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‘Ramseyfying’ Probabilistic Comparativism

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Comparativism is the view that comparative confidences (e.g., being more confident that P than that Q) are more fundamental than degrees of belief (e.g., believing that P with some strength x). I outline the basis for a new, nonprobabilistic version of comparativism inspired by a suggestion made by Frank Ramsey in “Probability and Partial Belief.” I show how, and to what extent, ‘Ramseyan comparativism’ might be used to weaken the (unrealistically strong) probabilistic coherence conditions that comparativism traditionally relies on.

1. Introduction. Beliefs come in degrees, or so it seems. Assuming they do, one important question concerns the basis of their numerical representation. It is typical to represent the varying strengths with which propositions might be believed using percentages, or real values between 0 and 1, or with intervals thereof. Moreover, it is typical to assume that these numbers encode more than merely ordinal information. For instance, it seems that we can meaningfully talk about intervals of strengths of belief: an agent—let’s call her α—might believe one proposition much more than she believes another, or she might believe it just a little more. Likewise for ratios: if α is 50% confident that the coin she flips will land heads, then most of us would be

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happy to say that she has half as much confidence in that event than she has in the coin landing either heads or tails. And, if she is even a little bit rational, then she will probably be at least twice as confident that it will land heads on the next toss than that it will land heads consistently on the next several tosses.

There is, in other words, a widespread prima facie commitment in our understanding of degrees of belief that they can be measured on a ratio scale or something much like it. Given this, we will assume for the remainder of this article that the numbers we use to represent the strengths of our beliefs can, at least in principle, carry *cardinal* (read: at least ratio and therefore also interval) information. Supposing that is correct, it is just the sort of thing that ought to be explained by any adequate account of what degrees of belief are. We do not get to posit cardinality for free—α's doxastic states do not come with little numbers attached to them, and they do not literally stand in numerical relationships with one another. Rather, they must have some nonnumerical structure that is in some way similar to and hence representable by the real values in the unit interval, in particular such that both the ordinal and relevant cardinal properties of and relations between those numbers represent something doxastically meaningful. That much is clear enough—the hard part consists in saying exactly what that structure is.

So how is it that we manage to get from the purely nonnumerical stuff in our heads through to numerical representations of our doxastic states that encode interesting cardinal information? A few answers to this question have been suggested. One long-standing tradition seeks to explain where the numbers come from and how they get their meaning, by considering how beliefs interact with *preferences* (e.g., Ramsey 1931b). Others have tried to extract numerical representations out of *comparative expectations*, a special kind of nonpropositional comparative attitude (e.g., Suppes and Zanotti 1976). Still other potential approaches have yet to be explored. For instance, if you like the idea that degrees of belief are really just outright beliefs about objective probabilities, then you might think that whatever cardinality they possess is derivative on the cardinal information possessed by those probabilities—wherever that comes from.

I am inclined to think that each of these possibilities is worth considering seriously; at least, none of them seem to me either obviously correct or irretrievably hopeless. I have argued elsewhere that the connection with preferences is one promising avenue to explore (Elliott 2019a). But in this article I want to focus on an entirely different kind of approach: *comparativism*.

For the sake of concreteness, I will take comparativism to be the view that the facts about an agent's degrees of belief supervene on, and indeed hold in virtue of, the facts about what we will call her *confidence comparisons*. These are purely ordinal comparative doxastic states such as *being more confident* that P than that Q, *being equally confident* that P as that

Q, or *being at least as confident* that *P* as that *Q*.[1] With that as their starting point, comparativists tend to see degrees of belief and the numerical representations thereof as a kind of theoretical tool, a way to represent and reason about sufficiently coherent systems of comparative confidence. Or to put that another way: the numbers we use to represent our beliefs ultimately describe a purely ordinal structure imposed over a set of propositions by our confidence comparisons, when those comparisons satisfy some minimum threshold of coherence.

On the face of it, comparativism might seem to struggle with providing any plausible explanation of the possibility of cardinal information. After all, individual confidence comparisons contain no more than purely ordinal information, so how could a system composed of nothing more than such comparisons possess anything more than that?[2] Nevertheless, comparativists have what is by now a standard explanation of how cardinality can be generated out of nothing more than ordinal confidence comparisons. By drawing on a well-worn analogy with the measurement of mass, length, and other extensive quantities, comparativists have managed to set down conditions (or axioms) under which meaningful cardinal information might be extracted out of a system of confidence comparisons.

That is the current state of play. However, the axioms to which comparativists typically appeal when addressing this kind of challenge are quite strong indeed. Essentially, they impose a comparative variety of probabilistic (and hence logical) coherence on the agents’ confidence comparisons. And this is a key limitation with the view in its most typical contemporary form: it lacks an adequate account of how ordinary agents—who do not live up to the very strict standards of probabilistic coherence—might nevertheless have beliefs that carry genuine cardinal information. Consequently, in this article I want to explore whether, and how, the standard ‘probabilistic’ axioms might be weakened, while maintaining the same basic strategy for extracting cardinality out of a system of comparative confidences.

1. It will not matter too much for what I have to say exactly how we define ‘comparativism’, and there of course are many other ways to precisify the general kind of idea that I am referring to. Most actual comparativists have taken a view that is at least in the vicinity of what I below characterize as probabilistic comparativism (e.g., de Finetti 1931; Koopman 1940; Savage 1954, chap. 3; Fine 1973, 68ff.; Hawthorne 2016; Stefánsson 2017, 2018); comparativist theories along these lines are also discussed in Krantz et al. (1971, 200) and Fishburn (1986). In some cases, a comparativist might focus on *quarternary* confidence comparisons (e.g., being more confident that *P* given *Q* than that *R* given *S*), rather than on binary comparisons like those I have described here. For the sake of brevity, I have limited my discussion to the relatively simple views that consider only binary confidence comparisons. Nevertheless, each of the main points of discussion in secs. 3 and 4 have fairly straightforward analogues for the typical case of the quarternary comparativist.

2. For a recent complaint along just these lines, see Meacham and Weisberg (2011, 659).

Let me say that again, for emphasis: the goal here is to explore whether, and to what extent, the usual probabilistic axioms can be weakened. This is a question of interest to proponents and opponents of comparativism alike and for those who might be on the fence. I stress however that my results are formal, not evaluative. The current article is not intended to be a defense of comparativism. (It would be woefully inadequate if so.) An evaluation of the overall merits and demerits of the comparativists' view is well beyond the scope of this discussion, and I will not try to address the tricky empirical question of whether and to what extent the weakened axioms are satisfied or even approximated by ordinary agents. Still less is this an article on what our comparative confidences should be like, so I will not have anything much to say about how Ramseyan comparativism relates to arguments for probabilism.

I begin my discussion by reviewing the standard account of how mass can be measured on a ratio scale and how probabilistic comparativism posits an essentially similar process for the measurement of belief (secs. 2 and 3). Following that, I discuss in a little more detail the motivations for seeking more general axioms under which cardinality can be extracted out of a system of confidence comparisons (sec. 4). Finally, I show that the axioms of what I will call *probabilistic comparativism* can be weakened to a significant extent, although not without limits. I do this by developing what I call *Ramseyan comparativism* (sec. 5). Moreover, I show that the Ramseyan axioms on confidence comparisons are in one important respect maximally weak: inasmuch as comparativists want to retain the analogy with the measurement of mass as it is usually understood, the Ramseyan axioms are as weak as they come.

2. The Measurement of Mass. Let a and b be any two concrete objects you like, and compare:

> Ordinal. a is more massive than b.
>
> Cardinal. a is twice as massive as b.

Cardinal obviously contains more information than Ordinal, and that information has to come from somewhere. Yet masses do not come with little numbers attached to them. Whatever it is that explains the extra information in Cardinal must ultimately be nonnumerical in nature. So how can we get from the nonnumerical facts on the ground through to numerical masses that encode interesting cardinal information?

The representational theory of measurement gives us a plausible answer.[3] First, note that Cardinal is true (roughly) if and only if (iff), if you were to take two disjoint objects each as massive as b (call them b_1 and b_2, b's

3. The locus classicus for this theory is Krantz et al. (1971).

duplicates) and join them together, then the resulting object would be just as massive as a. Call the operation of joining objects together *concatenation*; we assume that no mass is gained or lost in the act of concatenating. Given this, it is plausible that there is nothing more to the truth of a claim like Cardinal than what we have just said—that is, 'a is twice as massive as b' just means something roughly to the effect of 'a is as massive as the concatenation of two duplicates of b'. By reference, then, to purely ordinal comparisons between duplicates and the concatenations thereof, we have been able to give straightforward nonnumerical meaning to Cardinal.

And we can easily generalize this idea to explain other rational ratio comparisons. For positive integers n, m, say that a is n/m times as massive as b whenever there is some object c such that

1. a is as massive as the concatenation of n duplicates of c, and
2. b is as massive as the concatenation of m duplicates of c.

Now let x designate c's mass in whatever units you like—let's say slugs (~14.6 kg). Intuitively, a must then have a mass of $n \cdot x$ slugs, and b must have a mass of $m \cdot x$ slugs. Hence, a is n/m times as massive as b. Indeed, with a little bit more work, we can generalize the idea even further to explain arbitrary real ratio comparisons. However, for the sake of simplicity we will stick with rational ratios throughout this discussion.

Hiding in the background is a crucial empirical assumption: that the operation of concatenation behaves as a kind of nonnumerical analogue of addition. We rely on exactly this assumption to move from, for example, 'a is as massive as the concatenation of n duplicates of an object with a mass of x slugs' to 'a has a mass of $n \cdot x$ slugs'—that is, we assume that the mass of a concatenation is just the sum of the masses of the concatenands. (Imagine if, instead, concatenation behaved like quaddition: whenever you concatenate up to 57 duplicates together, things are as usual, but concatenate more and the result is always as massive as five duplicates. We could have then used concatenations to define our way up to one object's being 57 times more massive than another but no further.)

Fortunately, the analogy between concatenation and addition is quite close. Where

$$a \succsim^m b \text{ iff } a \text{ is at least as massive as } b,$$

$$a \sim^m b \text{ iff } a \text{ is exactly as massive as } b,$$

$$a \oplus b = \text{the concatenation of } a \text{ and } b,$$

then it is plausible that $\succsim^m$ is transitive and complete, and $\sim^m$ is its symmetric part. Furthermore, $\oplus$ behaves with respect to $\succsim^m$ a lot like $+$ behaves with respect to $\geq$: for all disjoint objects a, b, c,

1. $a \oplus b \succsim^m b$.
2. $a \oplus b \sim^m b \oplus a$.
3. $a \oplus (b \oplus c) \sim^m (a \oplus b) \oplus c$.
4. $a \succsim^m b$ iff $a \oplus c \succsim^m b \oplus c$.

Now compare these with the following properties of $+$ in relation to $\geq$, where n and m are nonnegative real numbers:

1. $n + m \geq m$.
2. $n + m = m + n$.
3. $n + (m + k) = (n + m) + k$.
4. $n \geq m$ iff, for any k, $n + k \geq m + k$.

Indeed, if we posit a rich enough space of concrete objects and make one further 'Archimedean' assumption (roughly, that no object is infinitely more massive than any other), then we can say something stronger still: if $\mathcal{O}$ is the set of $\sim^m$-equivalence classes of concrete objects and $\mathbb{R}^+$ the positive reals, then the relational system $\langle \mathcal{O}, \succsim^m, \oplus \rangle$ has essentially the same structure as $\langle \mathbb{R}^+, \geq, + \rangle$. Thus, we can assign a number to each object in such a way that $\succsim^m$ is represented by $\geq$, and $\oplus$ is represented by $+$. And with that in hand, we can start to define up ratios of masses, numerical differences in mass, ratios of differences in mass, and so on. In other words, we have all the basic resources needed to explain how numerical representations of mass manage to carry all sorts of interesting cardinal information.

The upshot: numerical masses represent a fully nonnumerical system of ordinal mass comparisons that have an 'additive' structure over concatenations. We are justified in treating ratios of masses as meaningful because there exists an operation on objects that is intuitively and formally like 'adding' masses together. And we can apply the same basic idea outlined here to account for the measurement of other (extensive) quantities: a is *twice as long as* b iff a is as long as two length duplicates of b laid end to end, a has *twice the volume of* b iff a has the same volume as two volume duplicates of b joined together, and an event e_1 has *twice the duration* of e_2 iff e_1 can be split into two disjoint events with the same duration as e_1.

To apply the same idea to the measurement of beliefs, comparativists have therefore historically sought an operation on the relata of confidence comparisons (i.e., propositions) that behaves, with respect to those comparisons, similarly enough to addition to justify treating it as a nonnumerical analogue thereof. As Krantz et al. put it, the strategy is "to treat the assignment of [subjective] probabilities as a measurement problem of the same fundamental character as the measurement of, e.g., mass or duration" (1971, 200). So let's see how that plays out in practice.

3. Probabilistic Comparativism. In this section I provide an overview of probabilistic comparativism. I begin by laying out some basic notation and assumptions (sec. 3.1), followed by the mathematical underpinnings of the view (sec. 3.2). Finally, I define two specific varieties of probabilistic comparativism—one 'precise' (sec. 3.3) and the other 'imprecise' (sec. 3.4).

3.1. Notation and Assumptions. Let α be an arbitrary thinking subject whose beliefs we are trying to represent. I will assume that the propositions regarding which α has beliefs can be modeled as subsets of some space of logically possible worlds, Ω. By 'logically possible', I mean no more than that the worlds are closed under a consequence relation at least as strong as that of classical propositional logic. So, you can assume that Ω includes metaphysically or even epistemically impossible worlds, if that is what floats your boat—as long as the worlds are classically logically consistent. (I talk more about this assumption in sec. 4.)

Next, let $\mathcal{B} \subseteq \wp(\Omega)$ denote that set of propositions regarding which α has beliefs. Without loss of generality, I assume throughout that $\mathcal{B}$ is a Boolean algebra of sets on Ω. So, $\mathcal{B}$ contains at least Ω and $\varnothing$, and it is closed under relative complements and binary intersections/unions. I also assume throughout that $\mathcal{B}$ is finite. Doing this will simplify much of the ensuing discussion and formalities.[4]

I assume that α's full system of confidence comparisons can be modeled with a single binary relation $\succsim$ defined over $\mathcal{B}$, where

$$P \succsim Q \text{ iff } \alpha \text{ believes } P \text{ at least as much as she believes } Q.$$

I refer to $\succsim$ as α's *confidence ranking*. Consequently, where $\succ$ and $\sim$ stand for the comparatives *more probable* and *equally probable*, respectively, I am in effect assuming that

$$P \sim Q \text{ iff } (P \succsim Q) \;\&\; (Q \succsim P),$$

$$P \succ Q \text{ iff } (P \succsim Q) \;\&\; \neg(Q \succsim P).$$

Nothing about this last assumption should be treated as obvious or trivial. For example, α might be at least as confident in P as in Q without being more confident in P than in Q or without being equally confident in P as

4. The finitude of $\mathcal{B}$ plays a minor (simplifying) role in relation to theorem 1. We can do without it if we instead make use of a more complicated version of definition 5. The finiteness assumption also plays a role in the existence proof of theorem 2. Where $\mathcal{B}$ is uncountable, additional 'continuity' assumptions can be placed on the comparative confidence relation that will guarantee the existence of the relevant type of representation. See Evren and Ok (2011) for discussion on these types of conditions.

in Q. Nevertheless, it will simplify the discussion, and nothing of great importance hangs on it.

Finally, where a function Cr assigns real numbers to the propositions in $\mathcal{B}$, I will say that Cr *almost agrees with* $\succsim$ iff, for all $P, Q \in \mathcal{B}$,

$$P \succsim Q \text{ only if } Cr(P) \geq Cr(Q),$$

and we will say that Cr *agrees with* $\succsim$ just in case

$$P \succsim Q \text{ iff } Cr(P) \geq Cr(Q).$$

For ease of expression, I treat *agreement* (but not *almost agreement*) as symmetric: $\succsim$ agrees with Cr just in case Cr agrees with $\succsim$.

3.2. Agreeing with Probabilities. Any Cr that agrees with confidence comparisons $\succsim$ is ipso facto at least an ordinal-scale representation of $\succsim$. Our task now is to lay out axioms under which such a function can be said to also carry cardinal information. This is where probabilities come in handy:

Definition 1. $Cr : \mathcal{B} \mapsto \mathbb{R}$ is a *probability function* iff, $\forall P, Q \in \mathcal{B}$,

1. $Cr(\Omega) = 1$,
2. $Cr(P) \geq 0$, and
3. if $P \cap Q = \varnothing$, then $Cr(P \cup Q) = Cr(P) + Cr(Q)$.

It follows immediately from the third criterion that if some probability function—any probability function—agrees with $\succsim$, then the union of disjoint sets is to $\succsim$ just as $\oplus$ is to $\succsim^m$, or as $+$ is to $\geq$. Great. That is exactly the kind of thing needed for the analogy with the measurement of mass to hold water.

Moreover, we have known for a long time the exact conditions under which a confidence ranking will agree with some probability function on $\mathcal{B}$. The following five axioms are individually necessary and jointly sufficient (see Scott 1964). For all $P, Q, R \in \mathcal{B}$,

Completeness. $P \succsim Q$ or $Q \succsim P$.

Preorder. (i) $P \succsim P$, and (ii) if $P \succsim Q$ and $Q \succsim R$, then $P \succsim R$.

Nontriviality. $\Omega \succ \varnothing$.

Nonnegativity. $P \succsim \varnothing$.

Scott's Axiom. Where $\mathbf{1}p$ denotes the indicator function of P, $(P_i)_{i=1}^n$ and $(Q_i)_{i=1}^n$ are finite sequences of propositions, and $(k_i)_{i=1}^n$ is a finite sequence of natural numbers, then if

1. $\Sigma_{i=1}^{n} k_i \cdot \mathbf{1}_{P_i}(\omega) = \Sigma_{i=1}^{n} k_i \cdot \mathbf{1}_{Q_i}(\omega)$ for all $\omega \in \Omega$, and
2. $P_i \succsim Q_i$, for $i = 1, \ldots, n - 1$,

then $Q_n \succsim P_n$.

Call the conjunction of the above five axioms the *Complete Package*.[5] Comparativists have frequently suggested that, when $\succsim$ conforms to the Complete Package, beliefs can be measured on a ratio scale with the union of disjoint sets playing the role of concatenation (e.g., Fine 1973, 68ff.; Stefánsson 2017, 2018).

It is possible to say something a little more general than this, though, and doing so will be useful in demonstrating a general continuity between probabilistic comparativism and the Ramseyan comparativisms that I develop below. First, note that if $\mathcal{C}r$ is a probability function, then if $\mathcal{C}r(P \cap Q) = 0$, then $\mathcal{C}r(P \cup Q) = \mathcal{C}r(P) + \mathcal{C}r(Q)$. That is to say: probability functions are also additive with respect to the union of what we will call *pseudodisjoint* propositions, where P and Q are pseudodisjoint for α just in case she has no confidence in their intersection. Or, more precisely,

Definition 2. For all $P \in \mathcal{B}$, P is

1. *minimal* iff $Q \succsim P$ for all $Q \in \mathcal{B}$,
2. *maximal* iff $P \succsim Q$ for all $Q \in \mathcal{B}$,
3. *middling* iff P is neither minimal nor maximal.

Definition 3. $\mathcal{P} \subseteq \mathcal{B}$ is a *set of pseudodisjoint propositions* iff, for any minimal Q and any $\mathcal{P}^\star \subseteq \mathcal{P}$ such that $|\mathcal{P}^\star| \geq 2$, $\bigcap \mathcal{P}^\star \sim Q$; furthermore, propositions $P_1, \ldots, P_n$ are *pairwise pseudodisjoint* iff there is a set of pseudodisjoint propositions $\mathcal{P}$ such that $P_1, \ldots, P_n \in \mathcal{P}$.

Assuming that α has exactly zero confidence in P whenever P is minimal, definition 3 plausibly characterizes in comparativist terms what it is for α to believe that at most one proposition from $P_1, \ldots, P_n$ is true.[6]

With all that in hand, we can note that the Complete Package implies that $\succsim$ is ‘Archimedean’ (roughly, no proposition is infinitely more probable than

5. In the context of the other axioms, Preorder is redundant, and Scott’s Axiom is equivalent to the slightly weaker formulation found in Scott (1964). See Harrison-Trainor et al. (2016). I have done it this way to make later discussions easier.

6. Definition 3 implies that every singleton set $\{P\} \in \mathcal{B}$ is trivially a ‘set of pseudodisjoint propositions’. This is a feature, not a bug. The rather tortured definition will be useful later when we generalize away from probability functions.

any other), and furthermore, where propositions P, Q, R are pairwise pseudodisjoint,

1. $(P \cup Q) \succsim Q$.
2. $(P \cup Q) \sim (Q \cup P)$.
3. $(P \cup (Q \cup R)) \sim ((P \cup Q) \cup R)$.
4. $P \succsim Q$ iff $(P \cup R) \succsim (Q \cup R)$.

Again, this is exactly what comparativists need to draw the analogy with the measurement of mass. So let's turn the foregoing mathematical points into a philosophical theory.

3.3. Precise Probabilistic Comparativism. Assuming that Cr agrees with α's confidence ranking, say henceforth that Cr constitutes a *fully adequate model* of α's beliefs whenever

$$\alpha \text{ believes } P \ \ \frac{n}{m} \text{ times as much as she believes } Q \text{ iff } Cr(P) = \frac{n}{m} \cdot Cr(Q).$$

I assume that full adequacy is worth striving for—after all, most theorists will be happy to make both of the following kinds of inferences:

1. α believes P to degree x and Q to degree y.
2. $x = n \cdot y$.

∴ α believes P n times as much as she believes Q.

And in the other direction,

1. α believes P n times as much as Q.
2. α believes P to degree y.

∴ α believes Q to degree $x = n \cdot y$.

Only full adequacy licenses inferences in both of these directions, and so I take it that full adequacy stands as an important desideratum for any comparativist theory. With that said, we can also say that Cr is *L-to-R adequate* iff the left-to-right direction of the above biconditional holds, and *R-to-L adequate* iff the right-to-left direction holds. Comparativists may well want to reject full adequacy in favor of mere L-to-R or R-to-L adequacy, provided that the rejection is well motivated and they are able to explain away any intuitions in support of full adequacy. (I say a little more about this in sec. 5.3.)

Next, let *precise probabilistic comparativism* denote any comparativist theory that is committed to the following conditional:

> Precise Probabilistic Comparativism. If $\mathcal{C}r$ is the unique probability function that agrees with α’s confidence ranking, then $\mathcal{C}r$ is a fully adequate model of α’s beliefs.

Note the stated requirement that the probability function be unique. This is needed to avoid contradiction: for any nontrivial algebra $\mathcal{B}$, there will always be some collection of probability functions on $\mathcal{B}$ that agree with one and the same confidence ranking—and since any two probability functions on the same domain will disagree on at least some ratios, any inference from ‘$\mathcal{C}r(Q) = n/m \cdot \mathcal{C}r(Q)$’ to ‘$\alpha$ believes P n/m times as much as Q’ will be valid only when the $\mathcal{C}r$ is unique in the relevant sense. In short, R-to-L adequacy presupposes uniqueness, which in turn requires further constraints on $\succsim$.

There are multiple ways to ensure uniqueness. Of particular note is what Stefánsson (2017, 2018; cf. also Savage 1954; Suppes 1969, 6–7) uses to ensure uniqueness in his recent defenses of probabilistic comparativism:

> Continuity. For all nonminimal P, Q, there are P', Q' such that $P \sim P'$, $Q \sim Q'$, and P' and Q' are each the union of some subset of a finite set of disjoint propositions $R_1, \dots, R_m$ such that $R_i \sim R_j$ for $i, j = 1, \dots, n$.

The interested reader can see Krantz et al. (1971, sec. 5.2) and Fishburn (1986) for other conditions sufficient to ensure uniqueness.

Now, probabilistic comparativism clearly has resources to put forward an account of how a system of confidence comparisons might end up carrying cardinal information, in the event that $\succsim$ satisfies the requisite axioms. In particular, consider the following principle, which in essence is just the comparative probability version of how we defined rational ratio comparisons for mass earlier in section 2:[7]

> General Ratio Principle. α believes P n/m times as much as Q if
>
> 1. for $0 < n \leq m$, there are m nonminimal, equiprobable pairwise pseudodisjoint propositions $R_1, \dots, R_m$ such that $Q \sim (R_1 \cup \dots \cup R_m)$ and $P \sim (R_1 \cup \dots \cup R_n)$, or
> 2. α believes P n'/m' times as much as R and believes R n''/m'' times as much as Q, where $n/m = n' \cdot n''/m' \cdot m''$.

7. The first clause of the General Ratio Principle is a close relative of Stefánsson’s (2018) Ratio Principle. The second (inductive) clause is new—in the context of a condition like Continuity it is redundant, but see sec. 4 for it put to work.

So, for instance, suppose that $Q \cap Q'$ is minimal. Then, α will take P to be twice as probable as Q inasmuch as $Q \sim Q'$ and $(Q \cup Q') \sim P$. In this case, Q and Q' are acting as 'duplicates' of one another, and $Q \cup Q'$ is their 'concatenation'.

3.4. Imprecise Probabilistic Comparativism. Say that Cr *confirms* the General Ratio Principle (GRP) just in case, whenever that principle implies that P is believed n/m times as much as Q, then $Cr(P) = {}^{n}/_{m} \cdot Cr(Q)$; otherwise, it *disconfirms* the GRP. It is easy to check that if any probability function almost agrees with $\succsim$, and $\varnothing$ is minimal, then that function will confirm the GRP. This means that it is possible to extend the account of ratio comparisons just given to incomplete confidence rankings.

For ordinary agents, the Completeness axiom is widely considered highly implausible. Consider the following statements, adapted from Fishburn (1986):

P = The global population in 2100 will be greater than 13 billion.
Q = The next card drawn from this old and incomplete deck will be a heart.

Are you more confident that P than that Q, or less, or just as confident in either? It is not clear that there must be a fact of the matter. Similar examples abound.[8]

There is a natural way of dealing with incompleteness to which comparativists can (and do) appeal. Where $\mathcal{F}$ is any set of real-valued functions on $\mathcal{B}$, say this time that the set $\mathcal{F}$ agrees with $\succsim$ just in case for all relevant P, Q,

$$P \succsim Q \quad \text{iff} \quad \forall\, Cr \in \mathcal{F} : Cr(P) \geq Cr(Q).$$

The idea behind a set-of-functions model is to recapture the structure of the confidence ranking by doing something like supervaluating over the functions in $\mathcal{F}$—only what is common to every such function is treated as having representative import. If P and Q are *incomparable* in terms of relative confidence, then $\mathcal{F}$ will contain at least one pair of probability functions that disagree on the relative ordering of P and Q—hence, we still manage to 'numerically' represent incomplete $\succsim$ rankings.

Alon and Lehrer (2014) have shown that a set of probability functions agrees with $\succsim$ just in case the latter satisfies the Complete Package minus the Completeness axiom (i.e., the *Noncomplete Package*). Furthermore,

8. You do not have to be convinced by the example, and here is not the place for a detailed discussion on whether we should expect 'gaps' in $\succsim$. What matters is just that there might be gaps, and many think that there are. Completeness may or may not be plausible for perfectly rational agents, but since our focus is on deidealizing the usual probabilistic theory that is neither here nor there.

while there will often be more than one set of probability functions F that agrees with $\succsim$, the union of all such sets will always agree with $\succsim$. In sum: whenever $\succsim$ satisfies the Noncomplete Package, there is guaranteed to be a unique set of probability functions that agrees with $\succsim$ and that is maximal with respect to inclusion.

Consequently, if we extend the definitions of full/L-to-R/R-to-L adequacy in the natural way (i.e., by inserting '$\forall\, \mathcal{C}r \in \mathcal{F}$' in the appropriate locations), we can characterize *imprecise probabilistic comparativism* by its commitment to:

> Imprecise Probabilistic Comparativism. If a nonempty set of probability functions $\mathcal{F}$ agrees with α's confidence ranking and $\mathcal{F}$ is maximal with respect to inclusion, then $\mathcal{F}$ is a fully adequate model of α's beliefs.

Imprecise probabilistic comparativism implies the precise version. More precisely, if we assume that $\mathcal{F}$ and $\mathcal{C}r$ are essentially the same representation whenever $\mathcal{F} = \{\mathcal{C}r\}$, then the two varieties of comparativism amount to one and the same thing whenever exactly one probability function agrees with $\succsim$.

Furthermore, every $\mathcal{C}r$ in a set $\mathcal{F}$ that agrees with $\succsim$ will itself almost agree with $\succsim$. So, if we also extend the definition of 'confirms the GRP' in the obvious way to sets of functions, it follows that if a set of probability functions $\mathcal{F}$ agrees with $\succsim$, then $\mathcal{F}$ confirms the GRP. The upshot is that both the precise and imprecise versions of probabilistic comparativism can extract cardinality from comparative confidences in basically the same way; the latter is a natural generalization of the former.

4. Why Generalize? We have seen now that conformity to the Noncomplete Package is sufficient for the union of pseudodisjoint sets to behave like addition. But it is by no means necessary. It is possible to weaken those axioms still further while maintaining the analogy, and I think it is of some importance for comparativism that this can be done. In this section I say why.

The basic reason is that the axioms of the Noncomplete Package are, in conjunction, quite strong—it is not likely that they are jointly satisfied by any ordinary agents. Since I think it is especially troubling, I will focus on one issue in particular: in the context of the (individually rather weak) axioms Nontriviality and Nonnegativity, Scott's Axiom immediately generates a probabilistic version of the classical problems of logical omniscience. Those three axioms entail that if $P \subseteq Q$ and $P, Q \in \mathcal{B}$, then $Q \succsim P$. Consequently,

> Logical Omniscience. If the worlds in Ω are closed under the consequence relation $\Rightarrow$, then for all $P, Q \in \mathcal{B}$, if $P \Rightarrow Q$, then $Q \succsim P$.

That is, any confidence ranking that is (i) defined over propositions taken from a space of worlds that is closed under ⇒ and (ii) agrees with a (set of) probability function(s) will ipso facto be 'coherent' with respect to ⇒ in the manner just described. In section 3.1 it was assumed that ⇒ is at least as strong as the consequence relation we find in classical propositional logic, and it is implausible that ordinary agents' confidence rankings are everywhere and always coherent with respect to that logic. I say more about that in a moment. But the point can also be put in a much more general way: we are (probably) not omniscient with respect to any very interesting logics, so unless ⇒ is extremely weak indeed, the confidence rankings of any ordinary agents will (probably) falsify at least one of Nontriviality, Nonnegativity, or Scott's Axiom.

How might a comparativist respond to this fact? Four obvious (but also obviously nonexhaustive) options are:

1. Argue that ordinary agents' comparative confidences do conform to the Noncomplete Package after all, because they are probabilistically coherent after all.
2. Argue that ordinary agents' comparative confidences do conform to the Noncomplete Package after all, once we define propositions over a richer space of worlds.
3. Argue that because ordinary agents' comparative confidences do not conform to the Noncomplete Package, they therefore do not ground any cardinal information (or not the same kind of information).
4. Accept that ordinary agents' comparative confidences do not conform to the Noncomplete Package and seek weaker axioms under which cardinality can be extracted from comparative confidences.

The fourth seems to me clearly the best option. After all, nothing about comparativism per se ties it irrevocably to specifically probabilistic representations of degrees of belief, and if more general conditions exist then it only makes sense for comparativists to find and use them. But if you prefer one of the others, or something else not listed, then so be it—there is no harm in developing ideas in many different directions. I will, however, here give some reasons to think that the fourth option should be preferred.

Regarding the first: I take it for granted in the following discussion that we are not (classically) logically omniscient. But maybe we are—sure, and I am not unsympathetic to the idea that we ordinary agents really are probabilistically coherent. But since this is usually met with an incredulous stare let's just move on already.

The second option seems to be the more common way of arguing that the Noncomplete Package can actually be satisfied by ordinary (and ordinarily irrational) agents. As I have noted, if the entailment relation ⇒ is weak enough, then logical omniscience might not look so bad. So what would happen if we remove the assumption that the worlds in Ω are closed under any interesting logic?

In a little more detail, the idea is this. If we help ourselves to a rich enough space of possible and impossible worlds, then it is well known that we can construct a probability function properly so-called on that enriched space that 'mimics' the behavior of a nonprobabilistic function defined over the smaller space of classical possible worlds.[9] So what looks like comparative confidences that are inconsistent with Nontriviality, Nonnegativity, or Scott's Axiom when they are defined for propositions qua sets of possible worlds can in fact be rerepresented using (sets of) probability functions, if we make use of enough impossible worlds. Hence, to apply the probabilistic comparativists' explanation of cardinality to ordinary agents, we do not need to weaken the axioms all. We can keep the Noncomplete Package as long as we just make sure to use enough impossible worlds.

That seems easy enough, but I do not think that this is a viable strategy for the comparativist to adopt. I will set out the reasons for this very briefly, since most of the relevant issues are discussed at length in Elliott (2019b). The problem is that once Ω includes enough impossible worlds for the strategy to work (roughly, for any impossibility, there is an impossible world that verifies it), then most subsets of Ω will be meaningless and consequently not representative of any proper contents of belief. Moreover, for any meaningful subset P of Ω, none of P's subsets or supersets will be meaningful, nor will any subset of $\Omega \setminus P$ be meaningful. In short, having too many impossible worlds in Ω renders useless for the purposes of comparativism any set-theoretic definition of 'concatenation' along the lines described in section 3. Furthermore, any algebra of propositions defined on a space of possible and impossible worlds that is rich enough to represent the contents of belief will contain only meaningful propositions just when the relevant space of worlds is closed under a consequence relation that is, for all intents and purposes, at least as strong as classical propositional logic.

Of course, comparativists do not have to define their concatenations set theoretically as I have done in section 3.2. But the only other place that we will plausibly find the structure required to define an appropriate concatenation operation is in the logical relations among the contents of the propositions. That is, we could define concatenations in terms of *disjunctions of inconsistent contents* (or disjunctions of contents whose conjunctions are minimal). But defining the concatenation operation in this way brings us

9. Where Ω is the space of classically possible worlds, $\mathcal{B} \subseteq \wp(\Omega)$, and $\mathcal{C}r : \mathcal{B} \mapsto [0, 1]$, then if Ω^+ is a rich enough extension of Ω into the space of impossible worlds, there is a probability function $\mathcal{C}r^+$ on an algebra of sets $\mathcal{B}^+ \subseteq \wp(\Omega^+)$ such that $\mathcal{C}r^+$ assigns x to the subset of Ω^+ that verifies φ iff $\mathcal{C}r$ assigns x to the subset of Ω that verifies φ (see Cozic 2006; Halpern and Pucella 2011; Elliott 2019b).

straight back to where we started vis-à-vis the problem of logical omniscience, and appealing to impossible worlds will be of absolutely no help here.

So there is no easy way to pursue either the first or the second route: if you want to tie the possibility of cardinality to the Noncomplete Package, then you will be tying it to very strong conditions of logical omniscience. And consequently you will need to face up to the empirical and intuitive evidence that ordinary agents just are not that good at classical logic.

Could we instead take the third route and argue that ordinary agents whose comparative confidences do not satisfy the Noncomplete Package cannot have beliefs that carry ratio and interval information? This does not strike me as very plausible. For example, the literature on the conjunction and disjunction fallacies already strongly suggests that ordinary agents do not have comparative confidences that respect even relatively simple bits of classical logic. So imagine that α has just committed the conjunction fallacy—she thinks it is more plausible that Linda is a bank teller (B) and active in the feminist movement (F) than that she's a bank teller. Are we going to say now that there is no meaningful way to answer the question of how much more α believes $B \cap F$ over B? Of course not. Similarly, *I* am not logically omniscient, and (like most people) I have probably fallen foul of various probabilistic fallacies before. My comparative confidences do not satisfy the Noncomplete Package. Maybe they do not even come close to satisfying those axioms. None of this prevents me from believing some things much more than other things, or at least twice as much as other things.

Our capacity to believe one proposition much more than another, or (at least) twice as much as another thing, and so on, is not hostage to any presupposition of logical coherence; still less should it depend on a condition of probabilistic representability. Most philosophers will see no inconsistencies at all in holding both that (*a*) ordinary agents' beliefs cannot be faithfully represented by (a set of) probability functions and that, (*b*) for arbitrary P and Q, an ordinary agent might believe P much more than Q, or (at least) twice as much as Q. These claims should be uncontroversial—only someone caught firmly in the grips of a deeply unrealistic picture of belief would think to deny it. Or at least I will say this: if you want to argue otherwise, then you will be facing a difficult uphill battle. Better, I think, to seek more general axioms under which cardinal information can be extracted from a system of comparative confidences.

5. The Ramseyan Alternatives. What I am calling Ramseyan comparativism is inspired by a brief remark from Frank Ramsey in "Probability and Partial Belief": "'Well, I believe it to an extent 2/3', i.e. (this at least is the most natural interpretation) 'I have the same degree of belief in it as in $P \vee Q$ when I think P, Q, R equally likely and know that exactly one of them is true'" (1931a, 95). In a recent paper, Weatherson (2016, 223–24) has also suggested that Ramsey's remark points toward a version of comparativism that is weaker than probabilistic comparativism. However, neither Ramsey nor Weatherson take their discussion beyond this initial suggestion, and (as we will soon see) there is a bit of work that needs to be done in order to flesh the idea out in full.

In the remainder of this article, I develop precise Ramseyan comparativism (secs. 5.1 and 5.2) and then an imprecise version (sec. 5.3). Following that, I prove an important result about the axioms under which Ramseyan comparativism supports the analogy with the measurement of mass (sec. 5.4).

5.1. The Main Ideas. First, it will be useful to introduce another definition (the term 'n-scale' comes from Koopman [1940]):

> DEFINITION 4. A set $\mathcal{P}$ of n pseudodisjoint propositions is an *n-scale of P* iff (i) $P \notin \mathcal{P}$, (ii) $\cup\, \mathcal{P} \sim P$, and (iii) for all Q, Q' in $\mathcal{P}$, $Q \sim Q'$.

We can take this as a comparativist characterization of what it is for an agent to think that Q is as likely as a disjunction of equiprobable propositions at most one of which is true. So, for example, if α thinks Q is as likely as $P \cup P'$, where P and P' are equiprobable and pseudodisjoint, then $\{P, P'\}$ is a 2-scale of Q. We will also assume that α is *certain* of P's truth just in case P is maximal, and we will represent certainty in P with $\mathcal{C}r(P) = 1$. This is something the Ramseyan view shares with probabilistic comparativism, where in order to fix the scales the values of the minimal and maximal propositions need to be stipulated.

In light of definition 4, Ramsey's idea can be recast as follows: α believes P to degree n/m when $P \sim (Q_1 \cup \ldots \cup Q_n)$, where the $Q_1, \ldots, Q_n$ belong to an m-scale $\{Q_1, \ldots, Q_n, \ldots, Q_m\}$ of some maximal proposition R. A good start—but there is a natural extension that will be helpful to incorporate into what follows.

Consider, to begin with, the following situation. Let $\mathcal{B}$ designate the power set of $\Omega = \{\omega_1, \omega_2, \omega_3\}$, and let $P_{\langle n \rangle}$ and $P_{\langle nm \rangle}$ designate the possible worlds propositions $\{\omega_n\}$ and $\{\omega_n, \omega_m\}$, respectively (e.g., $P_{\langle 12 \rangle} = \{\omega_1, \omega_2\}$). Suppose now that $\succsim$ is transitive and reflexive, and (where the square brackets indicate equiprobability)

$$\Omega \succ \begin{bmatrix} P_{\langle 13 \rangle} \\ P_{\langle 23 \rangle} \end{bmatrix} \succ \begin{bmatrix} P_{\langle 12 \rangle} \\ P_{\langle 3 \rangle} \end{bmatrix} \succ \begin{bmatrix} P_{\langle 1 \rangle} \\ P_{\langle 2 \rangle} \end{bmatrix} \succ \varnothing.$$

We can represent $\succsim$ with figure 1, where the relative sizes of the boxes containing the ω_i correspond to the order of propositions in the confidence ranking.

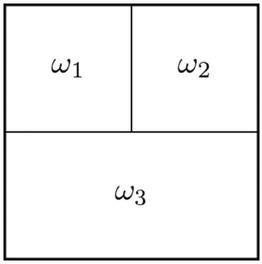

Figure 1. Indirect R-Scalability.

Now Ω is maximal, and $\{P_{\langle 12\rangle}, P_{\langle 3\rangle}\}$ is a 2-scale of Ω, so Ramsey would say that

$$\mathcal{C}r(P_{\langle 3\rangle}) = \mathcal{C}r(P_{\langle 12\rangle}) = \frac{1}{2}.$$

However, $P_{\langle 1\rangle}$ and $P_{\langle 2\rangle}$ do not belong to any n-scale of Ω, so Ramsey's idea does not yet give us any strength with which they are believed. But since $\{P_{\langle 1\rangle}, P_{\langle 2\rangle}\}$ is a 2-scale of $P_{\langle 12\rangle}$, it is only reasonable to say that

$$\mathcal{C}r(P_{\langle 1\rangle}) = \mathcal{C}r(P_{\langle 2\rangle}) = \frac{1}{4}.$$

We can capture the foregoing by means of the following definition:

Definition 5. For integers n, m such that $m \geq n \geq 0$, $m > 0$, P is

1. $0/m$ *valued* if P is minimal and m/m *valued* if P is maximal, and
2. n/m *valued* if $P \sim (Q_1 \cup \ldots \cup Q_{n'})$, where the $Q_1, \ldots, Q_{n'}$ belong to an m'-scale of an n''/m''-valued proposition, and $n' \cdot n''/m' \cdot m'' = n/m$.

The new, generalized version of Ramsey's idea now amounts to the claim that α believes P to degree n/m if P is n/m valued. As such, define a *Ramsey function* as follows:

Definition 6. $\mathcal{C}r : \mathcal{B} \mapsto [0, 1]$ is a *Ramsey function* (relative to $\succsim$) iff, for all $P \in \mathcal{B}$, if P is n/m valued, then $\mathcal{C}r(P) = n/m$.

The close connection between Ramsey functions and the GRP should at this point be apparent, and it should likewise already be clear that the way Ramsey proposes to measure degrees of belief is not too different from the strategy the probabilistic comparativists want to adopt. In fact, in the present terminology, the first (noninductive) clause of the GRP essentially states that

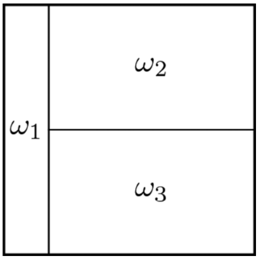

Figure 2. Failure of R-Scalability.

for $m \geq n$, P is believed n/m times as much as Q whenever $\mathcal{P}$ is an m-scale of Q, and $\mathcal{P}' \subseteq \mathcal{P}$ is an n-scale of P. In this case, for any Ramsey function $\mathcal{C}r$, $\mathcal{C}r(P) = n/m \cdot \mathcal{C}r(Q)$. With respect to n/m-valued propositions, Ramsey functions always confirm the GRP.

Essentially, a Ramsey function either directly or indirectly scales every middling n/m-valued proposition relative to some maximal proposition, which has a stipulated value. With respect to pairs of propositions that cannot be so scaled, however, a Ramsey function may disconfirm the GRP. An especially clear example where this would occur can be seen in figure 2. Where

$$\Omega \succ P_{\langle 23 \rangle} \succ \begin{bmatrix} P_{\langle 12 \rangle} \\ P_{\langle 13 \rangle} \end{bmatrix} \succ \begin{bmatrix} P_{\langle 2 \rangle} \\ P_{\langle 3 \rangle} \end{bmatrix} \succ P_{\langle 1 \rangle} \succ \varnothing.$$

In this case, the only nontrivial n-scale is the 2-scale $\{P_{\langle 2 \rangle}, P_{\langle 3 \rangle}\}$ of $P_{\langle 23 \rangle}$. According to the GRP, then, we should be able to say

$$\mathcal{C}r(P_{\langle 2 \rangle}) = \mathcal{C}r(P_{\langle 3 \rangle}) = \frac{1}{2} \cdot \mathcal{C}r(P_{\langle 23 \rangle}).$$

However, since $P_{\langle 23 \rangle}$ cannot be scaled relative to Ω, Ramsey's suggestion gives us no means of fixing values for $P_{\langle 2 \rangle}$, $P_{\langle 3 \rangle}$ and $P_{\langle 23 \rangle}$.

Call any proposition that is n/m valued *R-scalable*. All of the propositions other than Ω and $\varnothing$ represented in figure 2 are not R-scalable. Ramsey says nothing about how to measure propositions that are not R-scalable, although perhaps this is not a very troubling gap in his proposal. One might simply assume that such cases do not exist. Let $\mathcal{N}$ designate the set of R-scalable propositions, then:

R-SCALABILITY. $\mathcal{N} = \mathcal{B}$.

R-Scalability is not implied by the Complete Package. However, given that package, it is equivalent to Continuity. (See the appendix for a proof.)[10] In

10. The proof rests in part on the assumption that $\mathcal{B}$ is closed under unions. Without that assumption, Continuity will imply R-Scalability but not vice versa.

other words, precise probabilistic comparativists do not seem to have anything to fear from an axiom like R-Scalability. (Nevertheless, I discuss below how the Ramseyan comparativist can do without it.)

R-Scalability merely guarantees that every proposition in $\mathcal{B}$ is R-scalable. Importantly, this is not yet enough to ground a minimally plausible comparativist theory. There are still two additional problems that can arise in the absence of further assumptions about the structure of $\succsim$:

1. We need to ensure that definition 6 is *consistent*. Without further assumptions, it is possible that, for example, $P \sim Q$, where for some R, P belongs to a 2-scale of R and Q belongs to a 3-scale of R. This is clearly unacceptable: α cannot believe P to the degrees 1/2 and 1/3 simultaneously. If Ramsey functions are to be well defined, we will need to ensure that if P is both n/m valued and n'/m' valued, then $n/m = n'/m'$.
2. We need to ensure that any Ramsey function relative to $\succsim$ will *agree* with $\succsim$. Without further assumptions, there is no guarantee that $\mathcal{C}r(P) \geq \mathcal{C}r(Q)$ if or only if $P \succsim Q$. For instance, P could be 1/2 valued and Q 1/4 valued, yet $Q \succsim P$. This is also undesirable: if the order of the values we assign propositions does not match up to the confidence ranking, then there can be no plausible sense in which those values are a measure of the strengths with which those propositions are believed.

In the presence of R-Scalability, we can kill these two birds with a single stone by adding the following rather strong axiom:

R-Coherence. If P is n/m valued and Q is n'/m' valued, $P \succsim Q$ iff $n/m \geq n'/m'$.

R-Coherence is sufficient to avoid both worries, as established by the following representation theorem:

Theorem 1. (i) $\succsim$ satisfies R-Coherence iff there exists a Ramsey function $\mathcal{C}r$ with respect to $\succsim$, and (ii) $\succsim$ also satisfies R-Scalability iff $\mathcal{C}r$ is the unique Ramsey function relative to $\succsim$ that agrees with $\succsim$.

The proofs for this theorem and the two that follow below can be found in the appendix.

5.2. Precise Ramseyan Comparativism. We will say from now on that one accepts *precise Ramseyan comparativism* just in case one accepts the following conditional:

> PRECISE RAMSEYAN COMPARATIVISM. If $\mathcal{C}r$ is the only Ramsey function relative to α's confidence ranking, then $\mathcal{C}r$ is a fully adequate model of α's beliefs.

We can now characterize precisely the respects in which precise Ramseyan comparativism is more lenient than probabilistic comparativism. To start with, it is easy to see that R-Coherence is implied already by the Complete Package. Indeed, if any probability function $\mathcal{C}r$ agrees with $\succsim$, then $\mathcal{C}r$ is also a Ramsey function relative to $\succsim$. Moreover, where the Complete Package plus R-Scalability holds, then the unique probability function that agrees with $\succsim$ just is the unique Ramsey function that agrees with $\succsim$. This is important, since (in light of what we said earlier) it means that precise Ramseyan comparativism is a generalization of any version of precise probabilistic comparativism that makes use of Continuity.

In the other direction, R-Scalability and R-Coherence together obviously imply Completeness and Preorder. However, they do not imply any of Nontriviality, Nonnegativity, or Scott's Axiom. For a simple (albeit extreme) example in which all three of those axioms fail, assume that $\Omega = \{w_1, w_2, w_3, w_4\}$, $\succsim$ is transitive and reflexive, and

$$\begin{bmatrix} P_{\langle 4 \rangle} \\ P_{\langle 24 \rangle} \\ P_{\langle 124 \rangle} \\ P_{\langle 234 \rangle} \end{bmatrix} \succ \begin{bmatrix} \varnothing \\ \Omega \\ P_{\langle 23 \rangle} \\ P_{\langle 34 \rangle} \end{bmatrix} \succ \begin{bmatrix} P_{\langle 2 \rangle} \\ P_{\langle 14 \rangle} \\ P_{\langle 123 \rangle} \end{bmatrix} \succ \begin{bmatrix} P_{\langle 1 \rangle} \\ P_{\langle 3 \rangle} \\ P_{\langle 12 \rangle} \\ P_{\langle 13 \rangle} \\ P_{\langle 134 \rangle} \end{bmatrix}.$$

It is straightforward (albeit a little tedious) to check that R-Scalability and R-Coherence are satisfied in this case. The only nontrivial n-scales (i.e., $n > 1$) that can be defined using this ranking are:

1. The 2-scale $\{P_{\langle 23 \rangle}, P_{\langle 34 \rangle}\}$ of the maximal propositions;
2. The 2-scale $\{P_{\langle 123 \rangle}, P_{\langle 14 \rangle}\}$ of $\varnothing$, Ω, $P_{\langle 23 \rangle}$, and $P_{\langle 34 \rangle}$;
3. The several n-scales composed out of minimal propositions, each of some other minimal proposition.

Consequently, $\mathcal{C}r(\Omega) = \mathcal{C}r(\varnothing) = 1/2$ because $\{\Omega\}$ and $\{\varnothing\}$ are 1-scales of $P_{\langle 23 \rangle}$ and $P_{\langle 34 \rangle}$, where the latter are 1/2 valued, and $\mathcal{C}r(P_{\langle 2 \rangle}) = 1/4$, because $\{P_{\langle 2 \rangle}\}$ is a 1-scale of $P_{\langle 14 \rangle}$ and $P_{\langle 123 \rangle}$, where the latter are 1/4 valued. Every other proposition is either maximal or minimal and assigned either 1 or 0 accordingly. That the example violates Nontriviality and Nonnegativity is obvious; to see that it violates Scott's Axiom it suffices to consider the two short sequences $P_{\langle 13 \rangle}$, $P_{\langle 24 \rangle}$ and $P_{\langle 12 \rangle}$, $P_{\langle 34 \rangle}$.

The interesting 'work' here is of course being done entirely by R-Coherence. This axiom imposes a limited kind of additive structure on $\succsim$, specifically with respect to confidence rankings between propositions constructed out of members of the same n-scale of any n'/m'-valued proposition. Roughly, within an n-scale, $\succsim$ behaves "pseudoprobabilistically"—but not every proposition is constructible out of the members of an appropriate n-scale, and across n-scales $\succsim$ can behave quite irrationally indeed.

5.3. Imprecise Ramseyan Comparativism. If we wanted to drop R-Scalability out of the picture, we could do so by adopting a set-of-functions representation of $\succsim$. For that, we will need to add back in the Preorder axiom. This is obviously necessary for any real-valued function or set thereof to agree with $\succsim$, and it is not implied by R-Coherence alone.

> Theorem 2. $\succsim$ satisfies Preorder and R-Coherence iff there is a nonempty set $\mathcal{F}$ of Ramsey functions relative to $\succsim$ such that $\mathcal{F}$ agrees with $\succsim$, and in such cases there will also be a unique such $\mathcal{F}$ that agrees with $\succsim$ that is maximal with respect to inclusion.

Given this, let's characterize the imprecise variety of Ramseyan comparativism by its commitment to:

> Imprecise Ramseyan Comparativism. If $\mathcal{F}$ is a nonempty set of Ramsey functions with respect to α's confidence ranking, which is maximal with respect to inclusion and agrees with $\succsim$, then $\mathcal{F}$ is an R-to-L adequate model of α's beliefs.

Note that imprecise Ramseyan comparativism only claims R-to-L adequacy. This is because (as we have seen) Preorder and R-Coherence are not sufficient for a (set of) Ramsey function(s) to confirm the GRP in full. This is a limitation with the imprecise Ramseyan comparativist's theory, but perhaps not a devastating one. In effect, R-to-L adequacy says that we will not go wrong whenever we read cardinal information off of the numbers, although there may be some interesting cardinal properties to one's degrees of belief that are not appropriately captured by their cardinal representation. Although it is not perfect, I suspect that many comparativists would be satisfied by this result—nobody said that our numerical representations had to be perfect after all.

Imprecise Ramseyan comparativism also agrees exactly with (precise and imprecise) probabilistic comparativism whenever the Complete Package plus R-Scalability are satisfied. We have already shown that this is so for precise Ramseyan comparativism, but if this is not obvious in the case of imprecise Ramseyan comparativism then consider this: if we assume the

Complete Package plus R-Scalability, then the probability function $\mathcal{C}r$ that agrees with $\succsim$ is the Ramsey function that agrees with $\succsim$; from imprecise Ramseyan comparativism, $\mathcal{C}r$ is R-to-L adequate, so $\mathcal{C}r$ determines a unique ratio comparison for every pair of nonminimal propositions; and finally, α cannot believe P n/m times as much as Q and n'/m' as much as Q, for $n/m \neq n'/m'$.

5.4. The Importance of R-Coherence. Importantly, we can show that Preorder and R-Coherence are individually necessary for coherence with the GRP. As far as Preorder is concerned, this is obvious for the reasons already mentioned. The more interesting result concerns R-Coherence. Given some very minimal scaling assumptions, violations of that axiom imply that any $\mathcal{C}r$ that agrees with $\succsim$ cannot confirm the GRP:

> Theorem 3. If (i) $\mathcal{C}r$ agrees with $\succsim$, (ii) there are P, Q such that $P \succ Q$, and (iii) $\mathcal{C}r(R) = 0$ whenever R is minimal, then $\mathcal{C}r$ confirms the GRP only if R-Coherence is satisfied.
>
> Corollary. Under the same assumptions, mutatis mutandis, any set of real-valued functions F will confirm the GRP only if R-Coherence is satisfied.

In other words, assuming just that $\succsim$ has some nontrivial structure, and that minimal propositions can be assigned value 0, that a function (or set of functions) confirms the GRP implies that any comparative ranking it agrees with will satisfy Preorder and R-Coherence. Thus, we have found two minimal axioms necessary for the union of pseudodisjoint sets to behave like addition with respect to $\succsim$.

6. Conclusion. Let's take stock. The standard comparativist strategy for explaining cardinality is based on a purported analogy with the measurement of certain extensive quantities like length or mass. So, for instance, to say that P is n times more likely than Q, we just need to be able to say that P is as likely as the union of n 'duplicates' of Q, where the 'duplicates' are propositions that are equiprobable and pairwise (pseudo)disjoint. The two Ramseyan varieties of comparativism I have outlined offer an account of when this kind of 'adding' is meaningful that generalizes the axioms assumed by the more common probabilistic comparativism, thus applying to a wide range of confidence rankings that are not probabilistically representable.

In particular, we have shown that comparativists can in principle do without any appeal to Nontriviality, Nonnegativity, and Scott's Axiom and can avoid the problems that those axioms bring in their wake. This is an interesting result by itself, since it establishes that comparativists can preserve their favorite explanation of cardinality without necessarily committing to the stronger conditions required for probabilistic representability. Moreover, we have been able to show that the union of (pseudo)disjoint sets behaves like addition only if the comparative confidence ranking satisfies Preorder and R-Coherence. Inasmuch as comparativists want to retain the analogy with the measurement and mass as it is usually understood (i.e., in terms of the union of either disjoint or pseudodisjoint propositions), then Ramseyan comparativism is as general as it gets.

It remains to be seen whether it is correct to say that an agent α considers P to be n times more likely than Q iff P is as likely for her as the union of n pseudodisjoint duplicates of Q. But we now know the minimal conditions required for the analogy with mass to hold, so we can ask: (*a*) Are Preorder and R-Coherence plausibly satisfied by actual agents—or at least, by the kinds of agents who we are happy to say have degrees of belief that carry cardinal information? And, (*b*) if so, does the GRP in those cases accurately predict our considered judgments about the degrees of belief of such agents? These are questions that I have not considered in this article, but they will need careful consideration in future discussions on the viability of the comparativist view.

Appendix

Proofs

Proof That, Given the Complete Package, Continuity Is Equivalent to R-Scalability. Assume the Complete Package throughout. For the left to right, assume Continuity. This entails that for every middling proposition P, $P \sim (Q_1 \cup \ldots \cup Q_n)$, where the $Q_1, \ldots, Q_n$ belong to some m-scale of Ω, which gives us R-Scalability.

For the right to left, assume R-Scalability and (for reductio) that there exists a nonminimal atom A in the algebra $\mathcal{B}$ such that for every other atom A', $A' \succsim A$, with '$\succsim$' replaced by '$\succ$' in at least one instance. (Equivalently: assume there are nonminimal atoms not equally ranked by $\succsim$.)

Since $\Omega \setminus A$ is middling, it is R-scalable only if $(\Omega \setminus A) \sim (Q_1 \cup \ldots \cup Q_n)$, for some $Q_1, \ldots, Q_n$ in an m-scale of some R-scalable proposition S such that $S \succ (\Omega \setminus A)$.[11] However, let $\mathcal{C}r$ be any probability function that agrees with $\succsim$; then

$$\mathcal{C}r(Q_1) + \ldots + \mathcal{C}r(Q_n) = \mathcal{C}r(\Omega) - \mathcal{C}r(A).$$

11. We can safely ignore the case in which $S \sim (\Omega \setminus A)$, since then S will be R-scalable only if $\Omega \setminus A$ is.

Furthermore, the Q_i must be more probable than A, since as there exist atoms more probable than A the union of any and all propositions that are as probable as A will be strictly less probable than $\Omega \setminus A$. So, $\mathcal{C}r(Q_i) > \mathcal{C}r(A)$, and thus

$$\mathcal{C}r(Q_1) + \ldots + \mathcal{C}r(Q_n) + \ldots + \mathcal{C}r(Q_m) > \mathcal{C}r(\Omega).$$

But there is no $S \succ \Omega$, so $\Omega \setminus A$ is not R-scalable, contradicting our assumption. R-Scalability therefore implies that $A \sim A'$ for any two nonminimal atoms A and A'; from this, Continuity straightforwardly follows. QED

Proof of Theorem 1. Part i: For the left to right, assume R-Coherence. If P is n/m valued and n'/m' valued, then $n/m = n'/m'$. So there exists a function $\mathcal{C}r$ that assigns to each $P \in \mathcal{N}$ a unique rational value in [0, 1], and $\mathcal{C}r$ will be a Ramsey function relative to $\succsim$ on $\mathcal{N}$. This function can then be extended to the whole of $\mathcal{B}$ in the event that $\mathcal{B} - \mathcal{N} \neq \varnothing$ in any way you like. The right to left is obvious.

Part ii: For the left to right, assume R-Coherence and R-Scalability. For any $P, Q \in \mathcal{N}$ $(=\mathcal{B})$, suppose first that $P \succsim Q$. Where P is n/m valued and Q is n'/m' valued, $n/m \geq n'/m'$, so for any Ramsey function $\mathcal{C}r$ relative to $\succsim$, $\mathcal{C}r(P) \geq \mathcal{C}r(Q)$. Next, suppose $\mathcal{C}r(P) \geq \mathcal{C}r(Q)$; since $\mathcal{C}r$ is a Ramsey function, P is n/m valued and Q is n'/m' valued, for $n/m \geq n'/m'$; by R-Coherence, therefore $P \succsim Q$. So from R-Coherence and R-Scalability, there is a Ramsey-function $\mathcal{C}r$ relative to $\succsim$ that agrees with $\succsim$. It is obvious from the definitions that the restriction of $\mathcal{C}r$ to $\mathcal{N}$ will always be the unique Ramsey function relative to $\succsim$ on $\mathcal{N}$, and in this case $\mathcal{N} = \mathcal{B}$.

For the right to left, the existence of the Ramsey-function $\mathcal{C}r$ already entails R-Coherence by part i. That its uniqueness condition also entails R-Scalability is obvious given the finitude of $\mathcal{B}$.

Proof of Theorem 2. The right to left of the existence part is obvious given part i of theorem 1. For the left to right of the existence part, assume henceforth Preorder and R-Coherence. We focus on the case in which $\mathcal{N} \subset \mathcal{B}$, as R-Scalability trivializes the proof.

From Preorder, at least one nonempty set $\mathcal{F} = \{f_i : \mathcal{B} \mapsto \mathbb{R} | i = 1, \ldots, n\}$ exists that agrees with $\succsim$ (see Evren and Ok 2011, 556, proposition 1). Suppose that $\mathcal{F}$ is maximal with respect to inclusion. We then just need that there is some nonempty $\mathcal{F}^* \subseteq \mathcal{F}$ such that $\mathcal{F}^*$ also agrees with $\succsim$ and $\forall f \in \mathcal{F}^*$, f has an order-preserving transformation f' that is a Ramsey function with respect to $\succsim$. (We will say that f' is an order-preserving transformation of f just in case $f(P) \geq f(Q)$ iff $f'(P) \geq f'(Q)$.) The set of all such transformations f' will then agree with $\succsim$.

There are three cases to consider: (i) $\mathcal{N}$ is empty; (ii) $\mathcal{N}$ contains only the minimal or maximal or both elements of $\mathcal{B}$; (iii) $\mathcal{N}$ contains some middling propositions. The first two are straightforward and omitted. For the third, note that if $\mathcal{F}$ agrees with $\succsim$ and $P \succ Q$, then

1. $f(P) \geq f(Q)$ for all $f \in \mathcal{F}$.
2. $f(P) > f(Q)$ for some, but not necessarily all, $f \in \mathcal{F}$.

For P, $Q \in \mathcal{N}$, R-Coherence requires however that for any Ramsey function $\mathcal{C}r$, if $P \succ Q$, then $\mathcal{C}r(P) > \mathcal{C}r(Q)$. Consequently, it is not true that if $\mathcal{F}$ agrees with $\succsim$, then every $f \in \mathcal{F}$ has an order-preserving transformation that is also a Ramsey function with respect to $\succsim$. But define $\mathcal{F}^*$ as follows:

$$\mathcal{F}^{\star} = \{f \in \mathcal{F} | \text{ if } P, Q \in \mathcal{N} \text{ and } P \succ Q, \text{ then } f(P) > f(Q)\},$$

where $\mathcal{F}^*$ will be nonempty and will agree with $\succsim$. Let $\mathcal{F}_{\mathcal{N}}$ denote the set of restrictions of every $f \in \mathcal{F}^{\star}$ to $\mathcal{N}$. Given this, the unique Ramsey function (denoted $\mathcal{C}r_{\mathcal{N}}$) on $\mathcal{N}$ is going to be an order-preserving transformation of every $f \in \mathcal{F}_{\mathcal{N}}$. So we just have to show that each $f \in \mathcal{F}^{\star}$ has an order-preserving transformation bounded by 0 and 1 that is an extension of $\mathcal{C}r^{\mathcal{N}}$ from $\mathcal{N}$ to the whole of $\mathcal{B}$. Since $\mathcal{B}$ is finite this is straightforward.

The proof of the uniqueness condition is obvious. If $\mathcal{F}$ and $\mathcal{F}'$ both agree with $\succsim$, then $\mathcal{F} \cup \mathcal{F}'$ will too. QED

Proof of Theorem 3. Suppose just that $\mathcal{C}r$ agrees with $\succsim$ and that $\succsim$ violates R-Coherence. So, there exist P, Q such that P is n/m valued, Q is n'/m' valued, and not

$$(P \succsim Q) \leftrightarrow \left(\frac{n}{m} \geq \frac{n'}{m'}\right).$$

There are three cases: (1) neither P nor Q is minimal, (2) both P and Q are minimal, or (3) exactly one of P or Q is minimal.

Start with case 1. Focus on P, and let max designate some maximal proposition. (If P is n/m valued and nonminimal, then max exists.) Here P is either (i) as probable as the union of n members of an m-scale of max or (ii) as probable as the union of n'' members of an m''-scale of . . . the union of n''' members of an m'''-scale of max. If i, $\mathcal{C}r$ confirms the GRP only if

$$\mathcal{C}r(P) = \frac{n}{m} \cdot \mathcal{C}r(\text{max}).$$

If ii, it confirms only if

$$\mathcal{C}r(P) = \frac{(n'' \cdot \ldots \cdot n''')}{(m'' \cdot \ldots \cdot m''')} \cdot \mathcal{C}r(\text{max}) = \frac{n}{m} \cdot \mathcal{C}r(\text{max}).$$

The same applies to Q, mutatis mutandis, so $\mathcal{C}r$ confirms the GRP only if

$$\mathcal{C}r(Q) = \frac{n'}{m'} \cdot \mathcal{C}r(\max).$$

Assume for reductio that $\mathcal{C}r$ confirms the GRP, and suppose $n/m \geq n'/m'$. Hence, $\mathcal{C}r(P) \geq \mathcal{C}r(Q)$, and therefore $P \succsim Q$. In the other direction, suppose $P \succsim Q$, so $\mathcal{C}r(P) \geq \mathcal{C}r(Q)$, and $n/m \geq n'/m'$. So,

$$(P \succsim Q) \leftrightarrow \left(\frac{n}{m} \geq \frac{n'}{m'}\right),$$

which violates our assumptions.

Now consider case 2. Assume for this case that there are $P, Q \in \mathcal{B}$ such that $P \succ Q$ and that if P is minimal, then $\mathcal{C}r(P) = 0$. If P, Q are both minimal, then $P \sim Q$, and if $\mathcal{C}r$ agrees with $\succsim$, then $\mathcal{C}r(P) = \mathcal{C}r(Q) > \mathcal{C}r(R)$, for any R such that $R \not\precsim P$ (and hence $R \succ P$). Since P, Q are $0/m$ valued by definition, R-Coherence is violated only if P or Q is also n/m valued, for $n > 0$. Suppose this of P; then by the earlier reasoning, $\mathcal{C}r$ confirms the GRP only if $\mathcal{C}r(P) = n/m \cdot \mathcal{C}r(\max)$. Since $n/m > 0$ and $\mathcal{C}r(\max) > 0$, this is false, so $\mathcal{C}r$ disconfirms the GRP.

Case 3 is then straightforward, and the proof of the corollary (for sets of functions) follows the same structure. Both proofs are omitted. QED

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