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ON RELATIVE COMPLETE REDUCIBILITY

CHRISTOPHER ATTENBOROUGH, MICHAEL BATE, MAIKE GRUCHOT, ALASTAIR LITTERICK, GERHARD RÖHRLE

ABSTRACT. Let K be a reductive subgroup of a reductive group G over an algebraically closed field k. The notion of relative complete reducibility, introduced in [4], gives a purely algebraic description of the closed K-orbits in G^n, where K acts by simultaneous conjugation on n-tuples of elements from G. This extends work of Richardson and is also a natural generalization of Serre's notion of G-complete reducibility. In this paper we revisit this idea, giving a characterization of relative G-complete reducibility which directly generalizes equivalent formulations of G-complete reducibility. If the ambient group G is a general linear group, this characterization yields representation-theoretic criteria. Along the way, we extend and generalize several results from [4].

1. INTRODUCTION

Let G be a (possibly non-connected) reductive linear algebraic group and let $n \in \mathbb{N}$. The group G acts by simultaneous conjugation on the n-fold Cartesian product G^n. In his seminal work [7, Thm. 16.4], Richardson characterized the closed G-orbits in G^n in terms of the subgroup structure of G. In [3, Thm. 3.1] Richardson's characterization was shown to be equivalent to a notion of Serre arising from representation theory, [8], and these ideas were further extended in [4] to give a characterization of the closed K-orbits in G^n for an arbitrary reductive subgroup K of G. This gave rise to the notion of relative complete reducibility, which we briefly recall now (see Section 2 for full definitions).

Let H be a subgroup of G and let K be a reductive subgroup of G. Recall that (when G is connected) the parabolic subgroups of G have the form P_λ where λ is a cocharacter of G. Following [4], we say that H is *relatively G-completely reducible with respect to K* if for every cocharacter λ of K such that H is contained in the subgroup P_λ of G, there exists a cocharacter μ of K such that $P_\lambda = P_\mu$ and $H \subseteq L_\mu$, a Levi subgroup of P_λ. For $K = G$, this definition coincides with the usual notion of G-complete reducibility due to Serre, cf. [3], [8].

The following algebraic characterization of the closed K-orbits in G^n in terms of relative G-complete reducibility was given in [4, Thm. 1.1]:

Theorem 1.1. *Let K be a reductive subgroup of G. Let H be the algebraic subgroup of G generated by elements $x_1, \dots, x_n \in G$. Then $K \cdot (x_1, \dots, x_n)$ is closed in G^n if and only if H is relatively G-completely reducible with respect to K.*

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Recall that in the particular case $K = G$ and $G = \mathrm{GL}(V)$, a subgroup H of G is G-completely reducible if and only if V is a completely reducible H-module. This latter property can be defined equivalently by either (i) every H-stable flag of subspaces of V admits an H-stable complement; (ii) every H-stable subspace of V admits an H-stable complement; or (iii) V is a direct sum of irreducible H-submodules. Our main result, Theorem 1.2 below, generalizes these equivalences firstly to other connected reductive algebraic groups G, and also to the relative setting. When $G = \mathrm{GL}(V)$ this characterizes relative $\mathrm{GL}(V)$-complete reducibility (with respect to K) in terms of certain flags of submodules of V.

In order to state our results, we need some notation. Recall that an R-parabolic subgroup of G is a subgroup of the form P_λ for a cocharacter λ of G (see Section 2). Let $\mathcal{P}$ be the poset of R-parabolic subgroups of G under inclusion. For a reductive subgroup K of G, write $\mathcal{P}_K$ for the set of R-parabolic subgroups P_λ with $\lambda \in Y(K)$. Then $\mathcal{P}_K$ is also a poset under inclusion; write $\mathcal{M}_K$ for its maximal elements. If $K = G$ is connected then $\mathcal{P}_K$ is the poset dual to the spherical building of G, and $\mathcal{M}_K$ is the set of maximal (proper) parabolic subgroups of G; in this case all members of $\mathcal{M}_K$ have the same rank (i.e. height in the poset $\mathcal{P}$). This need not hold in general, cf. Example 4.3.

Conjugation gives a natural action of G and its subgroups on $\mathcal{P}$. Recall that two R-parabolic subgroups are called *opposite* if their intersection is an R-Levi subgroup of G. It follows easily from the definitions that a subgroup H of G is relatively G-completely reducible with respect to K precisely when each member of $\mathcal{P}_K$ containing H has an opposite in $\mathcal{P}_K$ containing H. The following is our main result.

Theorem 1.2. *Let $K \leq G$ be reductive algebraic groups with G connected, and let H be a subgroup of G. Then the following are equivalent:*

(i) *H is relatively G-completely reducible with respect to K.*
(ii) *Each member of $\mathcal{M}_K$ containing H has an opposite in $\mathcal{M}_K$ containing H.*
(iii) *There is an R-Levi subgroup L_μ with $\mu \in Y(K)$, such that $H \leq L_\mu$ and H is relatively L_μ-irreducible with respect to $K \cap L_\mu$.*

Remarks 1.3.

(i) Conditions (i)–(iii) specialize to the representation-theoretic notions discussed above if we take $K = G = \mathrm{GL}(V)$.
(ii) A result of Serre [8, Thm. 2.2] states that when $K = G$ is a connected reductive group, a subgroup is G-completely reducible precisely when it lies in a Levi subgroup of each *maximal* parabolic subgroup containing it. This result is central to the study of complete reducibility via the spherical building of G, as outlined in *loc. cit.* The equivalence of (i) and (ii) in Theorem 1.2 can be recast as a relative analogue of this building-theoretical result, generalizing Serre's result.
(iii) In the case $G = \mathrm{GL}(V)$, Theorem 1.2 provides very concrete criteria for relative complete reducibility. The general form of these criteria is given in Corollary 1.5 below, and is illustrated with explicit examples in Corollary 1.7 (when $K = \mathrm{SO}(V)$ or $\mathrm{Sp}(V)$) and in Appendix A (when K is simple of type G_2).

The proof of Theorem 1.2 makes use of the fact that G is connected. However, the implications (i) $\Leftrightarrow$ (iii) $\Rightarrow$ (ii) all hold without this assumption. For the missing implication,

essentially the only issue is that in a non-connected group, not every R-parabolic subgroup in $\mathcal{P}_K$ need arise as the intersection of members of $\mathcal{M}_K$ (see Example 2.3). However, this does hold when G is connected (Lemma 2.2). Theorem 1.2 is therefore a consequence of our next result. For this, define the set

$$\mathcal{P}'_K := \left\{ P \in \mathcal{P}_K \mid P = \bigcap \{Q \in \mathcal{M}_K \mid P \leq Q\} \right\}.$$

Theorem 1.4. *Let $K \leq G$ be reductive algebraic groups, and let H be a subgroup of G. Then each member of $\mathcal{M}_K$ containing H has an opposite in $\mathcal{M}_K$ containing H if and only if each member of $\mathcal{P}'_K$ containing H has an opposite in $\mathcal{P}'_K$ containing H.*

As a corollary, the conclusion of Theorem 1.2 also holds for (possibly non-connected) reductive groups G such that $\mathcal{P}_K = \mathcal{P}'_K$.

In the particular case $G = \mathrm{GL}(V)$, Theorem 1.2 gives a representation-theoretic characterization of relative complete reducibility, as follows. Recall that a parabolic subgroup of $\mathrm{GL}(V)$ is the stabilizer of a flag of subspaces in V. The poset of flags in V is the dual of the poset of parabolic subgroups in $\mathrm{GL}(V)$, i.e., we set $f \preceq f'$ provided $\mathrm{Stab}_G(f) \supseteq \mathrm{Stab}_G(f')$. For K a reductive subgroup of $\mathrm{GL}(V)$, we denote by $\mathcal{F}_K$ the set of flags in V which stem from K, i.e., which correspond to parabolic subgroups P_λ for λ a cocharacter of K. A flag f in $\mathcal{F}_K$ is called *minimal in $\mathcal{F}_K$* provided $f' \preceq f$ for f' in $\mathcal{F}_K$ implies $f' = f$. Let $\mathcal{MF}_K \subseteq \mathcal{F}_K$ be the set of minimal flags in $\mathcal{F}_K$. Again, Example 4.3 shows that members of $\mathcal{MF}_K$ may have varying lengths. Of course, $\mathcal{MF}_{\mathrm{GL}(V)}$ is the set of flags of length 1 in V, corresponding to the set of maximal parabolic subgroups in $\mathrm{GL}(V)$.

Corollary 1.5. *Let H and K be closed subgroups of $\mathrm{GL}(V)$ with K reductive. Then the following are equivalent:*

(i) *H is relatively $\mathrm{GL}(V)$-completely reducible with respect to K.*
(ii) *Every H-stable flag in $\mathcal{MF}_K$ has an H-stable opposite in $\mathcal{MF}_K$.*
(iii) *There is a maximal torus S of $C_K(H)$ such that H preserves the direct-sum decomposition of V into simultaneous S-eigenspaces, and this decomposition gives a flag which is maximal among H-stable flags in $\mathcal{F}_K$.*

When $K = G = \mathrm{GL}(V)$, the possibilities for the torus S in (iii) are products of the centres of the $\mathrm{GL}(V_i)$ corresponding to a direct-sum decomposition of V into irreducible modules $V = V_1 \oplus \ldots \oplus V_r$ $(r \geq 1)$; this gives the usual representation-theoretic characterization.

The next result more closely mirrors the representation-theoretic statement that V is a completely reducible H-module if and only if every H-submodule has a complement. For this, however, we require an additional hypothesis. Assume once more that $G = \mathrm{GL}(V)$, and write $\mathcal{S}_K$ for the set of subspaces of V which arise in flags from $\mathcal{F}_K$.

Corollary 1.6. *Let H, K be subgroups of $\mathrm{GL}(V)$ with K reductive. Suppose $\mathcal{MF}_K \subseteq \mathcal{MF}_{\mathrm{GL}(V)}$. Then the following are equivalent:*

(i) *H is relatively $\mathrm{GL}(V)$-completely reducible with respect to K.*
(ii) *For each $U \in \mathcal{S}_K$ which is stabilized by H there exists $W \in \mathcal{S}_K$ such that H stabilizes W and $V = U \oplus W$, as an H-module.*

Corollary 1.6 readily follows from Corollary 1.5 and Corollary 4.2. Note that the implication (i) $\Rightarrow$ (ii) can fail without the hypothesis on $\mathcal{MF}_K$, see Example 4.3. The essential problem is that an H-stable member of $\mathcal{S}_K$ need not arise from an H-stable flag in $\mathcal{F}_K$.

Corollary 1.6 may be viewed as a generalization of [4, Prop. 5.1]. The latter gives a representation-theoretic characterization of relative $\mathrm{GL}(V)$-complete reducibility in the case that $K = \mathrm{GL}(U)$ for a subspace U of V, which is closely related to the condition in Corollary 1.6, see Lemma 4.5 and Corollary 4.2.

Particularly natural candidates for the subgroup K in $\mathrm{GL}(V)$ are the classical groups $\mathrm{SO}(V)$ and $\mathrm{Sp}(V)$. This is the theme of our next result, which gives a characterization of relative $\mathrm{GL}(V)$-complete reducibility with respect to $\mathrm{SO}(V)$ or $\mathrm{Sp}(V)$ in terms of totally singular or totally isotropic subspaces.

Corollary 1.7. *Let H be a subgroup of $\mathrm{GL}(V)$ and let $K = \mathrm{Sp}(V)$ (resp. $\mathrm{SO}(V)$). Then the following are equivalent:*

(i) *H is relatively $\mathrm{GL}(V)$-completely reducible with respect to K.*
(ii) *Whenever H stabilizes a totally isotropic (resp. totally singular) subspace U and its annihilator $U^\perp$, there exists a totally isotropic (resp. totally singular) subspace $W \subseteq V$ such that H stabilizes W and $W^\perp$, and $V = W \oplus U^\perp = U \oplus W^\perp$ as H-modules.*

In the setting of Corollary 1.7, flags in $\mathcal{F}_K$ have the form

$$U_1 \subseteq \ldots \subseteq U_r \subseteq U_r^\perp \subseteq \ldots \subseteq U_1^\perp \subseteq V,$$

so that minimal flags in $\mathcal{F}_K$ are of the form $U \subseteq U^\perp \subseteq V$ for U a totally isotropic (resp. totally singular) subspace, hence the result is immediate from Corollary 1.5.

Note that we cannot relax the requirement in Corollary 1.7(ii) that H stabilizes $U^\perp$ and $W^\perp$, since H does not need to leave the form on V invariant (i.e. H need not be a subgroup of K).

In Section 5, we present a brief investigation of the notion of relative G-complete reducibility over an arbitrary field, obtaining rational versions of Theorems 1.2 and 1.4. In Appendix A we give an illustrative example of Theorem 1.2, with K simple of exceptional type and $G = \mathrm{GL}(V)$ for a K-module V.

2. Preliminaries

We work over a field k, which is taken to be algebraically closed except in Section 5. Let G be a reductive algebraic group defined over k – we allow the possibility that G is not connected. Let H be a closed subgroup of G. We write H° for the identity component of H.

For the set of cocharacters (one-parameter subgroups) of G we write $Y(G)$.

Suppose G acts on a variety X and let $x \in X$. Then for each cocharacter $\lambda \in Y(G)$ we define a morphism of varieties $\phi_{x,\lambda} : k^* \to X$ via $\phi_{x,\lambda}(a) = \lambda(a) \cdot x$. If this morphism extends to a morphism $\overline{\phi}_{x,\lambda} : k \to X$, then we say that the limit $\lim_{a\to 0} \lambda(a)\cdot x$ exists and set this limit equal $\overline{\phi}_{x,\lambda}(0)$. For each cocharacter $\lambda \in Y(G)$, let $P_\lambda = \{g \in G \mid \lim_{a\to 0} \lambda(a)g\lambda(a)^{-1} \text{ exists}\}$ and $L_\lambda = \{g \in G \mid \lim_{a\to 0} \lambda(a)g\lambda(a)^{-1} = g\}$. Following [3, §6], we call P_λ an *R-parabolic*

subgroup of G and L_λ an *R-Levi subgroup* of G. As mentioned previously, if G is connected then these R-parabolic subgroups and their R-Levi subgroups are precisely the parabolic subgroups and their Levi subgroups. If K is a reductive subgroup of G and $\lambda \in Y(K)$, we always denote by P_λ the R-parabolic subgroup of G attached to λ; if we need to consider the corresponding R-parabolic subgroup of K we write $P_\lambda(K)$ (and similarly for R-Levi subgroups). We recall the following central notions from [4].

Definition 2.1. Let H and K be subgroups of G with K reductive. We say that H *is relatively G-completely reducible with respect to K* if, for every $\lambda \in Y(K)$ such that H is contained in P_λ, there exists $\mu \in Y(K)$ such that $P_\lambda = P_\mu$ and H is contained in L_μ. We sometimes use the abbreviations *relatively G-cr with respect to K*. We say that H *is relatively G-irreducible with respect to K* if H is not contained in any subgroup P_λ with $\lambda \in Y(K)$.

Note that H is relatively G-completely reducible with respect to K if and only if H is relatively G-completely reducible with respect to K°, and similarly for relative G-irreducibility. In the case when $K = G$, Definition 2.1 coincides with the usual definitions, cf. [3].

Lemma 2.2. *Let $K \leq G$ be reductive groups with G connected, and let $\mathcal{P}_K$, $\mathcal{M}_K$ and $\mathcal{P}'_K$ be as in the introduction.*

(i) *Let $P \in \mathcal{P}_K$ and let Q be a maximal R-parabolic subgroup of G containing P. Then there exists $P' \in \mathcal{M}_K$ such that $P \leq P' \leq Q$.*
(ii) *Let $P_\lambda \in \mathcal{P}_K$ with $\lambda(k^*) \leq S$, a fixed maximal torus of K. Then $P = \bigcap_{i=1}^m P_{\lambda_i}$ for some $\lambda_i \in Y(S)$ such that $P_{\lambda_i} \in \mathcal{M}_K$ for all i. In particular, $\mathcal{P}_K = \mathcal{P}'_K$.*

Proof. (i) Note that maximal chains of R-parabolic subgroups in the posets $\mathcal{P}$ and $\mathcal{P}_K$ are finite; we argue by induction on the height of P in $\mathcal{P}_K$. If $P \in \mathcal{M}_K$ then there is nothing to prove. Now suppose that P is not maximal in $\mathcal{P}_K$. Then there exist λ, $\nu \in Y(K)$ such that $P = P_\lambda \leq P_\nu$ and $P_\nu \in \mathcal{M}_K$. If $P_\nu \leq Q$ then we are done. So we assume that $P_\nu \not\leq Q$.

Let $\mu \in Y(G)$ such that $Q = P_\mu$. There is a maximal torus T of G in P_λ such that $T \cap K$ is a maximal torus of K and such that $T \leq P_\lambda \leq P_\nu$. Let $T \leq B \leq P_\lambda$ be a Borel subgroup of G.

Let Φ be the set of roots of G with respect to T and let Φ^+ be the set of positive roots of G with respect to B, with corresponding set of simple roots Δ. For $I \subseteq \Delta$, we denote by P_I the standard parabolic subgroup of G generated by T and the root groups corresponding to roots in $-I \cup \Phi^+$. By the proof of [9, Prop. 8.4.5], setting $I_\xi := \{\alpha \in \Delta \mid \langle \alpha, \xi \rangle = 0\}$ for $\xi \in Y(T)$, where $\langle \alpha, \xi \rangle$ denotes the standard pairing between characters and cocharacters of T, if P_ξ contains B then $P_{I_\xi} = P_\xi$. Because of this, we have $I_\lambda \subseteq I_\nu$ and $I_\lambda \subsetneq I_\mu$. Note also that since P_μ is a maximal parabolic, we have $\Delta \setminus I_\mu = \{\alpha_0\}$, a single simple root. Since P_ν is not contained in P_μ, we have $\alpha_0 \in I_\nu$. Hence $\Delta = I_\mu \cup I_\nu$.

Given any $\alpha \in I_\mu \setminus I_\lambda$, we have $\langle \alpha, \lambda \rangle > 0$. Further, since I_ν is not contained in I_μ, there is at least one choice of $\alpha \in I_\mu \setminus I_\lambda$ with $\langle \alpha, \nu \rangle > 0$ as well. Let $n_1, n_2 \in \mathbb{N}$ be such that $\frac{n_1}{n_2} \in \mathbb{Q}_{>0}$ is the maximum of all ratios $\frac{\langle \alpha, \nu \rangle}{\langle \alpha, \lambda \rangle}$ over all $\alpha \in I_\mu \setminus I_\lambda$, and set $\chi := n_1\lambda - n_2\nu$. Note that $\chi \in Y(K)$ and thus $P_\chi \in \mathcal{P}_K$. We show in the following paragraph that P_χ is a standard parabolic subgroup of G properly containing P_λ and contained in P_μ. This is enough to complete the proof, by induction.

To see that P_χ is standard, let $\beta \in \Delta$. Then $\beta \in I_\mu$ or $\beta \in I_\nu \setminus I_\mu$, because $\Delta = I_\mu \cup I_\nu$, as observed above. In the first case, we have $\langle \beta, \chi \rangle \geq 0$, by the choice of $\frac{n_1}{n_2}$ above. In the second case, $\langle \beta, \nu \rangle = 0$, so we have $\langle \beta, \chi \rangle = n_1 \langle \beta, \lambda \rangle \geq 0$. Hence $\langle \beta, \chi \rangle \geq 0$ for all $\beta \in \Delta$, which shows that P_χ is standard. Now, to see that P_λ is properly contained in P_χ it suffices to check that I_λ is properly contained in I_χ. The containment is clear because $I_\lambda \subseteq I_\nu$ and χ is a combination of λ and ν. Moreover, if we let $\alpha \in I_\mu \setminus I_\lambda$ be such that the maximum value $\frac{n_1}{n_2}$ above is attained, then $\langle \alpha, \lambda \rangle > 0$, but $\langle \alpha, \chi \rangle = 0$, so the containment is proper. Finally, to see that $P_\chi \subseteq P_\mu$, we only need to see that $\langle \chi, \alpha_0 \rangle > 0$, where $\alpha_0 \in \Delta$ is the unique simple root outside I_μ, as above. But we have argued that $\alpha_0 \in I_\nu$, and $\alpha_0 \notin I_\lambda$ since P_λ is proper in P_μ. Hence $\langle \alpha_0, \chi \rangle = n_1 \langle \alpha_0, \lambda \rangle > 0$, and we are done.

Thus, by induction there exists $P' \in \mathcal{M}_K$ such that $P_\chi \leq P' \leq P_\mu = Q$, which completes the proof of (i).

(ii) Since G is connected, the R-parabolic subgroup P_λ is equal to the intersection of all maximal R-parabolic subgroups of G which contain it. By (i), for each such maximal R-parabolic subgroup Q_i $(i = 1, \ldots, m)$ we can find an R-parabolic subgroup $P_i \in \mathcal{M}_K$ with $P_\lambda \leq P_i \leq Q_i$. Then $P_\lambda = \bigcap_{i=1}^m P_i$. Each subgroup P_i has the form P_{λ_i} for some cocharacter λ_i of K. Now $S \leq P_\lambda \leq P_{\lambda_i}$ for each i, so S is a maximal torus of $P_{\lambda_i}(K)$. Since $\lambda_i(k^*)$ is a sub-torus of $P_{\lambda_i}(K)$, it is $P_{\lambda_i}(K)$-conjugate to a subgroup of S. So we can replace λ_i by an appropriate P_{λ_i}-conjugate so that $\lambda_i \in Y(S)$ for each i, as required. □

The following example shows that Lemma 2.2(ii) can fail when G is not connected.

Example 2.3. Let T be a 1-dimensional torus, let $\langle x \rangle$ be cyclic of order 8 and let $G = K = T^8 \rtimes \langle x \rangle$, with x permuting the factors of T^8 in the obvious manner. Then each R-parabolic subgroup of G contains $T^8 = G^\circ$, and every subgroup $G^\circ \leq P \leq G$ arises as an R-parabolic subgroup P_λ, depending on whether the 1-dimensional torus $\lambda(k^*)$ is centralized by x, x^2, x^4 or none of these. Then $T^8 \rtimes \langle x^2 \rangle$ is the unique maximal proper R-parabolic subgroup of G, and in particular its subgroup $T^8 \rtimes \langle x^4 \rangle$ is not the intersection of the maximal R-parabolic subgroups of G containing it.

Lemma 2.4. *Let G be reductive, and let λ_i $(i = 1, \ldots, m)$ be pairwise commuting cocharacters of G such that there exists a Borel subgroup B of G with $B \subseteq P_{\lambda_i}$ for each i. Then there exist postive integers n_i $(i = 1, \ldots, m)$ such that $P_\lambda = \bigcap_{i=1}^m P_{\lambda_i}$, where $\lambda = \sum_{i=1}^m n_i \lambda_i$.*

Proof. If G is connected, this follows quickly from the well-understood theory of standard parabolic subgroups (parabolic subgroups containing a fixed Borel subgroup); the subgroups P_{λ_i} correspond to choosing subsets of simple roots of G, and P_λ corresponds to choosing the union of these sets (independently of the choice of positive integers n_i). For general G, by [3, Lem. 6.2(iii)] it suffices to show that $R_u(P_\lambda) = R_u\left(\bigcap_{i=1}^m P_{\lambda_i}\right)$ and $L_\lambda = \left(\bigcap_{i=1}^m L_{\lambda_i}\right)$ for some choice of integers n_i. Moreover, it suffices to treat the case $m = 2$, and the general case then follows by an easy induction. Now, since $R_u(P) = R_u(P^\circ) = R_u(P \cap G^\circ)$ for every R-parabolic subgroup P, the equality $R_u(P_\lambda) = R_u\left(P_{\lambda_1} \cap P_{\lambda_2}\right)$ follows from the corresponding result for connected G and [3, Lem. 6.2(iii)]. Finally [3, Lem. 6.2(i)] tells us that the equality $L_\lambda = L_{\lambda_1} \cap L_{\lambda_2}$ holds for sufficiently large n_1. □

The following example shows that the positive integers n_i cannot be chosen arbitrarily when G is not connected. We thank Dr. Tomohiro Uchiyama for pointing this out.

Example 2.5. Let $G = \mathrm{SL}_3(k)\langle\sigma\rangle$, where σ is the inverse-transpose automorphism composed with conjugation by $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Then σ normalises the diagonal maximal torus T and the upper triangular Borel subgroup B, swapping the root groups U_α and U_β corresponding to the $(1,2)$ and $(2,3)$ matrix coordinates. Let $\lambda(c) = \mathrm{diag}(c, c, c^{-2})$, $\mu(c) = \mathrm{diag}(c^2, c^{-1}, c^{-1})$ for $c \in k^*$, so that $\lambda, \mu \in Y(T)$ with $P_\lambda^\circ = \langle T, U_{\pm\alpha}, U_\beta\rangle$ and $P_\mu^\circ = \langle T, U_{\pm\beta}, U_\alpha\rangle$. It is evident that σ does not normalise P_λ° or P_μ°, hence is not contained in P_λ or in P_μ. On the other hand, it is easily checked that σ centralises $(\lambda+\mu)(k^*)$; in particular $P_{\lambda+\mu}$ contains σ hence is strictly larger than $P_\lambda \cap P_\mu$. On the other hand, for any choice of *distinct* positive integers n_1, n_2, it is the case that $P_{n_1\lambda+n_2\mu} = P_\lambda \cap P_\mu = B$.

We require the following useful result, which is [4, Lem. 3.3].

Lemma 2.6. *Let $K \leq G$ be reductive groups.*

(i) *Let $\lambda, \mu \in Y(K)$ such that $P_\lambda = P_\mu$ and $u \in R_u(P_\lambda(K))$ such that $uL_\lambda(K)u^{-1} = L_\mu(K)$. Then $uL_\lambda u^{-1} = L_\mu$.*

(ii) *Let H be a subgroup of G. Then H is relatively G-completely reducible with respect to K if and only if for every $\lambda \in Y(K)$ such that $H \subseteq P_\lambda$ there exists $u \in R_u(P_\lambda(K))$ such that $H \subseteq L_{u\cdot\lambda}$.*

The following is the final ingredient needed in the proof of Theorems 1.2 and 1.4.

Lemma 2.7. *Let $K \leq G$ be reductive groups. If $P \in \mathcal{M}_K$ (resp. $P \in \mathcal{P}'_K$) and $Q \in \mathcal{P}_K$ is opposite to P, then $Q \in \mathcal{M}_K$ (resp. $Q \in \mathcal{P}'_K$).*

Proof. We prove firstly that if $P \in \mathcal{M}_K$, and if $Q \in \mathcal{P}_K$ is opposite to P, then $Q \in \mathcal{M}_K$. Suppose not. Then there exists $Q' \in \mathcal{M}_K$ such that $Q \lneq Q'$. Let T be a maximal torus of K contained in Q and Q'. Then there exits $\lambda \in Y(T)$ such that $P = P_\lambda$ and $Q = P_{-\lambda}$. Since $T \leq P_{-\lambda} < Q'$, there exists $\mu \in Y(T)$ such that $Q' = P_\mu$. Let $P' = P_{-\mu} \in \mathcal{P}_K$. Then $H \leq P_\lambda < P_{-\mu}$, contradicting $P \in \mathcal{M}_K$.

Now suppose $P \in \mathcal{P}'_K$. Since G is noetherian, we can write P as a finite intersection $P = \bigcap_{i=1}^m P_i$ with $P_i \in \mathcal{M}_K$ for each i. Let $Q \in \mathcal{P}_K$ be opposite to P, and let T be a maximal torus of K contained in $P \cap Q$. Then for each i we have $P_i = P_{\lambda_i}$ for some $\lambda_i \in Y(T)$, and by Lemma 2.4 we have $P = P_\lambda$ where $\lambda = \sum_{i=1}^m n_i\lambda_i$ for some positive integers n_i. By Lemma 2.6 and the uniqueness of opposite R-parabolic subgroups containing a given R-Levi subgroup [3, Lem. 6.11], we can conjugate λ (and each λ_i) by some fixed element of $R_u(P_\lambda(K))$ such that $Q = P_{-\lambda} = \bigcap_{i=1}^m P_{-\lambda_i}$. By the paragraph above, each subgroup $P_{-\lambda_i} \in \mathcal{M}_K$ for each i, hence $Q \in \mathcal{P}'_K$. □

3. Proof of Theorems 1.2 and 1.4

We now prove Theorems 1.2 and 1.4. We begin with the proof of Theorem 1.4, from which Theorem 1.2 follows in short order.

Recall that G is a (not necessarily connected) reductive algebraic group and H and K are subgroups of G, with K also reductive (but not necessarily connected). Also $\mathcal{P}_K$ is the set of R-parabolic subgroups of G of the form P_λ with $\lambda \in Y(K)$, $\mathcal{M}_K$ is the set of maximal members of $\mathcal{P}_K$ under inclusion, and $\mathcal{P}'_K$ is the set of members of $\mathcal{P}_K$ which can be realized as intersections of members of $\mathcal{M}_K$.

Proof of Theorem 1.4. Suppose that each member of $\mathcal{P}'_K$ which contains H admits an opposite in $\mathcal{P}'_K$ containing H. Since $\mathcal{M}_K \subseteq \mathcal{P}'_K$ by definition, if $P \in \mathcal{M}_K$ and $P \geq H$ then $P \in \mathcal{P}'_K$, hence admits an opposite $Q \in \mathcal{P}'_K$ which contains H. By Lemma 2.7 we have $Q \in \mathcal{M}_K$, as required.

Conversely, suppose that each member of $\mathcal{M}_K$ containing H admits an opposite in $\mathcal{M}_K$ which contains H, and let $P \in \mathcal{P}'_K$ such that $P \geq H$. By definition of $\mathcal{P}'_K$ we can write $P = \bigcap_{i=1}^m P_i$ for some $m \in \mathbb{N}$ and $P_i \in \mathcal{M}_K$ for each i. Taking m to be minimal, we proceed by induction on m, the case $m = 1$ being our starting hypothesis.

By Lemma 2.6 we can fix a maximal torus T of K contained in P such that $P = P_\lambda$ and $P_i = P_{\lambda_i}$ for some cocharacters λ and $\lambda_i \in Y(T)$ for each i. Write this intersection as $P_\lambda = P_{\lambda_1} \cap \bigcap_{i=2}^m P_{\lambda_i}$. By Lemma 2.4 we are free to write $\bigcap_{i=2}^m P_{\lambda_i} = P_\nu$ where $\nu = n_2\lambda_2 + \ldots + n_m\lambda_m$ for some positive integers n_i, and we are also free to replace λ (without changing P_λ) such that $\lambda = n_1\lambda_1 + n_\nu\nu$ for some positive integers n_1, n_ν.

Since $P_{\lambda_1} \in \mathcal{M}_K$, by hypothesis H lies in some R-Levi subgroup of P_{λ_1}. Since $P_\lambda \leq P_{\lambda_1}$ we have $R_u(P_{\lambda_1}(K)) \leq R_u(P_\lambda(K))$. Thus we are free to replace H by an $R_u(P_{\lambda_1}(K))$-conjugate so that $H \leq L_{\lambda_1}$, and this does not change whether H lies in an R-Levi subgroup of P_λ corresponding to a cocharacter of K. Moreover, replacing H with such a conjugate, we still have $H \leq P_\lambda \leq P_\nu$. Now since $H \leq L_{\lambda_1}$, we have $\lambda_1 \in Y(C_K(H) \cap L_\nu(K)) = Y(C_{L_\nu(K)}(H))$.

Now, we can apply the induction hypothesis to P_ν, so that there exists $\sigma \in Y(K)$ such that $P_\sigma \in \mathcal{P}'_K$ is opposite to P_ν and $H \leq L_\sigma$. Then there exists $u \in R_u(P_\nu(K))$ such that $u \cdot \nu \in Y(L_\sigma(K))$. So we set $\tau = -(u \cdot \nu)$ as cocharacters of some maximal torus of $L_\sigma(K)$. By the uniqueness of opposite parabolic subgroups containing a fixed maximal torus [3, Lem. 6.11] we have $P_\tau = P_\sigma$, $L_\tau = L_\sigma$. Since τ, $\lambda_1 \in Y(C_{P_\nu(K)}(H))$, τ is $C_{P_\nu(K)}(H)$-conjugate to a cocharacter ρ which commutes with λ_1. Then $P_\rho \cap P_\nu = P_\tau \cap P_\nu$ so P_ρ is still opposite to P_ν, and $H \leq L_\rho$. Now ρ and ν both commute with λ_1, and so the images of these cocharacters are all contained in some maximal torus of K, call it S. Since $\lambda = n_1\lambda_1 + n_\nu\nu$, the image of λ also lies in S.

Now, again by the uniqueness of opposite R-parabolic subgroups among those containing a given maximal torus of K, it follows that we can scale ρ so that $\rho = -n_\nu\nu$ and thus $-\lambda = -n_1\lambda_1 - n_\nu\nu = -n_1\lambda_1 + \rho$ as elements of $Y(S)$. This shows that $P_{-\lambda} = P_{-\lambda_1} \cap P_\rho$ is opposite to P_λ, lies in $\mathcal{P}'_K$ and contains H, as required. □

Proof of Theorem 1.2. The equivalence of the conditions (i) and (iii) follows immediately from [4, Prop. 3.17(ii)]. Now suppose (i) holds and let $P \in \mathcal{M}_K$ with $P \geq H$. By hypothesis, there exists an R-parabolic subgroup $Q \in \mathcal{P}_K$ which is opposite to P and contains H, and then $Q \in \mathcal{M}_K$ by Lemma 2.7, so condition (ii) holds. Note that we have not yet used the hypothesis that G is connected.

Finally, suppose that (ii) holds, so that each $P \in \mathcal{M}_K$ which contains H admits an opposite in $\mathcal{M}_K$ which contains H. Since G is connected by hypothesis, we have $\mathcal{P}_K = \mathcal{P}'_K$ by Lemma 2.2(ii). Then Theorem 1.4 tells us that each $P \in \mathcal{P}_K$ containing H has an opposite in $\mathcal{P}_K$ containing H, hence condition (i) holds. □

4. The case $G = \mathrm{GL}(V)$

We take this opportunity to note some interesting special cases of our results in the case $G = \mathrm{GL}(V)$. The first follows immediately from Lemma 2.2(i). Recall that $\mathcal{F}_K$ denotes the set of flags arising from parabolic subgroups corresponding to cocharacters of K, and $\mathcal{MF}_K$ denotes the minimal members of $\mathcal{F}_K$ (i.e. those whose stabilizers lie in $\mathcal{M}_K$).

Corollary 4.1. *Let $G = \mathrm{GL}(V)$, let $K \leq G$ be a reductive subgroup, let f be a flag in $\mathcal{F}_K$ and let U be a subspace in f. Then there is a flag f' in $\mathcal{MF}_K$ such that $f' \preceq f$ and U appears in f'.*

Recall that $\mathcal{S}_K$ is the set of subspaces of V which appear in flags from $\mathcal{F}_K$. The following is an immediate consequence of Corollary 4.1.

Corollary 4.2. *Let K be a reductive subgroup of $\mathrm{GL}(V)$. Then the following are equivalent:*

(i) $\mathcal{S}_K = \{U \subseteq V \mid (U \subseteq V) \in \mathcal{F}_K\}$.
(ii) $\mathcal{MF}_K \subseteq \mathcal{MF}_{\mathrm{GL}(V)}$.

Next, we note that the implication (i) ⇒ (ii) of Corollary 1.6 fails without the hypothesis on $\mathcal{MF}_K$, as the following example illustrates.

Example 4.3. Let $G = \mathrm{GL}_4(k)$ and let K be the subgroup of diagonal matrices of the form $\mathrm{diag}(t, s, s^{-1}, t^{-1})$ with $s, t \in k^*$. Let $e_1, \ldots, e_4$ be the standard basis of k^4 and $U = \langle e_1, e_2, e_3 \rangle$. Suppose that H is the parabolic subgroup of G corresponding to the flag $U \subseteq V$. Since the flags from $\mathcal{F}_K$ have subspaces of dimension $(2, 4)$, $(1, 3, 4)$ and $(1, 2, 3, 4)$, the group H is not contained in P_λ for any $\lambda \in Y(K) \setminus \{1\}$. Hence trivially, H is relatively G-cr with respect to K. Note that $U \in \mathcal{S}_K$ and H stabilizes U. One checks that the complement to U in the set $\mathcal{S}_K$ is $W = \langle e_4 \rangle$. But H does not stabilize W.

Remark 4.4. Let $G = \mathrm{GL}(V)$. If $\mathcal{MF}_K \not\subseteq \mathcal{MF}_G$ then there exists a subgroup H of G such that H is relatively G-cr with respect to K and H stabilizes a subspace $U' \in \mathcal{S}_K$ but does not stabilize any complement to U'. To see this, note that since $\mathcal{S}_K \neq \{U \subseteq V \mid (U \subseteq V) \in \mathcal{F}_K\}$, by Corollary 4.2, there exists a U' in $\mathcal{S}_K$ such that $(U' \subseteq V) \notin \mathcal{F}_K$. Set $H := \mathrm{Stab}_G(U' \subseteq V)$. Then H is not contained in P_λ for any $\lambda \in Y(K) \setminus \{1\}$. Trivially, H is relatively G-cr with respect to K. Note that H stabilizes U' in $\mathcal{S}_K$ but does not stabilize any complement to U', since H is a maximal parabolic subgroup of G.

For a fixed subspace $U \subseteq V$, we show in the following lemma that $K = \mathrm{GL}(U)$ satisfies the condition in Corollary 4.2(i). A maximal torus of G also satisfies the condition. So Corollary 1.6 applies in these instances, thanks to Corollary 4.2.

Lemma 4.5. *Let $G = \mathrm{GL}(V)$ and let $U \subseteq V$. Fix a complement $\widetilde{U}$ to U in V. Let $K = \mathrm{GL}(U) \leq G$, embedded via the decomposition $V = U \oplus \widetilde{U}$. Then $\mathcal{S}_K = \{W \subseteq V \mid (W \subseteq V) \in \mathcal{F}_K\}$.*

Proof. Let $(W_1 \subseteq \ldots \subseteq W_m \subseteq V)$ be in $\mathcal{F}_K$. One sees by inspection that for each $1 \leq i \leq m$ we have $W_i \subseteq U$ or $\widetilde{U} \subseteq W_i$. On the other hand, suppose that W is a subspace contained in U. Then we can find a complement W' to W containing $\widetilde{U}$ and the cocharacter which acts with weight 1 on W and weight 0 on W' lies in $Y(K)$ and affords the flag $(W \subseteq V) \in \mathcal{F}_K$. Similarly, all flags $(W \subseteq V)$ with $\widetilde{U} \subseteq W$ are in $\mathcal{F}_K$. Hence

$$\mathcal{F}_K = \{(W_1 \subseteq \ldots \subseteq W_m \subseteq V) \in \mathcal{F}_G \mid W_i \subseteq U \text{ or } \widetilde{U} \subseteq W_i \text{ for } 1 \leq i \leq m, \text{ for some } m\},$$

and so

$$\mathcal{S}_K = \{W' \subseteq V \mid W' \subseteq U \text{ or } \widetilde{U} \subseteq W'\} = \{W \subseteq V \mid (W \subseteq V) \in \mathcal{F}_K\}, \tag{4.6}$$

as claimed. □

In view of (4.6) and Corollary 4.2, Corollary 1.6 and Lemma 4.5 imply [4, Prop. 5.1]. So Corollary 1.6 may be viewed as a generalization of the special case treated in [4, Prop. 5.1].

Corollary 1.7 considers situations when K acts irreducibly on V. We close this section with a characterization of relative $\mathrm{GL}(V)$-complete reducibility in case V decomposes as a direct sum of K-modules. We first note a consequence of Lemma 2.6 which characterizes relative G-complete reducibility when G and K admit compatible direct-product structures.

Corollary 4.7. *For $i = 1, 2$, let $K_i \subseteq G_i$ be reductive groups, $G := G_1 \times G_2$ and $K := K_1 \times K_2$. Let $H \subseteq G$ be a subgroup. Then H is relatively G-completely reducible with respect to K if and only if H is relatively G-completely reducible with respect to K_i for $i = 1, 2$.*

Proof. Let $\lambda \in Y(K_1)$ such that $H \leq P_\lambda$. By the proof of [3, Lem. 2.12], the parabolic subgroups of G arising from cocharacters of K have the form $P_{\lambda_1} \times P_{\lambda_2}$ with $\lambda_i \in Y(K_i)$ for $i = 1, 2$, since $G = G_1 \times G_2$ and $K = K_1 \times K_2$. Hence $P_\lambda = P_\lambda(G_1) \times G_2$. Since H is relatively G-completely reducible with respect to K, there exists a $u = (u_1, u_2) \in R_u(P_\lambda(K))$ such that $H \leq L_{u \cdot \lambda} = L_{u_1 \cdot \lambda}(G_1) \times G_2$, by Lemma 2.6(ii). Therefore, $u_1 \in R_u(P_\lambda(K_1))$. It follows that H is relatively G-cr with respect to K_1, by Lemma 2.6(ii). The proof for K_2 is analogous.

For the reverse implication let $\lambda = (\lambda_1, \lambda_2) \in Y(K) = Y(K_1) \times Y(K_2)$ such that $H \leq P_{\lambda_1} \times P_{\lambda_2}$. Since H is relatively G-completely reducible with respect to K_i for $i = 1, 2$, there exits a $u_i \in R_u(P_{\lambda_i}(K_i))$ such that $H \leq L_{u_1 \cdot \lambda_1}(G_1) \times G_2$ resp. $H \leq G_1 \times L_{u_2 \cdot \lambda_2}(G_2)$. Therefore, we obtain

$$H \leq (L_{u_1 \cdot \lambda_1}(G_1) \times G_2) \cap (G_1 \times L_{u_2 \cdot \lambda_2}(G_2)) = L_{u \cdot \lambda}$$

for $u = (u_1, u_2) \in R_u(P_\lambda(K))$. Once again, by Lemma 2.6(ii), H is relatively G-cr with respect to K. □

The following result is now immediate from Corollary 4.7 and [4, Cor. 3.6].

Corollary 4.8. *Let $G = \mathrm{GL}(V)$ and suppose that both H and K preserve a direct-sum decomposition $V = \bigoplus_{i=1}^n V_i$. Suppose also that $K = K_1 \times \cdots \times K_n$ where $K_i \leq \mathrm{GL}(V_i)$ for each i. Then H is relatively G-completely reducible with respect to K if and only if H is relatively G-completely reducible with respect to K_i for all i.*

Both implications in Corollary 4.8 fail in general, as illustrated by our next example.

Example 4.9. Let $G = \mathrm{GL}(k^4)$, $K = \{\mathrm{diag}(t, s, s^{-1}, t^{-1}) \mid t, s \in k^*\}$, and $\{e_1, e_2, e_3, e_4\}$ is the canonical basis for k^4. Set $V_1 = \langle e_1, e_2\rangle$ and $V_2 = \langle e_3, e_4\rangle$. Let K_i be the image of the projection from K to $\mathrm{GL}(V_i)$ for $i = 1, 2$.

Let H be the stabilizer of $U := \langle e_2, e_4\rangle$ in G and note that $(U \subseteq k^4)$ belongs to $\mathcal{F}_K$. Thus H is a maximal parabolic subgroup of G and corresponds to a cocharacter of K, and as such it is not relatively G-cr with respect to K. However, H does not correspond to a cocharacter of K_i, and by maximality H is not contained in any parabolic subgroup of G correspond to a cocharacter of K_i $(i = 1, 2)$. Hence H is relatively G-irreducible with respect to K_i so it is relatively G-cr with respect to K_i, for $i = 1, 2$.

Now let $\widetilde{H}$ be the stabilizer of $\widetilde{U} := \langle e_1\rangle$ in G. Note that $\widetilde{H}$ is a maximal parabolic subgroup of G and corresponds to a cocharacter of K_1, thus it is not relatively G-cr with respect to K_1. However, since $\widetilde{H}$ does not correspond to a cocharacter of K, it is relatively G-irreducible with respect to K, in particular it is relatively G-cr with respect to K.

5. Rationality Questions

In this section k denotes an arbitrary field, G is a reductive k-defined group and K is a reductive k-defined subgroup of G. For a k-defined closed subgroup M of G, write $Y_k(M)$ for the k-defined cocharacters of M, and let $M(k)$ denote the group of k-points of M. First, we recall the definition of relative G-complete reducibility over k from [4, Def. 4.1], and also define the analogue of relative G-irreducibility over k.

Definition 5.1. Let H be a subgroup of G. We say that H *is relatively G-completely reducible over k with respect to K* if for every $\lambda \in Y(K)$ such that P_λ is k-defined and H is contained in P_λ, there exists $\mu \in Y(K)$ such that $P_\lambda = P_\mu$, H is contained in L_μ and L_μ is k-defined. We also say that H *is relatively G-irreducible over k with respect to K* if H is not contained in any k-defined parabolic subgroup P_λ with $\lambda \in Y(K)$.

Remark 5.2. By [4, Lem. 4.8], a subgroup is relatively G-cr over k with respect to K if and only if for every $\lambda \in Y_k(K)$ such that $H \leq P_\lambda$, there exists $\mu \in Y_k(K)$ such that $P_\lambda = P_\mu$ and $H \leq L_\mu$. By identical arguments to those in the proof of [4, Lem. 4.8], a subgroup is relatively G-irreducible over k with respect to K if and only if it is not contained in any R-parabolic subgroup P_λ with $\lambda \in Y_k(K)$.

Analogous to Theorem 1.1, we have a geometric characterization of relative G-complete reducibility over k. We recall some definitions [2, Def. 1.1], [5, Def. 5.4].

Definition 5.3. Let G be reductive and k-defined.

(i) Let G act k-morphically on an affine k-variety X, and let $x \in X$. The orbit $G(k) \cdot x$ is called *cocharacter-closed over k* if for all $\lambda \in Y_k(G)$ such that $\lim_{a\to 0} \lambda(a) \cdot x$ exists, then this limit lies in the $G(k)$-orbit $G(k) \cdot x$.

(ii) A *generic tuple* for a subgroup H of G is an n-tuple $(h_1, \ldots, h_n) \in G^n$ such that H and $\{h_1, \ldots, h_n\}$ generate the same associative subalgebra of $\mathrm{Mat}_{m\times m}(\overline{k})$, for some embedding $G \to \mathrm{GL}_m(\overline{k})$.

Note that a generic tuple for H always exists since $\mathrm{Mat}_{m\times m}(\overline{k})$ is a finite-dimensional $\overline{k}$-algebra. The following summarizes part of [4, Thm. 4.12(iii)].

Theorem 5.4. *Let K be a reductive subgroup of G, let $H \leq G$ and let $\mathbf{h} \in G^n$ be a generic tuple for H. Then H is relatively G-completely reducible over k with respect to K if and only if $K(k)\cdot\mathbf{h}$ is cocharacter-closed over k.*

We now generalize Theorems 1.2 and 1.4 to the rational setting. First, we need a rational analogue of [4, Prop. 3.17].

Proposition 5.5. *Let K be a k-defined reductive subgroup of G, and let $H \leq G$.*

(i) *Suppose that $H \leq L := L_\lambda$ where $\lambda \in Y_k(K)$. Then H is relatively G-completely reducible over k with respect to K if and only if H is relatively L-completely reducible over k with respect to $K \cap L$.*

(ii) *If L is minimal among subgroups of the form L_λ with $\lambda \in Y_k(K)$ containing H, then H is relatively G-completely reducible over k with respect to K if and only if H is relatively L-irreducible over k with respect to $K \cap L$.*

Proof. Noting that the image of λ is a k-split torus, part (i) follows directly from [2, Thm. 5.4(ii)] applied to the K-orbit of $\mathbf{h} \in G^n$, where $\mathbf{h} \in G^n$ is a generic tuple for H. For part (ii), we have shown that H is relatively L-cr over k with respect to $K \cap L$. But also, from the minimality of L, it follows that any k-defined cocharacter of $K \cap L$ centralized by H is central in L, hence H is relatively L-irreducible over k with respect to $K \cap L$. □

We are now in a position to prove the rational counterparts of Theorems 1.2 and 1.4. Define the following analogues of the sets $\mathcal{P}_K$, $\mathcal{M}_K$ and $\mathcal{P}'_K$:

$$\begin{aligned}
\mathcal{P}_{K,k} &= \{P_\mu \mid \mu \in Y_k(K)\}, \\
\mathcal{M}_{K,k} &= \{\text{inclusion-maximal members of } \mathcal{P}_{K,k}\}, \\
\mathcal{P}'_{K,k} &= \left\{P \in \mathcal{P}_{K,k} \mid P = \bigcap\{Q \in \mathcal{M}_{K,k} \mid P \leq Q\}\right\}.
\end{aligned}$$

With these definitions, the obvious analogue of Lemma 2.2 holds for connected G. In the proof, one needs to work with the relative root system of G with respect to a maximal k-split torus [6, V.21], [9, §15, 16], but otherwise the argument goes through *mutatis mutandis*. The conclusion of Lemma 2.4 also holds when considering only k-defined cocharacters. Since $\mathcal{M}_{K,k} = \mathcal{M}_K \cap \mathcal{P}_{K,k}$ and $\mathcal{P}'_{K,k} = \mathcal{P}'_K \cap \mathcal{P}_{K,k}$, Lemma 2.7 immediately implies its own rational analogue. Finally, [4, Lem. 4.6] shows that each time we use Lemma 2.6 to conjugate a subgroup or cocharacter, the conjugating element can be taken to be a k-point.

The following are now the rational versions of Theorem 1.2 and 1.4.

Theorem 5.6. *Let $K \leq G$ be reductive k-defined algebraic groups with G connected, and let H be a subgroup of G. Then the following are equivalent.*

(i) *H is relatively G-completely reducible over k with respect to K.*

(ii) *Every member of $\mathcal{M}_{K,k}$ containing H has an opposite in $\mathcal{M}_{K,k}$ containing H.*

(iii) *There is an R-Levi subgroup L_μ with $\mu \in Y_k(K)$, such that $H \leq L_\mu$ and H is relatively L_μ-irreducible over k with respect to $K \cap L_\mu$.*

Theorem 5.7. *Let $K \leq G$ be reductive k-defined algebraic groups, and let H be a subgroup of G. Then each member of $\mathcal{M}_{K,k}$ containing H has an opposite in $\mathcal{M}_{K,k}$ containing H if and only if each member of $\mathcal{P}'_{K,k}$ containing H has an opposite in $\mathcal{P}'_{K,k}$ containing H.*

Remark 5.8. By Remark 5.2, we could equally well take $\mathcal{P}_{K,k}$ to be the set of k-defined R-parabolic subgroups of the form P_λ with $\lambda \in Y(K)$ and $\mathcal{M}_{K,k}$ to be the set of maximal members of this $\mathcal{P}_{K,k}$. However, working with R-parabolic and R-Levi subgroups corresponding to elements of $Y_k(K)$ allows our proofs to be more naturally generalized.

Proof of Theorems 5.7 and 5.6. The rational analogue of Lemma 2.7 gives one direction of Theorem 5.7. For the reverse direction, the rational analogues of Lemmas 2.4 and 2.6 are almost sufficient for the proof of Theorem 1.4 to go through *mutatis mutandis*. The only subtle point is when conjugating a cocharacter by an element of '$C_{P_\nu(K)}(H)$'; we need the conjugating element to lie in $C_{P_\nu(K)}(H)(k)$. That we can guarantee this follows from the conjugacy of maximal k-split tori in [2, Lem. 2.12].

Thus Theorem 5.7 holds. In Theorem 5.6, the equivalence of conditions (i) and (iii) follows from Proposition 5.5. The implication (i) $\Rightarrow$ (ii) follows from the rational analogue of Lemma 2.7. As in the algebraically closed case, these implications do not use the fact that G is connected. Finally, if G is connected then the rational version of Lemma 2.2 tells us that $\mathcal{P}_{K,k} = \mathcal{P}'_{K,k}$, and so the implication (ii) $\Rightarrow$ (i) follows from Theorem 5.7. □

Appendix A. Extended example: G_2 in GL_7

We close by considering a non-trivial example of Corollary 1.5 (and hence Theorem 1.2) when V is a faithful irreducible module for a simple algebraic group of exceptional type.

Fix the algebraically closed field k and let K be a simple algebraic group of type G_2 over k. Then K can be realized as the group of invertible linear transformations of a 7-dimensional k-vector space V which preserve an alternating trilinear form and associated bilinear form (or quadratic form if k has characteristic 2) [1, §3].

By [1, Thms. 1,3], the R-parabolic subgroups $P_\lambda(K)$ of K, with $\lambda \in Y(K)$, are precisely the stabilizers in K of *doubly singular* subspaces of V of dimension 1 or 2, where *doubly singular* means singular with respect to both the trilinear form and the bilinear (or quadratic) form. By [1, Thm. 2], K is transitive on such subspaces of each dimension.

Now let $G = \mathrm{GL}(V)$. Elementary calculations with high weights show that a given maximal torus of K is G-conjugate to the subtorus

$$S = \{\mathrm{diag}(s, t, st^{-1}, 1, s^{-1}t, t^{-1}, s^{-1}) \mid s, t \in k^*\}$$

of G, with respect to an appropriate basis of V. It then follows quickly that flags of subspaces of V corresponding to a cocharacter $\lambda \in Y(S)$ involve intermediate subspaces of dimensions $\{2, 5\}$, $\{1, 3, 4, 6\}$ and $\{1, 2, 3, 4, 5, 6\}$. In the first two cases, $P_\lambda(K)$ is a maximal parabolic subgroup of K and $P_\lambda \in \mathcal{M}_K$. In the third case $P_\lambda(K)$ and P_λ are respectively Borel subgroups of K and of G. The flags of type $\{2, 5\}$ consist of doubly singular subspaces of V and their annihilators, and K is transitive on these. By [1, §§4.2] the flags of type $\{1, 3, 4, 6\}$ can be described as those of the form $U \subseteq \Delta(U) \subseteq \Delta(U)^\perp \subseteq U^\perp$, where $U = \langle x \rangle$ is 1-dimensional and doubly singular and the 3-dimensional subspace $\Delta(U)$ is defined as the

radical of the bilinear form $(u, v) = f(x, u, v)$ with f the K-stable trilinear form. Since K is transitive on the subspaces U it is also transitive on these flags.

Explicitly writing out condition (ii) of Theorem 1.2 yields the following criteria for relative complete reducibility in this scenario.

Theorem A.1. *With the above notation, a subgroup H of $G = \mathrm{GL}(V)$ is relatively G-completely reducible with respect to the subgroup K of type G_2 if and only if the following two conditions hold.*

(i) *If H stabilizes a flag $U \subseteq U^\perp$ where U is 2-dimensional and doubly singular, then H also stabilizes a flag $W \subseteq W^\perp$ where W is 2-dimensional and doubly singular, and $V = U \oplus W^\perp = W \oplus U^\perp$.*

(ii) *If H stabilizes a flag $U \subseteq \Delta(U) \subseteq \Delta(U)^\perp \subseteq U^\perp$ where U is 1-dimensional and doubly singular, then H stabilizes another such flag $W \subseteq \Delta(W) \subseteq \Delta(W)^\perp \subseteq W^\perp$ with*

$$V = U \oplus W^\perp = W \oplus U^\perp = \Delta(U) \oplus \Delta(W)^\perp = \Delta(W) \oplus \Delta(U)^\perp.$$

For example, let $U \subseteq U^\perp$ be a flag as in (i) and let H be any Levi subgroup of the corresponding parabolic subgroup of $\mathrm{GL}(V)$. Then $H \cong \mathrm{GL}_2 \times \mathrm{GL}_3 \times \mathrm{GL}_2$ and H does not stabilize a flag of subspaces of the form in (ii). Thus H is relatively G-cr with respect to G_2 if and only if the complementary flag stabilized by H also has the form described in (i).

Remark A.2. The construction of Chevalley groups of type G_2 given in [1] holds for arbitrary fields k, and produces the same description of the maximal parabolic subgroups thereof. Thus for split groups of type G_2 over any field k, the analogue of Theorem A.1 holds, characterising relative complete reducibility over k. We thank the anonymous referee for pointing this out.

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