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## A VISCOELASTIC THEORY OF TURBULENT FLUID PERMEATED WITH FIBRIL MAGNETIC FIELDS

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### ABSTRACT

The solar convection zone is a turbulent plasma interacting with a magnetic field. Its magnetic field is often described as fibrillar since it consists of slender flux tubes occupying a small fraction of the total volume. It is well known that plasma flow will exert a force on these magnetic fibrils, but few models have accounted for the back-reaction of the fibrils on the flow. We present a model in which the back-reaction of the fibrils on the flow is manifest as viscoelastic properties. On short timescales the fibrils react elastically with a shear modulus proportional to their overall magnetic energy density. On longer timescales they produce an effective viscosity resulting from collective aerodynamic drag. The viscosity due to flux tubes in the solar convection zone can be comparable to that attributed to turbulence there. These forces might have observable effects on the convection zone flows.

*Subject headings:* convection — magnetic fields — MHD — Sun: interior — Sun: magnetic fields — turbulence

### 1. INTRODUCTION

The interaction between a turbulent plasma and a magnetic field is a long-standing problem in astrophysics. A kinematic treatment of this problem studies the evolution of the magnetic field in the presence of a prescribed plasma flow. A well-known result of such an approach is the  $\alpha$ -effect, whereby kinetic helicity in a turbulent flow leads to amplification of the magnetic field; the system is a magnetic dynamo. A typical derivation invokes *mean field electrodynamics* in which the magnetic field is decomposed into a dominant smooth component plus small-scale, small-amplitude fluctuations (Moffatt 1978; Krause & Rädler 1981) representing turbulence. Averaging the magnetic induction equation over the rapidly fluctuating velocity yields the  $\alpha$ -effect.

Were the  $\alpha$ -effect to persist, it is believed that the amplified magnetic field would become strong enough to affect the fluid flow, violating the kinematic assumption. This hypothetical “back-reaction” on the flow by the magnetic field has been the subject of several investigations, some of which predict significant back-reaction even by very weak fields (Vainshtein & Cattaneo 1992; Kulsrud & Anderson 1992; Bhattacharjee & Yuan 1995). If the back-reaction has the effect of quenching the dynamo action before it can generate a significant field, then these results cast serious doubt on the viability of either the  $\alpha$ -effect or mean field electrodynamics, when confronted with the observed existence of relatively strong dynamo-generated magnetic fields.

The solar convection zone (CZ) is undoubtedly the best known and most thoroughly studied astrophysical example

of a turbulent plasma with a magnetic field. Where it is most easily measured, at the solar surface, the magnetic field is highly intermittent and seems poorly approximated by mean field electrodynamics. It is perhaps more accurate to describe the photosphere as a *fibril field*, in light of its concentration into features ranging from active regions on the largest scale down to  $\sim 10^{17}$  Mx flux elements (Stenflo 1973) at the smallest scales yet resolved. Many properties of sunspots and active regions have been successfully interpreted by describing these surface phenomena as the manifestations of strands of magnetic flux, called “flux tubes,” extending through all or most of the CZ (Parker 1955).

In its idealized form a fibril field is one where all magnetic field occurs in such discrete flux tubes, each separated by a sharp interface from a surrounding field-free plasma (Parker 1955; Roberts & Webb 1978; Spruit 1981). This assumed field structure stands in contrast to *continuum* models, such as mean field electrodynamics or full magnetohydrodynamics (MHD), where  $\mathbf{B}(\mathbf{x})$  is a smooth function, vanishing only at isolated points or in small regions. Theoretical arguments predict that high Reynolds number turbulence will concentrate the magnetic field into fibrils with relatively small filling factor (Weiss 1966; Parker 1984). Numerical solutions of nonlinear MHD equations, at the highest Reynolds numbers practical, have confirmed this tendency for concentration, but not to the extent that flux tubes become truly isolated (Nordlund et al. 1992; Matthews, Hughes, & Proctor 1995; Brummell, Cline, & Cattaneo 2002). Moreover, very sensitive measurements show that the regions surrounding photospheric flux tubes are not truly field-free (Lin & Rimmele 1999). Thus, the fibril state appears to be an idealization that is not likely to occur in reality. Nevertheless, its highly intermittent nature suggests that the solar magnetic field might be more

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reasonably approximated as a fibril field than as a field whose small-scale fluctuations are perturbations on a smooth component.

Dynamical equations for a single flux tube follow from ideal MHD applied to the plasma *inside* the tube (Roberts & Webb 1978; Spruit 1981). The results are a set of equations, called the “thin flux tube model,” describing the dynamics of a one-dimensional space curve (the tube’s axis), which are formally similar to the equations of motion for a buoyant elastic string. Versions of the thin flux tube equations typically applied to the solar CZ include the effects of gravity (i.e., buoyancy), magnetic tension, and the Coriolis force. The external medium couples across the tube’s interface through an aerodynamic drag force (Parker 1979; Choudhuri & Gilman 1987), much as it would to a string at high Reynolds number. In a stationary atmosphere this drag counteracts the buoyancy so an active region flux tube rises to the solar surface at its terminal velocity. Solutions of this case have shown remarkable agreement with the observed properties of active regions and sunspot groups (D’Silva & Choudhuri 1993; Fan, Fisher, & DeLuca 1993; D’Silva & Howard 1993; Fan, Fisher, & McClymont 1994; Caligari, Moreno-Insertis, & Schüssler 1995). Rising through a medium with turbulent velocity, a tube’s axis develops perturbations that match the statistical scatter present in the same observations (Longcope & Fisher 1996; Longcope, Fisher, & Pevtsov 1998; Longcope & Choudhuri 2002).

To date, fibril fields have been studied primarily in the kinematic regime where they are affected by the unmagnetized background but exert no back-reaction on it. It seems unlikely that a single flux tube would significantly alter flows in the CZ, making the kinematic limit a reasonable one. The combined forces of all the flux tubes in the CZ might, however, be enough to affect the CZ flow. Unfortunately, no formalism yet exists by which such a back-reaction might be quantified.

The back-reaction of a fibril field will take a form very different from that of a continuum field. To see this, consider their differing reactions to a simple prescribed shear flow  $\mathbf{v} = v_0 \sin(ky)\hat{\mathbf{x}}$ . A continuum field transverse to this flow has no equilibrium under the ideal induction equation,

$$\mathbf{B} = B_0\hat{\mathbf{y}} + B_0v_0kt \cos(ky)\hat{\mathbf{x}}. \quad (1)$$

This field produces an opposing Lorentz force proportional to the growing fluid displacement  $F_x \sim -v_x t$ , a reaction analogous to an elastic restoring force. It is clear from this fact alone that even a field that is initially very weak can eventually produce enough back-reaction to stop any flow.

In contrast to a continuum field, a *flux tube* transverse to the flow will reach an equilibrium state. In this equilibrium, magnetic tension force balances the aerodynamic drag from the flow. The unmagnetized medium flows around the flux tube so the Lorentz force does not grow indefinitely. Instead, the flux tube will produce a steady back-reaction opposing the shear flow. When summed over a local ensemble of tubes, this back-reaction will constitute a stress on the fluid and thereby produce an effective viscosity. Furthermore, rather than a simple Newtonian viscosity, the effective back-reaction is more accurately described as a viscoelasticity, since cessation of the driving flow  $\mathbf{v}$  will not result in the immediate collapse of the shear stress arising from magnetic tension. Instead, the stress will relax in a finite time, depending on the flux and drag on the fibrils.

The form of the back-reaction offered by a fibril field can be anticipated by an analogy to the physics of polymer solutions. Fluids permeated with microscopic polymers have been studied by a variety of investigators (Doi & Edwards 1986; Bird, Armstrong, & Hassager 1987) over many years. Each polymer strand is buffeted by the thermal fluctuations of the fluid, much as magnetic flux tubes are buffeted by plasma turbulence. The back-reactions of the polymers appear as viscoelastic effects in the dynamics of the fluid. Over short timescales the polymers exert an elastic restoring force, while after longer times the force becomes viscous. Thus, a steady shear flow is possible in a polymer solution. The coefficients dictating this viscoelastic behavior can be derived from the basic physics of the polymers and their coupling to the fluid (Doi & Edwards 1986). In the present work we apply this technique to thin flux tubes permeating a turbulent plasma, in order to derive viscoelastic coefficients describing a fibril field’s back-reaction.

Magnetohydrodynamic viscoelastic effects were recently proposed in a model of magnetized accretion disks (Ogilvie 2001). The proposed model uses a continuum magnetic field, similar to mean field electrodynamics, from which the back-reaction is the Lorentz force. To recover a viscous regime, the author hypothesizes magnetic reconnection with a short relaxation time, similar in effect to a large resistivity. Our approach will be fundamentally different since we assume a fibril field and make no specific assumption about magnetic reconnection.

The next section derives dynamical equations for an isolated magnetic flux tube immersed in a turbulent plasma. The turbulent buffeting drives the flux tube into a statistical equilibrium state in which its magnetic tension is capable of balancing the drag. In § 3 we present the results of nonlinear numerical solutions of the thin flux tube equations under the same conditions. These solutions corroborate the theoretical methods and fix unknown coefficients. In § 4 we derive viscoelastic coefficients for a collection of individual flux tubes in statistical steady state. In § 5 we estimate the viscosity for steady state flows in the solar CZ. Under certain assumptions we find that the flux tubes may produce a viscosity comparable to the well-known turbulent viscosity, which most theoretical models already account for. In the final section we speculate on how the viscoelastic forces may affect flows in the CZ, including the differential rotation.

## 2. THE MODEL

### 2.1. *The Dynamics of a Thin Flux Tube*

Nonlinear, dynamical equations for the evolution of a thin flux tube were proposed by Spruit (1981) and have been refined and extended by subsequent authors (Choudhuri 1990; Cheng 1992; Moreno-Insertis & Emonet 1996). These models are most frequently used to study the buoyancy effect of an active region flux tube through the solar CZ (see, e.g., Fisher et al. 2000 and references therein). To simplify our analysis, we consider a nonbuoyant flux tube immersed in an unstratified turbulent plasma. Without gravitation or rotation the tube experiences only the force of aerodynamic drag,  $\mathbf{F}_d$ , and its own magnetic tension (DeLuca, Fisher, & Patten 1993). The forces cause an element of tube to accelerate at

$$\frac{d\mathbf{v}}{dt} = \frac{B^2}{4\pi\rho} \frac{\partial^2 \mathbf{r}}{\partial s^2} + \frac{1}{8\pi\rho} \frac{\partial B^2}{\partial s} \frac{\partial \mathbf{r}}{\partial s} + \mathbf{a}_d, \quad (2)$$

where  $B(s)$  is the magnetic field strength and  $\mathbf{a}_d \equiv \mathbf{F}_d/\rho$  is the local acceleration due to drag.

A cylindrical tube segment moving at velocity  $\mathbf{v}$ , through an external flow with velocity  $\mathbf{v}^{(e)}$ , feels a drag force per unit volume

$$\mathbf{F}_d = -\frac{C_D\rho}{\sqrt{\pi\Phi/B}}|\mathbf{v}_\perp - \mathbf{v}_\perp^{(e)}|(\mathbf{v}_\perp - \mathbf{v}_\perp^{(e)}), \quad (3)$$

where  $\mathbf{v}_\perp$  and  $\mathbf{v}_\perp^{(e)}$  are the components of  $\mathbf{v}$  and  $\mathbf{v}^{(e)}$  perpendicular to the local tangent vector of the tube's axis  $\hat{\mathbf{s}} \equiv \partial\mathbf{r}/\partial s$ . Although the form of equation (3) renders the dynamics nonlinear in the general case, a good linear approximation can be made in the case in which the time-scales of fibril dynamics greatly exceed those of the turbulence.

The external fluid velocity  $\mathbf{v}^{(e)}(\mathbf{x}, t)$  will be replaced here by a random function whose statistics represent those of the turbulence. We will make the approximations that  $\mathbf{v}^{(e)}$  is isotropic and has negligibly small correlation time  $\tau_c$  compared with the dynamical times of the flux tube (i.e., it is white noise). Under these assumptions the drag term can be replaced by a general expression of the Langevin type

$$\mathbf{a}_d \simeq \mathbf{f}_\perp(s, t) - \zeta\mathbf{v}_\perp, \quad (4)$$

where averages over the turbulent fluctuations give  $\langle \mathbf{f}_\perp \rangle = 0$  and  $\langle \zeta \rangle = \zeta$ . The specific drag  $\zeta$  is found by averaging the definition given by equation (4),

$$\zeta \equiv -\left\langle \frac{\mathbf{a}_d \cdot \mathbf{v}_\perp}{|\mathbf{v}_\perp|^2} \right\rangle = \sqrt{\frac{B}{\pi\Phi}} \frac{\langle \mathbf{v}_\perp \cdot (\mathbf{v}_\perp - \mathbf{v}_\perp^{(e)}) |\mathbf{v}_\perp - \mathbf{v}_\perp^{(e)}| \rangle}{|\mathbf{v}_\perp|^2}, \quad (5)$$

where we have taken  $C_D = 1$ , the high Reynolds number limit (Batchelor 1967).

One limit that can be treated analytically is that of a small tube velocity,  $|\mathbf{v}_\perp^{(e)}| \gg |\mathbf{v}_\perp|$ , in which the drag is proportional to the average turbulent velocity and independent of the velocity  $\mathbf{v}_\perp$ . The expansion of equation (5) in powers of  $|\mathbf{v}_\perp|/|\mathbf{v}_\perp^{(e)}|$  yields two leading-order terms whose sum is

$$\zeta \simeq \frac{3\pi}{8} \sqrt{\frac{B}{\pi\Phi}} \langle |\mathbf{v}^{(e)}| \rangle, \quad (6)$$

where isotropy has been used to express  $\langle |\mathbf{v}_\perp^{(e)}| \rangle = (\pi/4)\langle |\mathbf{v}^{(e)}| \rangle$ . In any case,  $\zeta^{-1}$  is the timescale on which coherent motions relax by virtue of the random driving.

The general two-point statistics of the stochastic forcing  $\mathbf{f}_\perp$  are complicated since they depend on  $\mathbf{v}_\perp(s)$  and  $\mathbf{v}_\perp(s')$ , as well as the tangent vectors  $\hat{\mathbf{s}}$  and  $\hat{\mathbf{s}}'$ . Fortunately, we will have need for only the scalar correlation, which we write as

$$\langle \mathbf{f}_\perp(s, t) \cdot \mathbf{f}_\perp(s', t') \rangle = \Gamma C(s - s')\tau_c\delta(t - t'). \quad (7)$$

We have introduced the correlation function  $C(\Delta s)$ , which is unity at  $\Delta s \equiv s - s' = 0$  and decreases with increasing separation. Its decrease is due to decorrelation of the turbulent velocity at increasing distances, as well as to the differences in perpendicular direction between  $s$  and  $s'$ . We will assume that  $C(\Delta s)$  has a single peak at  $\Delta s = 0$  with half-width  $\sim l_t$ , the turbulent correlation length.

The product  $\tau_c\delta(t)$  in equation (7) is a shorthand for the time correlation of the turbulence. By assumption the turbulence has a very short correlation time; however, the single-time correlation is still finite and enters appropriately when integrating over the correlation of the noise.

The amplitude factor in equation (7) is set by the single-point, single-time moment

$$\begin{aligned} \Gamma &\equiv \langle |\mathbf{a}_d|^2 \rangle - \zeta^2 |\mathbf{v}_\perp|^2 \\ &= \frac{B}{\pi\Phi} \left[ \langle |\mathbf{v}_\perp - \mathbf{v}_\perp^{(e)}|^4 \rangle - \frac{\langle \mathbf{v}_\perp \cdot (\mathbf{v}_\perp - \mathbf{v}_\perp^{(e)}) |\mathbf{v}_\perp - \mathbf{v}_\perp^{(e)}|^2 \rangle}{|\mathbf{v}_\perp|^2} \right]. \end{aligned} \quad (8)$$

In the small tube velocity limit  $|\mathbf{v}_\perp| \ll |\mathbf{v}_\perp^{(e)}|$  the factor takes the simple form

$$\Gamma \simeq \frac{8}{15} \frac{B}{\pi\Phi} \langle |\mathbf{v}^{(e)}|^4 \rangle, \quad (9)$$

where isotropy has been used to express  $\langle |\mathbf{v}_\perp^{(e)}|^4 \rangle = (8/15)\langle |\mathbf{v}^{(e)}|^4 \rangle$ .

The tube's inertia is negligible under many circumstances, and the primary dynamics is that of turbulent advection being opposed by magnetic tension. In this situation it is possible to neglect the left-hand side of equation (2). We will define a Lagrangian coordinate  $s_0$ , which is the length coordinate of the material point in some reference configuration. The velocity of the tube may then be written as  $\mathbf{v} = \partial\mathbf{r}(s_0, t)/\partial t$ , and the tube's equation of motion becomes

$$\zeta \left| \frac{\partial\mathbf{r}}{\partial t} \right|_\perp = \frac{B^2}{4\pi\rho} \frac{\partial^2\mathbf{r}}{\partial s^2} + \frac{1}{8\pi\rho} \frac{\partial B^2}{\partial s} \frac{\partial\mathbf{r}}{\partial s} + \mathbf{f}_\perp. \quad (10)$$

## 2.2. The Mean Field Theory

Equation (10) describes the dynamics of a thin flux tube buffeted by the turbulent medium in which it is immersed. The random driving  $\mathbf{f}_\perp$  will induce random distortions in the axis  $\mathbf{r}(s_0, t)$ . It is the statistics of these distortions we wish to analyze. We will perform this analysis in the mean field limit, replacing the randomly fluctuating functions  $B(s)$  and  $\partial s/\partial s_0$  by their average values. Assuming that the statistics of the axis are homogeneous and isotropic, the mean field equation reduces to

$$\zeta \frac{\partial\mathbf{r}}{\partial t} = U \frac{\partial^2\mathbf{r}}{\partial s_0^2} + \mathbf{f}_\perp, \quad (11)$$

where the tension coefficient

$$U \equiv \frac{B^2}{4\pi\rho} \left( \frac{\partial s}{\partial s_0} \right)^{-2} \quad (12)$$

is a constant that, like  $\zeta$ , depends on the statistical properties of the solution. The right-hand side of equation (11) is now perpendicular to  $\hat{\mathbf{s}}$ , so the velocity  $\mathbf{v}$  will be so automatically.

Equation (11) is that of an elastic string driven by random noise. Models of this type have been studied extensively in the field of polymer physics. Because the equation is linear in the unknown, we can propose a Fourier decomposition,

$$\mathbf{r}(s_0, t) = \sum_q \mathbf{r}_q(t) e^{iqs_0}, \quad (13)$$

and solve immediately for its coefficients,

$$\mathbf{r}_q(t) = \int_{-\infty}^t \exp\left[-\frac{Uq^2}{\zeta}(t-t')\right] \mathbf{f}_q(t') \frac{dt'}{\zeta}. \quad (14)$$

(For simplicity we assume that the flux tube is a closed curve and thereby use discrete Fourier transforms.) Thus, the distortions of the flux tube depend on the random driving function  $\mathbf{f}_\perp$  over all past times.

Under the isotropic mean field approximation the general correlation matrix of the stochastic driving is

$$\langle \mathbf{f}_\perp \mathbf{f}_\perp \rangle = \frac{1}{3} \Gamma C(s-s') \tau_c \delta(t-t') I. \quad (15)$$

Introducing the Fourier decomposition of  $\mathbf{f}_\perp$  yields mode correlation statistics

$$\langle \mathbf{f}_q(t) \mathbf{f}_q^*(t') \rangle = \frac{1}{3} \Gamma C_q \tau_c \delta(t-t') \delta_{q,q'} I, \quad (16)$$

where  $C_q$  is the Fourier transform of the correlation function  $C(\Delta s)$ . The correlation function itself is defined in terms of extensible coordinates  $s$  and  $s'$ , which are appropriate for describing separations in the turbulent flow. Nevertheless, its Fourier transform is performed with respect to Lagrangian coordinates  $s_0$  and  $s'_0$  in which the dynamical solutions are cast. Care must be taken to transform between these coordinates using the mean field extension factor  $\partial s / \partial s_0$ .

Forming the general correlation matrix between axis modes  $\mathbf{r}_q$  and using equation (16) gives

$$\langle \mathbf{r}_q(t) \mathbf{r}_q^*(t') \rangle = \frac{1}{6} \frac{\Gamma C_q \tau_c}{\zeta q^2 U} \exp\left(-\frac{U q^2}{\zeta} |t-t'|\right) I \delta_{q,q'}. \quad (17)$$

Axis modes remain correlated over much longer times than the noise that drives them. A mode with wavelength  $\lambda = 2\pi(\partial s / \partial s_0) / q$  decorrelates over a time

$$\tau_\lambda = \frac{\zeta}{U q^2} = \frac{\zeta}{4\pi^2 v_A^3} \lambda^2, \quad (18)$$

where  $v_A = B / (4\pi\rho)^{1/2}$  is the local Alfvén speed.

It is now possible to calculate the coefficients in our mean field equation in terms of the average properties of the axis modes  $\mathbf{r}_q$ . Tube elements labeled by  $s_0$  and  $s'_0$  are separated by a vector

$$\Delta \mathbf{r} \equiv \mathbf{r}(s_0, t) - \mathbf{r}(s'_0, t). \quad (19)$$

The second moment of this displacement vector is

$$\langle |\Delta \mathbf{r}|^2 \rangle = \frac{4\Gamma \tau_c}{\zeta U} \sum_q C_q \frac{\sin^2(q\Delta s_0/2)}{q^2}, \quad (20)$$

where  $\Delta s_0 \equiv s_0 - s'_0$ . If the two points are sufficiently separated,  $|\Delta s_0| \gg l_t$ , then the sum becomes

$$\sum_q C_q \frac{\sin^2(q\Delta s_0/2)}{q^2} \simeq \frac{1}{4} \Delta s_0 \int_{-\infty}^{\infty} C(s) ds_0 \simeq \frac{1}{2} \Delta s_0 l_t \left(\frac{\partial s}{\partial s_0}\right)^{-1}, \quad (21)$$

since  $C_q \simeq C_0$  over the support of the factor  $\sin^2(q\Delta s_0/2) / q^2$  in this limit. The sum of the factor may be replaced by an integral with respect to  $dq / \Delta q$ , where the spacing between discrete Fourier modes  $\Delta q \ll 1 / \Delta s_0$ . In this limit, the section of tube between  $s$  and  $s'$  is a random walk of uncorrelated steps  $l_t$ . According to classical theory of random walks, the total number of such steps is  $N = \langle |\Delta \mathbf{r}|^2 \rangle / l_t^2$ . Thus, the total length of the tube segment is

$$\Delta s = \frac{\langle |\Delta \mathbf{r}|^2 \rangle}{l_t} \simeq \frac{2\Gamma \tau_c}{\zeta U} \left(\frac{\partial s}{\partial s_0}\right)^{-1} \Delta s_0. \quad (22)$$

The mean field extension factor  $\partial s / \partial s_0$  can now be equated to the extension  $\Delta s / \Delta s_0$  found from the randomly deformed axis. Doing so and using equation (12) for  $U$  yields a relationship

$$\frac{B^2}{4\pi\rho} = \frac{2\Gamma \tau_c}{\zeta}. \quad (23)$$

In the context of the polymer physics of Brownian motion, this expression is a version of the fluctuation dissipation theorem, linking drag and noise. In the present context, however, it becomes the determining relation for the field strength of the flux tubes at equilibrium.

The right-hand side of equation (23) depends on  $B$  through  $\Gamma$  and  $\zeta$ . Using the small flux tube velocity ( $|\mathbf{v}_\perp| \ll |\mathbf{v}_\perp^{(e)}|$ ) equations (6) and (9), it is possible to solve explicitly for

$$B = 3.454 \left( \rho \tau_c \frac{\langle |\mathbf{v}^{(e)}|^4 \rangle}{\langle |\mathbf{v}^{(e)}| \rangle} \right)^{2/3} \Phi^{-1/3}. \quad (24)$$

If we admit tube velocities comparable to the typical turbulent velocity  $v_t$ , then the general expressions for  $\zeta$  and  $\Gamma$ , i.e., equations (5) and (8), depend explicitly on  $B$ , as well as on averages of the relative velocities. If each of the velocity averages scales only as the appropriate power of  $v_t$  and is independent of  $B$  or  $\Phi$ , then equation (23) gives the same scaling

$$B \simeq K' (\rho \tau_c v_t^3)^{2/3} \Phi^{-1/3}, \quad (25)$$

but with a factor  $K'$  that depends on the rescaled velocity averages.

Equation (25) gives the field strength achieved by the flux tube under the action of turbulence. This represents a statistical equilibrium, where the flux tube's tension is strong enough to balance the aerodynamic drag of the turbulence. Remarkably, the field strength scales inversely with the tube's flux to the one-third power, so smaller flux tubes will tend to be stronger. This is a natural consequence of the inverse scaling of the drag force with cross-sectional area: thinner tubes couple more effectively to the turbulence, and they must be stronger before their tension can balance the drag force.

DeLuca et al. (1993) found a scaling similar to equation (25) for a closed ring of flux immersed in a steady cellular flow pattern (an ABC flow). While the Eulerian correlation time in such a flow is infinite, the tube sees a variation over times  $\tau_c \sim l_t / v$  due to its own motion. Using this in equation (25) gives a scaling identical to that found by DeLuca et al. (1993).

### 3. NUMERICAL SIMULATIONS

To verify the foregoing analysis, we perform numerical simulations of flux tube dynamics. The basic thin flux tube model equation (2), including inertia, is solved subject to the constraint that the magnetic tension coefficient  $B^2 / 4\pi\rho$  is uniform along the tube (i.e., that the second term in eq. [2] vanishes); this same constraint follows from the mean field approximation as discussed in § 2.2. This constraint also follows if the pressure fluctuations within the tube are assumed to be smoothed out much more quickly than the tube configuration changes, which is expected when

$c_s/v_A \simeq \beta^{1/2} \gg 1$ , where  $c_s$  is the plasma's sound speed. In the simulations described in this paper, the timescale on which pressure perturbations (sausage waves) within the tube travel a length  $l_t$  is much shorter than the turbulent timescale  $\tau_t$ , so this should be an excellent approximation. The resulting equation involves only inertia, the magnetic tension coefficient, the flux tube curvature vector, and the aerodynamic drag force (eq. [3]) due to turbulent motions acting to buffet the tube.

The primary agent driving evolution of the tube is then the external velocity field  $\mathbf{v}^{(e)}$  appearing in equation (3). In our formulation, the average properties of this flow field depend on only two quantities: a characteristic velocity amplitude  $v_t$  and a characteristic *eddy size* (or correlation length)  $l_t$ . An *eddy turnover* time can then be defined as  $\tau_t = l_t/v_t$ , which will turn out not to be equivalent to the correlation time of the noise driving the tube. To be consistent with our white-noise assumption, we will seek to keep  $\tau_t$  smaller than all other dynamical scales. At any given time,  $\mathbf{v}^{(e)}$  is a function of the Lagrangian arc length  $s_0$ , which can be assumed without loss of generality to vary from 0 to 1.

At a fixed time, the velocity at a point  $s = s_i = i(\delta s)$  is produced according to the recipe

$$\mathbf{v}_i^{(e)} = v_t \sum_{j=1}^N \mathbf{w}_j A\left(\frac{\delta s}{l_t}\right) \exp\left[\frac{-(i-j)^2(\delta s/N)^2}{2l_t^2}\right], \quad (26)$$

where  $\delta s$  is the arc length separation between adjacent mesh points ( $\delta s \equiv N^{-1}\partial s/\partial s_0$ ) and  $A(\delta s/l_t)$  is a normalization factor for the weighting. The velocity is endowed with randomness through an uncorrelated white-noise vector  $\mathbf{w}_i$ . Each 3-vector  $\mathbf{w}_i$  is a random variable with unit variance and is completely uncorrelated with any other point:  $\langle \mathbf{w}_i \mathbf{w}_j \rangle = I \delta_{ij}$ . The velocities at points separated by  $\Delta s < l_t$  are correlated by virtue of common terms in their respective sums, even though all terms themselves are mutually uncorrelated.

The time correlation of  $\mathbf{v}_i^{(e)}$  follows from correlations of the random vectors  $\mathbf{w}_i$ , which follow from an evolution equation. The vector  $\mathbf{w}_i$  is updated in time according to the prescription (Longcope & Fisher 1996)

$$\mathbf{w}_i^{\text{new}} = \exp\left(-\frac{\Delta t}{\tau_t}\right) \mathbf{w}_i^{\text{old}} + \sqrt{1 - \exp\left(-\frac{2\Delta t}{\tau_t}\right)} N_i, \quad (27)$$

where  $\Delta t$  is the time step and  $N_i$  is a freshly generated 3-vector of normal deviates (i.e., zero mean and unit variance). According to this prescription,  $\mathbf{w}_i$  will become uncorrelated after a time  $\sim \tau_t$ .

The numerical methods used to solve equation (2) are the same as those used to solve for the flux tube evolution described in DeLuca et al. (1993). Briefly, the first and second partial derivatives with respect to Lagrangian arc length are converted to a centered finite difference form with  $N$  uniformly spaced mesh points in  $s_0$ . Two additional ghost points are added to each end of the tube with values determined by periodic boundary conditions, since the tube forms a closed ring. The resulting finite difference equations form a set of  $6N$  coupled ordinary differential equations, which are advanced in time via a standard fourth-order Runge-Kutta technique. These equations result in a solution for the tube node locations and components of velocity normal to the tube's tangent vector.

We perform a set of runs in which  $\Phi$ ,  $v_t$ , and  $l_t$  are varied around nominal values. The nominal values are initial field strength  $B_0 = 10^5$  G, correlation length  $l_t = 10^9$  cm, turbulent velocity  $v_t = 10^4$  cm s<sup>-1</sup>, and magnetic flux  $\Phi = 10^{18}$  Mx. These parameters were chosen to resemble the flux and field strength for a small "elemental" flux tube near the base of the solar CZ, with values of  $v_t$  and  $l_t$  corresponding roughly to those from mixing length CZ parameters. The plasma density in these calculations was assumed to be  $\rho = 0.2$  g cm<sup>-3</sup>, which is the value near the base of the CZ (Böhm-Vitense 1958).

Because we assume that the flow inside the tube is incompressible and we ignore sausage waves along the tube, an auxiliary mass conservation equation must also be solved to derive the velocity component parallel to the magnetic tube direction. This is described in DeLuca et al. (1993).

Runs explore variations in  $\Phi$ , the turbulent velocity  $v_t$ , and the correlation length  $l_t$ , while all using the same initial length, mass density, and initial field strength. All runs start from an initially circular ring with length  $L_0 = 2\pi \times 10^{10}$  cm, designed to sample a large number ( $\sim 60$ ) of correlation lengths. The Alfvén transit time around the ring,  $\tau_A \equiv L/v_A$ , is an invariant due to incompressibility and is  $\tau_A = 10^6$  s for all runs.

A single simulation is run for a period of  $(6-10) \times 10^7$  s, or  $\sim 60\tau_A - 100\tau_A$ . For the nominal values of  $l_t$  and  $v_t$  this corresponds to  $\sim 600\tau_t - 1000\tau_t$ . The amount of computing resources necessary to carry out these simulations varied widely, ranging from roughly 10 minutes on a laptop computer to over a day on a high performance workstation, depending on the required resolution and the assumed values of  $v_t$ ,  $l_t$ , or  $\Phi$ . Resolution varied from 300 to 2500 mesh points, depending on the values of the above parameters. We encountered some numerical stability problems for high values of  $v_t$  or low values of  $l_t$ .

The numerical simulations form three different series; in each series a single parameter was varied, while all others retained their nominal value. The series explore the effects that changes in  $\Phi$ ,  $v_t$ , and  $l_t$  have on average global properties and on time evolution. In the first series of simulations, the magnetic flux assumed the values  $\Phi = 10^{16}$ ,  $10^{17}$ ,  $10^{18}$ , and  $10^{19}$  Mx. In the second series  $l_t$  assumed values of  $5 \times 10^8$ ,  $2 \times 10^9$ , and  $4 \times 10^9$  cm, in addition to its nominal value  $1 \times 10^9$  cm. In the third series  $v_t$  assumed values of  $5 \times 10^3$ ,  $2 \times 10^4$ , and  $4 \times 10^4$  cm s<sup>-1</sup>, in addition to the nominal value of  $1 \times 10^4$  cm s<sup>-1</sup>.

To facilitate discussion of the numerical results, we introduce the abbreviated notation Pii-Ljj-Vkk to refer to a simulation whose values of  $\Phi$ ,  $l_t$ , and  $v_t$ , are given by  $ii = \log_{10} \Phi$ ,  $jj = l_t/10^8$  cm, and  $kk = v_t/10^3$  cm s<sup>-1</sup>. The simulation with nominal values of all parameters is P18-L10-V10. Table 1 summarizes the parameters of each simulation.

Figure 1 displays a snapshot of a typical configuration at a point of statistical steady state. This particular case is run P18-L10-V40 at the end of the run, after roughly  $60\tau_A$  or  $\sim 2400\tau_t$  of elapsed time. Generally, flux tube configurations with larger values of  $\Phi$  or smaller values of  $v_t$  have a less complex looking configuration than the one shown here.

Figure 2 shows the variation of field strength with time for the four values of magnetic flux in the  $\Phi$  series. The evolution in all cases shows an initial transient behavior (the "blowup" phase, which can result from either an expansion or contraction from the initial length),

TABLE 1  
SUMMARY OF THE MOST IMPORTANT INPUT AND DERIVED PARAMETERS FROM THE SERIES OF FLUX TUBE SIMULATIONS

Run	$N_g^a$	$t_{\text{sim}}^b$ (s)	$\Phi$ (Mx)	$l_t$ (cm)	$v_t$ (cm s <sup>-1</sup> )	$\langle L \rangle / L_0 = \langle B \rangle / B_0^c$	$[\text{Var}(B)]^{1/2} / B_0^d$
P16-L10-V10.....	500	$1 \times 10^8$	$10^{16}$	$1 \times 10^9$	$1 \times 10^4$	2.36	0.251
P17-L10-V10.....	400	$1 \times 10^8$	$10^{17}$	$1 \times 10^9$	$1 \times 10^4$	1.31	0.197
P18-L10-V10.....	300	$1 \times 10^8$	$10^{18}$	$1 \times 10^9$	$1 \times 10^4$	0.745	0.111
P19-L10-V10.....	300	$1 \times 10^8$	$10^{19}$	$1 \times 10^9$	$1 \times 10^4$	0.395	0.093
P18-L05-V10.....	1500	$6.8 \times 10^7$	$10^{18}$	$5 \times 10^8$	$1 \times 10^4$	0.483	0.091
P18-L10-V10.....	300	$1 \times 10^8$	$10^{18}$	$1 \times 10^9$	$1 \times 10^4$	0.745	0.111 <sup>e</sup>
P18-L20-V10.....	500	$1 \times 10^8$	$10^{18}$	$2 \times 10^9$	$1 \times 10^4$	0.883	0.217
P18-L40-V10.....	500	$1 \times 10^8$	$10^{18}$	$4 \times 10^9$	$1 \times 10^4$	0.993	0.344
P18-L10-V05.....	300	$1 \times 10^8$	$10^{18}$	$1 \times 10^9$	$5 \times 10^3$	0.272	0.093
P18-L10-V10.....	300	$1 \times 10^8$	$10^{18}$	$1 \times 10^9$	$1 \times 10^4$	0.745	0.111 <sup>e</sup>
P18-L10-V20.....	500	$1 \times 10^8$	$10^{18}$	$1 \times 10^9$	$2 \times 10^4$	1.53	0.216
P18-L10-V40.....	2500	$6.2 \times 10^7$	$10^{18}$	$1 \times 10^9$	$4 \times 10^4$	3.74	0.377

NOTES.—The first four rows represent variations in  $\Phi$ , the next four rows show variations in  $l_t$ , and the last four rows show variations in  $v_t$ . The mass density  $\rho$  was  $0.2 \text{ g cm}^{-3}$  for all cases.

<sup>a</sup> Number of grid points along the tube in the simulation.

<sup>b</sup> Total simulated time.

<sup>c</sup>  $\langle L \rangle$  and  $\langle B \rangle$  represent the value of the loop length and field strength, respectively, averaged over the last half of the simulation.  $L_0$  is the initial loop length, equal to  $2\pi \times 10^{10}$  cm in all cases.  $B_0$  is the initial field strength, equal to  $10^5$  G in all cases.

<sup>d</sup> This column is the square root of the field strength variance about its average value  $\langle B \rangle$  divided by  $B_0$  during the last half of each simulation. This is the rms fluctuation of the field strength about its average value, in units of  $B_0$ . It is also the rms fluctuation in  $L$  about its average, in units of  $L_0$ .

<sup>e</sup> This row is repeated from the third row of this table.

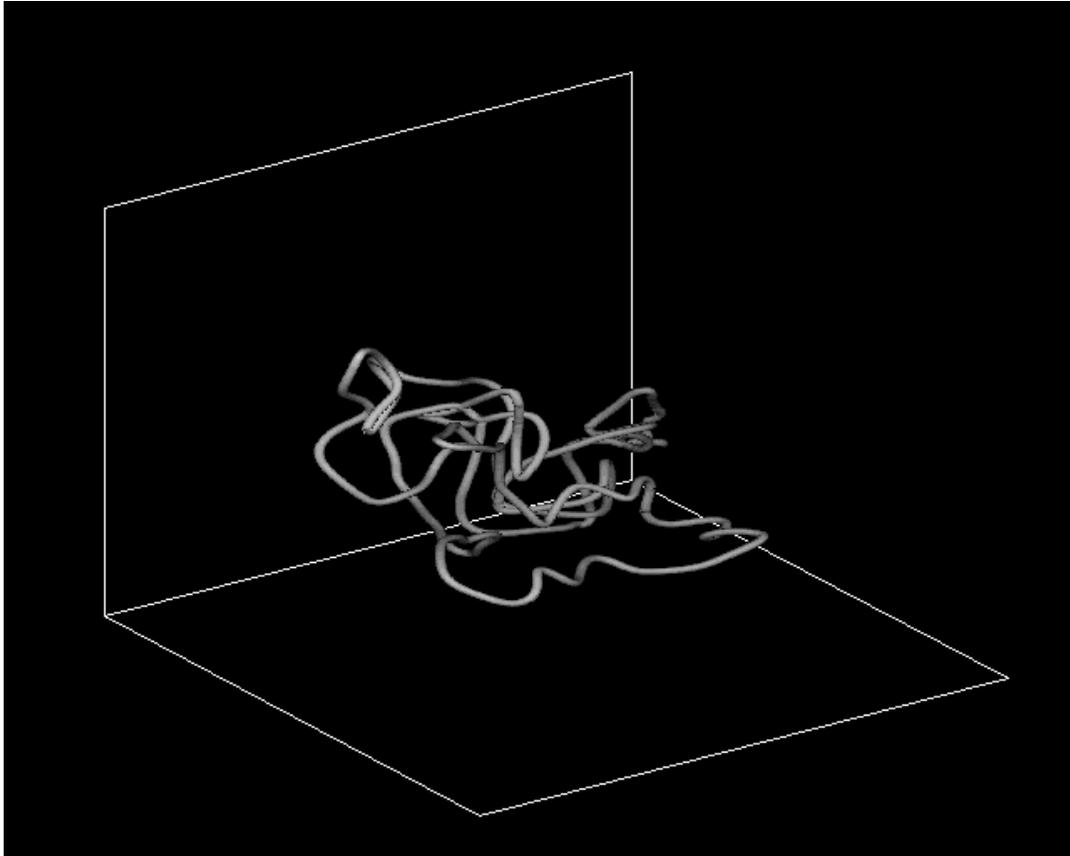


FIG. 1.—Snapshot from run P18-L10-V40

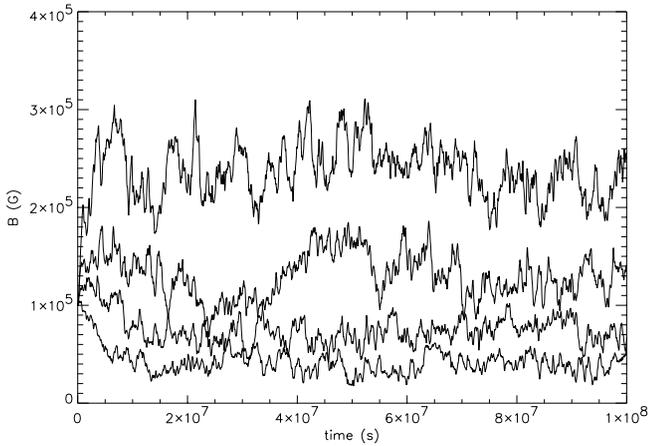


FIG. 2.—Variation of  $B$  over time for each of the runs in the  $\Phi$  series

followed by attainment of an approximately steady value with random fluctuations about that value. It is interesting to note that a simulation attempted with  $\Phi = 10^{20}$  Mx did not result in such a behavior; instead, the magnetic tension force overpowered the stochastic driving by the drag force and the flux tube collapsed with no statistical steady state being achieved. This “flux ring collapse” behavior was also seen in the simulations of DeLuca et al. (1993). We also find flux ring collapse using nominal values for  $\Phi$  and  $v_t$  and  $l_t \sim 2.5 \times 10^8$  cm.

The mean field strength,  $B$ , is found by averaging over the last half of each simulation. Figure 3a shows the variation in this average over the  $\Phi$  series, P16-L10-V10, P17-L10-V10, P18-L10-V10, and P19-L10-V10, which is well fitted by the power law  $B \sim \Phi^{-1/4}$ . Figure 3b shows the variation over the  $v_t$  series P18-L10-V05, P18-L10-V10, P18-L10-V20, and P18-L10-V40. The values are well fitted by the power law  $B \sim v_t^{5/4}$ . Finally,  $B$  varies slightly with  $l_t$  but not enough to yield a convincing power law.

The numerical simulations reveal that magnetic field strength  $B$  scales with magnetic flux  $\Phi$  and turbulent velocity  $v_t$ . Using density  $\rho$  and turbulent length  $l_t$  to recover

the correct dimensions gives a scaling

$$B = K \rho^{5/8} l_t^{1/2} v_t^{5/4} \Phi^{-1/4}, \quad (28)$$

where  $K \simeq 2.0$ .

This empirically derived behavior can be reconciled with the analytic expression given by equation (25) if the correlation time of the drag force is the geometric mean between the turbulent correlation time  $l_t/v_t$  and the Alfvén time  $l_t \rho^{1/2}/B$ . Substituting this time  $\tau_c \sim \rho^{1/4} l_t v_t^{-1/2} B^{-1/2}$  into equation (25) yields the observed scaling given by equation (28). The hybrid nature of  $\tau_c$  might arise from inertia, which was neglected in the analysis of § 2 but is included in the numerical simulation. A more detailed analysis, including comparisons of the spatial and temporal spectra  $\langle |r_q|^2 \rangle$ , would take us too far from our immediate objective of modeling the back-reaction of the tubes. We henceforth adopt equation (28) as the defining relationship between the turbulence properties and the equilibrium magnetic field strength.

The equipartition field strength,  $B_{\text{eq}} \equiv (4\pi)^{1/2} \rho^{1/2} v_t$ , is defined to yield an energy density equal to the average kinetic energy density of the turbulence. The equilibrium magnetic field (eq. [28]) can be written as

$$B = \frac{K}{\sqrt{4\pi}} B_{\text{eq}} \left( \frac{\Phi}{\Phi_t} \right)^{-1/4},$$

where  $\Phi_t \equiv \rho^{1/2} l_t^2 v_t$  is the characteristic flux set by the turbulence. Any flux tube with  $\Phi < 0.1 \Phi_t$  will attain field strength in excess of the equipartition strength as a result of the turbulence. It seems paradoxical, on its face, that turbulence can amplify a magnetic field above the equipartition strength. Upon reflection, however, the aerodynamic drag allows the flux tube to couple to flows outside itself, so local densities are not the appropriate quantities to compare.

#### 4. THE VISCOELASTIC PROPERTIES

Up to this point the turbulent fluid has been modeled as a (randomly) prescribed velocity field. This kind of passive driving, where the fluid affects the flux tubes but is not affected in return, is standard in flux tube models. The Lorentz force is present in the flux tube equation, but it acts only on the fluid

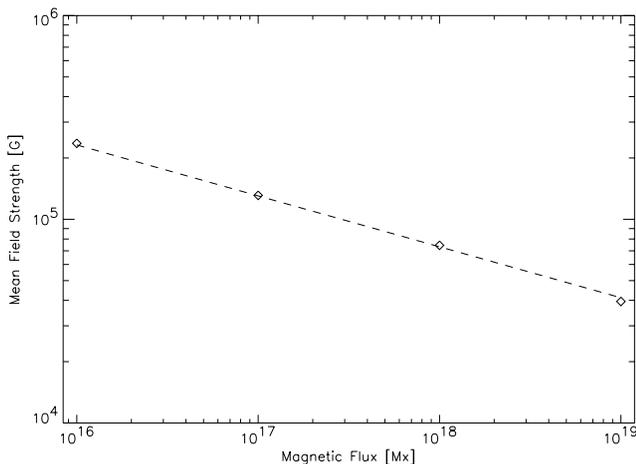


FIG. 3a

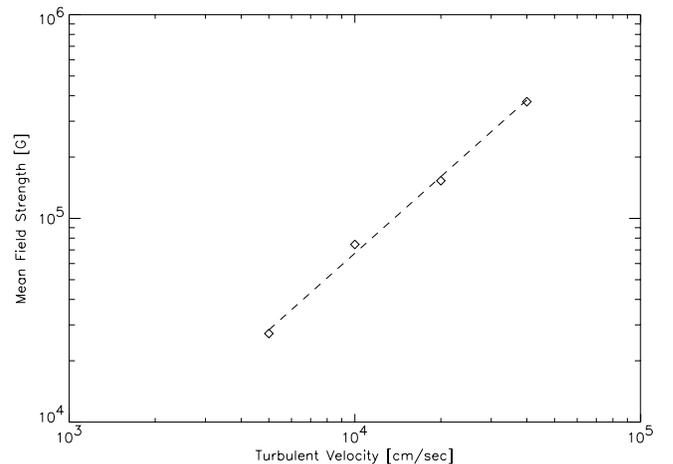


FIG. 3b

FIG. 3.—Variation in  $B$  with (a)  $\Phi$  and (b)  $v_t$ . The results of the numerical runs are shown as diamonds. The power-law fits are dashed lines.

within the tube. Here we derive the force exerted by an ensemble of flux tubes on the unmagnetized fluid.

We will assume that fluid is permeated with magnetic flux tubes, each maintained in statistical steady state by turbulence. The fractional volume occupied by tubes with fluxes in the range  $(\Phi, \Phi + d\Phi)$  will be denoted  $f(\Phi)d\Phi$ . Because of the assumption of incompressibility, these flux tubes will maintain the same fractional volume even as they evolve, so  $f(\Phi)$  will remain constant over time.

All aspects of the back-reaction will depend on the distribution function  $f(\Phi)$ , about which we can make only a few conjectures. The fraction of volume, or *filling factor*, occupied by all flux tubes is

$$f_\Phi = \int f(\Phi)d\Phi \leq 1. \quad (29)$$

In order for the dynamical equations of an isolated tube to apply to each flux tube, we should assume that  $f_\Phi \ll 1$ . The magnetic energy density within a tube is  $B^2/8\pi$ , where the field strength is maintained by the turbulence at its equilibrium value  $B(\Phi)$ , given by equation (28). Thus, the magnetic energy density of the entire fibril fields is

$$\epsilon_M = \frac{1}{8\pi} \int B^2 f(\Phi) d\Phi = \frac{K^2}{8\pi} \rho^{5/4} l_t v_t^{5/2} \langle \Phi^{-1/2} \rangle_\Phi f_\Phi, \quad (30)$$

where  $\langle \rangle_\Phi$  is a volume-weighted average over flux tubes only; it is the integral with respect to  $f(\Phi)d\Phi/f_\Phi$ . Comparing this to the kinetic energy density of the turbulence yields the inverse of the turbulent plasma  $\beta$ ,

$$\beta_t^{-1} \equiv \frac{2\epsilon_M}{\rho v_t^2} = \frac{K^2}{4\pi} \Phi_t^{1/2} \langle \Phi^{-1/2} \rangle_\Phi \Phi^{-1/2} f_\Phi, \quad (31)$$

which is expected to be less than 1.

#### 4.1. Shear Modulus

The fluid exerts a force on the magnetic flux tube through aerodynamic drag. According to Newton's third law, the flux tube must exert an equal and opposite force back on the fluid. In models with only a single flux tube this back-reaction is usually ignored. In a fluid permeated with flux tubes the back-reaction must be taken into account. By assumption, the aerodynamic drag in each tube is balanced by magnetic tension. Thus, the force felt by the fluid is exactly that of the magnetic tension forces within the collection of flux tubes. The fluid will feel a force equal to the sum of all the tension forces from the ensemble of flux tubes.

The magnetic tension in a thin flux tube is a manifestation of the Lorentz force, which can be written as the divergence of Maxwell stress tensor. The Maxwell stress tensor includes an isotropic piece  $-B^2\delta_{ij}/8\pi$ , which gives rise to a magnetic pressure gradient. Each flux tube is assumed to be in pressure balance with the unmagnetized fluid, so the magnetic pressure gradient is always offset by a gradient in plasma pressure. Thus, we need only consider the anisotropic part of the remaining piece of the Maxwell stress tensor,

$$\mathbf{T} = \frac{B^2}{4\pi} \hat{\mathbf{s}}\hat{\mathbf{s}}, \quad (32)$$

where the magnetic field has been taken to be parallel to the tube's axis.

Summing the contributions of all flux tubes within a given element of volume gives a net stress tensor<sup>2</sup>

$$\boldsymbol{\sigma} = \int \mathbf{T}f(\Phi)d\Phi = 2\epsilon_M \langle \hat{\mathbf{s}}\hat{\mathbf{s}} \rangle_\Phi, \quad (33)$$

where we have assumed that the directions,  $\hat{\mathbf{s}}$ , are distributed independently of the fluxes. For an isotropic distribution of  $\hat{\mathbf{s}}$  the stress tensor becomes that of a negative pressure  $\frac{2}{3}\epsilon_M\delta_{ij}$ . Evidently the collected tensions of the isotropic ensemble of flux tubes cancel two-thirds of the magnetic pressure. As a result of the very high  $\beta$ , it is likely that the response to this would be a slight compression of the plasma to compensate.

The plasma pressure cannot compensate any anisotropic contribution to  $\boldsymbol{\sigma}$  that may arise as a result of an anisotropic distribution of  $\hat{\mathbf{s}}$ . We began by assuming that the flux tube was buffeted by isotropic turbulence driving it to an isotropic steady state. If there were a shear flow in addition to the turbulence's, the flux tubes would develop an anisotropy.

Consider the stress at the  $z = 0$  plane when the fluid is instantaneously sheared horizontally. Prior to the shearing the flux tubes are in equilibrium with the isotropic turbulence, and so  $\hat{\mathbf{s}} = \hat{\mathbf{s}}_0$  is isotropically distributed. An instantaneous shearing,  $x \rightarrow x + \gamma z$ , will distort all flux tubes. After the shearing the directions will have an additional horizontal component

$$\hat{\mathbf{s}} \simeq \hat{\mathbf{s}}_0 + \gamma \hat{\mathbf{x}}(\hat{\mathbf{z}} \cdot \hat{\mathbf{s}}_0).$$

Averaging a product of these directions and using the fact that  $\hat{\mathbf{s}}_0$  is isotropic provides an off-diagonal shear stress

$$\boldsymbol{\sigma} = \frac{2}{3}\gamma\epsilon_M(\hat{\mathbf{x}}\hat{\mathbf{z}} + \hat{\mathbf{z}}\hat{\mathbf{x}}). \quad (34)$$

This stress creates a horizontal force  $\boldsymbol{\sigma} \cdot \hat{\mathbf{z}} = 2\hat{\mathbf{x}}\gamma\epsilon_M/3$  on the horizontal plane.

Expressing the example above in terms of the symmetric strain tensor  $\mathbf{e} = \gamma(\hat{\mathbf{x}}\hat{\mathbf{z}} + \hat{\mathbf{z}}\hat{\mathbf{x}})/2$  gives

$$\boldsymbol{\sigma} = \frac{4}{3}\epsilon_M\mathbf{e}. \quad (35)$$

The coefficient in such a stress-strain relation is the *elastic shear modulus*  $G = (4/3)\epsilon_M$ . Thus, the magnetic flux tubes endow the turbulent fluid with elastic properties, at least on very short timescales.

#### 4.2. Shear Viscosity

We summarize here a well-known argument in polymer physics for the emergence of viscoelastic contributions to the viscosity. The instantaneous shear considered above perturbs the isotropic equilibria of the flux tubes. The isotropic turbulence will then buffet the flux tubes, causing them to relax back to their isotropic equilibrium distribution, and its contribution to the stress will vanish. Each flux

<sup>2</sup> This definition is consistent with "stress tensors" in electrodynamics (Jackson 1975), elasticity (Landau & Lifshitz 1986), and Newtonian fluid mechanics (Batchelor 1967), whereby the force density is given by  $+\mathbf{V} \cdot \boldsymbol{\sigma}$ . The force on a volume element includes contributions  $\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}$  over its closed surface, where  $\hat{\mathbf{n}}$  is the *outward normal*. Since hydrodynamic pressure acts inward, it will contribute  $-p\delta_{ij}$  to  $\sigma_{ij}$ . In a few instances the term "stress tensor" is defined with the opposite sign, making it synonymous with the momentum flux tensor  $\Pi_{ij}$  (Bird et al. 1987).

tube will relax on a timescale  $\tau_G$  as each of the Fourier modes in its perturbation relaxes. The stress tensor  $\sigma$  will thus decay similarly. To recover a Newtonian viscosity, we consider the mean decay time for all modes of the tube.

The limit opposite to the impulsive shear is one in which the fluid is continuously sheared. This will lead to a persistent anisotropy in which the rate of strain  $\dot{\epsilon}$  is balanced by the relaxation

$$\langle \hat{s}\hat{s} \rangle = \frac{1}{3}I + \tau_G \dot{\epsilon}, \tag{36}$$

where the average here is over the axes of a given kind of flux tube. Placing this in equation (33) and retaining only the anisotropic terms gives a linear relation  $\sigma = \mu \dot{\epsilon}$ , where the coefficient

$$\mu = \frac{1}{4\pi} \int f(\Phi) B^2 \tau_G d\Phi \tag{37}$$

is the *dynamic shear viscosity*.<sup>3</sup>

The timescale  $\tau_G$  describes the relaxation of a given flux tube to its statistical equilibrium. This will depend on how the flux tubes within the tangle interact with one another. The dynamical interaction between high- $\beta$  flux tubes, including full and partial reconnection, is still being actively studied (Linton, Dahlburg, & Antiochos 2001) and cannot yet be easily incorporated into a model like ours. Nevertheless, it is possible to proceed under two different assumptions, which conceptually bracket all foreseeable cases. The *Rouse limit* obtains when flux tubes can reconnect so readily that they pass through one another effortlessly. The *entangled limit* obtains when flux tubes can never reconnect and bounce off each other instead. We expect the viscoelastic parameters found in these two limiting cases to bound those in cases in which flux tubes reconnect in more complicated ways.

<sup>3</sup> It is customary in MHD to denote the dynamic shear viscosity by  $\mu$  in order to reserve  $\eta$  for the magnetic diffusivity. We follow this custom here even though we will always assume a perfectly conducting plasma. Nor will this cause confusion with the elastic shear modulus, which we denote  $G$ .

4.2.1. *The Rouse Model*

Since flux tubes in the Rouse limit do not interact with one another, each obeys the dynamical equations of a single tube. An impulsive shear introduces perturbations  $r_q$  at all wavelengths up to one extending over the tube's entire length  $\lambda = L$ . A mode of wavelength  $\lambda$  will relax over a time  $\tau_\lambda$  given by equation (18). We will define this as  $\tau_\lambda = \tau_L (q_L/q)^2$ , where the wavenumber  $q = (\partial s/\partial s_0) 2\pi/\lambda$ ,  $q_L$  is the wavenumber for a mode covering the whole tube, and  $\tau_L$  is its relaxation time. The net effect of the dynamics of all these modes on the effective time-dependent decay of the shear modulus takes the form

$$G(t) = G(0) \frac{q_L}{q_t} \sum_{p=1}^{q_t/q_L} \exp\left(-\frac{2t}{\tau_L/p^2}\right), \tag{38}$$

where  $q_t$  is the wavenumber corresponding to the  $l_t$ , the smallest perturbation. The relaxation time for the entire tube is that which characterizes the decay of the shear modulus (Doi & Edwards 1986),

$$\tau_G = \int_0^\infty \frac{G(t)}{G(0)} dt \simeq \frac{l_t}{L} \frac{\tau_L}{2} \sum_{p=1}^\infty \frac{1}{p^2} = \frac{\zeta L l_t}{48 v_A^2}, \tag{39}$$

using equation (18) for  $\tau_L = \tau_\lambda(L)$ . This quantity depends on total flux  $\Phi$ , as well as on the total length of the tube  $L$ .

For concreteness we take our flux tubes to be a set of closed curves with a distribution of total lengths  $L$ . The total length is not a constant of the motion for a thin flux tube, but under the assumption of incompressibility, the total volume  $V$  is. Thus, our population of flux tubes will be described by their distribution of fluxes  $\Phi$  and volumes  $V$ .

A more convenient constant of the motion to describe the flux tube population is the volume per flux

$$\Upsilon \equiv \frac{V}{\Phi} = \frac{L}{B}. \tag{40}$$

The distribution of flux tubes may have arisen through processes of fragmentation, self-reconnection, and their inverses (see Fig. 4). None of these processes are described

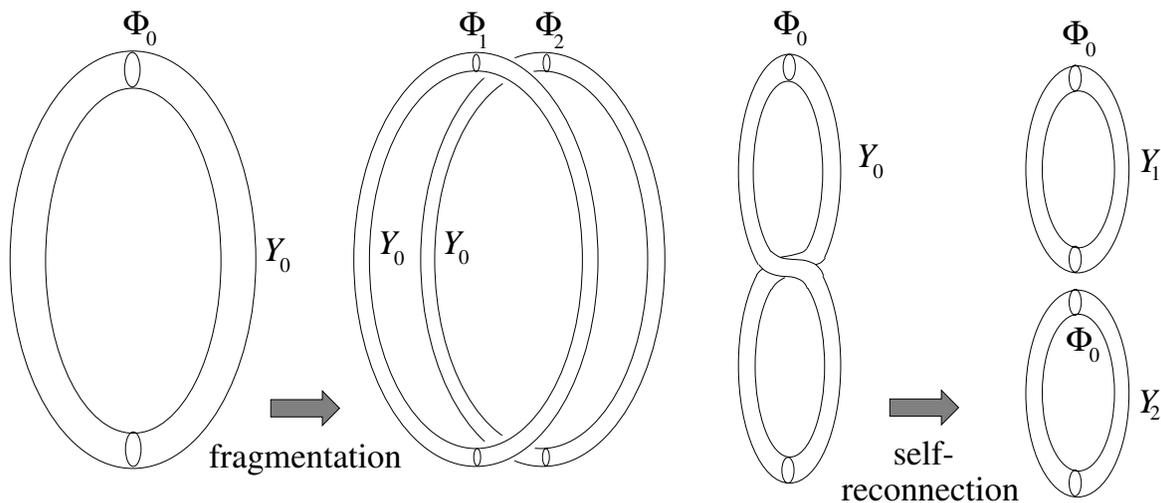


FIG. 4.—Processes of fragmentation and self-reconnection by which a single closed flux tube may break into two. Under fragmentation (*left*) a tube of flux  $\Phi_0$  breaks into tubes with  $\Phi_1 + \Phi_2 = \Phi_0$ , and the volume per flux  $\Upsilon_0$  is preserved. Under self-reconnection (*right*) a tube with volume per flux  $\Upsilon_0$  breaks into tubes such that  $\Upsilon_1 + \Upsilon_2 = \Upsilon_0$ , and the flux  $\Phi_0$  is preserved.

by equation (2), nor do they preserve flux or volume. A closed tube fragments when its cross section breaks into two pieces, giving rise to two closed flux rings of equal length. The resulting tubes have fluxes that sum to the flux of the original tube, but  $\Upsilon$  in each tube is the same as in the original;  $\Upsilon$  is preserved by fragmentation. A closed tube can break into two shorter tubes by crossing itself and reconnecting at the crossing point. The  $\Upsilon$ -values of the resulting tubes sum to that of the original, but the flux  $\Phi$  is preserved in each piece. After many fragmentations, self-reconnections, and inverses the set of tubes will have a distribution of  $\Phi$ - and  $\Upsilon$ -values. If the rates of the individual processes were independent of the conserved quantity, then  $\Phi$  and  $\Upsilon$  would be uncorrelated, and the fractional volume would be a product

$$f(\Phi, \Upsilon) = f(\Phi)p(\Upsilon), \quad (41)$$

where  $f(\Phi)$  is as before and  $p(\Upsilon)$  is the distribution of volume per mass among individual flux tubes.

Using the relaxation time given by equations (39) and (37) and averaging over both  $\Phi$  and  $\Upsilon$  yields a viscosity

$$\mu_R = \frac{K^{3/2}\sqrt{\pi}}{128} \rho^{3/2} v_t^2 \Phi_t^{7/8} f_\Phi \langle \Upsilon \rangle_\Phi \langle \Phi^{-7/8} \rangle_\Phi, \quad (42)$$

where the equilibrium magnetic field strength, equation (28), has been used. In order to obtain quantitative estimates, we use the small-velocity limit of  $\zeta$ , equation (6), here and in later expression. It is not clear that this approximation describes every Fourier mode in a general flux tube, but the full set of numerical simulations required to derive the scaling of  $\zeta$  is beyond the scope of this paper. The small-velocity limit does apply to the longest wavelengths of a tube, and we thus adopt it to facilitate the calculation of viscoelastic properties.

When a shear flow is initiated, the strain  $e$  will increase linearly in time. As it does the elastic stress  $Ge$  increases as well. After a time

$$\bar{\tau}_r = \frac{\mu_R}{G} = \frac{3\pi^{3/2}}{64K^{1/2}} \Phi_t^{3/8} \langle \Phi^{-7/8} \rangle_\Phi \langle \Phi^{-1/2} \rangle'_\Phi \rho^{1/2} \langle \Upsilon \rangle_\Phi, \quad (43)$$

the stress will approach the viscous stress, to which it will ultimately asymptote. This is the average relaxation rate for all of the flux tubes. If the distribution contains an accumulation of vanishingly small flux tubes so that  $\langle \Phi^{-7/8} \rangle_\Phi$  diverges, then this asymptote will never occur. Smaller flux tubes have greater aerodynamic drag and couple more effectively to the fluid, and it appears that when their density is sufficiently great this strong coupling prevents the kind of yielding that typifies individual thin flux tubes. This state of affairs is similar to the case of continuum MHD, which contains a purely elastic back-reaction force in the form of the Lorentz force. Moreover, an infinite relaxation time  $\bar{\tau}_r$  implies that the stress tensor increases without bound and any flow will eventually be quenched by the elastic stresses of the flux tubes. Conversely, when there are few enough of these arbitrarily small flux tubes that  $\langle \Phi^{-7/8} \rangle_\Phi$  remains bounded, the stress will cease its elastic growth after finite time  $\sim \bar{\tau}_r$ .

#### 4.2.2. The Entangled Model

Opposite to the Rouse limit, where flux tubes can pass effortlessly through one another, is the entangled limit,

where the flux tubes form mutually impenetrable barriers. The resulting gas of entangled, flexible chains is far more difficult to analyze than the noninteracting Rouse system. A single flux tube finds itself confined within an ever evolving tangled mesh composed of all other tubes. The starting point for analysis is the length density

$$\rho_L = \int f(\Phi) \frac{B}{\Phi} d\Phi = \frac{K f_\Phi}{l_t^2} \frac{\langle \Phi^{-5/4} \rangle_\Phi}{\Phi_t^{-5/4}}, \quad (44)$$

characterizing the ensemble of tubes. The length density is the length of tubes (regardless of flux) per unit volume of permeated fluid. A plane segment of area  $A$  will be pierced an average of  $A\rho_L$  times by tubes of different fluxes.

The tubes offer severe mutual constraints on dynamics, termed “entanglements” in polymer physics. Of significance for this work, the presence of the entanglement constraints greatly increases relaxation times and thus the effective fluid viscosity. Tubes undergoing interacting (entangled) dynamics will endow the fluid with higher viscosity than tubes with an identical distribution of  $\Phi$  and  $\Upsilon$  undergoing noninteracting (Rouse) dynamics.

Surprisingly, the presence of these topological entanglement constraints introduces a new length scale called the entanglement length,  $l_e$ . Tube segments smaller than  $l_e$  will be almost unaffected by the entanglement constraints, while longer segments experience strong constraints on their dynamics. The entanglement length scale is much longer than the typical distance between tubes because the motion of a “test tube” must couple to several others in order to “feel” a strong topological constraint. Therefore, even in a dense melt of entangled polymer chains, segments that just span a single entanglement length  $l_e$  can consist of up to  $10^2$  monomer units.

In spite of its importance, a quantitative theory for the entanglement length  $l_e$  is still an unsolved problem in polymer physics. Recent contributions of scaling theory (Colby & Rubinstein 1990), simulations (Everaers 1998), and careful empirical deductions from systematic data (Fetters et al. 1999) have led to the following current view: (i) When the chains are dense, the entanglement length is a near universal  $O(10)$  multiple of the “packing length” of the monomer units (in our case this is the turbulent correlation length  $l_t$ ). (ii) When the chains are not dense, the entanglement length scales as a power of the length density  $l_e \sim (\rho_L l_t^2)^{-\alpha}$  with the exponent  $\alpha$  taking values close to 1, depending on the exact physics of the interactions (candidate models have suggested values of exactly 1 and  $4/3$ ; McLeish 2003). Combining these two criteria provides an estimate of the entanglement length for our problem,

$$l_e \simeq 10 l_t^{-5/3} \rho_L^{-4/3} = 10 K^{-4/3} l_t f_\Phi^{-4/3} \Phi_t^{-5/3} \langle \Phi^{-5/4} \rangle_\Phi^{-4/3}. \quad (45)$$

Since the total filling factor  $f_\Phi \ll 1$ , the entanglement length will be many times  $l_t$ .

At present, the most successful model for entangled dynamics considers the set of entanglements to form a constraining “tunnel”<sup>4</sup> of radius  $l_e$  about each flexible chain. This forces it into curvilinear motions along the

<sup>4</sup> The polymer physics literature invariably uses the term “tube” here. We have chosen to substitute the term “tunnel” since our fundamental chain is also called a “tube” (i.e., of flux).

constraining tunnel at scales of  $l_e$  and longer. The resulting dynamics is characterized by a relaxation time of the full chain that scales a power of  $l_e$ ,

$$\tau_r^{(e)}(L) \simeq \tau_R(l_e) \left(\frac{L}{l_e}\right)^\xi = \tau_R(L) \left(\frac{L}{l_e}\right)^{\xi-2}. \quad (46)$$

Since the constrained relaxation cannot proceed faster than the unconstrained (Rouse model) relaxation, we see from the final expression that  $\xi \geq 2$ .

One well-studied example of constrained motion is the simple curvilinear diffusion, termed “reptation” (Doi & Edwards 1986), of linear chains (i.e., chains with two free ends) for which  $\xi > 3$  (Doi & Edwards 1986). It is not clear whether magnetic flux tubes are better modeled as linear chains (with free ends) or as closed *rings*. There has been a considerable, although more restricted, investigation in the polymer physics literature into the dynamics of entangled ring polymers experimentally (Roovers 1988) and theoretically (Rubinstein 1986; Obhukov, Rubinstein, & Duke 1994; Brown, Lenczycki, & Szamel 2001). If the rings are not mutually interlinked, then, in the presence of an effective *network* of permanent constraints, they are thought to form self-similar structures known from percolation theory as “lattice animals” (Cates & Deutsch 1986). These have the property that their radius of gyration  $R_g \sim L^{1/4}$  rather than  $L^{1/2}$  for linear chains. The diffusion of loops at all length scales around the entangled rings endows the terminal relaxation time with a scaling intermediate between that of unentangled chains and entangled linear chains, so  $2 < \xi < 3$  in equation (46).

The behavior of rings in a melt of other rings is considerably more complex, as it involves a strong contribution of cooperative effects from the finite lifetime of the entanglement constraints. The scaling of the effective viscosity with flux tube length  $L$ , however, is certainly bounded below by the unentangled Rouse case and above by the entangled linear polymer case. A single flux tube of length  $L$  will then contribute to the viscosity

$$\mu_e(L) = \mu_R(l_e) \left(\frac{L}{l_e}\right)^\xi, \quad (47)$$

where  $\mu_R(l_e) \equiv B^2 \tau_G(l_e)/4\pi$  is the integrand of equation (37) with  $L$  replaced by  $l_e$ . Guided by the results from polymer physics, we hereafter adopt the value  $\xi = 5/2$  appropriate for an ensemble of closed rings.

Summing up the contributions given by equation (47) over the entire distribution of lengths,  $L = B\Upsilon$ , yields a Newtonian viscosity

$$\begin{aligned} \mu_e &= \frac{\sqrt{\pi} \rho v_t l_e^{-3/2}}{128} \int f(\Phi, \Upsilon) B^3 \Upsilon^{5/2} \Phi^{-1/2} d\Phi d\Upsilon \\ &= \frac{K^5 \sqrt{\pi}}{1280 \sqrt{10}} \rho^{9/4} l_e^{-3/2} v_t^{7/2} f_\Phi^3 \Phi_i^{15/4} \langle \Phi^{-5/4} \rangle_\Phi^3 \langle \Upsilon^{5/2} \rangle_\Phi, \end{aligned} \quad (48)$$

using the small-velocity limit equation (6) for  $\zeta$  and equation (45) for the entanglement length  $l_e$ . It is noteworthy that the viscosity can diverge in the entangled limit even if it would not in the Rouse limit, since it is possible for  $\langle \Phi^{-7/8} \rangle_\Phi$  to be bounded while  $\langle \Phi^{-5/4} \rangle_\Phi$  is not. As predicted, the

entangled viscosity grows much faster with filling factor than does the Rouse limit:  $\sim f_\Phi^3$  rather than  $\sim f_\Phi$ .

## 5. VISCOELASTICITY OF THE SOLAR CONVECTION ZONE

The foregoing presented a general formalism for the dynamics of a turbulent fluid permeated with flux tubes. It assumed that each flux tube was in statistical equilibrium with homogeneous, isotropic turbulence characterized by length scale  $l_t$  and velocity  $v_t$ . The back-reaction of the entire collection of flux tubes took the form of a viscoelastic response characterized by an elastic shear modulus  $G$  and a dynamic shear viscosity  $\mu$ . These quantities depended on the turbulent fluid through  $l_t$ ,  $v_t$ , and  $\rho$  and on the properties of the flux tube distribution.

In order to estimate the possible significance of flux tubes on the structure of the solar CZ, we compare their viscosity to the traditional turbulent viscosity  $\mu_t \equiv \rho v_t l_t$ . Each viscosity will depend on the local turbulence at a given depth, as modeled by mixing length theory (Böhm-Vitense 1958). We take the local values of  $l_t$  and  $v_t$  from the mixing length model and apply these to the homogeneous, isotropic calculation at that depth. Figure 5 shows the variation of the mixing length  $l_t$  along with the quantities  $\Phi_i \equiv \rho^{1/2} l_t^2 v_t$  and  $B_{\text{eq}} = (4\pi)^{1/2} \rho^{1/2} v_t$  over the CZ using the CZ model of Spruit (1974).

Viscoelastic properties also depend on the distribution of fluxes and volumes of the flux tubes, about which very little is known. It might be possible to infer the flux distribution from photospheric observation. Deep magnetograms of the quiet Sun show surface flux distributions  $N(\Phi) \sim \exp(-\Phi/\Phi)$ , where  $\Phi \simeq 3 \times 10^{18}$  Mx (Schrijver et al. 1997). Exponential distributions of this form can be explained theoretically as the result of continued fragmentation, merging, and emergence of flux tubes (Schrijver et al. 1997). Any plane passing through the hypothetical isotropic, homogeneous flux tube distribution would be pierced by tubes in the range  $(\Phi, \Phi + d\Phi)$  with a density

$$N(\Phi) d\Phi = \frac{B}{\Phi} f(\Phi) d\Phi \sim \Phi^{-5/4} f(\Phi) d\Phi, \quad (49)$$

after using equation (28) for  $B(\Phi) \sim \Phi^{-1/4}$ . Regarding the quiet-Sun photosphere to be just such a sampling plane implies that  $f(\Phi) \sim \Phi^{5/4} \exp(-\Phi/\Phi)$ .

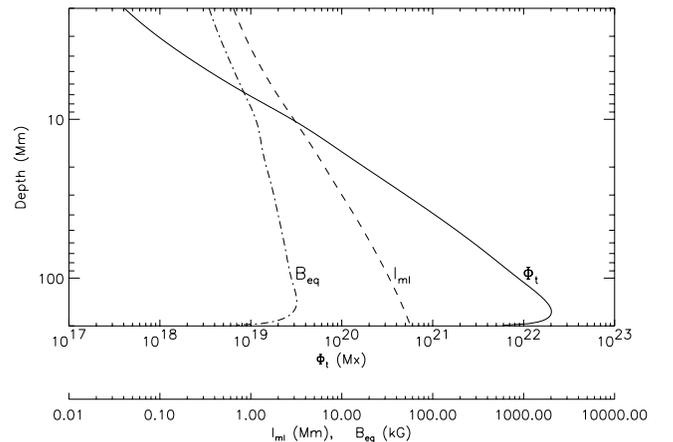


FIG. 5.—Mixing length  $l_t$  and the quantities  $\Phi_i$  and  $B_{\text{eq}}$  plotted against the depth running vertically downward.

Moments of the exponential distribution above all scale with the mean flux  $\bar{\Phi}$ ,

$$\langle \Phi^p \rangle_{\Phi} = \frac{\Gamma(p+9/4)}{\Gamma(9/4)} \bar{\Phi}^p, \quad (50)$$

as long as  $p > -9/4$ . Other well-behaved distributions  $f(\Phi)$  will have similar scalings, but with different constants. For concreteness we adopt the exponential form where  $\bar{\Phi}$  depends on depth, achieving its observed value  $\bar{\Phi} = 3 \times 10^{18}$  Mx at the solar surface.

The average of the flux distribution,  $\bar{\Phi}$ , is expected to vary with depth, perhaps in a fashion similar to the characteristic flux  $\Phi_t$  shown in Figure 5. At the solar surface, where  $\bar{\Phi}$  is actually observed,  $\Phi_t \simeq 4 \times 10^{17}$  Mx, so  $\bar{\Phi}/\Phi_t \sim 10$ . Flux tubes at the very base of the CZ are believed to be the source of bipolar active regions. If observed active regions form a representative sampling of this population, then we expect  $\bar{\Phi} \sim \Phi_t \sim 10^{22}$  Mx. It is entirely possible, even very likely, that only the large end of the population satisfies the requirements for emergence so  $\bar{\Phi}/\Phi_t < 1$  at the base of the CZ.

The volumetric filling factor of the flux tubes in the solar CZ,  $f_{\Phi}$ , is difficult to estimate since its definition depends on the assumed dichotomy between field-free plasma and flux tubes. We use equation (31) to replace  $f_{\Phi}$  with values of the turbulent  $\beta$ , which has been estimated to be a few percent (Parker 1984).

Of all the properties of the flux tube mixture, the distribution of volume to flux,  $\Upsilon$ , is least related to observable quantities. To cast estimates in more intuitive terms, we work in terms of the average of flux tube lengths

$$\langle L \rangle_{\Phi} = K \rho^{1/2} v_t \Phi_t^{1/4} \langle \Phi^{-1/4} \rangle_{\Phi} \langle \Upsilon \rangle_{\Phi}, \quad (51)$$

found by performing the same flux tube-only average to both sides of the expression  $L = \Upsilon B$  and assuming that  $\Phi$  and  $\Upsilon$  are independently distributed, as we argued previously they might be.

The largest flux tubes will be those toroidal tubes, found at the base of the CZ, whose emergence produces active regions. According to the Babcock dynamo picture, these tubes will be wound up by differential rotation, achieving a total length of 10 circumferences after one solar cycle. Using the mixing length,  $l_t = 60$  Mm, at the base of the CZ gives  $\langle L \rangle_{\Phi} \sim 500 l_t$ . Much less is known about the lengths of flux tubes in the middle or top of the CZ, so this is the best estimate we can offer. A flux tube must span several eddies in order to be maintained by turbulence against tension-induced self-collapse. Thus, no single flux tube in the distribution can be shorter than  $\sim 10 l_t$  (having a diameter of  $3 l_t$ ), and the average of *all* tubes could not be much smaller than twice this minimum,  $\langle L \rangle_{\Phi} \sim 20 l_t$ .

In the Rouse limit the effective viscosity is given by equation (42). Dividing this by the local turbulent viscosity and making the substitutions described above gives

$$\begin{aligned} \frac{\mu_R}{\mu_t} &= \frac{\pi^{3/2}}{32 K^{3/2}} \Phi_t^{1/8} \langle \Phi^{-7/8} \rangle_{\Phi} \langle \Phi^{-1/2} \rangle_{\Phi} \langle \Phi^{-1/4} \rangle_{\Phi} \frac{\langle L \rangle_{\Phi}}{l_t} \beta_t^{-1} \\ &\simeq 0.067 \left( \frac{\bar{\Phi}}{\Phi_t} \right)^{-1/8} \frac{\langle L \rangle_{\Phi}}{l_t} \beta_t^{-1}, \end{aligned} \quad (52)$$

using  $K = 2.0$  found in the simulations. This shows that the effectiveness of the flux tubes in redistributing momentum depends on their length but very little on their fluxes. Adopting the values  $\beta_t^{-1} = 0.03$  and  $\langle L \rangle_{\Phi}/l_t = 500$ , discussed above, gives  $\mu_R \simeq \mu_t$ . This demonstrates that even tubes that do not interact with one another (i.e., the Rouse model) can endow the fluid with an additional viscosity of the same magnitude as the turbulent viscosity. The effectiveness of flux tubes depends on their average length; a distribution of the shortest plausible flux tubes,  $\langle L \rangle_{\Phi} \sim 20 l_t$ , would create a viscosity only 4% of the turbulent viscosity, and their effectiveness would increase above this minimum value.

The entangled limit obtains when flux tubes are mutually impenetrable and extend over distances much greater than the entanglement length  $l_e$ . The ratio of the average tube to the entanglement length, equation (45), is

$$\begin{aligned} \frac{\langle L \rangle_{\Phi}}{l_e} &\simeq \frac{(4\pi)^{4/3}}{10 K^{4/3}} \Phi_t \langle \Phi^{-5/4} \rangle_{\Phi}^{4/3} \langle \Phi^{-1/2} \rangle_{\Phi}^{4/3} \frac{\langle L \rangle_{\Phi}}{l_t} \beta_t^{-4/3} \\ &= 1.3 \left( \frac{\bar{\Phi}}{\Phi_t} \right) \frac{\langle L \rangle_{\Phi}}{l_t} \beta_t^{-4/3}. \end{aligned} \quad (53)$$

For values  $\beta_t^{-1} = 0.03$ ,  $\bar{\Phi} = \Phi_t$ , and  $\langle L \rangle_{\Phi}/l_t = 500$  this ratio will be 6, on the verge of being much greater than unity. Thus, we expect flux tubes at the base of the CZ to exist just within the entangled limit. However, in the lower limit,  $\langle L \rangle_{\Phi} \sim 20 l_t$ , the tubes are safely out of the entangled limit and the Rouse limit viscosity given by equation (52) clearly applies to such short distributions.

In the entangled limit, the flux tubes collide with one another, permitting a more effective redistribution of momentum. The importance of interaction means that the effective viscosity scales as a higher power of the filling factor and thus a higher power of  $\beta_t^{-1}$ . The ratio of viscosities is

$$\begin{aligned} \frac{\mu_e}{\mu_t} &= \frac{\pi^{7/2}}{\sqrt{4000} K^{7/2}} \Phi_t^{13/8} \langle \Phi^{-5/4} \rangle_{\Phi}^3 \langle \Phi^{-5/8} \rangle_{\Phi} \langle \Phi^{-1/2} \rangle_{\Phi}^3 \\ &\quad \times \frac{\langle L^{5/2} \rangle_{\Phi}}{l_t^{5/2}} \beta_t^{-3} \\ &= 0.125 \left( \frac{\bar{\Phi}}{\Phi_t} \right)^{-13/8} \frac{\langle L^{5/2} \rangle_{\Phi}}{l_t^{5/2}} \beta_t^{-3}. \end{aligned} \quad (54)$$

Like the Rouse limit viscosity, this expression scales with tube lengths and inversely with turbulent  $\beta$  but also depends significantly on the average flux,  $\bar{\Phi}$ . The values used above yield a viscosity far greater than the turbulent value  $\mu_e \simeq 19 \mu_t$ .

If the addition of such a substantial viscosity could quench the dynamo action, then we might argue that the mean field strength will be limited to that value at which  $\mu_e \simeq \mu_t$ . Inverting equation (53) gives the restriction

$$\begin{aligned} \beta_t^{-1} &\leq \frac{K^{7/6} 4000^{1/6}}{\pi^{7/6}} \frac{\langle \Phi^{-5/8} \rangle_{\Phi}^{1/3} \langle \Phi^{-1/2} \rangle_{\Phi}}{\Phi_t^{13/24} \langle \Phi^{-5/4} \rangle_{\Phi}} \frac{l_t^{5/6}}{\langle L^{5/2} \rangle_{\Phi}^{1/3}} \\ &= 2.0 \left( \frac{\bar{\Phi}}{\Phi_t} \right)^{13/24} \frac{l_t^{5/6}}{\langle L^{5/2} \rangle_{\Phi}^{1/3}}. \end{aligned} \quad (55)$$

Taking  $\bar{\Phi} = \Phi_t$  and  $\langle L^{5/21/3} \rangle_{\Phi} = (500 l_t)^{5/6}$  gives a maximum value for the inverse turbulent  $\beta$  of  $\beta_t^{-1} = 0.013$ .

## 6. DISCUSSION

We have formulated a theory for a turbulent fluid permeated with fibril magnetic fields. The back-reaction of the fields on the turbulence will be a viscoelastic one. Over short timescales the field will resist deformation elastically, just as it does in conventional MHD. Unlike a conventional magnetofluid, a fibril field is not “frozen” to the fluid it permeates and will permit steady flow across its mean direction. The coupling between the fluid and the field provides an effective viscosity.

To demonstrate this claim, we developed a model of a collection of flux tubes embedded in homogeneous, isotropic turbulence. The turbulence drives each flux tube toward an equilibrium magnetic field strength at which its tension can, on average, balance the aerodynamic drag. Numerical simulations verify this fact and show how the equilibrium field strength scales with the tube’s flux and the properties of the turbulence. In equilibrium, the fibril field will react with an elastic modulus proportional to its energy density.

The effective viscosity of the fibril field depends on the rate at which the mesh of flux tubes relaxes back to its statistical equilibrium. If the mesh is even moderately dense, this relaxation process will involve interaction among the tubes. Thus, the effective viscosity depends on how colliding tubes interact. We considered two extreme cases, which conceptually bracket any realistic interaction physics: noninteracting tubes (the Rouse limit) or impenetrable tubes (the entangled limit). Modeling the CZ turbulence with the mixing length model, we found that the noninteracting limit (Rouse) produced a viscosity roughly equal to that due to turbulence. The most restrictive interactions possible (entangled) produce significant viscosity even when the magnetic field has 1% of the energy. This viscosity may well be one of the factors that limits the strength of dynamo-produced magnetic fields.

Much more study will be necessary before we can say with any confidence how the Sun’s fibril field affects the circulation in the solar CZ. Our preliminary results do, however, hold promise for some of the area’s more important questions. For example, observational evidence suggests that magnetic field strengths in active region flux tubes *exceed* that of equipartition with the turbulent energy density. This observation is at odds with continuum MHD models that face “dynamo quenching” for  $\beta_t^{-1} \gtrsim Rm^{-1/2} \sim 10^{-6}$  (Vainshtein & Cattaneo 1992). Our results show that viscoelastic response will affect the flow only for much larger energy densities of  $\beta_t^{-1} \sim 10^{-2}$ . Thus, it seems quite likely

that a mean field dynamo operating on fibril fields can produce flux tubes whose field strength locally exceeds turbulence equipartition.

It is also possible that the fibril fields in the solar CZ produce an anisotropic viscoelastic medium. Since the present work provides the first formulation of fibril-field interactions, we opted to treat the simplest case. We therefore assumed an isotropic distribution of flux tubes and ignored buoyancy. The result was a simple viscous strain involving a scalar coefficient of viscosity. Were the flux tubes aligned with a preferred direction, they would produce more strain in response to flows perpendicular to this: this viscous stress would involve a tensor coefficient. Indeed, their intrinsic buoyancy is likely to endow solar flux tubes with a radial preference, and large-scale orientation (Hale’s polarity law) will give them a toroidal preference. We expect that the result will be for the fibril field to distribute momentum most effectively along conical surfaces of constant latitude. If this turns out to be the case, then it might play a significant role in establishing the observed pattern of differential rotations, in which conical surfaces corotate.

There is one other reason to expect strong anisotropy of the field that arises from the viscoelastic nature of the fibril medium. In viscoelastic fluid flow, another dimensionless number, the *Weissenberg number*, arises from the product of the shear rate and the characteristic relaxation time of the stress. When  $\dot{\gamma}\tau_G$  is small, fibril structure is perturbed only marginally from its equilibrium statistical average, so apart from effects such as buoyancy, it will be nearly isotropic. However, when this number is large, flux tubes are strongly convected by the flow in a mean relaxation time and will therefore be strongly anisotropic. In the entangled limit, the values of  $\tau_G$  that arise from our physical estimates would indeed lead to large Weissenberg number flows at shear rates characteristic of the CZ. This opens up the possibility of other phenomena from nonlinear viscoelastic fluid dynamics such as elastic instability and strong “normal stresses” (Doi & Edwards 1986).

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