x))^2| dx$$

$$+ \int_{\mathbb{R}^d} |h^k_s(x)|^2 |\alpha_\delta(u^\epsilon_s(x)) - \alpha_\delta(u_s(x))| dx \to 0.$$

for each $\omega \in \Omega$ and $s \in [0,T]$. Moreover,

$$\Big| \int_{\mathbb{R}^d} \alpha_\delta(u^\epsilon_s(x))|h^{k\epsilon}_s(x)|^2 dx \Big| \leq |h^k_s|^2_H,$$

where the right-hand side is almost surely integrable on $[0,T]$. Hence the almost sure convergence of the fourth term in the right-hand side of (2.5) follows. By the inequalities in (2.2), similar arguments show the convergence of the last two terms in probability. We conclude that for each $t \in [0,T]$ equation (2.6) holds almost surely. Since the stochastic processes in both sides of (2.6) are càdlàg processes, equation (2.6) holds almost surely for all $t \in [0,T]$.

Now by letting $\delta \to 0$ in (2.6), using arguments similar to the previous ones, and keeping in mind the inequalities (2.2) and the fact that for all $v \in V$

$$|\beta_\delta(v) - v^+|_V \to 0, \ |\beta_\delta(v)|_V \leq |v|_V,$$

we can finish the proof of the theorem for $Q = \mathbb{R}^d$.

We reduce the case of a bounded Lipschitz domain Q to that of the whole space by using the sequence ϕ_n from Lemma 2.3. Remember that ϕ_n has compact support in Q. Thus for a function η on Q we denote by $\phi_n\eta$, not only the function defined on Q by the multiplication of ϕ_n and η, but also its extension to zero outside of Q. Notice that when u satisfies (2.1) on Q, then $\phi_n u$ satisfies

$$\phi_n u_t = \phi_n u_0 + \int_{(0,t]} \phi_n v^*_s ds + \int_{(0,t]} \phi_n h^k_s dw^k_s$$

$$+ \int_{(0,t]} \int_Y \phi_n K_s(z) \tilde{N}(ds,dz)$$

on the whole $\mathbb{R}^d$, where the functional $\phi_n v^*$ is defined by

$$\langle \phi_n v^*_s, g \rangle := \langle v^*_s, \phi_n g \rangle_Q$$

for $g \in H^1(\mathbb{R}^d)$. The notation $\langle \cdot,\cdot \rangle_Q$ means the duality product between $H^1_0(Q)$ and $H^{-1}(Q)$. Notice that $\langle v^*_s, \phi_n g \rangle_Q$ is well defined, since the restriction of $\phi_n g$ to Q belongs to $H^1_0(Q)$. Then by the result in the case of the whole space, we have

$$\int_Q \phi_n^2 |u^+_t|^2 dx = \int_Q |\phi_n u^+_0|^2 dx + 2\int_{(0,t]} \langle v^*_s, \phi_n^2 u^+_s \rangle_Q ds$$

$$+2\int_{(0,t]}\int_Q \phi_n^2 h_s^k u_s^+ dx dw_s^k + \int_{(0,t]}\int_Q \sum_k |\mathbb{I}_{\{\phi_n u_s>0\}}\phi_n h_s^k|^2 dx ds$$
$$+\int_{(0,t]}\int_Z\int_Q 2K_s(z)\phi_n^2 u_{s-}^+ dx\tilde{N}(ds,dz)$$
$$+\int_{(0,t]}\int_Z\int_Q |\phi_n(u_{s-}+K_s(z))^+|^2 - |\phi_n u_{s-}^+|^2 - 2K_s(z)\phi_n^2 u_{s-}^+ dx N(dz,ds),$$

since ϕ_n is supported in Q. It is now easy to take $n\to\infty$ here to finish the proof of the theorem. We only note that for the second term on the right-hand side we have by Lemma 2.3 and Remark 2.1

$$\langle v_s^*, \phi_n^2 u_s^+\rangle_Q \to \langle v_s^*,\ u_s^+\rangle_Q \quad \text{for all } \omega, s,$$

and for a constant C,

$$\langle v_s^*, \phi_n^2 u_s^+\rangle_Q \le C|v_s^*|_{V^*}|u_s|_V \quad \text{for all } n.$$

□

3. Comparison Theorems

In this section we present our comparison theorems for two types of equations. Together with the space $(Z,\mathcal{Z})$, we consider another measurable space $(F,\mathfrak{F})$, a quasi left-continuous, adapted point process $(\bar{p}_t)_{t\in[0,T]}$ in F, and two $\sigma-$finite measures $\pi^{(1)}$, $\pi^{(2)}$ on F. Let $M(dt,d\zeta)$ be the corresponding random measure on $[0,T]\times F$. We assume that its compensator is $dt\pi^{(2)}(d\zeta)$ and we write

$$\tilde{M}(dt,d\zeta) = M(dt,d\zeta) - dt\pi^{(2)}(d\zeta).$$

First we consider the equation

$$\begin{aligned} du_t(x) =& \{L_t u_t(x) + f_t(x,u_t(x),\nabla u_t(x))\}\,dt \\ &+ G_t^k(u)(x)dw_t^k + \int_Z g_t(x,z,u_{t-}(x))\tilde{N}(dt,dz), \end{aligned} \tag{3.1}$$

for $(t,x)\in[0,T]\times Q$, with initial condition

$$u_0(x) = \psi(x), \quad x\in Q, \tag{3.2}$$

where

$$L_t u(x) = D_j(a_t^{ij}(x)D_i u(x)) + \mathcal{I}_t^{(1)}u_t(x),$$
$$\mathcal{I}_t^{(1)}u(x) = \int_F [u(x+c_t(x,\zeta)) - u(x) - c_t(x,\zeta)\cdot\nabla u(x)]m_t(x,\zeta)\pi^{(1)}(d\zeta),$$
$$G_t^k(u)(x) = \phi_t^{ik}(x)D_i u(x) + \sigma_t^k(x,u(x)).$$

We make the following assumptions. Let $K>0$ denote a constant.

Assumption 3.1.

i) The coefficients a^{ij}, are real-valued $\mathscr{P}\times\mathscr{B}(Q)$ measurable functions on $\Omega\times[0,T]\times Q$ and are bounded by K for every $i,j=1,...,d$. The coefficient $\phi^i=(\phi^{ik})_{k=1}^{\infty}$ is an l_2-valued $\mathscr{P}\times\mathscr{B}(Q)$-measurable function on $\Omega\times[0,T]\times Q$ for every $i=1,2,...,d$, such that

$$\sum_i\sum_k|\phi_t^{ik}(x)|^2\leq K\quad\text{for all }\omega,\ t\text{ and }x.$$

ii) f is a real valued $\mathscr{P}\times\mathscr{B}(Q)\times\mathscr{B}(\mathbb{R})\times\mathscr{B}(\mathbb{R}^d)$ -measurable function on $\Omega\times[0,T]\times Q\times\mathbb{R}\times\mathbb{R}^d$, and $\sigma=(\sigma^k)_{k=1}^{\infty}$ is a $\mathscr{P}\times\mathscr{B}(Q)\times\mathscr{B}(\mathbb{R})$-measurable function on $\Omega\times[0,T]\times Q\times\mathbb{R}$, with values in l_2. The function g is defined on $\Omega\times[0,T]\times Q\times Z\times\mathbb{R}$ with values in $\mathbb{R}$ and it is $\mathscr{P}\times\mathscr{B}(Q)\times\mathcal{Z}\times\mathscr{B}(\mathbb{R})$-measurable. We assume that there exists a predictable process $\bar{h}_t$ with values in $L_2(Q)$, such that almost surely $\bar{h}\in L_2([0,T]\times Q)$, and for all ω,t,x,z,r,r'

$$|f_t(x,r,r')|^2+\sum_k|\sigma_t^k(x,r)|^2+\int_Z|g_t(x,z,r)|^2\nu(dz)$$
$$\leq K|r|^2+K|r'|^2+|\bar{h}_t(x)|^2.$$

iii) ψ is an $\mathscr{F}_0$-measurable random variable in $L_2(Q)$.

iv) There exists a constant $\varkappa>0$ such that for all ω,t,x and for all $\xi=(\xi_1,...\xi_d)\in\mathbb{R}^d$ we have

$$a_t^{ij}(x)\xi_i\xi_j-\frac{1}{2}\phi_t^{ik}(x)\phi_t^{jk}(x)\xi_i\xi_j\geq\varkappa|\xi|^2.$$

v) For all ω,t,x,z,r_1,r_2

$$\sum_k|\sigma_t^k(x,r_1)-\sigma_t^k(x,r_2)|^2\leq K|r_1-r_2|^2.$$

Assumption 3.2. The function $f_t(x,r,r')$ is continuous in r, for each ω,t,x and r'.

Assumption 3.3. For all $\omega,t,x,r_1,r_2,r_1',r_2'$

$$2(r_1-r_2)(f_t(x,r_1,r_1')-f_t(x,r_2,r_1'))$$
$$+\int_Z|g_t(x,z,r_1)-g_t(x,z,r_2)|^2\nu(dz)\leq K|r_1-r_2|^2,$$

and

$$|f_t(x,r_1,r_1')-f_t(x,r_1,r_2')|\leq K|r_1'-r_2'|.$$

Assumption 3.4. The function $r+g_t(x,z,r)$ is non-decreasing in r for all ω,t,x,z.

Assumption 3.5. The function c maps $\Omega\times[0,T]\times\mathbb{R}^d\times F$ into $\mathbb{R}^d$, it is $\mathscr{P}\times\mathscr{B}(\mathbb{R}^d)\times\mathfrak{F}$-measurable, and there exists an $\mathfrak{F}$-measurable real function $\bar{c}$ on F such that

(i) $|c_t(x,\zeta)|\leq\bar{c}(\zeta)$, for all ω,t,x,ζ,

(ii) $\int_F\bar{c}^2(\zeta)\pi^{(1)}(d\zeta)\leq K$,

(iii) $|c_t(x,\zeta)-c_t(y,\zeta)|\leq\bar{c}(\zeta)|x-y|$, for all ω,t,x,y,ζ.

Assumption 3.6. The function m maps $\Omega \times [0,T] \times \mathbb{R}^d \times F$ into $\mathbb{R}$, it is $\mathscr{P} \times \mathscr{B}(\mathbb{R}^d) \times \mathfrak{F}$- measurable, and we have
(i) $0 \le m_t(x,\zeta) \le K$, for all ω, t, x, ζ,
(ii)$|m_t(x,\zeta) - m_t(y,\zeta)| \le K|x-y|$, for all ω, t, x, y, ζ.

Assumption 3.7. The functions $c_t^l(x,\zeta)$, $l = 1, ..., d$, are twice continuously differentiable in x, for each ω, t, ζ, and
(i) $|D_i c_t^l(x,\zeta)| \le K$, $|D_{ij} c_t^l(x,\zeta)| \le K$, for all $i, j, l = 1, ..., d$,
(ii) $K^{-1} \le |det(\mathbb{I} + \theta \nabla c_t(x,\zeta))|$
for all ω, t, x, ζ and $\theta \in [0,1]$, where $\mathbb{I}$ denotes the identity matrix.

Remark 3.1. Denote by $T_{\theta,t,\zeta}$ the mapping $x \mapsto x + \theta c_t(x,\zeta)$, for fixed ω, t, θ and ζ. By virtue of the inverse function theorem, it follows from (ii) of Assumption 3.7 that $T_{\theta,t,\zeta}$ is a *local* diffeomorphism. In addition, by the first inequality in (i) and by (ii) of Assumption 3.7, there exists a constant $\gamma > 0$, such that the norm of the matrix $(\mathbb{I} + \theta \nabla c_t(x,\zeta))^{-1}$ is uniformly bounded by γ. Hence, by Hadamard's theorem (see, eg, Theorem 5.1.5 in [1]), $T_{\theta,t,\zeta}$ is a *global* diffeomorphism, for fixed ω, t, θ and ζ. We denote by $J_{\theta,t,\zeta}$ the inverse of $T_{\theta,t,\zeta}$. Notice that for fixed ω, t, θ, ζ and for all $j = 1, ..., d$, the functions $J^j_{\theta,t,\zeta}(x)$ are twice continuously in x, and their first and second order derivatives are uniformly bounded.

Remark 3.2. Under Assumptions 3.5 through 3.7, $\mathcal{I}_t^{(1)}$ is a bounded linear operator from $H_0^1(Q)$ into $H^{-1}(Q)$ for fixed (ω, t), and for all $u, v \in H_0^1(Q)$ the process $\langle \mathcal{I}_t^{(1)} u, v\rangle$ is predictable. To see this, consider first the case $Q = \mathbb{R}^d$. For $u \in C_c^\infty(\mathbb{R}^d)$(even for $u \in W_2^2(\mathbb{R}^d)$) one can easily see that $\mathcal{I}_t^{(1)} u(x)$ is a function in $L_2(\mathbb{R}^d)$. Then for $v \in C_c^\infty(\mathbb{R}^d)$ we have by Taylor's formula

$$\begin{aligned}
&(\mathcal{I}_t^{(1)} u, v) = \\
&\int_0^1 (1-\theta) \int_F \int_{\mathbb{R}^d} D_{ki} u(T_{\theta,t,\zeta}) c_t^i(x,\zeta) c_t^k(x,\zeta) m_t(x,\zeta) v(x)\, dx \pi^{(1)}(d\zeta)\, d\theta \\
&= \int_0^1 (\theta - 1) \int_F \int_{\mathbb{R}^d} D_i u(x + \theta c_t(x,\zeta)) D_j(q_t^{ij}(x,\zeta,\theta) v(x))\, dx \pi^{(1)}(d\zeta)\, d\theta,
\end{aligned} \tag{3.3}$$

where the last equality is obtained by integration by parts, and q^{ij} is given by

$$q_t^{ij}(x,\zeta,\theta) := \sum_{l=1}^d c_t^l(x,\zeta) c_t^i(x,\zeta) m_t(x,\zeta) D_l J^j_{\theta,t,\zeta}(T_{\theta,t,\zeta}(x)).$$

Due to Assumptions 3.5 through 3.7 for a constant $N = N(d,K)$,

$$(\mathcal{I}_t^{(1)} u, v) \le N |u|_{H^1(\mathbb{R}^d)} |v|_{H^1(\mathbb{R}^d)},$$

which shows that $\mathcal{I}_t^{(1)}$ extends uniquely to a bounded linear operator from H^1 to H^{-1}, and the duality product $\langle \mathcal{I}_t^{(1)} u, v\rangle$ is given by the right-hand side of (3.3). In case Q is a bounded Lipschitz domain, one can define the action of $\mathcal{I}_t^{(1)}u$ on $v \in H_0^1(Q)$ again by (3.3), where u and v this time are extended to zero outside of Q. For further study of these operators we refer to [7].

Definition 3.1. A strongly càdlàg adapted process u with values in $L_2(Q)$ is called a solution of the problem (3.1)-(3.2) if

i) $u_t \in H_0^1(Q)$ for $dP \times dt$ almost every $(\omega, t) \in \Omega \times [0,T]$,
ii) $\int_{(0,T]} |u_t|^2_{H_0^1} dt < \infty$ (*a.s.*),
iii) for all $\varphi \in H_0^1(Q)$ we have almost surely

$$(u_t, \varphi) = (\psi, \varphi) + \int_{(0,t]} \{-(a_s^{ij} D_i u_s, D_j \varphi) + (f_s(u_s, \nabla u_s), \varphi) + \langle \mathcal{I}_s^{(1)} u_s, \varphi \rangle\} ds$$

$$+ \int_{(0,t]} \{(\phi_s^{ik} D_i u_s, \varphi) + (\sigma_s^k(u_s), \varphi)\} dw_s^k + \int_{(0,t]} \int_Z (g_s(z, u_{s-}), \varphi) \tilde{N}(dz, ds)$$

for all $t \in [0,T]$, where $(\cdot, \cdot)$ is the inner product in $L_2(Q)$.

Theorem 3.1. *Let Assumptions 3.1 through 3.3 and 3.5 through 3.7 hold. Then there exists a unique solution of the problem* (3.1)-(3.2).

After some preliminaries we will see that Theorem 3.1 follows easily from Theorems 2.9 and 2.10 from [11].

Together with (3.1)-(3.2) let us also consider the problem

$$\begin{aligned} dv_t(x) =& \{L_t v_t(x) + F_t(x, v_t(x), \nabla v_t(x))\} dt \\ &+ G_t^k(v)(x) dw_t^k + \int_Z g_t(x, z, v_{t-}(x)) \tilde{N}(dt, dz), \qquad (3.4) \\ v_0(x) =& \Psi(x), \qquad (3.5) \end{aligned}$$

where F satisfies ii) from Assumption 3.1 and Ψ is an $\mathscr{F}_0$-measurable random variable in $L_2(Q)$.

Theorem 3.2. *Suppose that Assumptions 3.1 and 3.4 through 3.7 hold. Let u and v be solutions of the problems* (3.1)-(3.2) *and* (3.4)-(3.5) *respectively. Suppose that either f or F satisfy Assumption 3.3. Let $f \leq F$ and $\psi \leq \Psi$. Then almost surely, for all $t \in [0,T]$ we have $u_t(x) \leq v_t(x)$ for almost every $x \in Q$.*

Remark 3.3. Assumption 3.4 cannot be omitted in Theorem 3.2. Consider for example the SDE

$$u_t = 1 - \int_{(0,t]} 2u_{s-} d\tilde{N}_s,$$

where N_t is a Poisson process with intensity one. Let τ be the time that the first jump of N occurs. Then $P(\tau \leq T) > 0$. Since $u_t = e^{-2t}$ on $[0, \tau)$, one can see that on the set $\{\tau \leq T\}$ we have $u(\tau) = -e^{-2\tau} < 0$.

The second equation that we will deal with is

$$
\begin{aligned}
du_t(x) = &\{\mathcal{L}_t u_t(x) + f_t(x, u_t(x), \nabla u_t(x))\}dt \\
&+ G_t^k(u_t)(x) dw_t^k + \int_F S_{t,\zeta} u_{t-}(x)\ \tilde{M}(ds, d\zeta)
\end{aligned}
\tag{3.6}
$$

for $(t, x) \in [0, T] \times \mathbb{R}^d$, with initial condition

$$
u_0(x) = \psi(x), \quad x \in \mathbb{R}^d, \tag{3.7}
$$

where

$$
\begin{aligned}
\mathcal{L}_t u(x) &= L_t u(x) + \mathcal{I}_t^{(2)} u(x), \\
\mathcal{I}_t^{(2)} u(x) &= \int_F [\lambda_t(x + b_t(\zeta), \zeta) u(x + b_t(\zeta)) - \lambda_t(x, \zeta) u(x) \\
&\quad - b_t(\zeta) \cdot \nabla(\lambda_t(x, \zeta) u(x))] \pi^{(2)}(d\zeta), \qquad (3.8) \\
S_{t,\zeta} u(x) &= \lambda_t(x + b_t(\zeta), \zeta) u(x + b_t(\zeta)) - \lambda_t(x, \zeta) u(x) \qquad (3.9) \\
&\quad + (\lambda_t(x, \zeta) - 1) u(x).
\end{aligned}
$$

Obviously, if we ask later for some of the previous assumptions to hold for equation (3.6), we mean with $g \equiv 0$.

Assumption 3.8. The function b maps $\Omega \times [0, T] \times F$ into $\mathbb{R}^d$, it is $\mathscr{P} \times \mathfrak{F}$-measurable, and there exists an $\mathfrak{F}$-measurable real function $\bar{b}$ on F, such that for all ω, t and ζ we have

$$
|b_t(\zeta)| \leq \bar{b}(\zeta), \quad \int_F \bar{b}^2(\zeta) \pi^{(2)}(d\zeta) \leq K.
$$

The function λ maps $\Omega \times [0, T] \times \mathbb{R}^d \times F$ to $[0, \infty)$, is $\mathscr{P} \times \mathscr{B}(\mathbb{R}^d) \times \mathfrak{F}$-measurable, it is twice continuously differentiable in x for all ω, t, ζ, and we have

$$
|\lambda_t(x, \zeta)| + |\nabla \lambda_t(x, \zeta)| + |\nabla^2 \lambda_t(x, \zeta)| \leq K,
$$

$$
|1 - \lambda_t(x, \zeta)| \leq \bar{b}(\zeta), \text{ for all } \omega, t, x, \zeta.
$$

It is easy to see that due to Assumption 3.8 for every $t \in [0, T]$ and $\omega \in \Omega$ the mapping $\mathcal{I}_t^{(2)}$, defined in the same way as $\mathcal{I}_t^{(1)}$, is a bounded linear operator from H^1 to H^{-1}, and $\langle \mathcal{I}^{(2)} \phi, \varphi \rangle$ is a predictable process for any $\phi, \varphi \in H^1$.

The solution of the problem (3.6)-(3.7) is understood in the same sense as that of (3.1)-(3.2), and we have the following existence and uniqueness result.

Theorem 3.3. *Let Assumptions 3.1 through 3.3 and 3.5 through 3.8 hold. Then there exists a unique solution of the problem* (3.6)-(3.7).

We also consider the problem

$$dv_t(x) =\{\mathcal{L}_t v_t(x) + F_t(x, v_t(x), \nabla v_t(x))\}dt$$
$$+ G_t^k(v)(x)dw_t^k + \int_F S_{t,\zeta} v(x)\ \tilde{M}(ds, d\zeta), \tag{3.10}$$
$$v_0(x) =\Psi(x), \tag{3.11}$$

where F and Ψ are as in (3.4)-(3.5).

Theorem 3.4. *Suppose that Assumptions 3.1, and 3.5 through 3.8 hold. Let u and v solve* (3.6)-(3.7) *and* (3.10)- (3.11) *respectively. Suppose that either f or F satisfy Assumption 3.3. Let $f \leq F$ and $\psi \leq \Psi$. Then almost surely, for all $t \in [0,T]$ we have $u_t(x) \leq v_t(x)$ for almost every $x \in \mathbb{R}^d$.*

4. Auxiliary Facts

In this section we present some lemmas that we will need for the proofs of Theorems 3.1 through 3.4. The following is well known (see, e.g., [18], or exercise 1.3.19 in [15], or some more general results in [20]).

Lemma 4.1. *Let $u \in W_p^1(Q)$. Let $u_n \in W_p^1(Q)$ such that $|u_n - u|_{W_p^1} \to 0$ as $n \to \infty$. Then we have $|u_n^+ - u^+|_{W_p^1} \to 0$.*

For the next three lemmas, we assume that Assumptions 3.5 through 3.8 hold. For $u \in C_c^\infty(\mathbb{R}^d)$, let us define the quantities,

$$\varrho_t(u) := \int_{\mathbb{R}^d}\int_F \left(\lambda_t\left(x + b_t\left(\zeta\right)\right) u\left(x + b_t\left(\zeta\right)\right)\right)^2 - (\lambda_t(x,\zeta)u(x))^2$$
$$-2b_t(z) \cdot \nabla\left(\lambda_t\left(x,\zeta\right) u\left(x\right)\right) \lambda_t\left(x,\zeta\right) u\left(x\right) \pi^{(2)}(d\zeta)dx,$$
$$\tilde{\varrho}_t(u) := \int_{\mathbb{R}^d}\int_F [(\lambda_t\left(x + b_t\left(\zeta\right)\right) u\left(x + b_t\left(\zeta\right)\right))^+]^2 - [(\lambda_t(x,\zeta)u(x))^+]^2$$
$$-2b_t(z) \cdot \nabla\left(\lambda_t\left(x,\zeta\right) u\left(x\right)\right) \lambda_t\left(x,\zeta\right) u^+\left(x\right) \pi^{(2)}(d\zeta)dx.$$

Lemma 4.2. *For any $u \in C_c^\infty(\mathbb{R}^d)$, $\omega \in \Omega$, $t \in [0,T]$ and $\varepsilon > 0$ we have*

$$\int_{\mathbb{R}^d} \mathcal{I}_t^{(1)} u^2(x)\ dx \leq \varepsilon |u|^2_{H^1(\mathbb{R}^d)} + N(\varepsilon)|u|^2_{L_2(\mathbb{R}^d)}, \tag{4.12}$$

$$\int_{\mathbb{R}^d} \mathcal{I}_t^{(1)} (u^+)^2(x)\ dx \leq \varepsilon |u^+|^2_{H^1(\mathbb{R}^d)} + N(\varepsilon)|u^+|^2_{L_2(\mathbb{R}^d)}, \tag{4.13}$$

$$\varrho_t(u) \leq \varepsilon |u|^2_{H^1(\mathbb{R}^d)} + N(\varepsilon)|u|^2_{L_2(\mathbb{R}^d)}, \tag{4.14}$$

$$\tilde{\varrho}_t(u) \leq \varepsilon |u^+|^2_{H^1(\mathbb{R}^d)} + N(\varepsilon)|u^+|^2_{L_2(\mathbb{R}^d)}, \tag{4.15}$$

where the constant $N(\varepsilon)$ depends only on ε, K and d.

Proof. We prove (4.13). For $\delta > 0$ let $\mathcal{I}^{(1\delta)}$ and $\bar{\mathcal{I}}^{(1\delta)}$ denote the operators defined as $\mathcal{I}^{(1)}$ with F replaced by

$$F_\delta = \{\xi \in F : \bar{c}(\xi) < \delta\}$$

and by $F_\delta^c = F \setminus F_\delta$, respectively. Then clearly,

$$\int_{\mathbb{R}^d} \mathcal{I}_t^{(1)}(u^+)^2(x)dx = \int_{\mathbb{R}^d} \mathcal{I}_t^{(1\delta)}(u^+)^2(x)\,dx + \int_{\mathbb{R}^d} \bar{\mathcal{I}}_t^{(1\delta)}(u^+)^2(x)\,dx. \quad (4.16)$$

The first term on the right-hand side is equal to

$$\begin{aligned}\int_0^1 (1-\theta) \int_{F_\delta} \int_{\mathbb{R}^d} & D_{ij}(u^+)^2(x+\theta c_t(x,\zeta)) \\ & \times c_t^i(x,\zeta)c_t^j(x,\zeta)m_t(x,\zeta)dx\pi^{(1)}(d\zeta)d\theta \\ & = E_1(t,\delta) + E_2(t,\delta),\end{aligned}$$

where

$$\begin{aligned}E_1(t,\delta) &= \int_0^1 (1-\theta) \int_{F_\delta} \int_{\mathbb{R}^d} 2D_i u^+(x+\theta c_t(x,\zeta))D_j u^+(x+\theta c_t(x,\zeta)) \\ &\qquad \times c_t^i(x,\zeta)c_t^j(x,\zeta)m_t(x,\zeta)dx\pi^{(1)}(d\zeta)d\theta, \\ E_2(t,\delta) &= \int_0^1 (1-\theta) \int_{F_\delta} \int_{\mathbb{R}^d} 2u^+(x+\theta c_t(x,\zeta))D_{ij}u(x+\theta c_t(x,\zeta)) \\ &\qquad \times c_t^i(x,\zeta)c_t^j(x,\zeta)m_t(x,\zeta)dx\pi^{(1)}(d\zeta)d\theta.\end{aligned}$$

Using Assumptions 3.5, 3.6 and 3.7, we see after a change of variables that

$$|E_1(t,\delta)| \le C(\delta)C|u^+|^2_{H^1(\mathbb{R}^d)},$$

where $C(\delta) = \int_{F_\delta} \bar{c}^2(\zeta)\pi(dz)$ and C is a constant depending only on K and d. For E_2 we have

$$\begin{aligned}E_2(t,\delta) = \int_0^1 (1-\theta) \int_{F_\delta} \int_{\mathbb{R}^d} & 2D_j(D_i u(x+\theta c_t(x,\zeta)) \\ & \times q_t^{ij}(x,\zeta,\theta)u^+(x+\theta c_t(x,\zeta))\,dx\pi^{(1)}(d\zeta)\,d\theta.\end{aligned}$$

By integration by parts and using the Assumptions 3.5, 3.6 and 3.7 again we see that

$$|E_2(t,\delta)| \le C(\delta)C|u^+|^2_{H^1(\mathbb{R}^d)}.$$

For the second term in (4.16) by Young's inequality and Assumptions 3.6, 3.5, we have

$$\int_{\mathbb{R}^d} \bar{\mathcal{I}}_t^{(1\delta)}(u^+)^2(x)\ dx \le \gamma|u|^2_{H^1(\mathbb{R}^d)} + C(\gamma)|u|^2_{L_2(\mathbb{R}^d)},$$

for all $\gamma > 0$, where $C(\gamma)$ depends only on γ and K. Putting these estimates together and choosing δ and γ sufficiently small, we finish the proof of (4.13). One can repeat the same calculation with c replaced by b, $m = 1$ and λu in place of u to get (4.15). Also (4.2) and (4.14) can be proved in the same way. □

Lemma 4.3. *For any $u \in H^1_0(Q)$, $\omega \in \Omega$, $t \in [0,T]$ and $\varepsilon > 0$ we have*

$$2\langle \mathcal{I}^{(1)}_t u, u\rangle \le \varepsilon |u|^2_{H^1_0(Q)} + N(\varepsilon)|u|^2_{L_2(Q)}, \tag{4.17}$$

$$2\langle \mathcal{I}^{(1)}_t u, u^+\rangle \le \varepsilon |u^+|^2_{H^1_0(Q)} + N(\varepsilon)|u^+|^2_{L_2(Q)}, \tag{4.18}$$

where the constant $N(\varepsilon)$ depends only on ε and K and d.

Proof. We prove (4.18). It suffices to prove it for $Q = \mathbb{R}^d$. Due to Lemma 4.1 and the continuity of the operator $\mathcal{I}^{(1)}_t : H^1 \to H^{-1}$, we may and will also assume that $u \in C^\infty_c(\mathbb{R}^d)$. Notice that for any $\alpha, \beta \in \mathbb{R}$

$$2(\beta - \alpha)\alpha^+ \le (\beta^+)^2 - (\alpha^+)^2 - (\beta^+ - \alpha^+)^2 \le (\beta^+)^2 - (\alpha^+)^2. \tag{4.19}$$

Consequently, for any $\alpha, \beta, \gamma \in \mathbb{R}$

$$2(\beta - \alpha - \gamma)\alpha^+ \le (\beta^+)^2 - (\alpha^+)^2 - 2\gamma\alpha^+.$$

Using this with $\alpha = u(x)$, $\beta = u(x + c_t(x,\zeta))$ and $\gamma = c_t(x,\zeta)\nabla u(x)$, and taking into account that $2\nabla u u^+ = \nabla (u^+)^2$, we can easily see that

$$2\langle \mathcal{I}^{(1)}_t u, u^+\rangle = 2(\mathcal{I}^{(1)}_t u, u^+) \le \int_{\mathbb{R}^d} \mathcal{I}^{(1)}_t (u^+)^2(x)\, dx.$$

Hence (4.18) follows from Lemma 4.2. One can prove (4.17) in the same way, by using the inequality $2(\beta - \alpha)\alpha \le \beta^2 - \alpha^2$, instead of (4.19). □

For $u \in H^1(\mathbb{R}^d)$ we set

$$\mu_t(u) := \int_F \int_{\mathbb{R}^d} [(\lambda_t(x + b_t(\zeta), \zeta)u(x + b_t(\zeta))]^2 - [u(x)]^2 \\ - 2u(x)[\lambda_t(x + b_t(\zeta), \zeta)u(x + b_t(\zeta)) - u(x)]dx\pi^{(2)}(d\zeta),$$

$$\rho_t(u) := 2\langle \mathcal{I}^{(2)}_t u, u\rangle + \mu_t(u), \tag{4.20}$$

$$\tilde{\mu}_t(u) := \int_F \int_{\mathbb{R}^d} [(\lambda_t(x + b_t(\zeta), \zeta)u(x + b_t(\zeta)))^+]^2 - [u^+(x)]^2 \\ - 2u^+(x)[\lambda_t(x + b_t(\zeta), \zeta)u(x + b_t(\zeta)) - u(x)]dx\pi^{(2)}(d\zeta),$$

$$\tilde{\rho}_t(u) := 2\langle \mathcal{I}^{(2)}_t u, u^+\rangle + \tilde{\mu}_t(u). \tag{4.21}$$

Using the simple inequality $|[(x + y)^+]^2 - [x^+]^2 - 2x^+ y| \le 2|y|^2$, and Assumption 3.8 one can see that $\tilde{\mu}_t(u)$ is continuous in $u \in H^1(\mathbb{R}^d)$. It can be shown similarly that $\mu_t(u)$ is continuous in $u \in H^1(\mathbb{R}^d)$.

Lemma 4.4. *For any $u \in H^1(\mathbb{R}^d)$, $(\omega, t) \in \Omega \times \mathbb{R}^+$ and $\varepsilon > 0$ we have*

$$\rho_t(u) \le \varepsilon |u|^2_{H^1(\mathbb{R}^d)} + N(\varepsilon)|u|^2_{L_2(\mathbb{R}^d)}, \tag{4.22}$$

$$\tilde{\rho}_t(u) \le \varepsilon |u^+|^2_{H^1(\mathbb{R}^d)} + N(\varepsilon)|u^+|^2_{L_2(\mathbb{R}^d)}. \tag{4.23}$$

Proof. Since (4.22) can be shown in the same way as (4.23), we only prove the latter one. Clearly it suffices to prove it for $u \in C_c^\infty(\mathbb{R}^d)$. A simple calculation shows that

$$\tilde{\rho}_t(u) = \tilde{\varrho}_t(u) + \int_F \int_{\mathbb{R}^d} (\lambda_t(x,\zeta) - 1)^2 [u^+(x)]^2 dx \pi^{(2)}(d\zeta)$$

$$+ \int_F \int_{\mathbb{R}^d} 2b_t(\zeta) \cdot \nabla(u(x)\lambda_t(x,\zeta)) u^+(x)(\lambda_t(x,\zeta) - 1) dx \pi^{(2)}(d\zeta)$$

By Young's inequality, Assumption 3.8 and (4.15) we get that

$$\tilde{\rho}_t(u) \le \varepsilon |u^+|^2_{H^1(\mathbb{R}^d)} + N(\varepsilon)|u^+|^2_{L_2(\mathbb{R}^d)}.$$

□

Lemma 4.5. *Let Assumption 3.3 hold. Then for all $(\omega, t) \in \Omega \times [0,T]$, $u \in H_0^1(Q)$ and $\varepsilon > 0$ we have*

$$2(f_t(u,\nabla u) - f_t(v,\nabla v), u - v) + \int_Z |g_t(z,u) - g_t(z,v)|^2_{L_2(Q)} \nu(dz)$$
$$\le \varepsilon |u - v|^2_{H_0^1(Q)} + N(\varepsilon)|u - v|^2_{L_2(Q)}, \tag{4.24}$$

$$2(f_t(u,\nabla u) - f_t(v,\nabla v), (u - v)^+) + \int_Z |I_{u>v}(g_t(z,u) - g_t(z,v))|^2_{L_2(Q)} \nu(dz)$$
$$\le \varepsilon |(u - v)^+|^2_{H_0^1(Q)} + N(\varepsilon)|(u - v)^+|^2_{L_2(Q)}, \tag{4.25}$$

where $N(\varepsilon)$ depends only on ε and K.

Proof. We show (4.25). Using the second part of Assumption 3.3 and Young's inequality we have

$$2(f_t(u,\nabla u) - f_t(v,\nabla v), (u - v)^+) \le \frac{K}{\varepsilon} |(u - v)^+|^2_{L_2(Q)} + \varepsilon |\nabla(u - v)^+|^2_{L_2(Q)}$$

$$+ \int_Q (f_t(x,u,\nabla u) - f_t(x,v,\nabla u))(u(x) - v(x))^+ dx.$$

This combined with Assumption 3.3 gives (4.25). Inequality (4.24) can be shown in the same way. □

5. Proof of Theorems 3.1, 3.2, 3.3 and 3.4

We are now ready to proceed with the proofs of the main theorems.

Proof of Theorems 3.1 and 3.3. We prove Theorem 3.1. It suffices to show that conditions I) through IV) from [11] are satisfied, and then the result follows immediately from Theorems 2.9 and 2.10 of the same article. The growth condition of the operator $L_t + f_t(\cdot)$ can be verified easily. Notice that for every ω, t and x, the function $f_t(x,r,r')$ is continuous in (r,r'). Using this, ii) from Assumption 3.1 and the fact that L_t is a bounded linear operator from $H_0^1(Q)$ into $H^{-1}(Q)$, we see that $L_t + f_t(\cdot)$ is semicontinuous (in the sense of [11]). Now, by ii) and iv) from Assumption 3.1, the boundedness of ϕ and (4.17) we see that for a $\theta > 0$ and a constant C we have

$$2\langle L_t u, u\rangle + 2(u, f_t(u, \nabla u)) + \sum_k |G_t^k(u)|^2_{L_2(Q)} + \int_Z |g_t(u)|^2_{L_2(Q)}\nu(dz)$$
$$\leq -\theta|u|^2_{H^1_0(Q)} + C|u|^2_{L_2(Q)} + C|\bar{h}_t|^2_{L_2(Q)}.$$

for all t, ω and $u \in H^1_0(Q)$. This shows that the coercivity condition is satisfied. Using i), iv), v) from Assumption 3.1 and (4.17) we see that for all (t,ω) and $\gamma > 0$

$$2\langle L_t u - L_t v, u - v\rangle + \sum_k |G^k(u) - G^k(v)|^2_{L_2(Q)}$$
$$\leq (\gamma - \varkappa)|u - v|^2_{H^1_0(Q)} + C(\gamma)|u - v|^2_{L_2(Q)},$$

for all $u, v \in H^1_0(Q)$, where $\varkappa$ is the ellipticity constant form part (iv) of Assumption 3.1. Combining this with (4.24) we have that the monotonicity condition is also satisfied. The proof of Theorem 3.3 goes in the same way. We omit the details, we only note that one also has to use (4.22). □

Proof of Theorem 3.2. Without loss of generality we can assume that Assumption 3.3 is satisfied by f. For the difference $h = u - v$ we have

$$h_t = h_0 + \int_{(0,t]} L_s h_s + f_s(u_s, \nabla u_s) - F_s(v_s, \nabla v_s)\, ds$$
$$+ \int_{(0,t]} \phi_s^{ki} D_i h_s + \sigma_s^k(u_s) - \sigma_s^k(v_s)\, dw_s^k$$
$$+ \int_{(0,t]}\int_Z g(s, z, u_{s-}) - g(s, z, v_{s-}))\tilde{N}(ds, dz).$$

By Theorem 2.1 we have

$$|h_t^+|^2_{L^2} = \int_{(0,t]} A_s^{(1)} + A_s^{(2)} + A_s^{(3)} + 2\langle \mathcal{I}_s^{(1)} h_s, h_s^+\rangle\ ds + m_t$$

for a local martingale m_t, where

$$A_s^{(1)} = \int_Q \Big\{ - 2a_s^{ij}(x) D_i h_s^+(x) D_j h_s^+(x)$$
$$+ \sum_k \Big| I_{h_s>0} \sum_i \phi_s^{ki}(x) D_i h_s(x) + I_{h_s>0}(\sigma_s^k(x, u_s(x)) - \sigma_s^k(x, v_s(x)))\Big|^2 \Big\} dx \tag{5.26}$$

$$A_s^{(2)} = 2\int_Q (f_s(x, u_s, \nabla u_s) - F_s(x, v_s, \nabla v_s)) h_s^+(x) dx \tag{5.27}$$

$$A_s^{(3)} = \int_Z\int_Q \{[h_s(x) + g_s(x, z, u_{s-}(x)) - g_s(x, z, v_{s-}(x))]^+\}^2 - |h_s(x)^+|^2$$
$$- 2h_s^+(x)[g_s(x, z, u_s(x)) - g_s(x, z, v_s(x))] dx\nu(dz).$$

One can easily see that for every $\varepsilon > 0$, there exist $C(\varepsilon) > 0$ depending only on ε, K and d, such that

$$A_s^{(1)} \le (-\varkappa + \varepsilon)|h_s^+|^2_{H_0^1(Q)} + C(\varepsilon)|h_s^+|^2_{L_2(Q)}.$$

By Assumption 3.4 we obtain

$$A_s^{(3)} = \int_Z \int_Q I_{h_s>0}|g_s(x,z,u_s) - g_s(x,z,v_s)|^2 dx\nu(dz).$$

Hence, by (4.25) we have

$$A_s^{(2)} + A_s^{(3)} \le \varepsilon|h_s^+|^2_{H_0^1(Q)} + C(\varepsilon)|h_s^+|^2_{L_2(Q)}.$$

Combining these estimates and using (4.18) we have a constant C such that, almost surely

$$|h_t^+|^2_{L^2(Q)} \le C\int_{(0,t]} |h_s^+|^2_{L^2(Q)}\, ds + m_t \quad \text{for all } t \in [0,T].$$

Let $(\tau_n)_{n\in\mathbb{N}}$ be stopping times such that $\int_0^{t\wedge\tau_n} |h_s^+|^2_{L^2(Q)}\, ds \le n$ and almost surely, $\tau_n = T$ for n large enough. By a standard localization argument and Fatou's lemma we get

$$EI_{t\le\tau_n}|h_t^+|^2_{L^2(Q)} \le C\int_{(0,t]} EI_{s\le\tau_n}|h_s^+|^2_{L^2(Q)}\, ds < \infty \quad \text{for all } t \in [0,T],$$

and the result follows by Gronwall's and Fatou's lemmas. □

Proof of Theorem 3.4. We assume again that Assumption 3.3 is satisfied by f. For the difference $h = u - v$ we have

$$h_t = h_0 + \int_{(0,t]} \{\mathcal{L}_s h_s + f_s(u_s, \nabla u_s) - F_s(v_s, \nabla v_s)\}\, ds$$

$$+ \int_{(0,t]} \{\phi_s^{ki} D_i h_s + \sigma_s^k(u_s) - \sigma_s^k(v_s)\}\, dw_s^k + \int_{(0,t]}\int_F S_{s,\zeta} h_{s-}\tilde{M}(ds, d\zeta)$$

By Theorem 2.1 we have

$$|h_t^+|^2_{L^2(\mathbb{R}^d)} = \int_{(0,t]} A_s^{(1)} + A_s^{(2)} + \tilde{\rho}_s(h_s) + \langle \mathcal{I}_s^{(1)} h_s, h_s^+\rangle\, ds + m_t$$

for a local martingale m_t. Here $A^{(1)}, A^{(2)}$ are as in (5.26), (5.27) (with the integration over $\mathbb{R}^d$ instead of Q), and $\tilde{\rho}$ is defined in (4.21). By using the same arguments as in the previous proof, this time also using (4.23), we bring the proof to an end. □

References

[1] Berger, Melvin S. Nonlinearity and functional analysis. Lectures on nonlinear problems in mathematical analysis. Pure and Applied Mathematics. Academic Press, New York-London, 1977.

[2] Z.Q. Chen, K.H. Kim, An L^p-theory of non-divergence form SPDEs driven by Lévy processes,arXiv:1007.3295v1 [math.PR]

[3] L. Denis, A. Matoussi and L. Stoica, Maximum principle and comparison theorem for quasi-linear stochastic PDE's. *Electron. J. Probab.* **14** (2009), 500530.

[4] L. Dennis and D. Matoussi, Maximum principle for quasilinear SPDE's on a bounded domain without regularity assumptions, arXiv:1201.1092 [math.PR].

[5] L. Denis, A. Matoussi and J. Zhang, Maximum Principle for Quasilinear Stochastic PDEs with Obstacle, arXiv:1210.3445 [math.PR].

[6] C. Donati-Martin, E. Pardoux, White noise driven SPDEs with reflection, *Probab. Theory Related Fields*, **95** (1993), 1-24.

[7] M.G. Garroni, J.L. Menaldi, Second order elliptic integro-differential problems, Chapman & Hall, CRC Research Notes in Mathematics, 430, 2002.

[8] Grigelionis, B.; Mikulevicius, R.(1-SCA) Nonlinear filtering equations for stochastic processes with jumps. The Oxford handbook of nonlinear filtering, 95128, Oxford Univ. Press, Oxford, 2011.

[9] Grigelionis, B. Stochastic nonlinear filtering equations and semimartingales. Nonlinear filtering and stochastic control (Cortona, 1981), 6399, Lecture Notes in Math., 972, Springer, Berlin-New York, 1982.

[10] I. Gyöngy and N.V. Krylov, On stochastic equations with respect to semimartingales II, *Stochastics*, **6** (1982), 153-173.

[11] I. Gyöngy, On stochastic equations with respect to semimartingales III, *Stochastics*, **7** (1982), 231-254.

[12] I. Gyöngy and N.V. Krylov, On stochastic equations with respect to semimartingales I, *Stochastics*, **4** (1980), 1-21.

[13] N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion processes*, North-Holland Publishing Co., Amsterdam-New York; Kodansha, Ltd., Tokyo, 1981.

[14] P. Kotelenez, Comparison methods for a class of function valued stochastic partial differential equations. *Probab. Theory Related Fields* **93** (1992), 1-19.

[15] Krylov, N. V. Lectures on elliptic and parabolic equations in Sobolev spaces. Graduate Studies in Mathematics, 96. American Mathematical Society, Providence, RI, 2008.

[16] N.V. Krylov, Maximum principle for SPDEs and its applications, in *Stochastic differential equations: theory and applications*, Interdisciplinary Mathematical Sciences, 2. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.

[17] N.V. Krylov, On the Itô-Wentzell formula for distribution-valued processes and related topics. *Probab. Theory Related Fields* **150** (2011), 295–319.

[18] M. Marcus and V.J. Mizel, Every superposition operator mapping one Sobolev space into another is continuous *J. Funct. Anal.* **33** (1979), no. 2, 217–229.

[19] Rozovski, B. L. Stochastic evolution systems. Linear theory and applications to nonlinear filtering. Mathematics and its Applications (Soviet Series), 35. Kluwer Academic Publishers Group, Dordrecht, 1990.

[20] T. Runst and W. Sickel, Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations (Programming Complex Systems), (De Gruyton Series in Nonlinear Analysis and Applications 3), Berlin, New York, 1996.

[21] R. Situ, *Theory of stochastic differential equations with jumps and applications. Mathematical and Analytical Techniques with Applications to Engineering*, Springer, New York, 2005.