Low Complexity DOA Estimation for Wideband Off-Grid Sources Based on Re-Focused Compressive Sensing with Dynamic Dictionary

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***Abstract*—Under the compressive sensing (CS) framework, a novel focusing based direction of arrival (DOA) estimation method is first proposed for wideband off-grid sources, and by avoiding the application of group sparsity (GS) across frequencies of interest, significant complexity reduction is achieved with its computational complexity close to that of solving a single frequency based direction finding problem. To further improve the performance by alleviating both the off-grid approximation errors and the focusing errors which are even worse for the off-grid case, a dynamic dictionary based re-focused off-grid DOA estimation method is developed with the number of extremely sparse grids involved in estimation refined to the number of detected sources, and thus the complexity is still very low due to the limited complexity increase introduced by iterations, while improved performance can be achieved compared with those fixed dictionary based off-grid methods.**

***Index Terms*—Off-grid, wideband, direction of arrival (DOA), compressive sensing (CS), underdetermined.**

I. Introduction

Direction of arrival (DOA) estimation has been an active research area over the decades with applications including radar, sonar, radio astronomy, navigation, acoustics, and wireless communications [1]–[3], and will continue playing a significant role in many other aspects in the future, such as internet of things (IoT) [4], wireless sensor networks (WSN) [5], and massive multiple-input multiple-output (MIMO) systems [6]. With the development of millimeter wave techonology, the Massive (also known as large-scale) MIMO communication system has attracted great attention in recent years, offering enhanced communication network capacity, broad coverage, improved link reliability, and high spectral and energy efficiency [7]–[9]. In a massive MIMO system, DOAs are essential for beamforming and downlink precoding [10] at the base station equipped with massive antenna arrays. Therefore, the performance of wireless communication systems is sensitive to the DOA estimation performance [11], and obviously high computational complexity is one fundamental challenge related to large antenna arrays [9], [11].

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Traditionally, multiple signal classification (MUSIC) [12] and estimation of signal parameters via rotational invariance techniques (ESPRIT) [13] are two classic subspace based methods for DOA estimation. Under the compressive sensing (CS) framework, the sparse signal reconstruction method is introduced for DOA estimation with the ability of dealing both coherent and uncorrelated sources, and good performance can still be achieved for a low input signal to noise ratio (SNR) and a small number of snapshots [14]. In [15], after presenting the CS-based DOA estimation method for a single snapshot, the ℓ_1-SVD method based on singular value decomposition (SVD) is proposed for multiple snapshots. Then in [16], a method based on a sparse representation of array covariance vectors (referred to as ℓ_1-SRACV) is proposed, while the Bayesian compressive sensing strategies are studied in [17].

It is well known that $N-1$ sources can be detected based on a uniform linear array (ULA) with N sensors by employing the aforementioned estimation methods, and sparse spatial sampling (sparse array) is one solution to resolve more sources than the sensor number in the underdetermined case. The minimum redundant array (MRA) [18] and the minimum hole array (MHA) [19] are two representative examples, but there is no closed-form expressions for their geometries and it is still a challenge for designing MRA and MHA for a large number of sensors. Based on the difference co-array concept, simple array configurations including nested arrays [20] and co-prime arrays [21], [22], and their extensions against mutual coupling [23]–[27] and extensions based on high order statistics [28]–[32], have been proposed with increased degrees of freedom (DOFs) for DOA estimation. Compared with the spatial smoothing based MUSIC (SS-MUSIC) method for underdetermined narrowband DOA estimation [20], [21], [33], the CS-based method [22], [34] achieves a higher number of DOFs and better DOA results due to exploitation of all the unique co-array lags in lieu of only the consecutive part [14], [22], [29], [32]. In [35], [36], the group sparsity (GS) based underdetermined wideband DOA estimation method is developed, and the computational complexity can be reduced by combining redundant difference co-arrays together without sacrificing the performance. Then in [37], the focused compressive sensing method for direction finding in the underdetermined wideband case is presented with significant complexity reduction achieved.

Although the CS-based methods bring benefits in DOA estimation especially for the underdetermined case where higher DOFs and better performance can be achieved, the dictionary mismatch problem caused by off-grid sources is one major issue associated with the CS framework, which compromises the performance [38], [39]. A straightforward solution to avoid the off-grid effect is to construct a large overcomplete sensing matrix/dictionary with a dense search grid based on which the underlying sources can be considered as approximately on-grid, thus leading to high computational complexity which is also a challenge to be tackled in massive MIMO systems. It is suggested that the search grids should not be too dense in case the adjacent bases (steering vectors) become strongly correlated [40]. Instead of the iterative grid refining approach [15], several off-grid estimation methods are proposed to alleviate the dictionary mismatch effect by transforming the non-convex off-grid optimization into joint sparse signal and parameter estimation problem based on finite grids [41]–[44], and then gridless methods motivated by the atomic norm are studied in [40], [45], [46]. For the undedetermined off-grid case, a joint sparse recovery method is presented for narrowband DOA estimation in [47], [48], while wideband solutions are given in [49]. However, the group sparsity involved for solving wideband problems indicates that the complexity is much higher than that of the narrowband case.

Although sparse arrays can be adopted for cost and complexity reduction, the number of sensors employed in massive MIMO system is still large with resultant heavy workload. For the wideband DOA estimation probelm, how to reduce the computational complexity without compromising the estimation performance is still a major challenge.

In this paper, the underdetermined DOA estimation problem for wideband off-grid sources is studied, and we will show that significant complexity reduction can be achieved without sacrificing the performance with an initial coarse search grid. For the two-step off-grid wideband DOA estimation (TS-OG) method in [49], the most time-consuming process is to jointly recover the DOAs across all frequencies of interest based on the group sparsity constraint given their same spatial support. We first apply focusing on the virtual array corresponding to the difference co-arrays instead of the physical array, and then the virtual signal model can be combined simply and the focused off-grid signal model can be established employing the Taylor expansion. A focusing based off-grid (F-OG) wideband method is then formed by estimating the DOAs over a predefined coarse search grid and the off-grid terms separately with its complexity close to a single frequency based DOA estimation problem.

Then, we further investigate the focusing errors as well as the Taylor expansion approximation errors for the off-grid case. For the focused model: 1) only the predefined coarse grids are involved for DOA estimation and therefore focusing with even the actual DOAs may not lead to a good performance due to the focusing errors at those predefined grids; 2) Taylor expansions of the steering matrix at all frequencies of interest are not ensured to be close to that at the reference frequency after focusing, and therefore the focusing errors are essential to this dictionary mismatch effect; 3) off-grid approximation errors based on Taylor expansions are associated with the off-grid biases, and thus the focusing errors at those expansions lead to further accumulated approximation errors, which will definitely result in significant performance degradation.

To tackle these challenges, an iterative re-focused wide-band off-grid DOA estimation method based on a dynamic dictionary (DD-F-OG) is proposed. The coarse grid is used for focusing to obtain the DOA results based on the focused off-grid model in the first iteration. Then in following iterations, the updated DOA estimation results are chosen as the extremely sparse grids for dictionary generation to alleviate the off-grid effect, while a grid refining strategy also based on the DOA results is used for re-focusing. In this way, the complexity introduced by iterations is quite limited since the number of grids involved in estimation is reduced to the number of sources detected, and the total approximation errors reaches the lowest by eliminating the off-grid effect gradually compared with those methods with a fixed dictionary.

This paper is structured as follows. The wideband signal model and a review of the GS-based method for direction finding is presented in Section II. The focusing based off-grid wideband DOA estimation method with significantly reduced complexity is proposed in Section III, and the developed dynamic dictionary based re-focused off-grid wideband DOA estimation method with improved performance is given in Section IV. Simulation results are provided in Section V, and conclusions are drawn in Section VI.

II. Wideband Signal Model and Group Sparsity Based DOA Estimation

A. Wideband Signal Model

Consider an arbitrary N-sensor linear array with its sensor position set denoted as

$$\mathbb{S} = \{\hbar_n d \mid n = 0, 1, \ldots, N-1\} \ , \tag{1}$$

where $\hbar_n d$ represents the position of the n-th sensor with d being the unit inter-element spacing.

Assume that there are K wideband source signals, and these wideband signals are mutually uncorrelated. Denote $s_k(t)$ as the k-th signal with incident angle θ_k, $k = 1, 2, \ldots, K$. Then for the n-th sensor, the observed signal $x_n(t)$ can be expressed as

$$x_n(t) = \sum_{k=1}^{K} s_k\left[t - \tau_n(\theta_k)\right] + \bar{n}_n(t) \ , \tag{2}$$

where $\bar{n}_n(t)$ is the additive white Gaussian noise, and $\tau_n(\theta_k)$ represents the time delay of the k-th source signal arriving at the n-th sensor with the position $0d$ as the reference.

After sampling with a frequency f_s, where f_s is larger than the bandwidth of the source signals, the discrete version of $x_n(t)$ is denoted by $x_n[i]$, and the received signal vector is stacked as

$$\mathbf{x}[i] = \left[x_0[i], x_1[i], \ldots, x_{N-1}[i]\right]^T \ , \tag{3}$$

where $\{\cdot\}^T$ denotes the transpose operation.

We divide the received signals into several non-overlapping groups with the length L, and the array output model in the frequency domain after applying L-point discrete Fourier transform (DFT) to the p-th group is given by

$$\mathbf{X}[l,p] = \mathbf{A}(l,\boldsymbol{\theta})\mathbf{S}[l,p] + \overline{\mathbf{N}}[l,p] \ , \tag{4}$$

where the signal model at each frequency bin is assumed to fulfill the narrowband assumption. $\mathbf{X}[l,p] = \left[X_0[l,p], X_1[l,p], \ldots, X_{N-1}[l,p]\right]^T$, $l = 0,1,\ldots,L-1$, is the $N \times 1$ column signal vector at the l-th frequency bin and the p-th DFT group with

$$X_n[l,p] = \sum_{i=0}^{L-1} x_n[L\cdot(p-1)+i]\cdot e^{-j\frac{2\pi}{L}il} \ . \tag{5}$$

Similarly, $\mathbf{S}[l,p]$ and $\overline{\mathbf{N}}[l,p]$ represent the source signal vector and the noise vector in the frequency domain, respectively. $\mathbf{A}(l,\boldsymbol{\theta}) = \left[\mathbf{a}(l,\theta_1), \mathbf{a}(l,\theta_2), \ldots, \mathbf{a}(l,\theta_K)\right]$ is the $N \times K$ steering matrix associated with frequency f_l at the l-th frequency bin, and its column vector $\mathbf{a}(l,\theta_k)$ is expressed as

$$\mathbf{a}(l,\theta_k) = [e^{-j\frac{2\pi\hbar_0 d}{\lambda_l}\sin(\theta_k)}, \ldots, e^{-j\frac{2\pi\hbar_{N-1}d}{\lambda_l}\sin(\theta_k)}]^T \ , \tag{6}$$

where $\lambda_l = c/f_l$, and c is the signal propagation speed.

B. Group Sparsity Based Underdetermined Wideband DOA Estimation

Definition 1: The set of the second-order difference co-array (also known as difference co-array) $\mathbb{C}$ of the given array structure in (1) is defined as [20]

$$\begin{aligned}\mathbb{C} &= \{c_1 - c_2 \mid c_1, c_2 \in \mathbb{S}\} \\ &= \{(\hbar_{n_1} - \hbar_{n_2})d \mid n_1, n_2 = 0,1,\ldots N-1\} \ .\end{aligned} \tag{7}$$

The correlation matrix at the l-th frequency bin can be calculated by

$$\begin{aligned}\mathbf{R_{xx}}[l] &= \mathrm{E}\left\{\mathbf{X}[l,p]\cdot\mathbf{X}^H[l,p]\right\} \\ &= \sum_{k=1}^{K}\sigma_k^2[l]\mathbf{a}(l,\theta_k)\mathbf{a}^H(l,\theta_k) + \sigma_{\bar{n}}^2[l]\mathbf{I}_N \ ,\end{aligned} \tag{8}$$

where $\mathrm{E}\{\cdot\}$ is the expectation operator and $\{\cdot\}^H$ denotes the Hermitian transpose operation. $\sigma_k^2[l]$ and $\sigma_{\bar{n}}^2[l]$ are the k-th impinging signal power and the noise power at the l-th frequency bin, respectively, and $\mathbf{I}_N$ is the $N \times N$ identity matrix.

Vectorizing $\mathbf{R_{xx}}[l]$ in (8), we obtain an array model with the second order difference co-arrays in $\mathbb{C}$ as its virtual sensor positions, shown as

$$\mathbf{z}[l] = \mathrm{vec}\left\{\mathbf{R_{xx}}[l]\right\} = \mathbf{B}(l,\boldsymbol{\theta})\mathbf{u}[l] + \sigma_{\bar{n}}^2[l]\widetilde{\mathbf{I}}_{N^2} \ , \tag{9}$$

where $\mathbf{B}(l,\boldsymbol{\theta}) = \left[\mathbf{b}(l,\theta_1), \mathbf{b}(l,\theta_2), \ldots, \mathbf{b}(l,\theta_K)\right]$ with the equivalent steering vector $\mathbf{b}(l,\theta_k) = \mathbf{a}^*(l,\theta_k) \otimes \mathbf{a}(l,\theta_k)$, while $\otimes$ represents the Kronecker product. $\widetilde{\mathbf{I}}_{N^2} = \mathrm{vec}(\mathbf{I}_N)$ returns an $N^2 \times 1$ column vector, and $\mathbf{u}[l] = \left[\sigma_1^2[l], \sigma_2^2[l], \ldots, \sigma_K^2[l]\right]^T$ is the equivalent source signal vector of the virtual array.

Then, a predefined search grid $\boldsymbol{\theta_g}$ consisting of K_g potential incident angles, i.e., $\theta_{g,0}, \ldots, \theta_{g,K_g-1}$, is employed to generate an overcomplete representation of the steering matrix $\mathbf{B}(l,\boldsymbol{\theta})$, given as $\mathbf{B}(l,\boldsymbol{\theta_g}) = \left[\mathbf{b}(l,\theta_{g,0}), \ldots, \mathbf{b}(l,\theta_{g,K_g-1})\right]$, and a block diagonal matrix is constructed by

$$\widetilde{\mathbf{B}}(\boldsymbol{\theta_g}) = \mathrm{blkdiag}\left\{\mathbf{B}(l_0,\boldsymbol{\theta_g}), \mathbf{B}(l_1,\boldsymbol{\theta_g}), \ldots, \mathbf{B}(l_{Q-1},\boldsymbol{\theta_g})\right\} \ , \tag{10}$$

where we assume that there are Q frequency bins occupied by the wideband signals of interest with indexes l_q, $q = 0,1,\ldots,Q-1$ and $Q \le L$.

Denote $\mathbf{u_g}[l]$ as a $K_g \times 1$ column vector holding potential source signals over the the search grid $\boldsymbol{\theta_g}$. The group sparsity based wideband DOA estimation method is formulated as [35], [36]

$$\begin{aligned}\min_{\tilde{\mathbf{u}}_{\mathbf{g}},\mathbf{v}} \quad & \left\|\hat{\mathbf{u}}_g^{\circ}\right\|_1 \\ \text{subject to} \quad & \left\|\tilde{\mathbf{z}} - \widetilde{\mathbf{B}}(\boldsymbol{\theta_g})\tilde{\mathbf{u}}_{\mathbf{g}} - \mathbf{W}\mathbf{v}\right\|_2 \le \varepsilon \ ,\end{aligned} \tag{11}$$

where $\|\cdot\|_1$ returns the ℓ_1 norm, while $\|\cdot\|_2$ the ℓ_2 norm. $\mathbf{W} = \mathrm{blkdiag}\left\{\widetilde{\mathbf{I}}_{N^2}, \widetilde{\mathbf{I}}_{N^2}, \ldots, \widetilde{\mathbf{I}}_{N^2}\right\}$ is a $QN^2 \times Q$ block diagonal matrix, and ε is the allowable error bound. It is noted that $\tilde{\mathbf{u}}_{\mathbf{g}} = \left[\mathbf{u}_{\mathbf{g}}^T[l_0], \mathbf{u}_{\mathbf{g}}^T[l_1], \ldots, \mathbf{u}_{\mathbf{g}}^T[l_{Q-1}]\right]^T$ and $\mathbf{v} = \left[\sigma_{\bar{n}}^2[l_0], \sigma_{\bar{n}}^2[l_1], \ldots, \sigma_{\bar{n}}^2[l_{Q-1}]\right]^T$ are considered as matrix/vector holding unknown variables to be estimated, and

$$\hat{\mathbf{u}}_{\mathbf{g}}^{\circ} = \left[\left\|\tilde{\mathbf{u}}_{\mathbf{g},0}\right\|_2, \left\|\tilde{\mathbf{u}}_{\mathbf{g},1}\right\|_2, \ldots, \left\|\tilde{\mathbf{u}}_{\mathbf{g},K_g-1}\right\|_2, \left\|\mathbf{v}\right\|_2\right]^T \ , \tag{12}$$

with $\tilde{\mathbf{u}}_{\mathbf{g},k_g}$ being the k_g-th row vector of the matrix $\widetilde{\mathbf{U}}_{\mathbf{g}} = \left[\mathbf{u}_{\mathbf{g}}[l_0], \mathbf{u}_{\mathbf{g}}[l_1], \ldots, \mathbf{u}_{\mathbf{g}}[l_{Q-1}]\right]$.

Remark 1: The first K_g elements in $\hat{\mathbf{u}}_{\mathbf{g}}^{\circ}$ represent the estimated DOA results over the K_g search grids. For each frequency bin, the variables to be estimated is reduced to $(K_g+1)\times 1$ after vectorization based on the co-array concept, leading significant complexity reduction compared with the ℓ_1-SVD proposed in [15] where only the narrowband case ($Q = 1$) is considered. Furthermore, more sources than the number of physical sensors can be resolved by the GS method when sparse arrays such as nested array and co-prime array are employed, and therefore it is possible to reduce the physical sensors (less complexity) or, in other words, increase the number of users for a certain communication network.

III. Focusing Based Wideband DOA Estimation for Off-Grid Sources

For the wideband off-grid case, the group sparsity based off-grid (GS-OG) method is proposed to jointly recover the DOA results based on the predefined search grid and the off-grid bias vector [49], while the two-step off-grid (TS-OG) method [49] estimates them one by one with improved performance as well as reduced complexity. However, extremely high computational complexity is still the main drawback for all the GS associated methods, where $\mathbf{u}_g[l_q], \forall q = 0,1,\ldots,Q-1$, are estimated simultaneously to achieve better DOA results under the GS constraint. The complexity increases sharply with the number of co-arrays in $\mathbb{C}$ (equal to the number of rows in $\mathbf{B}(l_q,\boldsymbol{\theta_g})$), which is related to the number of physical sensors. Therefore, it is important to develop low complexity off-grid wideband DOA estimation method under the CS framework, especially for applications where a large array is employed. In this section, a focusing based off-grid wideband DOA estimation method under the CS framework is proposed, and its complexity is close to the co-array based DOA estimation problem for a single frequency.

A. Focusing on the Virtual Array

By applying the focusing algorithm, the signal sub-spaces across the frequencies of interest are aligned to a reference frequency with the generated focusing matrices [50], and then those signal models within the frequency bins of interest can be simply combined due to having nearly the same steering matrix, leading to less complexity required for the direction finding problem.

Denote f_r associated with the l_r-th frequency bin as the reference frequency, and the relationship $d \leq \frac{\lambda_{l_r}}{2} = \frac{c}{2f_r}$ should be satisfied to ensure the focused model is aliasing-free. f_r is commonly chosen as the center frequency within the bandwidth of interest to ensure good estimation performance. By applying the focusing algorithm of rotational signal-subspace (RSS) [51] on the virtual array with $\boldsymbol{\theta_F}$, we can obtain the $N^2 \times N^2$ RSS focusing matrix $\mathbf{T}[l]$ by solving the optimization problem as follows:

$$\begin{aligned} \min \quad & \|\mathbf{B}(l_r, \boldsymbol{\theta_F}) - \mathbf{T}[l]\mathbf{B}(l, \boldsymbol{\theta_F})\|_F \\ \text{subject to} \quad & \mathbf{T}^H[l]\mathbf{T}[l] = \mathbf{I}_{N^2} \ , \end{aligned} \tag{13}$$

and its solution is [37], [51]

$$\mathbf{T}[l] = \mathbf{V}[l]\mathbf{U}^H[l] \ , \tag{14}$$

where $\mathbf{B}(l, \boldsymbol{\theta_F})$ is constructed using $\boldsymbol{\theta_F}$, and $\|\cdot\|_F$ is the Frobenius norm. The column vectors in $\mathbf{U}[l]$ and $\mathbf{V}[l]$ are the left and right singular vectors of $\mathbf{B}(l, \boldsymbol{\theta_F})\mathbf{B}^H(l_r, \boldsymbol{\theta_F})$, respectively.

Then the virtual array model in (9) at the l-th frequency can be transformed as

$$\begin{aligned} \mathbf{y}[l] &= \mathbf{T}[l]\mathbf{z}[l] \\ &= \mathbf{T}[l]\mathbf{B}(l, \boldsymbol{\theta})\mathbf{u}[l] + \sigma_{\bar{n}}^2[l]\mathbf{T}[l]\widetilde{\mathbf{I}}_{N^2} \\ &\approx \mathbf{B}(l_r, \boldsymbol{\theta})\mathbf{u}[l] + \sigma_{\bar{n}}^2[l]\mathbf{T}[l]\widetilde{\mathbf{I}}_{N^2} \ . \end{aligned} \tag{15}$$

Remark 2: As illustrated in [37], better performance can be achieved by applying focusing on the virtual array model described in (9) directly in lieu of the physical array model in (4) since the accumulated model mismatch error in virtual array generation is avoided. The focusing performance is sensitive to $\boldsymbol{\theta_F}$, a short discussion about the selection of $\boldsymbol{\theta_F}$ will be given in *Remark 3-(2)*, and further analysis together with improvement on the wideband DOA estimation performance via dynamic dictionary based re-focused off-grid algorithm will be presented in Section IV.

B. Focused Off-Grid Compressive Sensing Solution

After focusing, a single wideband model can be obtained by averaging the signals at all frequency bins of interest given their shared equivalent steering matrix as follows

$$\bar{\mathbf{y}} = \frac{1}{Q}\sum_{q=0}^{Q-1}\mathbf{y}[l_q] \ . \tag{16}$$

For on-grid sources where their DOAs fall exactly on the predefined grids, this single wideband model under the CS framework with $\boldsymbol{\theta_g}$ can be rewritten as

$$\begin{aligned} \bar{\mathbf{y}} &= \frac{1}{Q}\sum_{q=0}^{Q-1}\mathbf{y}[l_q] \\ &= \mathbf{B}(l_r, \boldsymbol{\theta_g})\bar{\mathbf{u}}_\mathbf{g} + \frac{1}{Q}\sum_{q=0}^{Q-1}\sigma_{\bar{n}}^2[l_q]\mathbf{T}[l_q]\tilde{\mathbf{I}}_{N^2} \\ &= \mathbf{B}(l_r, \boldsymbol{\theta_g})\bar{\mathbf{u}}_\mathbf{g} + \overline{\mathbf{T}}\tilde{\mathbf{I}}_{N^2} \ , \end{aligned} \tag{17}$$

where $\bar{\mathbf{u}}_\mathbf{g} = \frac{1}{Q}\sum_{q=0}^{Q-1}\mathbf{u}_\mathbf{g}[l_q]$ is the $K_g \times 1$ column vector consisting of the potential equivalent signals to be estimated. For the Gaussian white noise assumption where $\sigma_{\bar{n}}^2[l_q] = \sigma_{\bar{n}}^2$, $\forall q = 0, 1, \ldots, Q-1$, we have $\overline{\mathbf{T}} = \frac{1}{Q}\sum_{q=0}^{Q-1}\{\mathbf{T}[l_q]\sigma_{\bar{n}}^2[l_q]\} = \frac{1}{Q}\sum_{q=0}^{Q-1}\mathbf{T}[l_q]\sigma_{\bar{n}}^2$.

Ideally $\bar{\mathbf{u}}_\mathbf{g}$ has the following form

$$\bar{u}_{g,k_g} = \begin{cases} \frac{1}{Q}\sum_{q=0}^{Q-1}\sigma_k^2[l_q], & \theta_{g,k_g} = \theta_k \ , \\ 0, & \text{others} \ , \end{cases} \tag{18}$$

where $k = 1, 2, \ldots, K$, and $\bar{u}_{g,k_g}$ is the k_g-th entry in $\bar{\mathbf{u}}_\mathbf{g}$.

In reality, the focused virtual signal model cannot be represented accurately with a finite number of grids. Instead of employing a very dense search grid which leads to extremely high computational complexity, we study the general case of off-grid sources and show that accurate DOA results can still be obtained via focused off-grid solutions based on a coarse grid, with significantly reduced complexity.

Denote θ_{g,m_k} as the nearest angle in the finite grid of the actual DOA θ_k, and then the steering vector at θ_k can be approximated by applying the Taylor expansion to θ_{g,m_k} by

$$\mathbf{b}(l, \theta_k) \approx \sum_{\mu=0}^{\infty}\frac{\partial^{(\mu)}\mathbf{b}(l, \theta_{g,m_k})}{\mu! \cdot \partial\theta_{g,m_k}^{(\mu)}}(\theta_k - \theta_{g,m_k})^\mu \ , \tag{19}$$

where $\frac{\partial^{(\mu)}\mathbf{b}(l,\theta_{g,m_k})}{\partial\theta_{g,m_k}^{(\mu)}}$ denotes the μ-th derivative of $\mathbf{b}(l, \theta_{g,m_k})$, $\mu!$ represents the factorial of μ, and $-\frac{r}{2} \leq \theta_k - \theta_{g,m_k} \leq \frac{r}{2}$ with $r = \theta_{g,k_g+1} - \theta_{g,k_g}$ being the step size of the adjacent search grid.

The focused wideband off-grid model exploiting the first-order Taylor expansion of the steering matrix $\mathbf{B}(l_r, \boldsymbol{\theta_g})$ can be approximated by

$$\bar{\mathbf{y}} \approx \big(\mathbf{B}(l_r, \boldsymbol{\theta_g}) + \mathbf{B}^{(1)}(l_r, \boldsymbol{\theta_g})\boldsymbol{\Delta_g}\big)\bar{\mathbf{u}}_\mathbf{g} + \overline{\mathbf{T}}\tilde{\mathbf{I}}_{N^2} \ , \tag{20}$$

where $\mathbf{B}^{(1)}(l_r, \boldsymbol{\theta_g}) = \big[\frac{\partial\mathbf{b}(l_r,\theta_{g,0})}{\partial\theta_{g,0}}, \ldots, \frac{\partial\mathbf{b}(l_r,\theta_{g,K_g-1})}{\partial\theta_{g,K_g-1}}\big]$, and $\boldsymbol{\Delta_g} = \mathrm{diag}\{\boldsymbol{\alpha_g}\}$ is a diagonal matrix with the perfect solution of the column bias vector $\boldsymbol{\alpha_g}$ given by

$$\alpha_{k_g} = \begin{cases} \theta_k - \theta_{g,k_g}, & k_g = m_k \ , \\ 0, & \text{others} \ , \end{cases} \tag{21}$$

for $k_g = 0, 1, \ldots, K_g - 1$, and α_{k_g} is the k_g-th entry of $\boldsymbol{\alpha_g}$.

The off-grid problem after focusing returns to a single frequency case at the reference frequency f_r, and therefore its complexity is significantly reduced without imposing the GS constraint. The wideband TS-OG method [49] with low complexity can be modified for DOA estimation based on the focused model. Note that it is a non-convex optimization problem to recover $\bar{\mathbf{u}}_\mathbf{g}$ and $\boldsymbol{\Delta}_{\boldsymbol{g}}$ jointly. As a result, a column vector $\boldsymbol{\beta}_{\boldsymbol{g}} = \boldsymbol{\Delta}_{\boldsymbol{g}}\bar{\mathbf{u}}_\mathbf{g} = \boldsymbol{\alpha}_{\boldsymbol{g}} \odot \bar{\mathbf{u}}_\mathbf{g}$ with $\odot$ representing the elementwise multiplication is defined, and we estimate $\bar{\mathbf{u}}_\mathbf{g}$ and $\boldsymbol{\alpha}_{\boldsymbol{g}}$ in lieu of $\boldsymbol{\Delta}_{\boldsymbol{g}}$ separately for convexity permission and also complexity reduction, leading to the following proposed focusing based wideband off-grid DOA estimation (referred to as F-OG) method

$$\begin{aligned} \text{Step 1:} \quad & \min_{\bar{\mathbf{u}}_\mathbf{g}, \sigma_{\bar{n}}^2} \quad \left\| \bar{\mathbf{u}}_\mathbf{g}^\circ \right\|_1 \\ & \text{subject to} \quad \left\| \bar{\mathbf{y}} - \mathbf{B}(l_r, \boldsymbol{\theta}_{\boldsymbol{g}})\bar{\mathbf{u}}_\mathbf{g} - \widetilde{\mathbf{T}}\sigma_{\bar{n}}^2 \tilde{\mathbf{I}}_{N^2} \right\|_2 \le \varepsilon, \\ \text{Step 2:} \quad & \min_{\boldsymbol{\alpha}_{\boldsymbol{g}}} \quad \left\| \Delta\bar{\mathbf{y}} - \mathbf{B}^{(1)}(l_r, \boldsymbol{\theta}_{\boldsymbol{g}})(\boldsymbol{\alpha}_{\boldsymbol{g}} \odot \bar{\mathbf{u}}_\mathbf{g}) \right\|_2 \\ & \text{subject to} \quad -\frac{r}{2}\mathbf{1}_{K_g} \preceq \boldsymbol{\alpha}_{\boldsymbol{g}} \preceq \frac{r}{2}\mathbf{1}_{K_g}, \end{aligned} \tag{22}$$

where $\widetilde{\mathbf{T}} = \frac{1}{Q}\sum_{q=0}^{Q-1}\mathbf{T}[l_q]$, $\Delta\bar{\mathbf{y}} = \bar{\mathbf{y}} - \mathbf{B}(l_r, \boldsymbol{\theta}_{\boldsymbol{g}})\bar{\mathbf{u}}_\mathbf{g} - \widetilde{\mathbf{T}}\sigma_{\bar{n}}^2\tilde{\mathbf{I}}_{N^2}$, and $\mathbf{1}_{K_g} = [1, 1, \dots, 1]^T$ is an $K_g \times 1$ column vector consisting of all ones. $\preceq$ represents $\le$ elementwise, and $\bar{\mathbf{u}}_\mathbf{g}^\circ = \left[\bar{\mathbf{u}}_\mathbf{g}^T, \sigma_{\bar{n}}^2\right]^T$.

Remark 3-(1): The CS-based formulation in the first step recovers the coarse DOAs $\bar{\mathbf{u}}_\mathbf{g}$ over the predefined search grid $\boldsymbol{\theta}_{\boldsymbol{g}}$ for the focused wideband model, followed by a minimization problem with a bounded constraint in the second step to estimate the off-grid bias vector $\boldsymbol{\alpha}_{\boldsymbol{g}}$ using the recovered $\bar{\mathbf{u}}_\mathbf{g}^\circ$. Denote $\tilde{\boldsymbol{\theta}}_K$ as the DOAs estimated in the first step and $\tilde{\boldsymbol{\alpha}}_K$ is the estimated bias vector corresponding to $\tilde{\boldsymbol{\theta}}_K$. The final DOA results are obtained by $\hat{\boldsymbol{\theta}}_K = \tilde{\boldsymbol{\alpha}}_K + \tilde{\boldsymbol{\theta}}_K$.

Remark 3-(2): Note that for the off-grid case, the exact grids corresponding to the actual DOAs may not be involved in $\boldsymbol{\theta}_{\boldsymbol{g}}$ for DOA estimation. In this way, focusing with even the actual DOAs may not lead to a good performance in (22). Towards this end, the search grid $\boldsymbol{\theta}_{\boldsymbol{g}}$ can be simply chosen for focusing to obtain a good estimation of $\bar{\mathbf{u}}_\mathbf{g}$ over the predefined grids, and further approximation errors will be analyzed and a dynamic dictionary based re-focusing solution will be proposed in Section IV.

IV. Re-Focused Wideband Off-Grid DOA Estimation Based on Dynamic Dictionary

The focusing error is sensitive to the initial DOAs $\boldsymbol{\theta}_{\boldsymbol{F}}$. The predefined search grid can be utilized as the initial DOAs to avoid the preliminary estimation of the DOAs. However, the model mismatch errors caused by focusing become worse for the off-grid case which is accumulated based on the off-grid approximation errors. In this section, we first reduce the number of grids involved in the second step of the F-OG method, and then a dynamic dictionary based re-focused wideband off-grid DOA estimation method (DD-F-OG) is proposed for performance improvement by alleviating both the focusing errors and off-grid model approximation errors.

A. Grids Reduction for the Focused Off-Grid Solution

In the second step of the proposed F-OG method, we can simply use a reduced grids with less number of potential incident angles instead of the full grids $\boldsymbol{\theta}_{\boldsymbol{g}}$, and lower complexity is achieved by the resultant minimization problem.

As a result, the focused wideband off-gird DOA estimation with reduced grids can be formulated as

$$\begin{aligned} \text{Step 1:} \quad & \min_{\bar{\mathbf{u}}_\mathbf{g}, \sigma_{\bar{n}}^2} \quad \left\| \bar{\mathbf{u}}_\mathbf{g}^\circ \right\|_1 \\ & \text{subject to} \quad \left\| \bar{\mathbf{y}} - \mathbf{B}(l_r, \boldsymbol{\theta}_{\boldsymbol{g}})\bar{\mathbf{u}}_\mathbf{g} - \widetilde{\mathbf{T}}\sigma_{\bar{n}}^2 \tilde{\mathbf{I}}_{N^2} \right\|_2 \le \varepsilon, \\ \text{Step 2:} \quad & \min_{\tilde{\boldsymbol{\alpha}}_K^1} \quad \left\| \Delta\bar{\mathbf{y}} - \mathbf{B}^{(1)}(l_r, \tilde{\boldsymbol{\theta}}_K^1)(\tilde{\boldsymbol{\alpha}}_K^1 \odot \tilde{\mathbf{u}}_K^1) \right\|_2 \\ & \text{subject to} \quad -\frac{r}{2}\mathbf{1}_K \preceq \tilde{\boldsymbol{\alpha}}_K^1 \preceq \frac{r}{2}\mathbf{1}_K, \end{aligned} \tag{23}$$

where $\tilde{\boldsymbol{\theta}}_K^1$ with the size $K \times 1$ denotes the estimated DOA results in the first step, $\tilde{\boldsymbol{\alpha}}_K^1$ is the bias vector related to $\tilde{\boldsymbol{\theta}}_K^1$, and $\tilde{\mathbf{u}}_K^1$ represents the values in $\bar{\mathbf{u}}_\mathbf{g}$ over the angles $\tilde{\boldsymbol{\theta}}_K^1$.

Obviously, the estimated DOA results are $\hat{\boldsymbol{\theta}}_K^1 = \tilde{\boldsymbol{\alpha}}_K^1 + \tilde{\boldsymbol{\theta}}_K^1$. The dimension of the variables to be estimated is reduced from K_g to the detected number of sources K, resulting in less complexity since $K_g \gg K$.

We can introduce an iteration strategy by adding the estimated $\tilde{\boldsymbol{\alpha}}_K$ related term into the first step in the next iteration. However, the focusing errors remain the same, and thus the DOA mismatches caused by the focusing part cannot be improved. To tackle this problem, dynamic dictionary based re-focused wideband off-grid estimation method is proposed as a solution.

B. Dynamic Dictionary Based Re-Focused Wideband Off-Grid DOA Estimation

For the off-grid case under the CS framework, the approximation of the signal model at the l-th frequency is expressed as

$$\mathbf{z} \approx \left(\mathbf{B}(l, \boldsymbol{\theta}_{\boldsymbol{g}}) + \mathbf{B}^{(1)}(l, \boldsymbol{\theta}_{\boldsymbol{g}})\boldsymbol{\Delta}_{\boldsymbol{g}}\right)\mathbf{u}_\mathbf{g}[l] + \sigma_{\bar{n}}^2[l]\tilde{\mathbf{I}}_{N^2} \,. \tag{24}$$

Note that $\boldsymbol{\Delta}_{\boldsymbol{g}}$ consisting of the bias vector is shared among all frequencies of interest.

After focusing, we have

$$\begin{aligned} \mathbf{y}[l] &= \mathbf{T}[l]\mathbf{z}[l] \\ &\approx \left[\mathbf{B}(l_r, \boldsymbol{\theta}) + \mathbf{T}[l]\mathbf{B}^{(1)}(l, \boldsymbol{\theta}_{\boldsymbol{g}})\boldsymbol{\Delta}_{\boldsymbol{g}}\right]\mathbf{u}_\mathbf{g}[l] + \sigma_{\bar{n}}^2[l]\mathbf{T}[l]\tilde{\mathbf{I}}_{N^2} \,. \end{aligned} \tag{25}$$

The focusing algorithm in (13) minimizes the focusing errors at angles $\boldsymbol{\theta}_{\boldsymbol{F}}$ ($\boldsymbol{\theta}_{\boldsymbol{F}} = \boldsymbol{\theta}_{\boldsymbol{g}}$ can be selected to avoid finding the DOAs in advance and to maintain a good estimation of $\bar{\mathbf{u}}_\mathbf{g}$ in step 1). However, $\mathbf{T}[l]\mathbf{B}^{(1)}(l, \boldsymbol{\theta}_{\boldsymbol{g}}) \approx \mathbf{B}^{(1)}(l_r, \boldsymbol{\theta}_{\boldsymbol{g}})$ cannot be satisfied, and it is difficult to ensure the relationship by imposing constraints due to the existence of unknown $\boldsymbol{\Delta}_{\boldsymbol{g}}$.

As a result, for the focused model: 1) only the coarse grids $\boldsymbol{\theta}_{\boldsymbol{g}}$ are involved for DOA estimation, and therefore focusing with even the actual DOAs may not lead to a good performance due to the focusing errors at those predefined grids; 2) Taylor expansions of the steering matrix after focusing $\mathbf{T}[l]\mathbf{B}^{(1)}(l, \boldsymbol{\theta}_{\boldsymbol{g}})$ are not ensured to be close to $\mathbf{B}^{(1)}(l_r, \boldsymbol{\theta}_{\boldsymbol{g}})$ at the reference frequency, and therefore the focusing errors are essential to this dictionary mismatch effect; 3) off-grid approximation errors in (20) are associated with the off-grid biases, and thus the focusing errors at $\mathbf{T}[l]\mathbf{B}^{(1)}(l,\boldsymbol{\theta_g})$ will definitely cause significant performance degradation due to further accumulated approximation errors.

An alternative solution is to modify the dictionary (search grid) iteratively based on the estimated DOA results, and the term $\mathbf{T}[l]\mathbf{B}^{(1)}(l,\boldsymbol{\theta_g})\boldsymbol{\Delta_g}$ can be ignored when $\boldsymbol{\Delta_g}\rightarrow\mathbf{0}$. Another advantage brought by this dynamic dictionary strategy is that, the off-grid model can be more accurately represented based on the first order Taylor expansion when $\boldsymbol{\Delta_g}\rightarrow\mathbf{0}$, and therefore better DOA results can be obtained.

Denote $\hat{\boldsymbol{\theta}}_K^{m-1}$ as the DOA estimates at the $(m-1)$-th iteration, and $\hat{\boldsymbol{\theta}}_K^{m-1}$ is considered as the search grid at the m-th iteration to generate the steering matrix $\mathbf{B}(l,\hat{\boldsymbol{\theta}}_K^{m-1})$ (also known as the sensing matrix or the dictionary) at the l-th frequency bin. Then, we set the focusing angles at the m-th iteration as

$$\hat{\boldsymbol{\theta}}_F^m=\left[\{\hat{\boldsymbol{\theta}}_K^{m-1}-r^m\}^T,\{\hat{\boldsymbol{\theta}}_K^{m-1}\}^T,\{\hat{\boldsymbol{\theta}}_K^{m-1}+r^m\}^T\right]^T , \tag{26}$$

where the focusing step size at the m-th iteration $r^m=\frac{r^{m-1}}{\eta}$ with $\eta\geq 1$ being a parameter for refining, and the focusing matrices are obtained by solving

$$\begin{aligned}\min\quad&\left\|\mathbf{B}(l_r,\hat{\boldsymbol{\theta}}_F^m)-\mathbf{T}_m[l]\mathbf{B}(l,\hat{\boldsymbol{\theta}}_F^m)\right\|_F\\ \text{subject to}\quad&\mathbf{T}_m^H[l]\mathbf{T}_m[l]=\mathbf{I}_{N^2}\ ,\end{aligned} \tag{27}$$

with

$$\mathbf{T}_m[l]=\mathbf{V}[l]\mathbf{U}^H[l]\ . \tag{28}$$

Remark 4: It is noted that better performance can be achieved by focusing at two adjacent angles around the estimated ones [51], and the angle interval r^m is refined in each iteration for performance improvement. Please also note that the number of entries in $\hat{\boldsymbol{\theta}}_F^m$ for focusing is $3K$, while only K sparse grids in $\hat{\boldsymbol{\theta}}_K^{m-1}$ are involved for DOA estimation.

Then, dictionaries at frequency bins of interest are re-focused to the reference frequency with the re-focused wideband virtual array model updated to

$$\begin{aligned}\mathbf{y}_m[l]&=\mathbf{T}_m[l]\mathbf{z}[l]\\&\approx\mathbf{B}(l_r,\hat{\boldsymbol{\theta}}_K^{m-1})\mathbf{u}[l]+\sigma_{\bar{n}}^2[l]\mathbf{T}_m[l]\widetilde{\mathbf{I}}_{N^2}\ ,\end{aligned} \tag{29}$$

and for the off-grid case, we have

$$\begin{aligned}\bar{\mathbf{y}}_m&=\frac{1}{Q}\sum_{l_q=0}^{Q-1}\mathbf{y}_m[l_q]\\ \approx&\big(\mathbf{B}(l_r,\hat{\boldsymbol{\theta}}_K^{m-1})+\mathbf{B}^{(1)}(l_r,\hat{\boldsymbol{\theta}}_K^{m-1})\boldsymbol{\Delta}_K^m\big)\tilde{\mathbf{u}}_K^m+\widetilde{\mathbf{T}}_m\sigma_{\bar{n}}^2\tilde{\mathbf{I}}_{N^2}\ ,\end{aligned} \tag{30}$$

where $\widetilde{\mathbf{T}}_m=\frac{1}{Q}\sum_{q=0}^{Q-1}\mathbf{T}_m[l_q]$. $\tilde{\mathbf{u}}_K^m$ is the vector to be estimated, reflecting the equivalent signals over the refined search grid $\hat{\boldsymbol{\theta}}_K^{m-1}$, and $\boldsymbol{\Delta}_K^m=\text{diag}\{\boldsymbol{\alpha}_K^m\}$ with $\boldsymbol{\alpha}_K^m$ representing the corresponding bias vector.

After re-focusing, the wideband off-grid model is updated with the refined dictionary, and the dynamic dictionary based re-focused wideband off-grid (DD-F-OG) DOA estimation method at the m-th iteration ($m\geq 2$) is formulated as

$$\begin{aligned}\text{Step 1:}\quad&\min_{\tilde{\mathbf{u}}_K^m,\sigma_{\bar{n}}^2}\quad\|\tilde{\mathbf{u}}_K^{m\circ}\|_1\\ &\text{subject to}\quad\left\|\bar{\mathbf{y}}_m-\mathbf{B}(l_r,\hat{\boldsymbol{\theta}}_K^{m-1})\tilde{\mathbf{u}}_K^m-\widetilde{\mathbf{T}}_m\sigma_{\bar{n}}^2\tilde{\mathbf{I}}_{N^2}\right\|_2\leq\varepsilon,\\ \text{Step 2:}\quad&\min_{\tilde{\boldsymbol{\alpha}}_K^m}\quad\left\|\Delta\bar{\mathbf{y}}_m-\mathbf{B}^{(1)}(l_r,\hat{\boldsymbol{\theta}}_K^{m-1})(\tilde{\boldsymbol{\alpha}}_K^m\odot\tilde{\mathbf{u}}_K^m)\right\|_2\\ &\text{subject to}\quad-\frac{r^m}{2}\mathbf{1}_K\preceq\tilde{\boldsymbol{\alpha}}_K^m\preceq\frac{r^m}{2}\mathbf{1}_K,\end{aligned} \tag{31}$$

where $\tilde{\mathbf{u}}_K^m$ is the estimated DOA results in the first step, $\tilde{\boldsymbol{\alpha}}_K^m$ is the bias vector related to $\hat{\boldsymbol{\theta}}_K^{m-1}$, $\Delta\bar{\mathbf{y}}_m=\bar{\mathbf{y}}_m-\mathbf{B}(l_r,\hat{\boldsymbol{\theta}}_K^{m-1})\tilde{\mathbf{u}}_K^m-\widetilde{\mathbf{T}}_m\sigma_{\bar{n}}^2\tilde{\mathbf{I}}_{N^2}$, and $\tilde{\mathbf{u}}_K^{m\circ}=\left[\{\tilde{\mathbf{u}}_K^m\}^T,\sigma_{\bar{n}}^2\right]^T$.

Finally, the DOA results estimated at the m-th iteration is $\hat{\boldsymbol{\theta}}_K^m=\tilde{\boldsymbol{\alpha}}_K^m+\tilde{\boldsymbol{\theta}}_K^{m-1}$.

Remark 5-(1): The DOA estimates $\hat{\boldsymbol{\theta}}_K^{m-1}$ at the $(m-1)$-th iteration is utilized as the search grid at the m-th iteration for dictionary generation, and therefore the off-grid biases in $\tilde{\boldsymbol{\alpha}}_K^m$ decreases with the increase of m. The smaller the $\tilde{\boldsymbol{\alpha}}_K^m$, the more accurate focusing approximation and also the off-grid approximation based on the first order Taylor expansion can be achieved, leading to better estimates of $\hat{\boldsymbol{\theta}}_K^m$, which is again translated to a smaller $\tilde{\boldsymbol{\alpha}}_K^{m+1}$ in the next iteration. Furthermore, the focusing mismatch error decreases with less number of entries in $\hat{\boldsymbol{\theta}}_F^m$ (reduced from K_g to $3K$ with more accurate grids) involved in the focusing process (27), which also leads to improved performance.

Remark 5-(2): The number of sources is not required for DOA estimation under the CS framework. Based on successful detections, the dimension of the sensing matrix (dictionary) reduces from $N^2\times K_g$ to $N^2\times K$ for the m-th ($m\geq 2$) iteration and also the second step in the first iteration, while the number of parameters to be estimated decreases from K_g to K. Therefore, the complexity associated with the m-th iteration is extremely low compared with that of the first iteration since $K_g\gg K$.

Remark 5-(3): An extremely dense search grid $\boldsymbol{\theta_F}$ can be employed for focusing in the first iteration to construct a frequency invariant transformation, where the focused model would be globally (at nearly all potential angles) close to the model at the reference frequency in the Frobenius manner and therefore the focusing matrices $T_1[l]$ can be used for the following iterations to avoid the re-focusing process. In this way, although the complexity is further reduced due to absence of the re-focusing process, the DOA estimation performance may not be better since the model errors at those source-related angles are not guaranteed to be smaller and on the contrary, they turn out to be larger due to the limited degrees of freedom of the system for minimising the focusing error in (13), which leads to worse performance, as will be shown in our simulations.

The procedure of the proposed DD-F-OG method with M iterations is summarized as follows:

1) Initialize $m=1$ and generate a coarse search grid $\boldsymbol{\theta_g}$ within the entire incident angles of interest with a large step size r.

2) Apply the focusing algorithm based on $\hat{\boldsymbol{\theta}}_F^1 = \boldsymbol{\theta_g}$ as in (13), and then estimate the wideband DOA results $\hat{\boldsymbol{\theta}}_K^1$ by applying the proposed DD-F-OG method with the first iteration in (23).
3) Set $m = m + 1$, and an updated dictionary is generated with $\hat{\boldsymbol{\theta}}_K^{m-1}$ employed as the refined search grid, while the updated $\hat{\boldsymbol{\theta}}_F^m$ is used for re-focusing.
4) Based on the re-focused wideband off-grid model in (29), solve the DOA estimation problem by applying the DD-F-OG method for the m-th ($m \geq 2$) iteration in (31) to obtain the estimates $\hat{\boldsymbol{\theta}}_K^m$.
5) Repeat steps 3) and 4) until $m = M$, and $\hat{\boldsymbol{\theta}}_K^M$ are the final estimation results of the wideband DOAs.

V. Simulation Results

A. Simulation Settings

In this section, an example of co-prime array with $N_1 = 3$ and $N_2 = 4$ is considered, where the inter-element spacing for the N_2-sensor sub-array is $N_1 d$, while $N_2 d$ is the spacing between adjacent sensors of the other sub-array with $2N_1 - 1$ sensors. The set of sensor positions is given by $\mathbb{S} = \{0, 3, 4, 6, 8, 9, 12, 16, 20\}\, d$ with the total number of physical sensors as $N = 2N_1 + N_2 - 1 = 9$. In the initialization step, a search grid $\boldsymbol{\theta_g}$ is generated within the full angle range from $-90°$ to $90°$ with the step size of r, and the number of grids $K_g = \frac{180}{r} + 1$.

Assume that there are $K = 12$ wideband source signals (more than the number of physical sensors) whose incident angles are uniformly distributed between $-59.25°$ and $58.75°$. The bandwidth of the impinging signals occupies $Q = 15$ frequency bins indexed from 17 to 31 after applying DFT with $L = 64$ points, and therefore the normalized frequency range is $[0.5\pi, \pi]$. The center frequency within the bandwidth of interest, i.e., normalized frequency 0.75π at the $l_r = 24$-th frequency bin with $f_r = \frac{l_r}{L} f_s$, is chosen as the reference frequency, and the unit spacing d is set as $d = \frac{\lambda_{l_r}}{2} = \frac{c}{2f_r}$. The software package CVX [52], [53] is used to solve the optimization problems for off-grid sources of the F-OG method in (22) and the DD-F-OG method (31), and the allowable error bound ε is chosen to give the best estimation results through trial-and-error in every experiment.

B. Complexity Comparison

In some applications such as massive MIMO communications, although sparse arrays can be employed for resolving more sources than the number of sensors, a large array is usually equipped and complexity reduction is always a big priority.

For further comparison with the well-known SS-MUSIC (also known as co-array MUSIC) [20], [21], [23], [33], [54] which is commonly used to deal with the underdetermined narrowband DOA estimation problem, we apply SS-MUSIC based on the focused wideband signal model in (17) to form its wideband extension, referred to as F-SS-MUSIC. Furthermore, the joint sparse recovery method [47], [48] for the underdetermined narrowband off-grid case can also be applied to the focused wideband signal model (17), leading to its straight forward wideband extension referred to as F-JS-OG.

We first compare the computational complexity of different wideband DOA estimation methods, and the number of parameters to be estimated are listed in Table I. Clearly, the number of parameters to be estimated in the GS method [36] and the TS-OG method [49] is nearly Q times larger than that of the proposed focusing based off-grid solutions, while the TS-OG method has the largest number of parameters. Furthermore, although the proposed DD-F-OG method at each iteration has the lowest complexity, more iterations are required for performance improvement.

The combination algorithm in [36] is utilized to merge the redundant co-arrays together for further complexity reduction, and the computation time, calculated by the MATLAB profiler under the environment of Intel CPU I5-4570S with the processor frequency 2.90 GHz and 8 GB RAM, is also listed in Table I. As expected, the computation time required by the GS method and the TS-OG method increases sharply with the number of grids K_g involved due to the group sparsity constraint across all frequencies of interest, while the complexities of the proposed F-OG method and the DD-F-OG method with the same iterations remain nearly the same for all step sizes employed. It is noted that for the subspace method F-SS-MUSIC, due to its extreme fast computation speed compared with those CS-based methods, a dense search grid with small step size can be employed to avoid the off-grid effect. It is also noted that for the F-JS-OG method, the computation time is larger than that of the F-OG method due to joint sparse recovery of the DOA results and the bias vector simultaneously, and its complexity increases significantly with the number of grids K_g.

C. Wideband DOA Estimation Results

For the first set of simulations, we set the input signal to noise ratio (SNR) as 0 dB, the number of snapshots at each frequency bin is 1000, and the step size r of the initial coarse grid is $3°$ for those CS-based methods. It is noted that the actual DOAs are used for focusing in F-SS-MUSIC and F-JS-OG to obtain a good estimation result, and a small step size $r = 0.05°$ is employed for the subspace method F-SS-MUSIC to ensure good performance without the off-grid effect. The DOA estimation results obtained by different underdetermined wideband methods for the off-grid case are shown in Fig. 1, where the solid lines in the figure represent the DOA estimates obtained, while the dotted lines are the actual incident angles of the off-grid sources. We can see clearly that all the 12 sources have been resolved successfully by all methods based on the 9-sensor co-prime array.

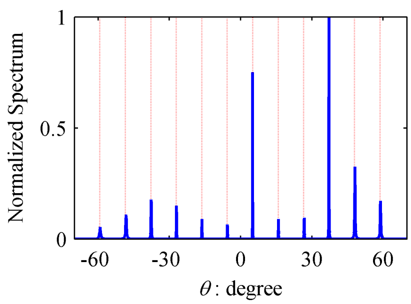
For the second set of simulations, we study the influence of the $\boldsymbol{\theta_F}$ for focusing application on the performance of the F-OG method. The root mean square error (RMSE) results versus the input SNRs and the number of snapshots are shown in Figs. 2 and 3, respectively, where the step size is $r = 3°$, and each point is based on 500 Menter Carlo simulation trials. The F-OG represents the proposed method with the coarse search

TABLE I
COMPUTATIONAL COMPLEXITY COMPARISON

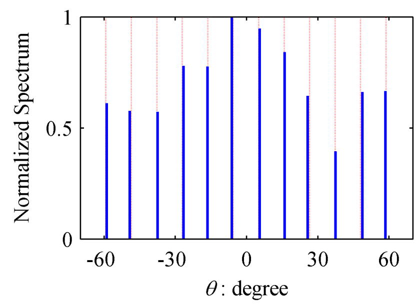
Number of parameters to be estimated							
Steps	GS [36]	TS-OG [49]	F-SS-MUSIC	F-JS-OG	F-OG	DD-F-OG: the 1-st iteration	DD-F-OG: the m-th iteration ($m \geq 2$)
Step 1	$(K_g+1)Q$	$(K_g+1)Q$	K_g	$2K_g+1$	K_g+1	K_g+1	$K+1$
Step 2	$0^{\dagger}$	K_g	$0^{\dagger}$	$0^{\dagger}$	K_g	K	K

Computation Time									
Step sizes	K_g	GS [36]	TS-OG [49]	F-SS-MUSIC	F-JS-OG	F-OG	DD-F-OG (2 iterations)	DD-F-OG (3 iterations)	DD-F-OG (4 iterations)
$r = 5^{\circ}$	37	1.7522 s	2.1253 s	0.0315 s	1.4509 s	1.0538 s	1.4431 s	1.7756 s	2.1416 s
$r = 3^{\circ}$	61	3.2160 s	3.6217 s	0.0380 s	1.8923 s	1.0641 s	1.4596 s	1.7881 s	2.1654 s
$r = 2^{\circ}$	91	4.2184 s	4.7818 s	0.0408 s	2.2028 s	1.0925 s	1.4730 s	1.8228 s	2.1914 s
$r = 1^{\circ}$	181	6.9508 s	8.4275 s	0.0571 s	3.4728 s	1.1270 s	1.5073 s	1.9063 s	2.2113 s
$r = 0.5^{\circ}$	361	14.9298 s	18.1965 s	0.0856 s	6.2890 s	1.1391 s	1.4910 s	1.8868 s	2.2539 s

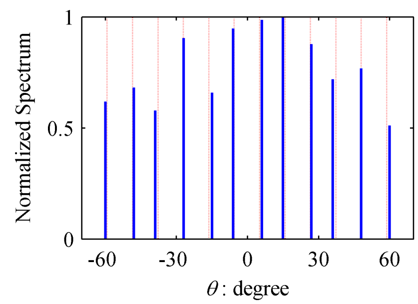
† The GS, F-SS-MUSIC, and F-JS-OG methods estimate the DOA results directly without Step 2.



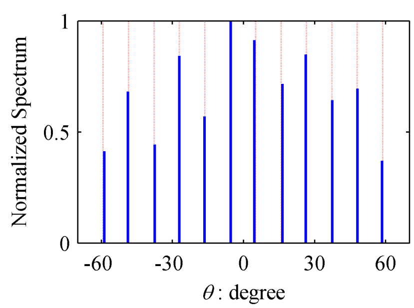
(a) DOA estimation results of F-SS-MUSIC.



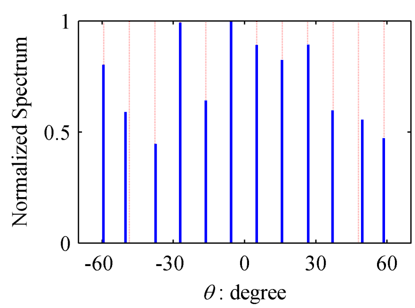
(b) DOA estimation results of the F-JS-OG method.



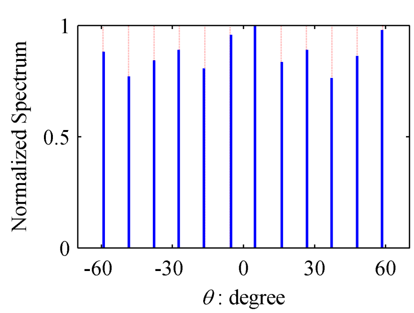
(c) DOA estimation results of the GS method.



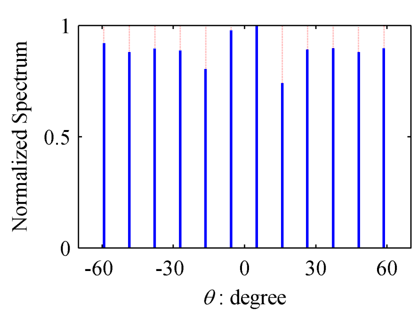
(d) DOA estimation results of the TS-OG method.



(e) DOA estimation results of the F-OG method.



(f) DOA estimation results of the DD-F-OG method with 2 iterations.



(g) DOA estimation results of the DD-F-OG method with 3 iterations.

Fig. 1. DOA estimation results obtained by different underdetermined wideband methods for the off-grid case.

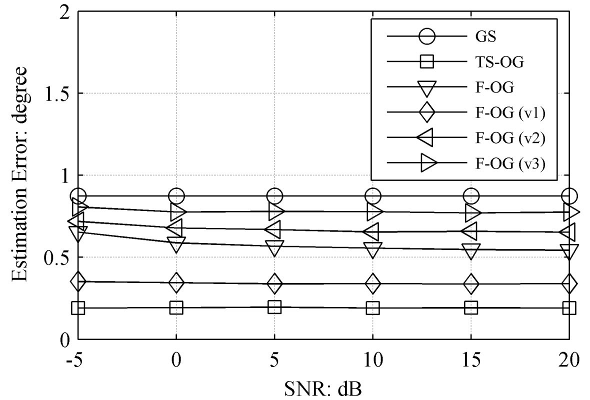


Fig. 2. RMSE results versus input SNRs for different $\boldsymbol{\theta_F}$ involved.

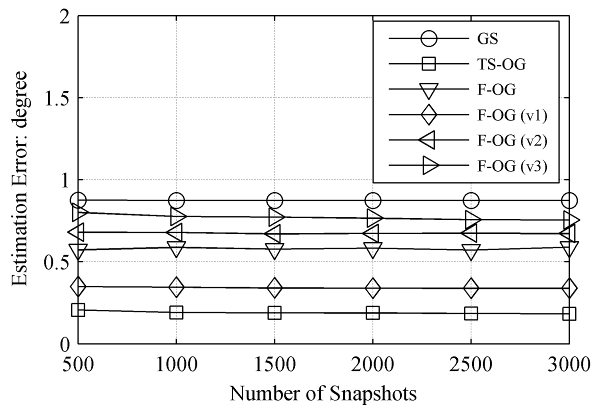


Fig. 3. RMSE results versus number of snapshots for different $\boldsymbol{\theta_F}$ involved.

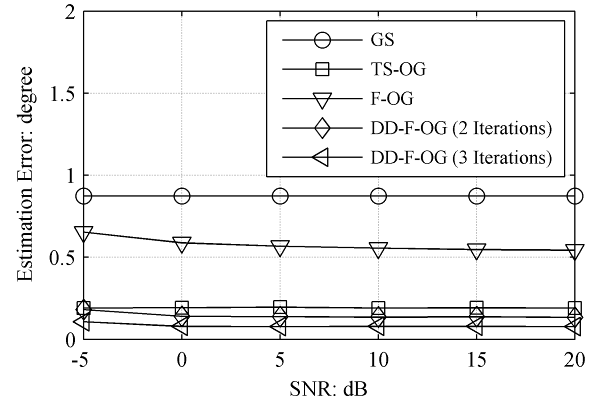


Fig. 4. RMSE results versus input SNRs for a fixed $r = 3°$.

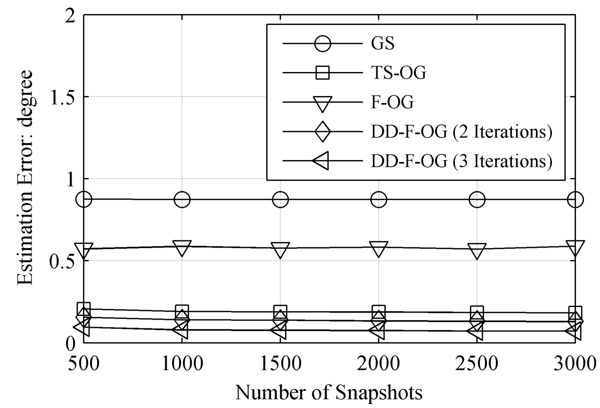


Fig. 5. RMSE results versus number of snapshots for a fixed $r = 3°$.

grid $\boldsymbol{\theta_g}$ utilized as the focusing angles with $\boldsymbol{\theta_F} = \boldsymbol{\theta_g}$, the F-OG (v1) represent the method employing $\boldsymbol{\theta_F} = \boldsymbol{\theta}$ with $\boldsymbol{\theta}$ being the actual DOAs, while an extremely dense search grid within the full range of $-90°$ to $90°$ based on a small step size $0.05°$ is used as $\boldsymbol{\theta_F}$ in F-OG (v2). Different from focusing on the virtual array in the aforementioned methods, we apply focusing on the physical array with actual DOAs in F-OG (v3). Obviously, the performance of those off-grid solutions is better than that of the GS method due to the off-grid calibration, and the performance of the F-OG (v3) with focusing on physical array based on actual DOAs is the worst among all off-grid solutions due to the accumulated system errors in generating a virtual array with more sensors. It is clear that the F-OG method outperforms the F-OG (v2) as discussed in *Remark 5-(3)*. Although the F-OG (v1) performs better than the F-OG method, it is still worse than the TS-OG method due to the accumulated system mismatch error by applying focusing to the off-grid approximation model as illustrated before, and in practice the actual DOAs are unknown parameters. That is why the DD-F-OG method is developed.

For the third set of simulations, we compare the RMSE results of different methods, and the RMSE results versus input SNRs are shown in Fig. 4, where the step size is fixed at $r = 3°$. Similarly, we can see that the GS method suffers a severe off-grid effect and has the largest estimation errors. Due to the focusing errors, although the F-OG method performs better than the GS method, its RMSE is still worse than that of the TS-OG method but with significant reduced complexity. The performance of the DD-F-OG method improves with iterations, and both DD-F-OG (2 iterations) and DD-F-OG (3 iterations) outperform other methods with low complexity achieved as verified in Table I, while DD-F-OG (3 iterations) is the best.

Fig. 5 gives the RMSE results with respect to the number of snapshots at each frequency bin, which again verifies the superior performance of the proposed DD-F-OG method.

For the fourth set of simulations, we further compare the proposed solution DD-F-OG with the F-SS-MUSIC and the F-JS-OG methods, and the RMSE results with respect to the input SNR and the number of snapshots are shown in Figs. 6 and 7, respectively, where $r = 3°$ is used for the DD-F-OG and F-JS-OG methods, while $r = 0.05°$ is employed for F-SS-MUSIC. For F-SS-MUSIC and F-JS-OG, it is worth nothing that the actual DOAs are used for focusing to ensure good performance, while refined focusing angles are adopted in the DD-F-OG method. Obviously, although the complexity of the F-SS-MUSIC is less than that of the DD-F-OG, the proposed DD-F-OG outperforms the F-SS-MUSIC consistently since all the unique co-array lags can be exploited by DD-F-OG, while F-SS-MUSIC method can only utilize the consecutive

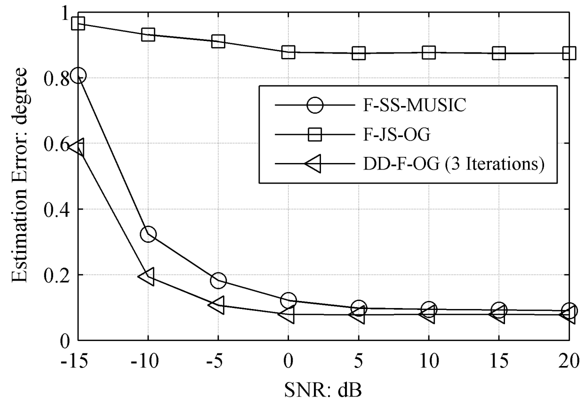


Fig. 6. RMSE results versus input SNRs of different methods.

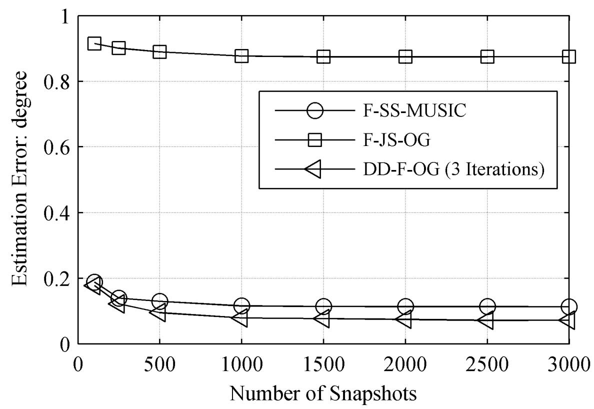


Fig. 7. RMSE results versus number of snapshots of different methods.

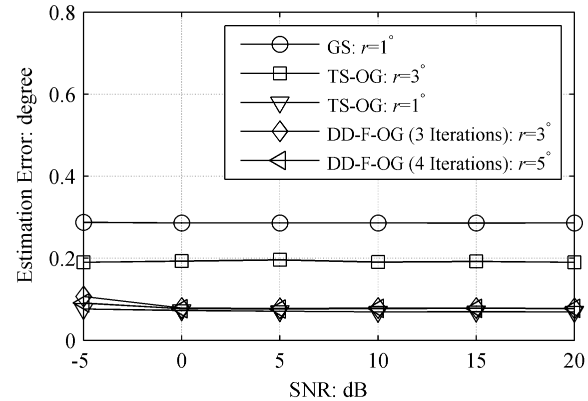


Fig. 8. RMSE results versus input SNRs for different step sizes.

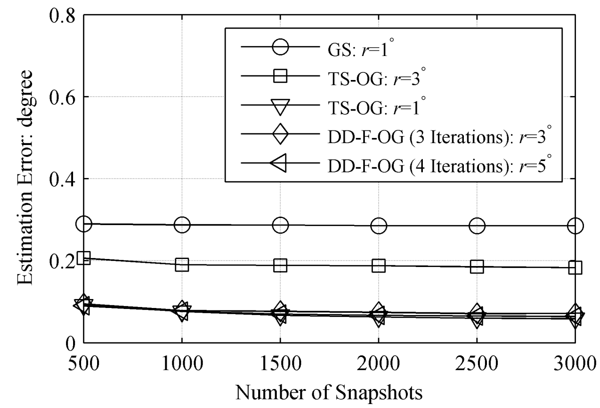


Fig. 9. RMSE results versus number of snapshots for different step sizes.

co-array lags for DOA estimation. This explains why the CS-based methods are presented and the low-complexity off-gird problems are studied in the underdetermined case [22], [29], [32]. Then for the F-JS-OG method, although the actual DOAs are employed for focusing, its performance as well as computational complexity is still the worst since the required joint recovery results in a more difficult optimization problem, and the focusing errors accumulated with the off-grid approximation error cannot be alleviated effectively with a fixed dictionary. Furthermore, for the narrowband underdetermined case, the Cramér-Rao Bound (CRB) converges to a constant value when the input SNR is sufficiently large (Theorem 4 in [55]), and this property is definitely inherited by the wideband underdetermined case. Moreover, there still exists focusing error (although it becomes extremely small in our proposed solution and the F-SS-MUSIC method) by applying the focusing algorithm, and when the SNR and the number of snapshots are large enough, the estimation performance will be mainly affected by the focusing approximation errors and therefore remain similar.

For the next set of simulations, we compare the RMSE results obtained by different wideband DOA estimation methods with different initial step sizes. The TS-OG method with $r = 3^\circ$ is redrawn as a benchmark, and the RMSEs versus the input SNR are shown in Fig. 8, while the RMSEs with respect to the number of snapshots are presented in Fig. 9. It can be concluded from the figures that GS with a smaller $r = 1^\circ$ is the worst among all methods considered, while the TS-OG method with a smaller $r = 1^\circ$ is better than TS-OG with $r = 3^\circ$ due to the reduced off-grid effect for a denser grid employed.

Furthermore, DD-F-OG (3 iterations) with a large $r = 3^\circ$, and DD-F-OG (4 iterations) with an even larger $r = 5^\circ$ share a similar good performance as the TS-OG with the smaller step size $r = 1^\circ$, verifying that the performance of the DD-F-OG method is relatively independent of the initial step size due to the iteratively refined dictionary and definitely more iterations are required for a larger step size to achieve a similar performance. As compared in Table I, it is worth nothing that without compromising the performance, only $1.7881s$ is required by the DD-F-OG (3 iterations) with $r = 3^\circ$ and $2.1416s$ for the DD-F-OG (4 iterations) with $r = 5^\circ$, both of which are quite smaller than the demand of the TS-OG method with $r = 1^\circ$ (8.4275 s). Therefore, the proposed DD-F-OG method is capable of achieving good performance with a significantly reduced complexity.

Finally, we set the step size $r = 5^\circ$, the SNR as 20 dB, and the number of snapshots as 1000. The RMSE results obtained by the proposed DD-F-OG method with respect to iteration number are given in Fig. 10, where clearly the performance

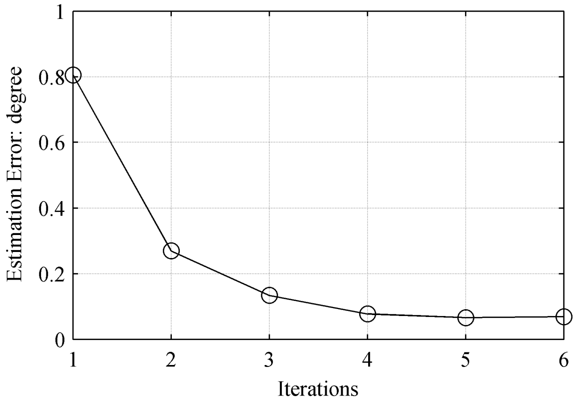


Fig. 10. RMSE results of the DD-F-OG method with respect to iteration number.

improves with iterations, and a similar performance is achieved for $m \geq 4$.

VI. Conclusions

In this paper, the DOA estimation problem with low complexity for wideband off-grid sources has been studied. The focusing based off-grid solution for the underdetermined case was first presented, where the focusing algorithm was applied to the difference co-arrays instead of the physical array, and its complexity was significantly reduced by removing the group sparsity constraint across all frequencies of interest due to their shared common spatial support. Then, after analyzing the focusing errors and the off-grid approximation errors, a re-focused wideband off-grid method based on a dynamic dictionary (DD-F-OG) was proposed to alleviate the system mismatch errors with its improved performance relatively independent of the initial coarse search grid employed, and the extra complexity associated with the iterative process is extremely low due to the lower number of refined sparse grids (equal to the number of detected sources) involved for both re-focusing and estimation. It has been shown by simulations that the proposed DD-F-OG method achieves the best performance with significantly reduced complexity compared with other wideband solutions with the same initial dictionary. It has also been shown by simulations that the performance of the proposed DD-F-OG method under a coarse dictionary is similar to the TS-OG method with dense grids (corresponding to an extremely heavy workload), and therefore less computation time is required.

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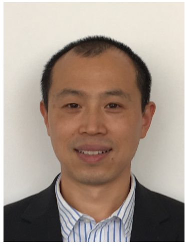

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