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Cirici, J., Egas Santander, D., Livernet, M. et al. (1 more author) (2020) Model category structures and spectral sequences. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 150 (6). pp. 2815-2848. ISSN 0308-2105

<https://doi.org/10.1017/prm.2019.45>

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MODEL CATEGORY STRUCTURES AND SPECTRAL SEQUENCES

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ABSTRACT. Let R be a commutative ring with unit. We endow the categories of filtered complexes and of bicomplexes of R -modules, with cofibrantly generated model structures, where the class of weak equivalences is given by those morphisms inducing a quasi-isomorphism at a certain fixed stage of the associated spectral sequence. For filtered complexes, we relate the different model structures obtained, when we vary the stage of the spectral sequence, using the functors shift and décalage.

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1. INTRODUCTION

Spectral sequences are important algebraic structures providing a means of computing homology groups by a process of successive approximations. They express intricate relationships among homotopy, homology, or cohomology groups arising from diverse situations. Since the introduction of spectral sequences by Leray in the nineteen-fifties, they have become essential in many branches of mathematics: spectral sequences are widely recognized as being fundamental and powerful computational tools in algebraic topology, algebraic geometry and homological algebra, at the same time as being useful techniques in analysis and mathematical physics (see [15] for examples in different contexts).

Two main algebraic sources for functorial spectral sequences are the categories of filtered complexes and of bicomplexes (also called double complexes). Given an object A in either of these two categories, its associated spectral sequence is a collection of r -bigraded complexes $\{E_r(A), \delta_r\}_{r \geq 0}$ with the property that $E_{r+1}(A) \cong H(E_r(A), \delta_r)$. Functoriality ensures that every morphism $f : A \rightarrow B$ will induce a morphism of r -bigraded complexes $E_r(f) : E_r(A) \rightarrow E_r(B)$ at each stage of the associated spectral sequence. For every $r \geq 0$ one may consider the class of morphisms f such that the induced map $E_r(f)$ at the r -stage, is a quasi-isomorphism of r -bigraded complexes. This gives a class of weak equivalences \mathcal{E}_r

2010 *Mathematics Subject Classification.* 18G55, 18G40.

Key words and phrases. filtered complex, bicomplex, spectral sequence, model category.

J. Cirici would like to acknowledge financial support from the DFG (SPP-1786), AGAUR (Beatriu de Pinós Program) and partial support from the AEI/FEDER, UE (MTM2016-76453-C2-2-P). D. Egas Santander would like to thank the Berlin Mathematical School and CRC 647 for partial financial support.

which contains all isomorphisms and satisfies the two-out-of-three property. Elements of \mathcal{E}_r are called *E_r -quasi-isomorphisms*. We have a chain of inclusions

$$\mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}_r \subseteq \mathcal{E}_{r+1} \subseteq \cdots .$$

Given a category \mathcal{C} with a class of weak equivalences \mathcal{E} , a central problem in homotopical algebra is to study the passage to the *homotopy category*: this is the localized category $\mathrm{Ho}(\mathcal{C}) = \mathcal{C}[\mathcal{E}^{-1}]$ obtained by making morphisms in \mathcal{E} into isomorphisms. Originally arising in the category of topological spaces, this is a problem of a very general nature, and central in many problems of algebraic geometry and topology. The classical approach to this problem is nowadays provided by Quillen's model categories. The verification of a set of axioms satisfied by three distinguished classes of morphisms (weak equivalences, fibrations and cofibrations) gives a reasonably general context to study the homotopy category. A particular type of model category is a cofibrantly generated one. In this case, cofibrations and trivial cofibrations are generated by sets of morphisms I and J and such categories enjoy particularly useful recognition theorems. They have good properties with respect to transfer of model structures along adjunctions. Important examples of cofibrantly generated model categories are model structures on the categories of topological spaces, of simplicial sets and of chain complexes of R -modules (see [12] and [11] for details).

Let \mathcal{C} be either the category of filtered complexes or the category of bicomplexes of R -modules, where R is a commutative ring with unit. By taking the class \mathcal{E}_r of E_r -quasi-isomorphisms, in this paper we study the *r -homotopy category* defined by inverting E_r -quasi-isomorphisms. There is a sequence of localization functors

$$\mathrm{Ho}_0(\mathcal{C}) \rightarrow \mathrm{Ho}_1(\mathcal{C}) \rightarrow \mathrm{Ho}_2(\mathcal{C}) \rightarrow \cdots .$$

We define sets I_r and J_r of generating cofibrations and trivial cofibrations and build cofibrantly generated model structures where the class of weak equivalences is given by E_r -quasi-isomorphisms.

The problem of studying the homotopy categories $\mathrm{Ho}_r(\mathcal{C})$ is not only of interest in the context of abstract homotopical algebra. Indeed, it relates to several homological and homotopical invariants of geometric and topological origin which highlight the interest of studying more flexible structures than the one provided by the initial stage \mathcal{E}_0 . We mention a few examples. In the mixed Hodge theory of Deligne [6], there are two filtrations associated to the complex of singular cochains of every complex algebraic variety: the Hodge filtration and the weight filtration. These filtrations are not well-defined but become proper invariants only up to E_0 -quasi-isomorphism (for the Hodge filtration) and E_1 -quasi-isomorphism (for the weight filtration). A second example is in the context of Sullivan's rational homotopy theory: Halperin and Tanré [10] developed a theory of minimal models of filtered differential graded algebras and defined a filtered homotopy type with respect to E_r -quasi-isomorphisms. Their theory has proven to be a useful tool in the rational homotopy theory of complex manifolds, via the Frölicher spectral sequence and the Borel spectral sequence of a principal holomorphic bundle (see [9]). Although they prove some lifting axioms for their minimal objects, the theory of Halperin and Tanré lacks an underlying model structure. Another example lies at the intersection of Deligne's mixed Hodge theory and Sullivan's rational homotopy: the rational homotopy type of a complex algebraic variety is entirely determined by the first stage of the multiplicative weight spectral sequence (see [16], [3]). Again, this is an invariant defined in the homotopy category of filtered algebras up to E_1 -quasi-isomorphism.

The homotopy theory of filtered complexes has been classically studied by considering as weak equivalences the class of morphisms of filtered complexes such that the restriction at each step of the filtration is a quasi-isomorphism. Note that this class of equivalences is

contained in \mathcal{E}_0 , and for bounded filtrations, the two classes agree (see Proposition 3.24). The first steps were done by Illusie (see Chapter V of [13]), who developed a theory of filtered injective resolutions for bounded below cochain complexes of filtered objects in an abelian category. An alternative approach in the context of exact categories was developed by Laumon [14]. More recently, Di Natale [7] provided the category of (unbounded) complexes of R -modules with non-negative decreasing filtrations, with a cofibrantly generated model structure, with the above weak equivalences. A generalization to higher stages of the results of Laumon and Illusie on filtered derived categories has been developed in [18] and [4] for bounded below filtered complexes with biregular filtrations. However, a model category approach accounting for the localization at higher stages of the spectral sequences was missing in the literature.

The homotopy theory of bicomplexes has recently been studied by Muro and Roitzheim in [17], by considering the total weak equivalences as well as the equivalences given after taking horizontal and vertical cohomology. This second class of equivalences corresponds to \mathcal{E}_1 in our setting. However, their techniques do not allow for a generalization to higher stages. Moreover, their approach is restricted to the case of bicomplexes sitting in the right half plane. Their methods do not extend to our setting, since they heavily use the fact that the spectral sequence of such a bicomplex is strongly convergent. To our knowledge, the present paper contains the first treatment of E_r -quasi-isomorphisms in the context of model categories. We next explain our main results.

Denote by \mathbf{FC}_R the category of unbounded filtered cochain complexes of R -modules. The spectral sequence of a filtered complex A may be written as a quotient $E_r(A) \cong Z_r(A)/B_r(A)$ where $Z_r(A)$ and $B_r(A)$ denote the r -cycles and r -boundaries respectively. Both Z_r and B_r are functorial for morphisms of filtered complexes. For each $r \geq 0$, we provide three different cofibrantly generated model structures for filtered complexes. These are summarized in the table below.

TABLE 1. Model structures for filtered complexes

		weak equivalences	fibrations
(A_r)	Theorem 3.14	E_r -quasi-isomorphisms	$Z_r(f)$ surjective
(B_r)	Theorem 3.16	E_r -quasi-isomorphisms	$Z_0(f)$ and $E_i(f)$ surjective for all $i \leq r$
(C_r)	Theorem 3.26	Z_r -quasi-isomorphisms	$Z_r(f)$ surjective

Model structure (B_r) is an easy consequence of (A_r) , and allows for a characterization of fibrations in terms of E_i for $i \leq r$ instead of Z_r , which may prove to be more convenient in particular situations. For each fixed r , the two model structures (A_r) and (B_r) are Quillen equivalent and (A_r) is a right Bousfield localization of (C_r) . Note that in (C_r) , weak equivalences are given by those morphisms $f : A \rightarrow B$ such that $Z_r(f)$ is a quasi-isomorphism of r -bigraded complexes. In particular, (C_0) has as weak equivalences the class of filtered quasi-isomorphisms (those morphisms inducing a quasi-isomorphism at each step of the filtration), classically considered in the study of filtered complexes.

The flexibility of filtered complexes allows comparisons of the above model structures when varying r as we next explain. Deligne introduced a pair of adjoint functors, called *shift* and *décalage*, defined in the category of filtered complexes. The spectral sequences associated to these functors are related by a shift of indexing. In Theorem 3.22 we show that shift and décalage give Quillen equivalences of the model categories

$$(A_0) \rightleftarrows (A_1) \rightleftarrows (A_2) \rightleftarrows \dots$$

and the same is true for (B_r) and (C_r) respectively, when varying $r \geq 0$.

Denote by bC_R the category of bicomplexes of R -modules. We consider the spectral sequence associated to a bicomplex defined as the spectral sequence associated to its total complex with the column filtration. (Of course, similar results hold for the row filtration.) This spectral sequence admits a very precise description in terms of certain complexes that we call *witness r -cycles* and *witness r -boundaries*, denoted by ZW_r and BW_r respectively (see Subsection 4.1). These functors have the advantage that they are representable in the category of bicomplexes. In fact, the representing complexes will play the role of the spheres and discs that are defined in the classical cofibrantly generated model structure for complexes of R -modules. For each $r \geq 0$, we provide two different cofibrantly generated model structures for bicomplexes. These are summarized in the following table.

TABLE 2. Model structures for bicomplexes

		weak equivalences	fibrations
(A'_r)	Theorem 4.37	E_r -quasi-isomorphisms	f and $ZW_r(f)$ surjective
(B'_r)	Theorem 4.39	E_r -quasi-isomorphisms	$E_i(f)$ surjective for all $i \leq r$

Again, (B'_r) is an easy consequence of (A'_r) and allows for a different characterization of fibrations. Note that (A'_r) and (B'_r) are the model category structures obtained in analogy to (A_r) and (B_r) for filtered complexes. An important difference from the case of filtered complexes is that, in the case of bicomplexes, we do not have the shift and décalage functors comparing the different structures (see Remark 4.41). This fact and the added difficulty in proving the main results for bicomplexes exhibit how these objects are much more rigid than filtered complexes. However, for a fixed r the two model structures (A'_r) and (B'_r) on the category of bicomplexes are Quillen equivalent.

The following table summarizes the key relationships between the various model structures.

TABLE 3. Relating the model structures

filtered complexes	$id : (A_r) \rightleftarrows (B_r) : id$	Quillen equivalence for each $r \geq 0$
	$S^l : (A_r) \rightleftarrows (A_{r+l}) : \text{Dec}^l$	Quillen equivalence for each $r, l \geq 0$
	$S^l : (B_r) \rightleftarrows (B_{r+l}) : \text{Dec}^l$	Quillen equivalence for each $r, l \geq 0$
	$S^l : (C_r) \rightleftarrows (C_{r+l}) : \text{Dec}^l$	Quillen equivalence for each $r, l \geq 0$
	$id : (A_r) \rightleftarrows (C_r) : id$	(A_r) is a right Bousfield localization of (C_r) for each $r \geq 0$
bicomplexes	$id : (A'_r) \rightleftarrows (B'_r) : id$	Quillen equivalence for each $r \geq 0$

The paper is organized as follows. Section 2 covers background material on the categories of filtered complexes and bicomplexes and on model structures. Section 3 presents the model structures on filtered complexes and Section 4 gives the model structures on bicomplexes.

Acknowledgments. We would like to thank Gabriel Drummond-Cole for illuminating comments regarding limits and colimits in filtered complexes and the anonymous referee for useful corrections. We would also like to thank the commune of Entrevaux for providing an atmosphere which fostered creativity.

Notation. Throughout this paper, we let R denote a commutative ring with unit. Complexes will be cohomologically graded.

2. PRELIMINARIES

In this preliminary section, we collect the main definitions and known results on filtered complexes, bicomplexes and model categories that we will use throughout the paper.

2.1. Bigraded complexes. Throughout this section we let $r \geq 0$ be an integer.

Definition 2.1. An r -bigraded complex is a (\mathbb{Z}, \mathbb{Z}) -bigraded R -module $A = \{A^{p,q}\}$ together with maps of R -modules $\delta_r : A^{p,q} \rightarrow A^{p-r, q+1-r}$ such that $\delta_r^2 = 0$. A *morphism of r -bigraded complexes* is a map of bigraded modules commuting with the differentials.

We denote by $r\text{-bC}_R$ the category of r -bigraded complexes. The cohomology of every r -bigraded complex is a bigraded R -module and it has a natural class of quasi-isomorphisms associated to it.

We will use the following homological algebra constructions.

Definition 2.2. The *translation* of an r -bigraded complex A is the r -bigraded complex $T(A)$ given by

$$T^{p,q}(A) := A^{p-r, q-r+1}.$$

Definition 2.3. Let $f : A \rightarrow B$ be a morphism of r -bigraded complexes. The *cone of f* is the r -bigraded complex $(C(f), D)$ given by

$$C^{p,q}(f) = T^{p,q}(A) \oplus B^{p,q} = A^{p-r, q-r+1} \oplus B^{p,q} \text{ with } D(a, b) = (da, f(a) - db).$$

An r -bigraded complex A is called *acyclic* if $H^{p,q}(A) = 0$ for all $p, q \in \mathbb{Z}$. Note that a morphism of r -bigraded complexes is a quasi-isomorphism if and only if its cone $C(f)$ is acyclic.

2.2. Filtered complexes. We will consider unbounded complexes of R -modules endowed with increasing filtrations indexed by the integers.

Definition 2.4. A *filtered R -module* (A, F) is an R -module A together with a family of submodules of A denoted $\{F_p A\}_{p \in \mathbb{Z}}$ indexed by the integers such that $F_{p-1} A \subseteq F_p A$ for all $p \in \mathbb{Z}$. A *morphism of filtered modules* is a morphism $f : A \rightarrow B$ of R -modules which is *compatible with filtrations*: $f(F_p A) \subseteq F_p B$ for all $p \in \mathbb{Z}$.

We will say that a filtered R -module (A, F) is *pure of weight p* if

$$0 = F_{p-1} A \subseteq F_p A = A.$$

Given a morphism of filtered modules $f : (A, F) \rightarrow (B, F)$ we will let $F_p f : F_p A \rightarrow F_p B$ denote the restriction of f to $F_p A$.

Definition 2.5. A *filtered complex* (A, d, F) is a cochain complex $(A, d) \in \mathbf{C}_R$ together with a filtration F of each R -module A^n such that $d(F_p A^n) \subseteq F_p A^{n+1}$ for all $p, n \in \mathbb{Z}$. Note in particular that $(F_p A, d|_{F_p})$ is a subcomplex of (A, d) . Denote by \mathbf{FC}_R the category of filtered complexes of R -modules. Its morphisms are given by morphisms of complexes compatible with filtrations.

Remark 2.6. The category \mathbf{FC}_R is complete and cocomplete. To see this let \mathbb{Z}_+ be the poset $\mathbb{Z} \cup \{*\}$ i.e., the poset of integers adjoin a terminal object denoted $*$. Since the category of cochain complexes \mathbf{C}_R is an abelian category, the diagram category $\mathbf{C}_R^{\mathbb{Z}_+}$ is complete and cocomplete. In particular, limits and colimits are computed object-wise. The

(co)completeness of \mathbf{FC}_R follows from [19, Theorem 4.5.15] and the fact that \mathbf{FC}_R is a reflective subcategory of $\mathbf{C}_R^{\mathbb{Z}^+}$ (i.e., the inclusion functor admits a left adjoint).

Moreover, [19, Theorem 4.5.15] gives precise constructions of limits and colimits which we describe here in our setting. Limits in \mathbf{FC}_R are computed level-wise. On the other hand, for the case of colimits let $D : I \rightarrow \mathbf{FC}_R$ be a small diagram and denote $D(i) = (D_\infty, F)$. Then we have diagrams $D_p : I \rightarrow \mathbf{C}_R$ and $D_\infty : I \rightarrow \mathbf{C}_R$ given by $D_p(i) := F_p(D(i))$ and $D_\infty(i) = D_\infty$. Note that for any p we have a canonical natural transformation $D_p \rightarrow D_\infty$ inducing a morphism of complexes between their respective colimits. Then

$$\operatorname{colim}_I(D) = (\operatorname{colim}_I(D_\infty), F)$$

where

$$F_p(\operatorname{colim}_I(D)) := \operatorname{Im}(\operatorname{colim}_I(D_p) \rightarrow \operatorname{colim}_I(D_\infty)).$$

In particular, if $f : (A, F) \rightarrow (B, F)$ is a morphism of filtered R -modules, then its kernel and cokernel are given by

$$F_p \operatorname{Ker} f = \operatorname{Ker} F_p f \text{ and } F_p \operatorname{Coker} f = F_p B / F_p B \cap f(A).$$

One can similarly show that the category of filtered modules is (co)complete.

Every filtered complex A has an associated spectral sequence $\{E_r(A), \delta_r\}_{r \geq 0}$. The r -stage $E_r(A)$ is an r -bigraded complex and may be written as the quotient

$$E_r^{p,q}(A) \cong Z_r^{p,q}(A) / B_r^{p,q}(A),$$

where the r -cycles are given by

$$Z_r^{p,n+p}(A) := F_p A^n \cap d^{-1}(F_{p-r} A^{n+1})$$

and the r -boundaries are given by $B_0^{p,n+p}(A) = Z_0^{p-1,n+p-1}(A)$ and

$$B_r^{p,n+p}(A) := Z_{r-1}^{p-1,n+p-1}(A) + dZ_{r-1}^{p+r-1,n+p+r-2}(A) \text{ for } r \geq 1.$$

Given an element $a \in Z_r(A)$, we will denote by $[a]_r$ its image in $E_r(A)$. For $[a]_r \in E_r(A)$, we have $\delta_r([a]_r) = [da]_r$. Note that both Z_r and B_r are functorial for morphisms of filtered complexes.

Definition 2.7. A morphism of filtered complexes $f : A \rightarrow B$ is called an E_r -quasi-isomorphism if the morphism $E_r(f)$ is a quasi-isomorphism of r -bigraded complexes.

Denote by \mathcal{E}_r the class of E_r -quasi-isomorphisms of \mathbf{FC}_R . This class contains all isomorphisms of \mathbf{FC}_R , satisfies the two-out-of-three property and is closed under retracts.

We will use the following result.

Lemma 2.8. *Let $r \geq 0$ and let $f : K \rightarrow L$ be a morphism of filtered complexes. The following are equivalent.*

- (1) *The maps $Z_r(f)$ and $Z_{r+1}(f)$ are bidegree-wise surjective.*
- (2) *The maps $Z_r(f)$ and $E_{r+1}(f)$ are bidegree-wise surjective.*

Proof. It suffices to prove (2) \Rightarrow (1). Let $b \in Z_{r+1}^{p,*}(L)$. The surjectivity of $E_{r+1}(f)$ gives $a \in Z_{r+1}^{p,*}(K)$ and $\beta \in B_{r+1}^{p,*}(L)$ such that $f(a) = b + \beta$. Write $\beta = x + dy$ with $x \in Z_r^{p-1,*}(L)$ and $y \in Z_r^{p+r,*}(L)$. Surjectivity of $Z_r(f)$ gives $u \in Z_r^{p-1,*}(K)$ and $v \in Z_r^{p+r,*}(K)$ such that $f(u) = x$ and $f(v) = y$, so that $f(a - u - dv) = b$. Note that one may see u as an element in $F_p K^n$ with $du \in F_{p-1-r} K^{n+1}$. This gives $u \in Z_{r+1}^{p,n+p}$. Also, $dv \in F_p K^n$ satisfies $ddv = 0$. Therefore $a - u - dv \in Z_{r+1}^{p,n+p}(K)$. \square

2.3. Bicomplexes. We consider (\mathbb{Z}, \mathbb{Z}) -bigraded R -modules $A = \{A^{i,j}\}$, where elements of $A^{i,j}$ are said to have bidegree (i, j) . The *total degree* of an element $a \in A^{i,j}$ is $|a| := j - i$. A morphism of bidegree (p, q) maps $A^{i,j}$ to $A^{i+p, j+q}$. We denote by bgMod_R the category whose objects are (\mathbb{Z}, \mathbb{Z}) -bigraded R -modules and morphisms are bidegree $(0, 0)$ maps.

Definition 2.9. The *total graded R -module* $\text{Tot}(A)$ of a bigraded R -module $A = \{A^{i,j}\}$ is given by

$$\text{Tot}(A)^n := \prod_i A^{i, n+i}.$$

The *column filtration* of $\text{Tot}(A)$ is the filtration given by

$$F_p \text{Tot}(A)^n := \prod_{i \leq p} A^{i, n+i} \text{ for all } p, n \in \mathbb{Z}.$$

Definition 2.10. A *bicomplex* (A, d_0, d_1) is a bigraded R -module $A = \{A^{i,j}\}$ together with two differentials $d_0 : A^{i,j} \rightarrow A^{i, j+1}$ and $d_1 : A^{i,j} \rightarrow A^{i-1, j}$ of bidegrees $(0, 1)$ and $(-1, 0)$ respectively, such that $d_0 d_1 = d_1 d_0$.

Definition 2.11. A *morphism of bicomplexes* $f : (A, d_0, d_1) \rightarrow (B, d_0, d_1)$ is a map of bigraded modules $f : A^{i,j} \rightarrow B^{i,j}$ of bidegree $(0, 0)$ such that $d_0 f = f d_0$ and $d_1 f = f d_1$. We denote by bC_R the category of bicomplexes.

The category bC_R is symmetric monoidal with the usual tensor product of bicomplexes.

Definition 2.12. The *total complex* of a bicomplex (A, d_0, d_1) is the cochain complex given by $(\text{Tot}(A), d)$, where $d : \text{Tot}(A)^* \rightarrow \text{Tot}(A)^{*+1}$ is defined by

$$d(a)_i := d_0(a_i) + (-1)^n d_1(a_{i+1}), \text{ for } a = (a_i)_{i \in \mathbb{Z}} \in \text{Tot}(A)^n.$$

Here $a_i \in A^{i, n+i}$ denotes the i -th component of a , and $d(a)_i$ is the i -th component of $d(a)$. Similarly, if $f : A \rightarrow B$ is a morphism of bicomplexes then it induces the morphism of cochain complexes $\text{Tot} f$ given by $(\text{Tot} f(a))_i = f(a_i)$.

The construction above yields a functor

$$\text{Tot} : \text{bC}_R \longrightarrow \mathbf{FC}_R,$$

where the total complex is endowed with a filtered complex structure by the column filtration. Of course, one could also construct such a functor using the row filtration, but we choose to fix our attention on the column filtration. Thus, every bicomplex (A, d_0, d_1) has an associated spectral sequence $\{E_r^{*,*}(A), \delta_r\}$, which is functorial for morphisms of bicomplexes. Moreover, for each $r \geq 0$, the E_r -term of the spectral sequence defines a functor

$$E_r : \text{bC}_R \longrightarrow r\text{-bC}_R.$$

In good cases, for example if the bicomplex is second quadrant, the spectral sequence converges to the cohomology of the total complex.

Remark 2.13. We have chosen to use the product version of the totalization functor, sometimes denoted Tot^{Π} . One could also use the direct sum version, Tot^{\oplus} , or the mixed version with

$$\text{Tot}^{\text{mix}}(A)^n := \prod_{i \leq 0} A^{i, n+i} \oplus \bigoplus_{i > 0} A^{i, n+i}.$$

For bicomplexes, all three give functors to complexes, and, endowed with the column filtration, to filtered complexes.

The mixed version is the only one that makes sense for certain generalizations of bicomplexes, such as (\mathbb{Z}, \mathbb{Z}) -bigraded twisted complexes (also known as multicomplexes). The

filtration of Tot^{mix} also has better properties, being both complete and exhaustive. On the other hand, viewed as functors to complexes, Tot^{mix} preserves neither limits nor colimits, whereas Tot^{Π} preserves limits and indeed is a right adjoint, and Tot^{\oplus} preserves colimits and is a left adjoint.

For the purposes of this paper, the distinction between these different versions of totalization is unimportant, since they all give rise to the same functorial spectral sequence.

The following result is well-known (see for example [5]).

Lemma 2.14. *Let (A, d_0, d_1) be a bicomplex. Then*

$$E_r^{p,q}(A) \cong Z_r^{p,q}(A)/B_r^{p,q}(A),$$

where

$$\begin{aligned} Z_0^{p,q}(A) &:= A^{p,q} \text{ and } B_0^{p,q}(A) := 0. \\ Z_1^{p,q}(A) &:= A^{p,q} \cap \text{Ker}(d_0) \text{ and } B_1^{p,q}(A) := A^{p,q} \cap \text{Im}(d_0). \end{aligned}$$

For $r \geq 2$, the r -cycles are given by

$$Z_r^{p,q}(A) := \left\{ a_0 \in A^{p,q} \mid \begin{array}{l} d_0 a_0 = 0 \text{ and there exist } a_i \in A^{p-i, q-i} \text{ for } 1 \leq i \leq r-1 \\ \text{with } d_1 a_{i-1} = d_0 a_i \text{ for all } 1 \leq i \leq r-1 \end{array} \right\}$$

and the r -boundaries are given by

$$B_r^{p,q}(A) := \left\{ x \in A^{p,q} \mid \begin{array}{l} \text{there exist } b_i \in A^{p+r-1-i, q+r-2-i} \text{ for } 0 \leq i \leq r-1 \\ \text{with } x = d_0 b_{r-1} + d_1 b_{r-2}, \\ \text{and } d_0 b_0 = 0, \\ \text{and } d_1 b_{i-1} = d_0 b_i, \text{ for all } 1 \leq i \leq r-2 \end{array} \right\}.$$

We have $\delta_0 = d_0$ and $\delta_1[a] = [d_1 a]$. For all $r \geq 2$ we have $\delta_r[a_0] = [d_1 a_{r-1}]$.

Definition 2.15. Let $r \geq 0$. A morphism of bicomplexes $f : (A, d_0, d_1) \rightarrow (B, d_0, d_1)$ is said to be an E_r -quasi-isomorphism if the morphism $E_r(f) : E_r(A) \rightarrow E_r(B)$ at the r -stage of the associated spectral sequence is a quasi-isomorphism of r -bigraded complexes (that is, $E_{r+1}(f)$ is an isomorphism).

Denote by \mathcal{E}_r the class of E_r -quasi-isomorphisms of bC_R . This class contains all isomorphisms of bC_R , satisfies the two-out-of-three property and is closed under retracts.

2.4. Model categories. We collect some definitions and results on cofibrantly generated model categories from [12].

Definition 2.16. Let \mathcal{C} be a complete and cocomplete category and I a class of maps in \mathcal{C} .

- (i) A morphism is called I -injective (resp. I -projective) if it has the right (resp. left) lifting property with respect to morphisms in I . We write

$$I\text{-inj} := \text{RLP}(I) \text{ and } I\text{-proj} := \text{LLP}(I).$$

- (ii) A morphism is called an I -fibration (resp. I -cofibration) if it has the right (resp. left) lifting property with respect to I -projective (resp. I -injective) morphisms. We write

$$I\text{-fib} := \text{RLP}(I\text{-proj}) \text{ and } I\text{-cof} := \text{LLP}(I\text{-inj}).$$

- (iii) A map is a *relative I -cell complex* if it is a transfinite composition of pushouts of elements of I . We denote by $I\text{-cell}$ the class of relative I -cell complexes.

Definition 2.17. A model category \mathcal{C} is said to be *cofibrantly generated* if there are sets I and J of maps such that the following conditions hold.

- (1) The domains of the maps of I are small relative to $I\text{-cell}$.
- (2) The domains of the maps of J are small relative to $J\text{-cell}$.

- (3) Fibrations are J -injective.
- (4) Trivial fibrations are I -injective.

The set I is called the *set of generating cofibrations*, and J the *set of generating trivial cofibrations*.

The following is a consequence of Kan's Theorem (cf. [11, Theorem 11.3.1] or [12, Theorem 2.1.19]) using compact domains in the sense of Di Natale in [7].

Theorem 2.18 (D. M. Kan). *Suppose \mathcal{C} is a category with all small colimits and limits. Let \mathcal{W} be a subcategory of \mathcal{C} and I and J sets of maps in \mathcal{C} . Then there is a cofibrantly generated model structure on \mathcal{C} with I as the set of generating cofibrations, J as the set of generating trivial cofibrations, and \mathcal{W} as the subcategory of weak equivalences if and only if the following conditions are satisfied.*

- (1) *The subcategory \mathcal{W} satisfies the two out of three property and is closed under retracts.*
- (2) *The domains of I are compact relative to I -cell.*
- (3) *The domains of J are compact relative to J -cell.*
- (4) *J -cof $\subseteq \mathcal{W}$.*
- (5) *I -inj = $\mathcal{W} \cap J$ -inj.*

Note that the categories of filtered complexes and bicomplexes we will consider satisfy the assumptions of this theorem as well as conditions (1), (2) and (3). Indeed, the category \mathbf{bC}_R of bicomplexes is abelian and thus is complete and cocomplete. The category of filtered complexes \mathbf{FC}_R is also complete and cocomplete by Remark 2.6.

3. MODEL CATEGORY STRUCTURES ON FILTERED COMPLEXES

In this section, we present three model categories for filtered complexes, each of them depending on an integer $r \geq 0$ fixing the stage of the spectral sequence at which we localize. We also compare the model categories obtained when we vary r , via the functors shift and décalage.

3.1. Representability of the cycles and boundaries functors. We next show that the functors Z_r and B_r defining the spectral sequence of a filtered complex, are representable by filtered complexes.

We will denote by $R_{(p)}$ the filtered R -module given by R concentrated in pure weight p . The notation $R_{(p)}^n$ means that we consider it in degree n within a filtered complex.

Definition 3.1. Let $p, n \in \mathbb{Z}$. For all $r \geq 0$ let

$$\mathcal{Z}_r(p, n) := \left(R_{(p)}^n \xrightarrow{1} R_{(p-r)}^{n+1} \right)$$

be the filtered complex whose only non-trivial degrees are n and $n+1$ and whose only non-trivial differential is given by the identity of R , and is compatible with filtrations. For all $r \geq 1$ define

$$\mathcal{B}_r(p, n) := \left(R_{(p+r-1)}^{n-1} \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} R_{(p)}^n \oplus R_{(p-1)}^n \xrightarrow{\begin{pmatrix} 0, 1 \end{pmatrix}} R_{(p-r)}^{n+1} \right).$$

For all $r \geq 1$ define a morphism of filtered complexes

$$\varphi_r : \mathcal{Z}_r(p, n) \longrightarrow \mathcal{B}_r(p, n)$$

via the following diagram:

$$\begin{array}{ccccc} & & R_{(p)}^n & \longrightarrow & R_{(p-r)}^{n+1} \\ & & \downarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} & & \downarrow 1 \\ R_{(p+r-1)}^{n-1} & \longrightarrow & R_{(p)}^n \oplus R_{(p-1)}^n & \longrightarrow & R_{(p-r)}^{n+1} \end{array}$$

The vertical arrows are defined via the identity on R and are easily seen to be compatible with filtrations.

The following two lemmas are direct consequences of the definitions.

Lemma 3.2. *For $r \geq 1$, we have*

$$\mathcal{B}_r(p, n) = \mathcal{Z}_{r-1}(p+r-1, n-1) \oplus \mathcal{Z}_{r-1}(p-1, n)$$

and the diagram

$$\begin{array}{ccc} \mathcal{Z}_r(p, n) & \longrightarrow & 0 \\ \varphi_r \downarrow & & \downarrow \\ \mathcal{B}_r(p, n) & \longrightarrow & \mathcal{Z}_r(p+r-1, n-1) \end{array}$$

is a pushout diagram.

Lemma 3.3. *Let $r \geq 0$ and let $p, n \in \mathbb{Z}$. Let A be a filtered complex.*

- (1) *Giving a map of filtered complexes $\mathcal{Z}_r(p, n) \rightarrow A$ is equivalent to giving $a \in Z_r^{p, n+p}(A)$.*
- (2) *Giving a map of filtered complexes $\mathcal{B}_r(p, n) \rightarrow A$ is equivalent to giving a pair (b, c) with $b \in Z_{r-1}^{p-1, n+p-1}(A)$ and $c \in Z_{r-1}^{p+r-1, n+p+r-2}(A)$.*
- (3) *Having a solid diagram of morphisms of filtered complexes*

$$\begin{array}{ccc} \mathcal{Z}_{r+1}(p, n) & \longrightarrow & A \\ \varphi_{r+1} \downarrow & \nearrow & \downarrow f \\ \mathcal{B}_{r+1}(p, n) & \longrightarrow & B \end{array}$$

is equivalent to having a triple (a, b, c) where $a \in Z_{r+1}^{p, n+p}(A)$, $b \in Z_r^{p-1, n+p-1}(B)$ and $c \in Z_{r-1}^{p+r-1, n+p+r-2}(B)$ are such that $f(a) = b + dc$.

- (4) *Having a lift in the above solid diagram is equivalent to having a pair (b', c') where $b' \in Z_r^{p-1, n+p-1}(A)$ and $c' \in Z_r^{p+r, n+p+r-1}(A)$ satisfy $a = b' + dc'$ with $f(b') = b$ and $f(c') = c$.*

Remark 3.4. All of these statements can be made functorial, so that, for example the functor $Z_r^{p, n+p}$ is the representable functor $\mathbf{FC}_R(\mathcal{Z}_r(p, n), -)$.

3.2. Some constructions in filtered homological algebra. We collect some basic homological algebra constructions for filtered complexes that we will use in the sequel.

Definition 3.5. Let $r \geq 0$. The r -translation of a filtered complex (A, d, F) is the filtered complex $(T_r(A), d, F)$ given by

$$F_p T_r^n(A) := F_{p-r} A^{n+1}.$$

This defines a functor T_r on filtered complexes, with inverse T_r^{-1} given by

$$F_p (T_r^{-1})^n(A) := F_{p+r} A^{n-1}.$$

The r -cone $(C_r(f), D, F)$ of a morphism of filtered complexes $f : A \rightarrow B$ is the filtered complex given by

$$F_p C_r(f)^n := F_p T_r^n(A) \oplus F_p B^n = F_{p-r} A^{n+1} \oplus F_p B^n \text{ with } D(a, b) = (da, f(a) - db).$$

Remark 3.6. Note that for a morphism of filtered complexes $f : A \rightarrow B$ we have

$$C^{p,q}(E_r(f)) = E_r^{p-r, q+1-r}(A) \oplus E_r^{p,q}(B),$$

and $D([a]_r, [b]_r) = ([da]_r, [f(a) - db]_r)$. The assignment $([a]_r, [b]_r) \mapsto [(a, b)]_r$ gives an isomorphism of r -bigraded complexes

$$C^{*,*}(E_r(f), D) \cong (E_r^{*,*}(C_r(f)), \delta_r).$$

In particular, f is an E_r -quasi-isomorphism if and only if the r -bigraded complex $E_r(C_r(f))$ is acyclic.

Notation 3.7. Given a filtered complex (A, d, F) we will denote by $\mathcal{M}_r(A) := T_r^{-1}C_r(1_A)$ the filtered complex given by the cone of the identity, shifted conveniently. We have

$$F_p \mathcal{M}_r^n(A) = F_p A^n \oplus F_{p+r} A^{n-1}$$

and the projection to the first component $\pi_1 : \mathcal{M}_r(A) \rightarrow A$ induces a bidegree-wise surjection $Z_k(\pi_1)$ for all $0 \leq k \leq r$. Note also that $E_r(\mathcal{M}_r(A))$ is acyclic.

Definition 3.8. Let $f, g : A \rightarrow B$ be two morphisms of filtered complexes. An r -homotopy from f to g is given by a degree preserving filtered map $h : A \rightarrow T_r^{-1}(B)$ such that $dh + hd = g - f$. This is equivalent to having a collection of morphisms of R -modules $h^n : A^n \rightarrow B^{n-1}$ such that $dh + hd = g - f$ and $h^n(F_p A^n) \subseteq F_{p+r} B^{n-1}$. We write $h : f \underset{r}{\simeq} g$.

The following result exhibits how r -homotopies are the right notion to consider when localizing with respect to E_r -quasi-isomorphisms.

Proposition 3.9 ([1], p. 321). *Let $f, g : A \rightarrow B$ be two morphisms of filtered complexes such that $f \underset{r}{\simeq} g$. Then $E_{r+1}(f) = E_{r+1}(g)$.*

3.3. Model category structures. Throughout this section we let $r \geq 0$ be an integer.

Definition 3.10. Let I_r and J_r be the sets of morphisms of \mathbf{FC}_R given by

$$I_r := \{\mathcal{Z}_{r+1}(p, n) \rightarrow \mathcal{B}_{r+1}(p, n)\}_{p, n \in \mathbb{Z}} \text{ and } J_r := \{0 \rightarrow \mathcal{Z}_r(p, n)\}_{p, n \in \mathbb{Z}}.$$

Proposition 3.11. *A morphism of filtered complexes f is J_r -injective if and only if $Z_r(f)$ is bidegree-wise surjective.*

Proof. It follows directly from (1) of Lemma 3.3. \square

Proposition 3.12. *We have $I_r\text{-inj} = \mathcal{E}_r \cap J_r\text{-inj}$.*

Proof. Assume first that $f : A \rightarrow B$ is I_r -injective. Lemma 3.2 and (1) of Lemma 3.3 imply that f is J_{r+1} -injective. Consider the solid diagram

$$\begin{array}{ccc} & & A \\ & \nearrow \gamma & \downarrow f \\ \mathcal{Z}_{r+1}(p, n) & \xrightarrow{\varphi_{r+1}} \mathcal{B}_{r+1}(p, n) & \xrightarrow{g} B \end{array}$$

Since f is J_{r+1} -injective, there exists a lift γ such that $f\gamma = g\varphi_{r+1}$. Since f is I_r -injective, there exists ψ such that $\psi\varphi_{r+1} = \gamma$ and $f\psi = g$. Hence by the first statement of Lemma 3.2, f is J_r -injective. Since $Z_{r+1}(f)$ is bidegree-wise surjective, so is $E_{r+1}(f)$. Let us prove that

$E_{r+1}(f)$ is injective. Let $a \in Z_{r+1}(A)$ such that $[f(a)] = [0]$, that is, there exist $b, c \in Z_r(B)$ such that $f(a) = b + dc$. This corresponds to the solid commutative diagram

$$(D) \quad \begin{array}{ccc} \mathcal{Z}_{r+1}(p, n) & \xrightarrow{a} & A \\ \varphi_{r+1} \downarrow & \searrow^{b'+dc'} & \downarrow f \\ \mathcal{B}_{r+1}(p, n) & \xrightarrow{b+dc} & B \end{array}$$

which admits a lift since f is I_r -injective. That is, there exists b', c' such that $a = b' + dc' \in B_{r+1}(A)$ hence $[a] = [0] \in E_{r+1}(A)$.

Conversely, assume $f \in \mathcal{E}_r \cap J_r$ -inj and consider the solid diagram (D) which amounts to consider elements $a \in Z_{r+1}^{p+*,*}(A), b + dc \in B_{r+1}^{p+*,*}(B)$ such that $f(a) = b + dc$. This gives $E_{r+1}(f)([a]) = [0]$ and the injectivity of $E_{r+1}(f)$ implies $a = b' + dc'$ for some $b' \in Z_r^{p-1,*}(A)$ and $c' \in Z_r^{p+r,*}(A)$. Applying f one gets the equation

$$b - f(b') = d(f(c') - c).$$

Note that $f(c') - c \in F_{p+r}B^{n-1}$ and the equation tells us that $d(f(c') - c) \in F_{p-1}B^n$. Therefore we have $f(c') - c \in Z_{r+1}^{p+r,*}(B)$. By Lemma 2.8, $Z_{r+1}(f)$ is bidegree-wise surjective, so there exists $u \in Z_{r+1}^{p+r,*}(A)$ so that $f(c') - c = f(u)$. Note that $du \in F_{p-1}A^n$ and that $b - f(b') = f(du)$. In conclusion setting $\beta = b' + du \in Z_r^{p-1,n}(A)$ and $\gamma = c' - u \in Z_r^{p+r,*}(A)$ one gets $a = \beta + d\gamma$ and $f(\beta) = b, f(\gamma) = c$. Finally, $\beta + d\gamma$ is the desired lift in the diagram. \square

Proposition 3.13. *For all $r \geq 0$ and all $0 \leq k \leq r$ we have J_k -cof $\subseteq \mathcal{E}_r$.*

Proof. We prove this by borrowing a technique used in [8]. Let $f : A \rightarrow B$ be a J_k -cofibration. By Proposition 3.11 this means that f has the left lifting property with respect to maps g such that $Z_k(g)$ is surjective. Consider the filtered complex $\mathcal{M}_r(B) = T_r^{-1}C_r(1_B)$ of Notation 3.7 and consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{\begin{pmatrix} id \\ 0 \end{pmatrix}} & A \oplus \mathcal{M}_r(B) \\ f \downarrow & & \downarrow (f, \pi_1) \\ B & \xrightarrow{=} & B \end{array}$$

Since $Z_k(\pi_1)$ is surjective, it follows that $Z_k(f, \pi_1)$ is also surjective, and so a lift $h : B \rightarrow A \oplus \mathcal{M}_r(B)$ exists in this diagram. Since $E_r(\mathcal{M}_r(B))$ is acyclic, applying E_{r+1} to the diagram we get that $f \in \mathcal{E}_r$. \square

Theorem 3.14. *For every $r \geq 0$, the category \mathbf{FC}_R admits a right proper cofibrantly generated model structure, where:*

- (1) *weak equivalences are E_r -quasi-isomorphisms,*
- (2) *fibrations are morphisms of filtered complexes $f : A \rightarrow B$ such that $Z_r(f)$ is bidegree-wise surjective, and*
- (3) *I_r and J_r are the sets of generating cofibrations and generating trivial cofibrations respectively.*

Proof. By Proposition 3.11, the class \mathcal{Fib}_r of r -fibrations is given by those morphisms f such that $Z_r(f)$ is bidegree-wise surjective. By Theorem 2.18 it suffices to check that $\mathcal{E}_r \cap J_r$ -inj = I_r -inj and that J_r -cof $\subseteq \mathcal{E}_r$. These follow from Propositions 3.12 and 3.13 (for the case $k = r$)

respectively. By [11] 13.1.3 right properness follows directly from the fact that all objects are fibrant. \square

In certain situations, it may be more practical to characterize fibrations via the surjectivity of E_r instead of Z_r .

Definition 3.15. Let I'_r and J'_r be the sets of morphisms of \mathbf{FC}_R given by

$$I'_r := \cup_{k=0}^{r-1} J_k \cup I_r \text{ and } J'_r := \cup_{k=0}^r J_k.$$

We have:

Theorem 3.16. *For every $r \geq 0$, the category \mathbf{FC}_R admits a right proper cofibrantly generated model structure, where:*

- (1) *weak equivalences are E_r -quasi-isomorphisms,*
- (2) *fibrations are morphisms of filtered complexes $f : A \rightarrow B$ such that $Z_0(f)$ is bidegree-wise surjective and $E_i(f)$ is bidegree-wise surjective for all $i \leq r$, and*
- (3) *I'_r and J'_r are the sets of generating cofibrations and generating trivial cofibrations respectively.*

Proof. By Lemma 2.8 and (2) of Lemma 3.3 we have that a map is J'_r -injective if and only if $Z_0(f)$ is bidegree-wise surjective and $E_i(f)$ is bidegree-wise surjective for all $i \leq r$. By Theorem 2.18 it suffices to show that

$$J'_r\text{-cof} \subseteq \mathcal{E}_r \text{ and } I'_r\text{-inj} = \mathcal{E}_r \cap J'_r\text{-inj}.$$

This follows from Propositions 3.12 and 3.13 together with the following comparison of classes of morphisms:

$$I'_r\text{-inj} = I_r\text{-inj} \cap \bigcap_{k=0}^{r-1} J_k\text{-inj} = \mathcal{E}_r \cap \bigcap_{k=0}^r J_k\text{-inj} = \mathcal{E}_r \cap J'_r\text{-inj}.$$

Right properness follows directly from the fact that all objects are fibrant. \square

Remark 3.17. Note that the fibrations of the model structure of Theorem 3.16 form a subclass of the class of fibrations of the model structure of Theorem 3.14. This may be checked using induction and Lemma 2.8. Moreover, since these two model structures have the same weak equivalences, the opposite is true for the cofibrations. Thus the identity functors

$$id : (\mathbf{FC}_R, I_r, J_r, \mathcal{E}_r) \rightleftarrows (\mathbf{FC}_R, I'_r, J'_r, \mathcal{E}_r) : id$$

give a Quillen equivalence between the two model structures.

3.4. Comparison of model structures via shift and décalage.

Definition 3.18. Let $r \geq 0$. The r -*shift* of a filtered complex (A, d, F) is the filtered complex $(A, d, S^r F)$ defined by

$$S^r F_p A^n := F_{p+rn} A^n.$$

This defines a functor $S^r : \mathbf{FC}_R \rightarrow \mathbf{FC}_R$ which is the identity on morphisms.

Note that $S^0 = id$ and that $S^r = S^1 \circ \overset{(r)}{\cdot} \circ S^1$. The r -shift functor has a right adjoint, called the décalage, which was first introduced by Deligne in [6].

Definition 3.19. Let $r \geq 0$. The r -*décalage* of a filtered complex (A, d, F) is the filtered complex $(A, d, \text{Dec}^r F)$ given by

$$\text{Dec}^r F_p A^n := F_{p-rn} A^n \cap d^{-1}(F_{p-r(n+1)} A^{n+1}) = Z_r^{p-rn, p-rn+n}(A).$$

This defines a functor $\text{Dec}^r : \mathbf{FC}_R \rightarrow \mathbf{FC}_R$ which is the identity on morphisms.

Note that $\text{Dec}^0 = \text{id}$ and that $\text{Dec}^r = \text{Dec}^1 \circ \overset{(r)}{\cdots} \circ \text{Dec}^1$. The following is easily verified.

Lemma 3.20 ([4]). *We have $\text{Dec}^r \circ S^r = \text{id}$ and $(S^r \text{Dec}^r F)_p = F_p \cap d^{-1}(F_{p-r})$. In particular, there is a natural transformation $S^r \circ \text{Dec}^r \rightarrow \text{id}$ and S^r is left adjoint to Dec^r :*

$$\text{Hom}(S^r A, B) = \text{Hom}(A, \text{Dec}^r B).$$

The functors shift and décalage allow us to compare weak equivalences as follows.

Lemma 3.21. *For all $k \geq 0$ we have $(S^r)^{-1}(\mathcal{E}_{k+r}) = \mathcal{E}_k$ and $\mathcal{E}_{k+r} = (\text{Dec}^r)^{-1}(\mathcal{E}_k)$.*

Proof. By definition of the shift it follows that $E_{k+1}^{p,p+n}(S^1 A) = E_k^{p+n,p+2n}(A)$ for all $k \geq 0$. By Proposition 1.3.4 of [6], the canonical map $E_{k+1}^{p,p+n}(\text{Dec}^1 A) \rightarrow E_{k+2}^{p-n,p}(A)$ is an isomorphism for all $k \geq 0$. These give isomorphisms of bigraded complexes. \square

Theorem 3.22. *For all $l, r \geq 0$ we have a Quillen equivalence*

$$S^l : (\mathbf{FC}_R, I_r, J_r, \mathcal{E}_r) \rightleftarrows (\mathbf{FC}_R, I_{l+r}, J_{l+r}, \mathcal{E}_{l+r}) : \text{Dec}^l.$$

Proof. To see that (S^l, Dec^l) is a Quillen adjunction, it suffices to check that

$$\text{Dec}^l(\mathcal{F}ib_{r+l}) \subseteq \mathcal{F}ib_r \text{ and } \text{Dec}^l(\mathcal{E}_{r+l}) \subseteq \mathcal{E}_r.$$

Indeed, $f : A \rightarrow B$ is an $(r+l)$ -fibration if and only if $\text{Dec}^{r+l} F_p f$ is degree-wise surjective for every $p \in \mathbb{Z}$. Since $\text{Dec}^{r+l} = \text{Dec}^r \circ \text{Dec}^l$ we have that $\text{Dec}^l f$ is an r -fibration. By Lemma 3.21 we have $\mathcal{E}_{r+l} = (\text{Dec}^l)^{-1}(\mathcal{E}_r)$. Therefore $\text{Dec}^l(\mathcal{E}_{r+l}) \subseteq \mathcal{E}_r$.

To show that (S^l, Dec^l) is a Quillen equivalence, it suffices to show that $S^l A \rightarrow B$ is in \mathcal{E}_{r+l} if and only if $A \rightarrow \text{Dec}^l B$ is in \mathcal{E}_r (see [11] 8.5.20 and 8.5.23). Assume $f : S^l A \rightarrow B$ is a map in \mathcal{E}_{r+l} . The induced map $A \rightarrow \text{Dec}^l B$ is obtained as the composite of $\text{Dec}^l f$ with the unit of the adjunction. Since $f \in \mathcal{E}_{r+l}$ we get $\text{Dec}^l f \in \mathcal{E}_r$. The unit of the adjunction $A \rightarrow \text{Dec}^l S^l A$ is the identity so it lives in \mathcal{E}_r . Conversely if $g : A \rightarrow \text{Dec}^l B$ lives in \mathcal{E}_r then the induced map $S^l A \rightarrow B$ is obtained as $S^l g$ composed with the counit of the adjunction. We already know that $S^l g \in \mathcal{E}_{r+l}$. We are left to prove that the counit ϵ of the adjunction is in \mathcal{E}_{r+l} . Let (A, d, F) be a filtered complex, and $\epsilon_A : S^l \text{Dec}^l A \rightarrow A$. Recall that ϵ_A is the identity on the cochain complex A . We have seen that $\epsilon_A \in \mathcal{E}_{r+l}$ if and only if $\text{Dec}^l(\epsilon_A) \in \mathcal{E}_r$, but $\text{Dec}^l \epsilon_A : \text{Dec}^l S^l \text{Dec}^l A \rightarrow \text{Dec}^l A$ is the map $\text{Dec}^l \text{id}_A$ which is the identity on every $\text{Dec}^l F_p A$, hence a quasi-isomorphism. \square

We end this section by considering a class of weak equivalences \mathcal{W}_r given by a stronger notion than E_r -quasi-isomorphism and which, for $r = 0$, coincides with the class of filtered quasi-isomorphisms: those morphisms of filtered complexes inducing a quasi-isomorphism at each step of the filtration.

Definition 3.23. Let $r \geq 0$. A morphism of filtered complexes $f : A \rightarrow B$ is called a Z_r -quasi-isomorphism if the induced morphism $Z_r(f) : Z_r(A) \rightarrow Z_r(B)$ is a quasi-isomorphism of r -bigraded complexes.

We will denote by \mathcal{W}_r the class of Z_r -quasi-isomorphisms. Note that $f \in \mathcal{W}_r$ if and only if $\text{Dec}^r f \in \mathcal{W}_0$ and that we have inclusions $\mathcal{W}_r \subseteq \mathcal{W}_{r+1}$ for all $r \geq 0$.

Proposition 3.24. *For all $r \geq 0$ we have $\mathcal{W}_r \subseteq \mathcal{E}_r$. Conversely, if $f : A \rightarrow B$ is a morphism of filtered complexes with bounded below filtrations, then every E_r -quasi-isomorphism is a Z_r -quasi-isomorphism.*

Proof. The inclusion $\mathcal{W}_0 \subseteq \mathcal{E}_0$ follows from the short exact sequence

$$0 \rightarrow F_{p-1}A \rightarrow F_pA \rightarrow Gr_p^F A \rightarrow 0,$$

where Gr_p^F denotes the p -graded functor given by $Gr_p^F(A) = F_pA/F_{p-1}A$.

Let $f \in \mathcal{W}_r$. Then $\text{Dec}^r f \in \mathcal{W}_0 \subseteq \mathcal{E}_0$. Since $\mathcal{E}_r = (\text{Dec}^r)^{-1}(\mathcal{E}_0)$, the result follows.

Let $f : A \rightarrow B \in \mathcal{E}_0$ be a morphism of filtered complexes with bounded below filtrations. Then there exists a sufficiently small k such that $F_k f = Gr_k^F f$. Therefore, $H^*(F_k f)$ is an isomorphism. By induction over $p \geq k$, via the five lemma applied to the long exact sequence

$$\cdots \rightarrow H^*(F_{p-1}f) \rightarrow H^*(F_p f) \rightarrow H^*(Gr_p^F f) \rightarrow H^*(F_{p-1}f) \rightarrow \cdots,$$

we get that $f \in \mathcal{W}_0$. For $r > 0$ the proof follows again using décalage. \square

An easy adaptation of the model structure constructed in Section 3.3 gives a cofibrantly generated model structure with \mathcal{W}_r as the class of weak equivalences. This extends Di Natale's result [7] for $r = 0$, to higher r and unbounded filtrations.

Definition 3.25. Let I_r'' and J_r'' be the sets of morphisms of \mathbf{FC}_R given by

$$I_r'' := \left\{ R_{(p-r)}^{n+1} \rightarrow \mathcal{Z}_r(p, n) \right\}_{p, n \in \mathbb{Z}} \quad \text{and} \quad J_r'' := \{0 \rightarrow \mathcal{Z}_r(p, n)\}_{p, n \in \mathbb{Z}}.$$

We have:

Theorem 3.26. *For every $r \geq 0$, the category \mathbf{FC}_R admits a right proper cofibrantly generated model structure, where:*

- (1) *weak equivalences are Z_r -quasi-isomorphisms,*
- (2) *fibrations are morphisms of filtered complexes $f : A \rightarrow B$ such that $Z_r(f)$ is bidegree-wise surjective, and*
- (3) *I_r'' and J_r'' are the sets of generating cofibrations and generating trivial cofibrations respectively.*

The analogue of Theorem 3.22 on the equivalence of model structures for varying r via shift and décalage is also true for the above model structure with \mathcal{W}_r weak equivalences. The proof is verbatim, using the following observation.

Lemma 3.27. *For all $k \geq 0$ we have $(S^r)^{-1}(\mathcal{W}_{k+r}) = \mathcal{W}_k$ and $\mathcal{W}_{k+r} = (\text{Dec}^r)^{-1}(\mathcal{W}_k)$.*

Proof. Since $\text{Dec}^{k+r} \circ S^r = \text{Dec}^k$, one has $f \in \mathcal{W}_k$ if and only if $S^r f \in \mathcal{W}_{k+r}$. Similarly, since $\text{Dec}^{k+r} = \text{Dec}^k \circ \text{Dec}^r$ one has $f \in \mathcal{W}_{k+r}$ if and only if $\text{Dec}^r f \in \mathcal{W}_k$. \square

Remark 3.28. By Proposition 3.24, for each $r \geq 0$, the weak equivalences of the model structure of Theorem 3.26 form a subclass of the class of weak equivalences of the model structure of Theorem 3.14. Moreover, these two model structures have the same fibrations. It follows, using [11, 3.3.19], that the identity functors

$$id : (\mathbf{FC}_R, I_r, J_r, \mathcal{E}_r) \rightleftarrows (\mathbf{FC}_R, I_r'', J_r'', \mathcal{W}_r) : id$$

give a Quillen adjunction with the right adjoint displaying the model structure of Theorem 3.14 as a right Bousfield localization of that of Theorem 3.26.

4. MODEL CATEGORY STRUCTURES ON BICOMPLEXES

In this section, we present our model structures on the category of bicomplexes. We begin with a detailed study of the r -cycles and r -boundaries of the spectral sequence, together with the notion of *witnesses* to how elements are such cycles and boundaries. These notions are defined and then shown to be given by representable functors. Subsequently, we develop a notion of r -cylinder and use it to define r -homotopy. Finally, we establish two different cofibrantly generated model structures for which the weak equivalences are the E_r -quasi-isomorphisms.

4.1. Witness cycles and witness boundaries. We next describe the terms of the spectral sequence associated with a bicomplex, in terms of witness r -cycles and witness r -boundaries.

Definition 4.1. Let (A, d_0, d_1) be a bicomplex and let $r \geq 0$.

Define the R -bigraded modules of *witness r -cycles* by $ZW_0^{p,q}(A) = Z_0^{p,q}(A) = A^{p,q}$ and for $r \geq 1$ by

$$ZW_r^{p,q}(A) = \left\{ (a_0, a_1, \dots, a_{r-1}) \mid \begin{array}{l} a_i \in A^{p-i, q-i}, d_0 a_0 = 0 \\ \text{and } d_1 a_{i-1} = d_0 a_i \text{ for all } 1 \leq i \leq r-1 \end{array} \right\}.$$

For $r \geq 1$, define an r -bigraded complex structure on $ZW_r(A)$ by

$$d_r(a_0, \dots, a_{r-1}) = (d_1 a_{r-1}, 0, \dots, 0).$$

Define a map of R -bigraded modules

$$z_r : ZW_r^{p,q}(A) \longrightarrow Z_r^{p,q}(A)$$

by $z_0 = id_A$ and, for $r \geq 1$, by letting

$$(a_0, \dots, a_{r-1}) \mapsto a_0.$$

Define the R -bigraded modules of *witness r -boundaries* by $BW_0^{p,q}(A) = 0$, $BW_1^{p,q}(A) = A^{p,q}$ and for $r \geq 2$ by

$$BW_r^{p,q-1}(A) = \left\{ (b_0, \dots, b_{r-2}; a; c_0, \dots, c_{r-2}) \mid \begin{array}{l} a \in A^{p, q-1}, \\ (b_0, \dots, b_{r-2}) \in ZW_{r-1}^{p+r-1, q+r-2}(A) \\ (c_0, \dots, c_{r-2}) \in ZW_{r-1}^{p-1, q-1}(A) \end{array} \right\}.$$

Define a map of R -bigraded modules of bidegree $(0, 1)$

$$b_r : BW_r^{p,q-1}(A) \rightarrow B_r^{p,q}(A)$$

by $b_0 = 0$, $b_1 = d_0$ and, for $r \geq 2$, by letting

$$(b_0, \dots, b_{r-2}, a, c_0, \dots, c_{r-2}) \mapsto d_0 a + d_1 b_{r-2}.$$

Lastly, define a map of R -bigraded modules of bidegree $(0, 1)$

$$w_r : BW_r^{p,q-1}(A) \longrightarrow ZW_r^{p,q}(A)$$

by $w_0 = 0$, $w_1 = d_0$ and, for $r \geq 2$, by letting

$$(b_0, \dots, b_{r-2}; a; c_0, \dots, c_{r-2}) \mapsto (d_0 a + d_1 b_{r-2}, d_1 a + c_0, c_1, \dots, c_{r-2}).$$

Given a morphism of bicomplexes $f : A \rightarrow B$, the morphisms of R -bigraded modules $ZW_r(f), BW_r(f)$ are defined componentwise, giving rise to functors $ZW_r, BW_r : \mathbf{bC}_R \rightarrow \mathbf{bgMod}_R$ and natural transformations z_r, b_r, w_r . In addition the functor ZW_r may be lifted to take values in the category $r\text{-bC}_R$ and $w_r(BW_r(A))$ is a sub r -bigraded complex of $ZW_r(A)$.

Remark 4.2. Note that $(a_0, a_1, \dots, a_{r-1}) \in ZW_r^{p,q}(A)$ corresponds to an element $a_0 \in Z_r^{p,q}(A)$, together with a sequence of elements (a_1, \dots, a_{r-1}) witnessing how a_0 is such an element. Note also that the natural transformations z_r and b_r are projections and that, for $r \geq 2$,

$$BW_r^{p,q-1}(A) = ZW_{r-1}^{p+r-1, q+r-2}(A) \oplus A^{p,q-1} \oplus ZW_{r-1}^{p-1, q-1}(A).$$

Proposition 4.3. *For every $r \geq 0$, there is a commutative diagram of natural transformations of functors from bC_R to bgMod_R*

$$\begin{array}{ccccc} BW_r & \xrightarrow{w_r} & ZW_r & & \\ b_r \downarrow & & z_r \downarrow & & \\ B_r & \hookrightarrow & Z_r & \twoheadrightarrow & E_r^{p,q} \end{array}$$

and the natural transformation $\pi_r : ZW_r \rightarrow E_r$ induced by the above diagram satisfies

$$\text{Ker } \pi_r(A) = \text{Im } w_r(A),$$

for every bicomplex A . In particular, we have

$$E_r^{p,q}(A) \cong ZW_r^{p,q}(A) / w_r(BW_r^{p,q-1}(A)),$$

giving rise to an isomorphism of functors from bC_R to $r\text{-bC}_R$.

Proof. A simple verification shows that the above diagram commutes. Let A be a bicomplex. Since $\pi_r \circ w_r = 0$, the inclusion $\text{Im } w_r(A) \subseteq \text{Ker } \pi_r(A)$ holds. Let $x = (a_0, \dots, a_{r-1})$ be in $\text{Ker } \pi_r(A)$. There exists (b_0, \dots, b_{r-1}) such that $a_0 = d_0 b_{r-1} + d_1 b_{r-2}$, $d_0 b_0 = 0$ and $d_1 b_{i-1} = d_0 b_i$, for all $1 \leq i \leq r-2$. Since $d_1 a_0 = d_0 a_1 = d_0 d_1 b_{r-1}$, the element $c_0 = a_1 - d_1 b_{r-1}$ satisfies $d_0 c_0 = 0, d_1 c_0 = d_1 a_1 = d_0 a_2$. Therefore we have that $y = (b_0, \dots, b_{r-2}; b_{r-1}; a_1 - d_1 b_{r-1}, a_2, \dots, a_{r-1}) \in BW_r(A)$ satisfies $w_r(y) = x$. \square

Lemma 4.4. *Let $f : K \rightarrow L$ be a morphism of bicomplexes and $r \geq 0$. Then, the following are equivalent.*

- (1) f induces a surjective morphism $ZW_k(f) : ZW_k(K) \rightarrow ZW_k(L)$, for all $0 \leq k \leq r$.
- (2) f induces a surjective morphism $Z_k(f) : Z_k(K) \rightarrow Z_k(L)$, for all $0 \leq k \leq r$.
- (3) f induces a surjective morphism $E_k(f) : E_k(K) \rightarrow E_k(L)$, for all $0 \leq k \leq r$.

Proof. For $r = 0$ the three assertions tell us that f is a surjective morphism. Assume $r \geq 1$. One has (1) \Rightarrow (2) \Rightarrow (3) because the natural transformations $ZW_k \rightarrow Z_k \rightarrow E_k$ are projections for every k . Let us prove (3) \Rightarrow (1) by induction on r .

Assume hypothesis (3) holds. Assume first that $r = 1$ and consider the diagram

$$\begin{array}{ccccccc} K & \xrightarrow{d_0} & ZW_1(K) & \longrightarrow & E_1(K) & \longrightarrow & 0 \\ d_0 f \downarrow & & ZW_1(f) \downarrow & & E_1(f) \downarrow & & \\ 0 & \longrightarrow & w_1(BW_1(L)) & \longrightarrow & ZW_1(L) & \longrightarrow & E_1(L) \longrightarrow 0 \end{array}$$

Note that since $f = E_0(f)$ is surjective by assumption, the map $d_0 f$ on the left is surjective into the image of $w_1 = d_0$. The snake lemma then gives the short exact sequence

$$0 \rightarrow \text{Coker}(ZW_1(f)) \rightarrow \text{Coker}(E_1(f)) \rightarrow 0.$$

Hence (3) implies (1). Assume $r > 1$. By induction hypothesis, the map f induces a surjective morphism $ZW_k(f)$ for every $0 \leq k \leq r-1$. Let $z \in ZW_r^{p,q}(L)$. By surjectivity

of $E_r(f)$ there exists $z' \in ZW_r^{p,q}(K)$ such that $[f(z')] = [z]$, that is $z - f(z') = w_r(u)$ for some $u \in BW_r^{p,q-1}(L)$. But

$$BW_r(L)^{p,q-1} = ZW_{r-1}^{p+r-1,q+r-2}(L) \oplus L^{p,q-1} \oplus ZW_{r-1}^{p-1,q-1}(L)$$

and f and $ZW_{r-1}(f)$ are surjective, therefore so is $BW_r(f)$. Hence, there exists $v \in BW_r^{p,q}(K)$ such that $z = f(z' + w_r(v))$. \square

Remark 4.5. The proof of the above lemma shows that for $f : K \rightarrow L$ a morphism of bicomplexes and $r \geq 1$, the following are equivalent.

- (1) The maps $ZW_r(f)$, $ZW_{r-1}(f)$ and f are surjective.
- (2) The maps $E_r(f)$ and $ZW_{r-1}(f)$ and f are surjective.

4.2. Representability of the witness cycles and boundaries functors. In this section we show that the functors ZW_r and BW_r are representable by bicomplexes $\mathcal{Z}\mathcal{W}_r$ and $\mathcal{B}\mathcal{W}_r$. These bicomplexes will play the role of the spheres and discs that we may find in a cofibrantly generated model category structure (see [12]).

We represent such a bicomplex A by a graph, where vertices represent finite direct sums of copies of R . Viewing elements of a finite direct sum as column vectors, an arrow in the graph corresponds to the differential $d_0^{i,j}$ or $d_1^{i,j}$ and is described using matrix notation. If there is no vertex at place (i, j) , it means that $A^{i,j} = 0$ and if there is no arrow, it means that the differential considered is 0.

Definition 4.6. The 0-disc at place (i, j) , $\mathbb{D}_0(i, j)$, is the bicomplex given by

$$\begin{array}{ccc} R^{i-1,j+1} & \xleftarrow{1} & R^{i,j+1} \\ \uparrow 1 & & \uparrow 1 \\ R^{i-1,j} & \xleftarrow{1} & R^{i,j} \end{array}$$

Definition 4.7. Define the bicomplex $\mathcal{Z}\mathcal{W}_0(i, j) = \mathbb{D}_0(i, j)$ and for $r \geq 1$, define the bicomplex $\mathcal{Z}\mathcal{W}_r(i, j)$ at place (i, j) by

$$\mathcal{Z}\mathcal{W}_r(i, j)^{p,q} = \begin{cases} R & \text{if } (p, q) = (i - k, j - k), 0 \leq k \leq r - 1, \\ R & \text{if } (p, q) = (i - k - 1, j - k), 0 \leq k \leq r - 1, \\ 0 & \text{else,} \end{cases}$$

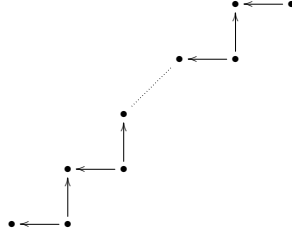
with differentials

$$\begin{aligned} d_0 : R^{i-k,j-k} &\xrightarrow{1} R^{i-k,j-k+1}, & \text{for } 1 \leq k \leq r - 1, \\ d_1 : R^{i-k,j-k} &\xrightarrow{1} R^{i-k-1,j-k}, & \text{for } 0 \leq k \leq r - 1, \end{aligned}$$

and all other differentials are 0.

We may depict $\mathcal{Z}\mathcal{W}_r(i, j)$ as a staircase graph with r horizontal steps as follows, where each bullet represents R , each arrow represents the identity map and the top-right bullet

has bidegree (i, j) .



Examples 4.8. $\mathcal{Z}\mathcal{W}_1(i, j)$ is the bicomplex given by:

$$R^{i-1, j} \xleftarrow{1} R^{i, j}.$$

$\mathcal{Z}\mathcal{W}_3(i, j)$ is given by:

$$\begin{array}{ccc} & & R^{i-1, j} \xleftarrow{1} R^{i, j} \\ & & \uparrow 1 \\ & R^{i-2, j-1} \xleftarrow{1} R^{i-1, j-1} & \\ & \uparrow 1 & \\ R^{i-3, j-2} \xleftarrow{1} R^{i-2, j-2} & & \end{array}$$

Definition 4.9. Define the bicomplex $\mathcal{B}\mathcal{W}_1(i, j-1) = \mathbb{D}_0(i, j-1)$ and for $r \geq 2$ the bicomplex $\mathcal{B}\mathcal{W}_r(i, j-1)$ by

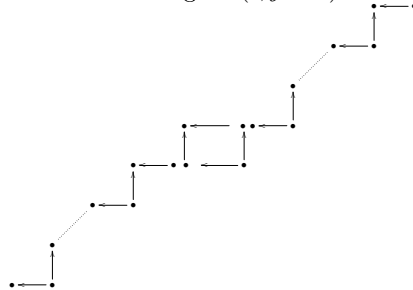
$$\mathcal{B}\mathcal{W}_r(i, j-1) = \mathcal{Z}\mathcal{W}_{r-1}(i-1, j-1) \oplus \mathbb{D}_0(i, j-1) \oplus \mathcal{Z}\mathcal{W}_{r-1}(i+r-1, j+r-2).$$

For $r = 1$, define a morphism of bicomplexes $\iota_1(i, j) : \mathcal{Z}\mathcal{W}_1(i, j) \longrightarrow \mathcal{B}\mathcal{W}_1(i, j-1) = \mathbb{D}_0(i, j-1)$ by letting $\iota_1(i, j)^{i, j} = \iota_1(i, j)^{i-1, j} = 1_R : R \rightarrow R$ and for $r \geq 2$, define a morphism of bicomplexes $\iota_r(i, j) : \mathcal{Z}\mathcal{W}_r(i, j) \longrightarrow \mathcal{B}\mathcal{W}_r(i, j-1)$ by letting

$$\begin{aligned} \iota_r(i, j)^{i, j} = \iota_r(i, j)^{i-1, j-1} : R &\xrightarrow{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} R \oplus R \\ \iota_r(i, j)^{p, q} : R^{p, q} &\xrightarrow{1} R^{p, q}, \end{aligned}$$

for every (p, q) such that $\mathcal{Z}\mathcal{W}_r(i, j)^{p, q} = \mathcal{B}\mathcal{W}_r(i, j-1)^{p, q} = R$.

With the conventions explained above, $\mathcal{B}\mathcal{W}_r(i, j-1)$ may be pictured as follows, where the bottom right corner of the box is in bidegree $(i, j-1)$.



Example 4.10. $\mathcal{B}\mathcal{W}_2(i, j-1)$ is the following bicomplex:

$$\begin{array}{ccccc}
 & & R^{i-1,j} & \xleftarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} & (R \oplus R)^{i,j} & \xleftarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & R^{i+1,j} \\
 & & \uparrow \begin{pmatrix} 0 & 1 \end{pmatrix} & & \uparrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} & & \\
 R^{i-2,j-1} & \xleftarrow{\begin{pmatrix} 1 & 0 \end{pmatrix}} & (R \oplus R)^{i-1,j-1} & \xleftarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} & R^{i,j-1} & &
 \end{array}$$

The map $\iota_2(i, j) : \mathcal{Z}\mathcal{W}_2(i, j) \rightarrow \mathcal{B}\mathcal{W}_2(i, j-1)$ of bicomplexes is depicted in the following diagram, where $\mathcal{Z}\mathcal{W}_2(i, j)$ appears with dotted arrows and $\mathcal{B}\mathcal{W}_2(i, j-1)$ with dashed arrows.

$$\begin{array}{ccccccc}
 & & & & R^{i-1,j} & \xleftarrow{\dots\dots\dots} & R^{i,j} \\
 & & & & \uparrow 1 & & \uparrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
 & & & & R^{i-1,j} & \xleftarrow{\dots\dots\dots} & (R \oplus R)^{i,j} & \xleftarrow{\dots\dots\dots} & R^{i+1,j} \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & R^{i-2,j-1} & \xleftarrow{\dots\dots\dots} & R^{i-1,j-1} & & \\
 & & & & \uparrow 1 & & \uparrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
 & & & & R^{i-2,j-1} & \xleftarrow{\dots\dots\dots} & (R \oplus R)^{i-1,j-1} & \xleftarrow{\dots\dots\dots} & R^{i,j-1}
 \end{array}$$

Directly from the definitions we get the following lemmas.

Lemma 4.11. *For $r \geq 1$ the bicomplex $\mathbb{D}_0(i, j-1)$ is a retract of $\mathcal{B}\mathcal{W}_r(i, j-1)$ and for $r \geq 2$ the bicomplex $\mathcal{Z}\mathcal{W}_{r-1}(i-1, j-1)$ is a retract of $\mathcal{B}\mathcal{W}_r(i, j-1)$.*

Lemma 4.12. *For $r \geq 1$, the diagram*

$$\begin{array}{ccc}
 \mathcal{Z}\mathcal{W}_r(i, j) & \longrightarrow & 0 \\
 \downarrow \iota_r & & \downarrow \\
 \mathcal{B}\mathcal{W}_r(i, j-1) & \longrightarrow & \mathcal{Z}\mathcal{W}_r(i+r-1, j+r-2)
 \end{array}$$

is a pushout diagram.

Remark 4.13. Note that for all $r \geq 1$, the E_r -term of the (column) spectral sequence associated to the bicomplex $\mathcal{Z}\mathcal{W}_r(i, j)$ is the r -bigraded complex given by

$$E_r(\mathcal{Z}\mathcal{W}_r(i, j)) : R^{i-r, j+1-r} \xleftarrow{\delta_r=1} R^{i, j}.$$

Therefore we have $E_{r+1}(\mathcal{Z}\mathcal{W}_r(i, j)) = 0$. Note that $E_1(\mathbb{D}_0(i, j)) = 0$ so that for $r \geq 0$, this gives $E_r(\mathcal{B}\mathcal{W}_r(i, j)) = 0$.

The following lemma is a direct consequence of the definitions of $\mathcal{Z}\mathcal{W}_r$ and $\mathcal{B}\mathcal{W}_r$.

Lemma 4.14. *Let $r \geq 0$ and let $(i, j) \in \mathbb{Z} \times \mathbb{Z}$.*

- (1) *Giving a morphism of bicomplexes $\mathbb{D}_0(i, j) \rightarrow A$ is equivalent to giving an element a in $A^{i, j}$.*
- (2) *Giving a morphism of bicomplexes $\mathcal{Z}\mathcal{W}_r(i, j) \rightarrow A$ is equivalent to giving an element in $Z\mathcal{W}_r^{i, j}(A)$.*
- (3) *Giving a morphism of bicomplexes $\mathcal{B}\mathcal{W}_r(i, j) \rightarrow A$ is equivalent to giving an element in $B\mathcal{W}_r^{i, j}(A)$.*

Under these correspondences, for $r \geq 1$, the map $\iota_r : \mathcal{Z}\mathcal{W}_r(i, j) \rightarrow \mathcal{B}\mathcal{W}_r(i, j-1)$ corresponds to the map $w_r : \mathcal{B}\mathcal{W}_r^{i, j-1}(A) \rightarrow \mathcal{Z}\mathcal{W}_r^{i, j}(A)$ so that a commutative diagram of the form

$$\begin{array}{ccc} \mathcal{Z}\mathcal{W}_r(i, j) & \longrightarrow & A \\ \iota_r \downarrow & & \downarrow f \\ \mathcal{B}\mathcal{W}_r(i, j-1) & \longrightarrow & B \end{array}$$

corresponds to a pair (a, b) , $a \in \mathcal{Z}\mathcal{W}_r(A)^{i, j}$, $b \in \mathcal{B}\mathcal{W}_r^{i, j-1}(B)$ such that $f(a) = w_r(b)$.

Remark 4.15. All of these statements can be made functorial, so that, for example $\mathcal{Z}\mathcal{W}_r(i, j) = \text{bC}_R(\mathcal{Z}\mathcal{W}_r(i, j), -)$.

4.3. r -cylinders and r -cones. We collect some homological algebra constructions for bi-complexes, leading to a notion of r -homotopy.

Definition 4.16. For $r = 0$, we define the 0-cylinder Cyl_0 as the bicomplex

$$\begin{array}{c} (R \oplus R)^{0,0} \\ \uparrow \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ R^{0,-1} \end{array}$$

For $r \geq 1$, define the r -cylinder Cyl_r as the bicomplex whose underlying bigraded module is $\mathcal{Z}\mathcal{W}_r(r, r-1) \oplus R^{0,0}$ and whose differentials coincide with those of $\mathcal{Z}\mathcal{W}_r(r, r-1)$ except for

$$d_1^{1,0} : R^{0,0} \oplus \mathcal{Z}\mathcal{W}_r(r, r-1) = (R \oplus R)^{0,0} \xleftarrow{\begin{pmatrix} -1 \\ 1 \end{pmatrix}} R^{1,0}.$$

For all $r \geq 0$, the morphisms of R -modules

$$R \oplus R \xrightarrow{id} R \oplus R \xrightarrow{\begin{pmatrix} 1 & 1 \end{pmatrix}} R$$

induce morphisms of bicomplexes

$$(R \oplus R)^{0,0} \xrightarrow{i} \text{Cyl}_r \xrightarrow{p} R^{0,0}$$

giving a factorization of the fold map.

Example 4.17. The 1-cylinder Cyl_1 is the bicomplex given by

$$(R \oplus R)^{0,0} \xleftarrow{\begin{pmatrix} -1 \\ 1 \end{pmatrix}} R^{1,0}.$$

The 2-cylinder Cyl_2 is given by

$$\begin{array}{ccc} & R^{1,1} & \xleftarrow{1} R^{2,1} \\ & \uparrow 1 & \\ (R \oplus R)^{0,0} & \xleftarrow{\begin{pmatrix} -1 \\ 1 \end{pmatrix}} & R^{1,0} \end{array}$$

An easy inspection shows that, for all $r \geq 0$, $E_r(\text{Cyl}_r)$ is the r -bigraded complex

$$d_r : R^{r, r-1} \rightarrow (R \oplus R)^{0,0} \text{ with } d_r = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Notation 4.18. For the sequel, for $r \geq 1$, we will denote by e_-, e_+ generators of $(\text{Cyl}_r)^{0,0}$, by $e_{i,i}$, for $1 \leq i \leq r-1$, and $e_{i,i-1}$, for $1 \leq i \leq r$, generators of $(\text{Cyl}_r)^{i,i}$ and $\text{Cyl}_r^{i,i-1}$ respectively, so that:

$$\begin{aligned} d_0(e_{i,i}) &= d_1(e_{i,i}) = 0, \\ d_0(e_{i,i-1}) &= d_1(e_{i+1,i}) = e_{i,i}, \quad \text{for } 1 \leq i \leq r-1, \\ d_0(e_{r,r-1}) &= 0, \\ d_1(e_{1,0}) &= e_+ - e_-. \end{aligned}$$

Definition 4.19. For A a bicomplex, the r -cylinder of A is the bicomplex $\text{Cyl}_r(A) := \text{Cyl}_r \otimes A$. We denote by

$$i : A \oplus A \rightarrow \text{Cyl}_r(A) \text{ and } p : \text{Cyl}_r(A) \rightarrow A$$

the maps induced by i and p defined on Cyl_r . We have that pi is the fold map. We denote by $i_-, i_+ : A \rightarrow \text{Cyl}_r(A)$ the maps obtained from i by composing with the injection on the first (second) component.

Definition 4.20. Let $f, g : A \rightarrow B$ be two morphisms of bicomplexes and let $r \geq 0$. An r -homotopy from f to g is given by a morphism of bicomplexes $h : \text{Cyl}_r(A) \rightarrow B$ such that $hi = (f, g)$. We use the notation $h : f \underset{r}{\simeq} g$.

Remark 4.21. For $r \geq 1$, we have

$$\text{Cyl}_r(A)^{p,q} = (Re_- \otimes A^{p,q}) \oplus \bigoplus_{i=1}^{r-1} (Re_{i,i} \otimes A^{p-i,q-i}) \oplus \bigoplus_{i=1}^r (Re_{i,i-1} \otimes A^{p-i,q+1-i}) \oplus (Re_+ \otimes A^{p,q}).$$

Suppressing the explicit generators of the free R -modules of rank 1, we write this as

$$\text{Cyl}_r(A)^{p,q} = A^{p,q} \oplus \bigoplus_{i=1}^{r-1} A^{p-i,q-i} \oplus \bigoplus_{i=1}^r A^{p-i,q+1-i} \oplus A^{p,q}$$

and we write an element in $\text{Cyl}_r(A)$ as $(a_0, (a_i)_{1 \leq i \leq r-1}, (b_i)_{1 \leq i \leq r}, b_0)$. With this notation, we have

$$d_0(a_0, (a_i)_i, (b_i)_i, b_0) = (d_0 a_0, ((-1)^i d_0 a_i + b_i)_i, ((-1)^{i-1} d_0 b_i)_i, d_0 b_0)$$

and

$$d_1(a_0, (a_i)_i, (b_i)_i, b_0) = (d_1 a_0 - b_1, ((-1)^i d_1 a_i + b_{i+1})_i, ((-1)^i d_1 b_i)_i, d_1 b_0 + b_1).$$

Proposition 4.22. Let $f, g : A \rightarrow B$ be two morphisms of bicomplexes such that $f \underset{r}{\simeq} g$. Then $E_{r+1}(f) = E_{r+1}(g)$.

Proof. Let $f, g : (A, d_0, d_1) \rightarrow (B, d_0, d_1)$ be two morphisms of bicomplexes. We consider first the case $r = 0$. A 0-homotopy from f to g corresponds to a morphism of bigraded modules $h : A \rightarrow B$ of bidegree $(0, -1)$ such that $d_0 h + h d_0 = g - f$ and $-d_1 h + h d_1 = 0$. In particular, this is a homotopy with respect to the differential d_0 . So $E_1(f) = E_1(g)$.

Now let $r \geq 1$. An r -homotopy $h : \text{Cyl}_r(A) \rightarrow B$ from f to g associates to $(a_0, (a_i)_i, (b_i)_i, b_0)$ the element

$$f(a_0) + \sum_{i=1}^{r-1} k_i(a_i) + \sum_{i=1}^r h_i(b_i) + g(b_0),$$

where $h_i : A \rightarrow B$ has bidegree $(i, i-1)$ and $k_i : A \rightarrow B$ has bidegree (i, i) . Writing the conditions satisfied by h to be a morphism of bicomplexes, we get that

$$k_i = d_0 h_i + (-1)^i h_i d_0 = d_1 h_{i+1} + (-1)^i h_{i+1} d_1, \quad \text{for } 1 \leq i \leq r-1$$

and that

$$d_0 h_r = (-1)^{r+1} h_r d_0 \text{ and } d_1 h_1 + h_1 d_1 = g - f.$$

Hence this amounts to having a collection of morphisms of bigraded modules $h_i : A \rightarrow B$ of bidegree $(i, i-1)$, with $1 \leq i \leq r$, such that,

$$\begin{aligned} d_1 h_{i+1} + (-1)^i h_{i+1} d_1 &= d_0 h_i + (-1)^i h_i d_0, & \text{for } 1 \leq i \leq r-1, \\ 0 &= (-1)^r d_0 h_r + h_r d_0, \\ d_1 h_1 + h_1 d_1 &= g - f. \end{aligned}$$

By setting $\widehat{h}_m = (-1)^{r+1} h_{r-m}$ for $0 \leq m \leq r-1$ and $\widehat{h}_m = 0$ for $m \geq r$ we get that the collection of morphisms $\widehat{h}_m : A \rightarrow B$ of bidegree $(r-m, r-1-m)$ satisfies for all $m \geq 0$

$$(H_{m1}) \quad \sum_{i+j=m} (-1)^{i+r} d_i \widehat{h}_j + (-1)^i \widehat{h}_i d_j = \begin{cases} 0 & \text{if } m < r, \\ g_{m-r} - f_{m-r} & \text{if } m \geq r, \end{cases}$$

where $f_i = \begin{cases} f & \text{if } i = 0, \\ 0 & \text{otherwise,} \end{cases}$ and similarly for g , and $d_i = 0$ for $i \neq 0, 1$.

This amounts to saying that the collection $(\widehat{h}_m)_m$ is an r -homotopy (of twisted complexes) from f to g as proven in [2, Proposition 3.18]. Hence $E_{r+1}(f) = E_{r+1}(g)$ follows from Proposition 3.24 of [2]. \square

Remark 4.23. It follows from the explicit description of r -homotopies in the proof above that \cong is an equivalence relation.

Definition 4.24. Let $f : A \rightarrow B$ and $g : A \rightarrow C$ be two morphisms of bicomplexes. The *double mapping r -cylinder* $\text{Cyl}_r(f, g)$ is the bicomplex obtained as the colimit of the diagram

$$\begin{array}{ccccc} & & A & & A & & \\ & & \swarrow & & \swarrow & & \\ & & f & & i_- & & \\ & & \searrow & & \searrow & & \\ B & & & & \text{Cyl}_r(A) & & C \\ & & & & \swarrow & & \\ & & & & i_+ & & \\ & & & & \swarrow & & \\ & & & & g & & \end{array}$$

Proposition 4.25. For $r = 0$, the bicomplex $\text{Cyl}_0(f, g)$ is described as

$$\text{Cyl}_0(f, g)^{p,q} = B^{p,q} \oplus A^{p,q+1} \oplus C^{p,q},$$

with

$$d_0(\beta, a, \gamma) = (d_0\beta - f(a), -d_0a, d_0\gamma + g(a))$$

and

$$d_1(\beta, a, \gamma) = (d_1\beta, d_1a, d_1\gamma).$$

For $r \geq 1$, the bicomplex $\text{Cyl}_r(f, g)$ is described as

$$\text{Cyl}_r(f, g)^{p,q} = B^{p,q} \oplus \bigoplus_{i=1}^{r-1} A^{p-i, q-i} \oplus \bigoplus_{i=1}^r A^{p-i, q+1-i} \oplus C^{p,q},$$

with

$$d_0(\beta, (a_i)_i, (b_i)_i, \gamma) = (d_0\beta, ((-1)^i d_0 a_i + b_i)_i, ((-1)^{i-1} d_0 b_i)_i, d_0\gamma)$$

and

$$d_1(\beta, (a_i)_i, (b_i)_i, \gamma) = (d_1\beta - f(b_1), ((-1)^i d_1 a_i + b_{i+1})_i, ((-1)^i d_1 b_i)_i, d_1\gamma + g(b_1)).$$

Proof. This is a consequence of the description of the bicomplex $\text{Cyl}_r(A)$. \square

Definition 4.26. Let $f : A \rightarrow B$ be a morphism of bicomplexes. For $r \geq 0$, the *mapping r -cone* of f is the object $\text{Cyl}_r(0, f)$, where $0 : A \rightarrow 0$. The *r -cone* of a bicomplex A is $\text{Cyl}_r(0, id_A)$ and is denoted $C_r(A)$.

Remark 4.27. For $r = 0$, if A is the bicomplex $R^{0,0}$ then $C_0(A)$ is the bicomplex

$$\begin{array}{c} R^{0,0} \\ \uparrow 1 \\ R^{0,-1} \end{array}$$

and moreover for any bicomplex A one has $C_0(A) = C_0(R^{0,0}) \otimes A$. Note that for every $p \in \mathbb{Z}$, $C_0(A)^{p,*}$ is the usual cone of the cochain complex $(A^{p,*}, d_0)$.

For $r \geq 1$, if A is the bicomplex $R^{0,0}$ then $C_r(A) = \mathcal{Z}\mathcal{W}_r(r, r-1)$ and moreover for any bicomplex A one has $C_r(A) = \mathcal{Z}\mathcal{W}_r(r, r-1) \otimes A$.

Denoting by $s_r A$ the bicomplex $Re_{r,r-1} \otimes A$ and projecting onto that component we get a morphism of bicomplexes

$$\phi_r : C_r(A) \rightarrow s_r A.$$

Explicitly, with the notation above for $r \geq 1$, $\phi_r((a_i)_i, (b_i)_i, \gamma) = b_r$.

Definition 4.28. A bicomplex A is *r -contractible* if $id_A \cong 0$.

Proposition 4.29. For $r \geq 0$ the *r -cone of a bicomplex is r -contractible*.

Proof. For the case $r = 0$ this follows from the standard statement in cochain complexes by Remark 4.27. Now let $r \geq 1$. For the purpose of the proof, we denote by S the bicomplex $\mathcal{Z}\mathcal{W}_r(r, r-1)$. We build first an r -homotopy $H : \text{Cyl}_r(S) \rightarrow S$ from the identity id_S to 0. The result will then follow for any bicomplex A . Indeed $\text{Cyl}_r(C_r(A)) = \text{Cyl}_r(S) \otimes A$ so that $H \otimes 1_A : \text{Cyl}_r(C_r(A)) \rightarrow C_r(A)$ will be a homotopy from the identity of $S \otimes A = C_r(A)$ to 0. Let us denote by $\beta_{i,i}$ the generators of $S^{i,i}$ for $0 \leq i \leq r-1$ and $\beta_{i,i-1}$ the generators of $S^{i,i-1}$ for $1 \leq i \leq r$. Let β be any generator of S . We define H on generators of $\text{Cyl}_r(S) = \text{Cyl}_r(S) \otimes S$ by

$$\begin{aligned} H(e_+ \otimes \beta) &= \beta, & H(e_- \otimes \beta) &= 0, & H(e_{k+1,k} \otimes \beta_{i+1,i}) &= 0 \\ (-1)^k H(e_{k,k} \otimes \beta_{i+1,i}) &= H(e_{k+1,k} \otimes \beta_{i,i}) = \begin{cases} \beta_{i+k+1,i+k}, & \text{if } i+k+1 \leq r \\ 0, & \text{if not,} \end{cases} \\ H(e_{k,k} \otimes \beta_{i,i}) &= \begin{cases} \beta_{i+k,i+k}, & \text{if } i+k \leq r-1 \\ 0, & \text{if not.} \end{cases} \end{aligned}$$

Then it is a matter of computation to check that H is a morphism of bicomplexes and that $Hi = id_S \oplus 0_S$. □

Corollary 4.30. Let A be a bicomplex and $r \geq 0$. Then $E_{r+1}(C_r(A)) = 0$.

Proof. Since $id_{C_r(A)} \cong 0$, this follows from Proposition 4.22. □

Remark 4.31. The suspension $s_r : \text{bC}_R \rightarrow \text{bC}_R$, given on objects by $A \mapsto s_r A$, is bijective and we will denote by $s_r^{-1} : A \mapsto s_r^{-1} A$ the inverse process. Any morphism $f : A \rightarrow B$ induces a morphism $s_r^{-1} f : s_r^{-1} A \rightarrow s_r^{-1} B$ and we will denote by $\psi_r : s_r^{-1} C_r(A) \rightarrow A$ the morphism $s_r^{-1} \phi_r$.

Proposition 4.32. The morphism $\psi_r : s_r^{-1} C_r(A) \rightarrow A$ satisfies $ZW_s(\psi_r)$ is surjective for $0 \leq s \leq r$.

Proof. The case $r = 0$ is trivial. Let us assume $r \geq 1$. We prove it for ϕ_r , which will imply the statement for ψ_r . We consider first the case $s = r$. Let $(a_0, a_1, \dots, a_{r-1})$ be an element of $ZW_r(s_r A)$, where $a_i \in A$, that is $(-1)^{r-1}d_0 a_0 = 0$ and $(-1)^r d_1 a_k = (-1)^{r-1}d_0 a_{k+1}$ for $0 \leq k \leq r-2$. We define the element $X_k = (x_1, \dots, x_{r-1}, y_1, \dots, y_r, z)$ of $C_r(A)$ where all the elements are zero except $y_{i+r-k} = a_i$ for $0 \leq i \leq k$.

It is a short computation to check that (X_0, \dots, X_{r-1}) is an element of $ZW_r(C_r(A))$ and that the induced map $ZW_r(\phi_r)$ on ZW_r satisfies

$$\phi_r(X_0, \dots, X_{r-1}) = (a_0, \dots, a_{r-1})$$

Note that since $(X_0, \dots, X_k) \in ZW_k(C_r(A))$ is defined from the data (a_0, \dots, a_k) , the same proof applies to $ZW_k(\phi_r)$, for $0 \leq k \leq r$. \square

4.4. Model category structures.

Definition 4.33. For $r \geq 0$, consider the sets of morphisms of bicomplexes

$$I_r = \left\{ \mathcal{Z}W_{r+1}(i, j) \xrightarrow{\iota_{r+1}} \mathcal{B}W_{r+1}(i, j-1) \right\}_{i, j \in \mathbb{Z}} \quad \text{and} \quad J_r = \left\{ 0 \longrightarrow \mathcal{Z}W_r(i, j) \right\}_{i, j \in \mathbb{Z}}.$$

Proposition 4.34. For each $r \geq 0$, a map f is J_r -injective if and only if $ZW_r(f)$ is surjective.

Proof. This follows from (2) of Lemma 4.14. \square

Proposition 4.35. For all $r \geq 0$ we have $I_r\text{-inj} = \mathcal{E}_r \cap J_0\text{-inj} \cap J_r\text{-inj}$.

Proof. Let $r \geq 0$. Assume first that $f : A \rightarrow B$ is I_r -injective. Lemma 4.12 and (2) of Lemma 4.14 imply that f is J_{r+1} -injective. Consider the following diagram.

$$\begin{array}{ccccc} & & & & A \\ & & & \nearrow \varphi & \downarrow f \\ & & & \psi & B \\ \mathcal{Z}W_{r+1}(i, j) & \xrightarrow{\iota_{r+1}} & \mathcal{B}W_{r+1}(i, j-1) & \xrightarrow{g} & B \end{array}$$

The map f is J_{r+1} -injective so φ exists such that $f\varphi = g\iota_{r+1}$. The map f is I_r -injective so ψ exists such that $\psi\iota_{r+1} = \varphi$ and $f\psi = g$. Hence by Lemma 4.11, $f \in J_0\text{-inj} \cap J_r\text{-inj}$. Since $ZW_{r+1}(f)$ is surjective in each bidegree so is $E_{r+1}(f)$. Let us prove that $E_{r+1}(f)$ is injective. Let $a \in ZW_{r+1}(A)$ such that $[f(a)] = [0]$, that is, there exists $b \in BW_{r+1}(B)$ such that $f(a) = w_{r+1}(b)$. This corresponds to the following solid commutative diagram

$$(1) \quad \begin{array}{ccc} \mathcal{Z}W_{r+1}(i, j) & \xrightarrow{a} & A \\ \iota_{r+1} \downarrow & \nearrow a' & \downarrow f \\ \mathcal{B}W_{r+1}(i, j-1) & \xrightarrow{b} & B \end{array}$$

which admits a lift since f is I_r -injective. That is, there exists $a' \in BW_{r+1}(A)$ such that $a = w_{r+1}(a')$ and $f(a') = b$. In particular $[a] = [0] \in E_{r+1}(A)$. Thus $E_{r+1}(f)$ is an isomorphism and $f \in \mathcal{E}_r$.

Conversely, assume $f \in \mathcal{E}_r \cap J_0\text{-inj} \cap J_r\text{-inj}$ and consider the solid diagram (1) which amounts to considering elements $a \in ZW_{r+1}(A), b \in BW_{r+1}(B)$ such that $f(a) = w_{r+1}(b)$. In consequence $E_{r+1}(f)([a]) = [0]$ and the injectivity of $E_{r+1}(f)$ implies $a = w_{r+1}(a')$ for some $a' \in BW_{r+1}(A)$, so that $b = f(a') \in \text{Ker } w_{r+1}(B)$.

Elements in $\text{Ker } w_{r+1}(B)$ are in natural 1-to-1-correspondence with elements of $ZW_{r+1}(B)$ through $(b_0, \dots, b_{r-1}; a; c_0, 0, \dots, 0) \mapsto (b_0, \dots, b_{r-1}, -a)$. The surjectivity of $f, ZW_r(f)$ and $E_{r+1}(f)$ together with Remark 4.5 imply $ZW_{r+1}(f)$ is surjective and so is f restricted to $\text{Ker } w_{r+1}$ and there exists $x \in \text{Ker } w_{r+1}(A)$ such that $f(x) = b - f(a')$. As a consequence one has $a = w_{r+1}(a' + x), f(a' + x) = b$ and $a' + x$ is the desired lift in the diagram. \square

Proposition 4.36. *For all $r \geq 0$ and all $0 \leq k \leq r$ we have $J_k\text{-cof} \subseteq \mathcal{E}_r$.*

Proof. The proof is parallel to that of Proposition 3.13 in the filtered setting. Let $f : X \rightarrow Y$ be such a map. Consider the following diagram.

$$\begin{array}{ccc} X & \xrightarrow{\begin{pmatrix} id \\ 0 \end{pmatrix}} & X \oplus s_r^{-1}(C_r(Y)) \\ f \downarrow & & \downarrow (f \quad \psi_r) \\ Y & \xrightarrow{=} & Y \end{array}$$

From Propositions 4.32 and 4.34 the righthand vertical map is J_s -injective for every $0 \leq s \leq r$ so there is a lift in the diagram. From Proposition 4.30 one has $E_{r+1}(C_r(Y)) = 0$. Applying the functor E_{r+1} to the diagram, we see that $E_{r+1}(f)$ is an isomorphism. \square

Theorem 4.37. *For every $r \geq 0$, the category bC_R admits a right proper cofibrantly generated model structure, where:*

- (1) *weak equivalences are E_r -quasi-isomorphisms,*
- (2) *fibrations are morphisms of bicomplexes $f : A \rightarrow B$ such that f and $ZW_r(f)$ are bidegree-wise surjective, and*
- (3) *I_r and $J_0 \cup J_r$ are the sets of generating cofibrations and generating trivial cofibrations respectively.*

Proof. Set $K_r = J_0 \cup J_r$. From Theorem 2.18 and Proposition 4.34 we have to prove that $K_r\text{-cof} \subseteq \mathcal{E}_r$ and $I_r\text{-inj} = \mathcal{E}_r \cap K_r\text{-inj}$. The first assertion is a direct consequence of Proposition 4.36 and the second one of Proposition 4.35. By [11, 13.1.3] right properness follows directly from the fact that all objects are fibrant. \square

As in the filtered complex case, in certain situations it may be easier to characterize fibrations if they are described in terms of surjectivity of E_r instead of ZW_r .

Definition 4.38. Let I'_r and J'_r be the sets of morphisms of FC_R given by

$$I'_r := \cup_{k=1}^{r-1} J_k \cup I_r \text{ and } J'_r := \cup_{k=0}^r J_k.$$

The proof of the following result is analogous to that of Theorem 3.16 for filtered complexes.

Theorem 4.39. *For every $r \geq 0$, the category bC_R admits a right proper cofibrantly generated model structure, where:*

- (1) *weak equivalences are E_r -quasi-isomorphisms,*
- (2) *fibrations are morphisms of bicomplexes $f : A \rightarrow B$ such that $E_i(f)$ is bidegree-wise surjective for every $0 \leq i \leq r$, and*
- (3) *I'_r and J'_r are the sets of generating cofibrations and generating trivial cofibrations respectively.*

Remark 4.40. Note that the fibrations of the model structure of Theorem 4.39 form a subclass of the class of fibrations of the model structure of Theorem 4.37. Moreover, since

the two model structures have the same weak equivalences, it follows just as in the filtered complex case that the identity functors

$$id : (\mathrm{bC}_R, I_r, J_0 \cup J_r, \mathcal{E}_r) \rightleftarrows (\mathrm{bC}_R, I'_r, J'_r, \mathcal{E}_r) : id$$

give a Quillen equivalence between these two model structures.

Remark 4.41. In the case of filtered complexes, we showed via the shift and décalage adjunction, that within each family the model categories are equivalent when we vary r . However, bicomplexes are much more rigid structures than filtered complexes. Directly transferring both the shift and décalage constructions to the case of bicomplexes does not give well-defined functors. Indeed, such constructions for a bicomplex change the bidegrees of the differentials, so their images land in categories of bicomplexes whose differentials have different bidegrees, hence outside of the original category. Thus the shift-décalage adjunction cannot be directly transferred to give an analogous comparison of model structures for varying r in the bicomplex case.

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