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ON THE GROUP OF RING MOTIONS OF AN H-TRIVIAL LINK

CELESTE DAMIANI AND SEIICHI KAMADA

Abstract. In this paper we compute a presentation for the group of ring motions of the split union of a Hopf link with Euclidean components and a Euclidean circle. A key part of this work is the study of a short exact sequence of groups of ring motions of general ring links in $\mathbb{R}^3$. This sequence allowed us to build the main result from the previously known case of the ring group with one component, which a particular case of the ring groups studied by Brendle and Hatcher. This work is a first step towards the computation of a presentation for groups of motions of H-trivial links with an arbitrary number of components.

1. Introduction

An *H-trivial link* of type (m, n) is a link in $\mathbb{R}^3$ which is ambiently isotopic to the split union of m Hopf links and n trivial knots. When $m = 0$, it is a trivial link with n components. H-trivial links are a generalization of trivial links, and play an important role in normal forms of immersed surface-links in $\mathbb{R}^4$ [KK17, KKKL17].

A *ring* in $\mathbb{R}^3$ is a circle in the strict Euclidian sense, *i.e.*, a round circle on a plane in $\mathbb{R}^3$. We call a link in $\mathbb{R}^3$ a *ring link* if each component is a ring. The *ring group* R_n (of a trivial ring link with n components) was introduced by Brendle and Hatcher [BH13] as the fundamental group of the space of all configurations of ring links which are equivalent, as ring links, [1] to a trivial ring link with n components. We generalize this notion to the ring group $R_{m,n}$ as the fundamental group of the space of all configurations of ring links which are equivalent, as ring links, to an H-trivial ring link of type (m, n). We give presentations of the ring groups $R_{m,n}$ for $(m, n) = (0, 1)$, $(1, 0)$, and $(1, 1)$. Some basic properties of the group $R_{m,n}$ are also given.

The paper is structured as follows: in Section 2 we give the basic definitions concerning ring motions, and we discuss some tools and properties. In Section 3 we review known results about the ring group R_n of a trivial link, discussing its relation with the motion group of a trivial link studied in [Dah62] and [Gol81], and recalling a complete presentation given in [BH13] (Proposition 3.4). In Section 4 we introduce an exact sequence for groups of ring motions of ring links (Proposition 4.1) on which we rely to find presentations for many of the considered groups. In Section 5 we focus on the particular case of the ring group $R_{0,1}$ of just one ring. Here we give an alternative argument for the proof of its presentation (Lemma 5.1). This serves as a strategy model for the case of the ring group $R_{1,0}$ of a Hopf link, treated in Section 6 (Lemma 6.5). Finally in Section 7 we join all preliminary results, and using standard techniques to write presentations for group extensions we give a

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[1] The original definition of R_n in [BH13] is the fundamental group of the space of all configurations of ring links which are equivalent *as links* to a trivial link with n components. If a ring link is equivalent as a link to a trivial ring link, then it is equivalent as a ring link. This fact is asserted in [BH13].

presentation for the group of ring motions $R_{1,1}$ of a H-trivial ring link of type $(1,1)$ in the main result of this paper (Theorem 7.7).

2. Ring motions and motions of links

Let M be a 3-manifold in $\mathbb{R}^3$. A link in M is called a *ring link* if each component is a ring. Two ring links L and L' in M are *equivalent* (as ring links in M) if there exists an isotopy $\{L_t\}_{t\in[0,1]}$ through ring links L_t in M with $L_0 = L$ and $L_1 = L'$.

For a ring link L in M, let $\mathcal{R}(M,L)$ be the space of all configurations of ring links which are equivalent, as ring links in M, to L. This space has L as base point. The *ring group* of L in M, denoted by $R(M,L)$, is the fundamental group $\pi_1(\mathcal{R}(M,L))$.

Let $L_{m,n}$ be a ring link in $\mathbb{R}^3$ which is a *split* union of m Hopf links and n trivial knots, namely, each Hopf link (and each trivial knot component) can be separated from the other by a convex hull in $\mathbb{R}^3$. The *ring group* $R_{m,n}$ is the ring group $R(\mathbb{R}^3, L_{m,n})$ of $L_{m,n}$, *i.e.*, the fundamental group of the space of all configurations of ring links which are equivalent, as ring links, to $L_{m,n}$. This group does not depend on the choice of a base point $L_{m,n}$.

A *ring motion* of a ring link L in M is a loop in the based space $\mathcal{R}(M,L)$, which is presented by a 1-parameter family $\{L_t\}_{t\in[0,1]}$ of ring links in M with $L = L_0 = L_1$. The *stationary motion* or the *trivial motion* of L is a ring motion $\{L_t\}_{t\in[0,1]}$ with $L = L_t$ for all $t \in [0,1]$. Two ring motions are said to be *equivalent* (as ring motions) or *homotopic* if they are homotopic through ring motions of L in M. The product of two ring motions are defined by concatenation. The set of equivalence classes of ring motions of L in M forms a group. This is, by definition, the ring group $R(M,L)$.

Ring groups are related to motion groups as introduced by Dahm [Dah62] and Goldsmith [Gol81, Gol82]. Let M be a 3-manifold and L a link in M. Roughly speaking, a *motion* of L in M is a 1-parameter family $\{L_t\}_{t\in[0,1]}$ of links in M with $L = L_0 = L_1$ such that there exists an ambient isotopy $\{f_t\}_{t\in[0,1]}$ of M with compact support and such that $L_t = f_t(L)$ for $t \in [0,1]$. Two motions are said to be *equivalent* (as motions) or *homotopic* if they are homotopic through motions of L in M. The product of two motions is defined by concatenation. The set of equivalence classes of motions of L in M forms a group, which is the *motion group* of L in M and is denoted by $\mathcal{M}(M,L)$. For a detailed treatment of motions and motion groups, we refer to Dahm [Dah62] and Goldsmith [Gol81, Gol82].

For a ring link L in a 3-manifold $M \subset \mathbb{R}^3$, there is a natural homomorphism

$$R(M,L) \to \mathcal{M}(M,L).$$

This map is an isomorphism when $M = \mathbb{R}^3$ and L is a trivial ring link [BH13, Theorem 1], [Dam17, Theorem 3.10].

3. The ring group and the motion group of a trivial link

In this section we recall some known results about the group $R_{0,n} = R_n = R(\mathbb{R}^3, L) \cong \mathcal{M}(\mathbb{R}^3, L)$ of a trivial link L with n components.

Let L be a link in $\mathbb{R}^3$. The *Dahm homomorphism* is a well-defined homomorphism

$$D\colon \mathcal{M}(\mathbb{R}^3, L) \longrightarrow \mathrm{Aut}(\pi_1(\mathbb{R}^3 \setminus L)),$$

defined as follows. Let $\{L_t\}_{t\in[0,1]}$ be a motion of L in $\mathbb{R}^3$, and p a base point far from the motion. Let $A \subset \mathbb{R}^3 \times [0,1]$ be the annulus with $A \cap \mathbb{R}^3 \times \{t\} = L_t \times \{t\}$ for $t \in [0,1]$. Consider the automorphism $(i_1)_*^{-1} \circ (i_0)_*\colon \pi_1(\mathbb{R}^3 \setminus L; p) \to \pi_1(\mathbb{R}^3 \setminus L; p)$, where i_k, for $(k = 0, 1)$, is the inclusion map of $\mathbb{R}^3 \setminus L = (\mathbb{R}^3 \setminus L) \times \{k\}$ to $\mathbb{R}^3 \times [0,1] \setminus A$. Then $D(\{L_t\}_{t\in[0,1]})$ is defined by this automorphism.

The Dahm homomorphism is also defined on the ring group $R(\mathbb{R}^3, L)$,

$$D\colon R(\mathbb{R}^3, L) \longrightarrow \mathrm{Aut}(\pi_1(\mathbb{R}^3 \setminus L)).$$

Let $n \geq 1$, and $C = C_1 \sqcup \cdots \sqcup C_n$ be a trivial (ring) link with n components in $\mathbb{R}^3$, with $C_i = \{(x, y, 0) \in \mathbb{R}^3 \mid (x - i)^2 + y^2 = (1/4)^2\}$ for $i = 1, \ldots, n$.

The fundamental group $\pi_1(\mathbb{R}^3 \setminus C)$ is the free group F_n of rank n generated by $x_1, \ldots, x_n$, where x_i is the element represented by a positively oriented meridian loop of C_k with respect to the counterclockwise orientation of C_k.

The two following results display some basic properties for the motion group $\mathcal{M}(\mathbb{R}^3, C)$ and the ring group R_n. These will lead to explaining the relation between the two, and to recalling a presentation for these groups.

Theorem 3.1 ([Gol81, Theorems 5.3 and 5.4])**.**

(1) The Dahm homomorphism

$$D\colon \mathcal{M}(\mathbb{R}^3, C) \longrightarrow \mathrm{Aut}(F_n)$$

is injective.

(2) The motion group $\mathcal{M}(\mathbb{R}^3, C)$ is generated by the following types of motions:

- *Permute the ith and the $(i+1)$st rings by pulling the ith ring through the $(i+1)$st ring.*
- *Permute the ith and the $(i+1)$st rings by passing the ith ring around the $(i+1)$st ring.*
- *Reverse the orientation of the ith ring by rotationg it by 180 degrees around the x-axis.*

(3) The above generators correspond to the following automorphisms of F_n:

$$\sigma_i : \begin{cases} x_i \mapsto x_{i+1}; \\ x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1}; \\ x_j \mapsto x_j, & \text{for } j \neq i, i+1. \end{cases} \tag{3.1}$$

$$\rho_i : \begin{cases} x_i \mapsto x_{i+1}; \\ x_{i+1} \mapsto x_i; \\ x_j \mapsto x_j, & \text{for } j \neq i, i+1. \end{cases} \tag{3.2}$$

$$\tau_i : \begin{cases} x_i \mapsto x_i^{-1}; \\ x_j \mapsto x_j, & \text{for } j \neq i. \end{cases} \tag{3.3}$$

(4) The image of the Dahm homomorphism, i.e., the subgroup of $\mathrm{Aut}(F_n)$ *generated by the above automorphisms, is the group of automorphisms of F_n of the form $\alpha\colon x_i \mapsto w_i^{-1} x_{\pi(i)}^{\pm 1} w_i$, where π is a permutation of the indices and w_i is a word in F_n (compare with the* group of conjugating automorphisms *[Sav96], also known as* group of permutation-conjugacy automorphisms *[SW17]).*

Theorem 3.2 ([BH13, Theorem 1])**.** *Let $\mathcal{R}_n$ be the configuration space of ring links which are equivalent to C and let $\mathcal{L}_n$ be the space of all smooth links equivalent to C. The inclusion of $\mathcal{R}_n$ into $\mathcal{L}_n$ is a homotopy equivalence.*

Leaning on Theorem 3.2 it is possible to show that there is a natural isomorphism between $R_n = R(\mathbb{R}^3, C)$ and $\mathcal{M}(\mathbb{R}^3, C)$ [Dam17, Theorem 3.10]. Thus the statement of Theorem 3.1 holds for the ring group R_n too.

Remark 3.3. Our notations σ_i, ρ_i, τ_i for the motions and the automorphisms in Theorem 3.1 are different from those used in [Gol81] or [BH13]. However they coincide with the ones used in [Dam17], where this group is called *extended loop braid group* LB_n^{ext}.

Proposition 3.4 ([BH13, Theorem 3.7]). *The group R_n admits a presentation given by the sets of generators $\{\sigma_i, \rho_i \mid i = 1, \dots, n-1\}$ and $\{\tau_i \mid i = 1, \dots, n\}$ subject to relations:*

$$\begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } i = 1, \dots, n-2 \\ \rho_i\rho_j = \rho_j\rho_i & \text{for } |i-j| > 1 \\ \rho_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\rho_{i+1} & \text{for } i = 1, \dots, n-2 \\ \rho_i^2 = 1 & \text{for } i = 1, \dots, n-1 \\ \rho_i\sigma_j = \sigma_j\rho_i & \text{for } |i-j| > 1 \\ \rho_{i+1}\rho_i\sigma_{i+1} = \sigma_i\rho_{i+1}\rho_i & \text{for } i = 1, \dots, n-2 \\ \sigma_{i+1}\sigma_i\rho_{i+1} = \rho_i\sigma_{i+1}\sigma_i & \text{for } i = 1, \dots, n-2 \\ \tau_i\tau_j = \tau_j\tau_i & \text{for } i \neq j \\ \tau_i^2 = 1 & \text{for } i = 1, \dots, n \\ \sigma_i\tau_j = \tau_j\sigma_i & \text{for } |i-j| > 1 \\ \rho_i\tau_j = \tau_j\rho_i & \text{for } |i-j| > 1 \\ \tau_i\rho_i = \rho_i\tau_{i+1} & \text{for } i = 1, \dots, n-1 \\ \tau_i\sigma_i = \sigma_i\tau_{i+1} & \text{for } i = 1, \dots, n-1 \\ \tau_{i+1}\sigma_i = \rho_i\sigma_i^{-1}\rho_i\tau_i & \text{for } i = 1, \dots, n-1. \end{cases} \tag{3.4}$$

4. Extensions and projections

Let L_1 and L_2 be ring links in a 3-manifold $M \subset \mathbb{R}^3$ with $L_1 \cap L_2 = \emptyset$.

We say that a ring motion $\{L_{1(t)}\}_{t\in[0,1]}$ of L_1 in M and a ring motion $\{L_{2(t)}\}_{t\in[0,1]}$ of L_2 in M are *disjoint* if $L_{1(t)} \cap L_{2(t)} = \emptyset$ for all $t \in [0,1]$. In this case, $\{L_{1(t)} \sqcup L_{2(t)}\}_{t\in[0,1]}$ is a ring motion of $L_1 \sqcup L_2$ in M. We denote this ring motion by $\{L_{1(t)}\}_{t\in[0,1]} \sqcup \{L_{2(t)}\}_{t\in[0,1]}$ and call it the *union* of the motions $\{L_{1(t)}\}_{t\in[0,1]}$ and $\{L_{2(t)}\}_{t\in[0,1]}$.

We denote by $R(M, L_1, L_2)$ the subgroup of the ring group $R(M, L_1 \sqcup L_2)$ consisting of equivalence classes of ring motions which can be written as the union of a motion of L_1 and a motion of L_2. It is a subgroup of index two if and only if there exists a ring motion of $L_1 \sqcup L_2$ in M which interchanges L_1 and L_2. Otherwise, $R(M, L_1, L_2) = R(M, L_1 \sqcup L_2)$.

For a ring motion $\{L_{2(t)}\}_{t\in[0,1]}$ of L_2 in $M \setminus L_1$, we have a ring motion $\{L_1\}_{t\in[0,1]} \sqcup \{L_{2(t)}\}_{t\in[0,1]}$ of $L_1 \sqcup L_2$ in M, where $\{L_1\}_{t\in[0,1]}$ is the stationary motion of L_1. We call it the *extension* of $\{L_{2(t)}\}_{t\in[0,1]}$ with L_1, and we denote it by $e(\{L_{2(t)}\}_{t\in[0,1]})$. We have a well-defined homomorphism

$$e\colon R(M \setminus L_1, L_2) \longrightarrow R(M, L_1, L_2)$$

with $e([\{L_{2(t)}\}_{t\in[0,1]}]) = [e(\{L_{2(t)}\}_{t\in[0,1]})]$.

Let

$$p_1\colon R(M, L_1, L_2) \longrightarrow R(M, L_1)$$

be the homomorphism sending $[\{L_{1(t)}\}_{t\in[0,1]} \sqcup \{L_{2(t)}\}_{t\in[0,1]}]$ to $[\{L_{1(t)}\}_{t\in[0,1]}]$.

Proposition 4.1. *Let L_1 and L_2 be disjoint ring links in $M \subset \mathbb{R}^3$. Consider the composition of e and p_1:*

$$R(M \setminus L_1, L_2) \xrightarrow{\ e\ } R(M, L_1, L_2) \xrightarrow{\ p_1\ } R(M, L_1). \tag{4.1}$$

Then Im $e \subset$ Ker p_1.

Proof. Follows directly from the definitions of the applications e and p_1. □

Although it seems likely that the sequence (4.1) is exact in many cases, few examples are known to the authors at this moment. For example, in the case of trivial links due to [BH13] and in the cases which we discuss in this paper, sequence (4.1) is exact.

Remark 4.2. Let L_1 and L_2 be disjoint links in a 3-manifold M. It is known [Gol81, Proposition 3.10] that the following sequence on motion groups is exact:

$$\mathcal{M}(M \setminus L_1, L_2) \xrightarrow{e} \mathcal{M}(M, L_1, L_2) \xrightarrow{p_1} \mathcal{M}(M, L_1).$$

5. The ring group of a ring

First we observe the ring group of a ring C in $\mathbb{R}^3$. Let C be the unit ring $\{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$. In [Gol81] and [BH13] it is shown that the ring group $R(\mathbb{R}^3, C)$ and the motion group $\mathcal{M}(\mathbb{R}^3, C)$ are cyclic groups of order 2 generated by the class of a ring motion of C rotating it 180 degrees about the y-axis.

Let $R_x(\varphi), R_y(\varphi), R_z(\varphi)$ denote (counterclockwise) rotations of $\mathbb{R}^3$ about the x-axis, the y-axis and the z-axis by angle φ. These are identified with special orthogonal matrices as follows:

$$R_x(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}, \quad R_y(\varphi) = \begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix}$$

$$R_z(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let τ_C be the element of $R(\mathbb{R}^3, C)$ represented by a ring motion $\{R_y(\pi t)(C)\}_{t\in[0,1]}$, *i.e.*, the 180 degrees rotation about the y-axis.

Lemma 5.1 ([BH13, Theorem 3.7], [Gol81, Theorem 5.3])**.** *The ring group $R(\mathbb{R}^3, C)$, which is isomorphic to the motion group $\mathcal{M}(\mathbb{R}^3, C)$, admits the presentation*

$$\langle \tau_C \mid \tau_C^2 = 1 \rangle.$$

Proof. We only show the case of $R(\mathbb{R}^3, C)$. Any ring L in $\mathbb{R}^3$ is determined uniquely by the position of the center $c(L) \in \mathbb{R}^3$, the radius $r(C) \in \mathbb{R}_{>0}$, and an element $g(C)$ of the Grassman manifold $G(2, 3)$ of unoriented 2-planes through the origin O in $\mathbb{R}^3$, which is obtained from the plane $H(C)$ containing C by sliding it along the vector $\overrightarrow{c(L)O}$. Thus the space of rings in $\mathbb{R}^3$ is identifies with $\mathbb{R}^3 \times \mathbb{R}_{>0} \times G(2, 3)$. There is a deformation retract to the subspace $\{O\} \times \{1\} \times G(2, 3) \cong G(2, 3)$. The fundamental group of $G(2, 3)$ is a cyclic group of order 2 generated by a loop which rotates the xy-plane by 180 degrees about the y-axis. This corresponds $\tau_C \in R(\mathbb{R}^3, C)$. □

The proof above suggests a strategy to deform a ring motion to a "standard" ring motion, which is used later for the case of a Hopf link.

6. The ring group of a Hopf link

Let H_1 and H_2 be unit rings in $\mathbb{R}^3$ with $H_1 = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ and $H_2 = \{(0, y, z) \in \mathbb{R}^3 \mid (y - 1)^2 + z^2 = 1\}$. The *positive standard rotation* of H_2 along H_1 is a ring motion $\{R_z(2\pi t)(H_2)\}_{t\in[0,1]}$ of H_2 in $\mathbb{R}^3 \setminus H_1$ or in $\mathbb{R}^3$, and the *negative standard rotation* of H_2 along H_1 is a ring motion $\{R_z(-2\pi t)(H_2)\}_{t\in[0,1]}$.

Lemma 6.1. *The ring group $R(\mathbb{R}^3 \setminus H_1, H_2)$ admits the presentation*

$$\langle \ell \mid \quad \rangle,$$

where ℓ is represented by the positive standard rotation of H_2 along H_1.

First we introduce the *rotation number* of a ring motion of H_2 in $\mathbb{R}^3 \setminus H_1$ such that we obtain a homomorphism $\mathrm{r}ot : R(\mathbb{R}^3 \setminus H_1, H_2) \to \mathbb{Z}$ with $\mathrm{r}ot(\ell) = 1$.

Given an orientation on H_2, note that H_2 always comes back to itself with the same orientation after any ring motion H_2 in $\mathbb{R}^3 \setminus H_1$. Thus, a ring motion of H_2 in $\mathbb{R}^3 \setminus H_1$ induces a continuous map from $H_2 \times S^1 \to \mathbb{R}^3 \setminus H_1$, and hence a homomorphism $H_2(H_2 \times S^1; \mathbb{Z}) \to H_2(\mathbb{R}^3 \setminus H_1; \mathbb{Z})$ on the 2nd homology groups. We call it the homomorphism on the 2nd homology groups induced from the motion of H_2. Note that if two ring motions are homotopic as ring motions then their induced homomorphisms are the same.

Note that $H_2(H_2 \times S^1; \mathbb{Z}) \cong \mathbb{Z}$ and $H_2(\mathbb{R}^3 \setminus H_1; \mathbb{Z}) \cong \mathbb{Z}$ and the homomorphism induced from the positive standard rotation of H_2 along H_1 sends a generator to a generator. Choose generators of $H_2(H_2 \times S^1; \mathbb{Z}) \cong \mathbb{Z}$ and $H_2(\mathbb{R}^3 \setminus H_1; \mathbb{Z}) \cong \mathbb{Z}$ so that the homomorphism induced from the positive standard rotation of H_2 along H_1 sends $1 \in \mathbb{Z}$ to $1 \in \mathbb{Z}$. The *rotation number* of the motion is the integer which is the image of 1 under the induced homomorphism on the 2nd homology groups. It yields the desired homomorphism $\mathrm{r}ot : R(\mathbb{R}^3 \setminus H_1, H_2) \to \mathbb{Z}$ with $\mathrm{r}ot(\ell) = 1$.

We call a ring motion $\{L_t\}_{t\in[0,1]}$ of H_2 in $\mathbb{R}^3 \setminus H_1$ a *normal* ring motion if there is a continuous map $\phi : [0,1] \to \mathbb{R}$ such that $L_t = R_z(2\pi\phi(t))(H_2)$ for all $t \in [0,1]$. For a normal ring motion, $\phi(1) - \phi(0) \in \mathbb{Z}$ is the rotation number. We have that the equivalence class of a normal ring motion is $\ell^{\phi(1)-\phi(0)} \in R(\mathbb{R}^3 \setminus H_1, H_2)$.

Proof of Lemma 6.1. It is sufficient to show that $R(\mathbb{R}^3 \setminus H_1, H_2)$ is generated by ℓ, by using the rotation number. Let $\{L_t\}_{t\in[0,1]}$ be a ring motion of H_2 in $\mathbb{R}^3 \setminus H_1$. We give H_2 the orientation induced from the yz-axis. We can give an orientation to L_t which is induced from the orientation of H_2. Let $c(L_t) \in \mathbb{R}^3$ be the center of L_t, $r(L_t) \in \mathbb{R}_{>0}$ the radius, and $g^+(L_t)$ an element of the Grassman manifold $G^+(2,3)$ of oriented 2-planes through the origin O in $\mathbb{R}^3$, which is obtained from the oriented plane $H(L_t)$ containing L_t by sliding it along the vector $\overrightarrow{c(L_t)O}$. Let $D(L_t)$ be the oriented disk in $\mathbb{R}^3$ bounded by L_t in the plane $H(L_2)$ and let $d(L_t)$ be the intersection point $D(L_t) \cap H_1$. Give H_1 an orientation induced from the xy-plane. Since each disk $D(L_t)$ intersects with H_1 on $d(L_t)$ in the positive direction, we can deform, up to homotopy as ring motions, the ring motion so that the normal vector of $D(L_t)$ at $d(L_t)$ is the positive unit tangent vector of H_1. Then each $H(L_t)$ becomes a 2-plane in $\mathbb{R}^3$ containing the z-axis. Next, we deform the ring motion so that the radius $r(L_t)$ is 1 for all $t \in [0,1]$. Finally, we deform the ring motion so that the center $c(L_t)$ is the intersection point $d(L_t)$. Now we see that any ring motion is homotopic as ring motions to a normal ring motion. This implies that $R(\mathbb{R}^3 \setminus H_1, H_2)$ is generated by ℓ. □

Now we discuss the ring group $R(\mathbb{R}^3, H_1, H_2)$. Let H denote the Hopf link $H_1 \sqcup H_2$. Let $\tau_H \in R(\mathbb{R}^3, H_1, H_2)$ be represented by $\{R_y(\pi t)(H)\}_{t\in[0,1]}$, *i.e.*, the rotation of 180 degrees about the y-axis and let $\ell \in R(\mathbb{R}^3, H_1, H_2)$ be represented by $\{R_z(2\pi t)(H_2)\}_{t\in[0,1]}$, *i.e.*, the positive standard rotation of H_2 along H_1.

Lemma 6.2. *In the ring group $R(\mathbb{R}^3, H_1, H_2)$, we have the following.*

(1) $\tau_H^2 = \ell$ and $\tau_H^4 = \ell^2 = 1$.
(2) $\tau_H \ell \tau_H^{-1} = \ell^{-1}$.
(3) The order of τ_H is 4 and the order of ℓ is 2.

Proof. (1) Let $f_{\tau_H} \colon [0,1] \to SO(3)$ and $f_\ell \colon [0,1] \to SO(3)$ be maps defined by

$$f_{\tau_H}(t) = R_y(\pi t) \in SO(3) \quad \text{and} \quad f_\ell(t) = R_z(2\pi t) \in SO(3).$$

Then $[f_{\tau_H} * f_{\tau_H}] = [f_\ell] = -1$ in $\pi_1(SO(3)) = \{1, -1\}$. This implies that $\tau_H^2 = \ell$ and $\tau_H^4 = \ell^2 = 1$ in $R(\mathbb{R}^3, H_1, H_2)$.

(2) $[f_{\tau_H}^{-1} * f_{\tau_H} * f_{\tau_H}] = [f_\ell^{-1}] = -1$ in $\pi_1(SO(3)) = \{1, -1\}$.

(3) Consider the image of τ_H in the motion group $\mathcal{M}(\mathbb{R}^3, H)$ under the homomorphisms $R(\mathbb{R}^3, H_1, H_2) \to R(\mathbb{R}^3, H) \to \mathcal{M}(\mathbb{R}^3, H)$. By using the double linking number defined in [CKSS02], it can be seen that the order of τ_H in $\mathcal{M}(\mathbb{R}^3, H)$ is 4. Thus the order of τ_4 in $R(\mathbb{R}^3, H_1, H_2)$ is 4. By (1), the order of ℓ is 2.

□

Lemma 6.3. *Let H_1 and H_2 be the unit rings as above. Consider the composition of e and p_1:*

$$R(\mathbb{R}^3 \setminus H_1, H_2) \xrightarrow{e} R(\mathbb{R}^3, H_1, H_2) \xrightarrow{p_1} R(\mathbb{R}^3, H_1). \tag{6.1}$$

(1) The sequence (6.1) *is exact.*
(2) The map p_1 is surjective.

Proof. (1) By Proposition 4.1, we have that I*m* $e \subset$ K*er* p_1. We show that K*er* $p_1 \subset$ I*m* e. Let $[\{L_{1(t)}\}_{t\in[0,1]} \sqcup \{L_{2(t)}\}_{t\in[0,1]}]$ belong to K*er* p_1. Then $[\{L_{1(t)}\}_{t\in[0,1]}] = 1$ in $R(\mathbb{R}^3, H_1)$. Thus the ring motion $\{L_{1(t)}\}_{t\in[0,1]}$ is homotopic to the stationary motion $\{H_1\}_{t\in[0,1]}$ of H_1. To obtain such a homotopy, we use the strategy in the proof of Lemma 5.1. Namely, first we change the ring motion $\{L_{1(t)}\}_{t\in[0,1]}$ so that the center $c(L_{1(t)})$ of the ring $L_{1(t)}$ is the origin O for every $t \in [0, 1]$, then change the radius $r(L_{1(t)})$, and change the element $g(L_{1(t)})$ of the Grassman manifold $G(2, 3)$. This procedure may change $\{L_{2(t)}\}_{t\in[0,1]}$ by a homotopy keeping $L_{2(t)}$ to be a ring for every t. Thus the ring motion $\{L_{1(t)}\}_{t\in[0,1]} \sqcup \{L_{2(t)}\}_{t\in[0,1]}$ is equivalent to a motion which is the union of the statinary motion of H_1 and a ring motion of H_2. Therefore, K*er* $p_1 \subset$ I*m* e.

(2) By Lemma 5.1, the ring group $R(\mathbb{R}^3, H_1)$ is generated by τ_{H_1}. The map p_1 sends $\tau_H \in R(\mathbb{R}^3, H_1, H_2)$ to $\tau_{H_1} \in R(\mathbb{R}^3, H_1)$. Thus p_1 is surjective. □

Theorem 6.4. *The ring group $R(\mathbb{R}^3, H_1, H_2)$ of the ordered Hopf link $H = H_1 \sqcup H_2$ admits the presentation*

$$\langle \tau_H \mid \tau_H^4 = 1 \rangle. \tag{6.2}$$

Proof. By Lemma 6.3, we have a short exact sequence:

$$1 \longrightarrow e(R(\mathbb{R}^3 \setminus H_1, H_2)) \xrightarrow{\iota} R(\mathbb{R}^3, H_1, H_2) \xrightarrow{p_1} R(\mathbb{R}^3, H_1) \longrightarrow 1. \tag{6.3}$$

Since $R(\mathbb{R}^3 \setminus H_1, H_2)$ is generated by $\ell \in R(\mathbb{R}^3 \setminus H_1, H_2)$ (Lemma 6.1), the image $e(R(\mathbb{R}^3 \setminus H_1, H_2))$ is generated by $\ell \in R(\mathbb{R}^3, H_1, H_2)$. By Lemma 6.2 the order of $\ell \in R(\mathbb{R}^3, H_1, H_2)$ is 2. Thus, we have

$$e(R(\mathbb{R}^3 \setminus H_1, H_2)) = \langle \ell \mid \ell^2 = 1 \rangle. \tag{6.4}$$

By Lemma 5.1, we have

$$R(\mathbb{R}^3, H_1) = \langle \tau_{H_1} \mid \tau_{H_1}^2 = 1 \rangle. \tag{6.5}$$

We choose τ_H as a set-theoretical lift of τ_{H_1} under p_1. By Lemma 6.2, we have

$$\tau_H^2 = \ell \quad \text{and} \quad \tau_H \ell \tau_H^{-1} = \ell^{-1}. \tag{6.6}$$

Using the short exact sequence (6.3), presentations (6.4) and (6.5), and relations (6.6), and applying a standard method to give presentations for group extensions [Joh97, Chapter 10] we have that

$$R(\mathbb{R}^3, H_1, H_2) = \langle \ell, \tau_H \mid \ell^2 = 1, \tau_H^2 = \ell, \tau_H \ell \tau_H^{-1} = \ell^{-1} \rangle, \tag{6.7}$$

which is reduced to the desired presentation (6.2). □

Now we discuss the ring group $R(\mathbb{R}^3, H)$. Let e_2 be the unit vector $(0,1,0)$ in $\mathbb{R}^3$.

We consider an element $s \in R(\mathbb{R}^3, H)$ which interchanges H_1 and H_2, represented by the ring motion realized by a sequence of isometries of $\mathbb{R}^3$ as follows: first slide H along $(-1/2)e_2$, apply the rotation by 45 degrees about the y-axis, apply the rotation by 180 degrees about the x-axis, apply the rotation by -45 degrees about the y-axis, and slide along $(1/2)e_2$. (This ring motion is equivalent to the following ring motion: first slide H along $-e_2$, apply the rotation by 180 degrees about the x-axis, and then apply the rotation by -90 degrees about the y-axis.)

Lemma 6.5. *In the group $R(\mathbb{R}^3, H)$, the following relations are satisfied.*

$$s^2 = \tau_H^2 \quad \textit{and} \quad s\tau_H s^{-1} = \tau_H^{-1} \quad \in R(\mathbb{R}^3, H). \tag{6.8}$$

Proof. We have $s^2 = \ell$ in $R(\mathbb{R}^3, H)$ (it is easily understood when we draw link diagrams on the yz-plane). Thus, by Lemma 6.2, we have $s^2 = \tau_H^2$. By the sequence of isometries of $\mathbb{R}^3$ in the definition of s, the y-axis is sent to itself with reversed orientation. Since τ_H is a motion of H realized by the rotation of 180 degrees along the y-axis, we have $s\tau_H s^{-1} = \tau_H^{-1}$. □

Theorem 6.6. *The ring group $R(\mathbb{R}^3, H)$ of the Hopf link admits the presentation*

$$\langle \tau_H, s \mid \tau_H^4 = 1,\ s^2 = \tau_H^2,\ s\tau_H s^{-1} = \tau_H^{-1} \rangle. \tag{6.9}$$

Remark that presentation (6.9) can be rewritten as

$$\langle \tau_H, s \mid \tau_H^2 = s^2 = (\tau_H s)^2 \rangle, \tag{6.10}$$

which is a famous presentation of the *quaternion group*.

Proof. The ring group $R(\mathbb{R}^3, H_1, H_2)$ is a subgroup of $R(\mathbb{R}^3, H)$ of index 2; consider the short exact sequence

$$1 \longrightarrow R(\mathbb{R}^3, H_1, H_2) \longrightarrow R(\mathbb{R}^3, H) \longrightarrow \mathbb{Z}_2 \longrightarrow 1. \tag{6.11}$$

As a set-theoretical lift of the generator of $\mathbb{Z}_2$ under $R(\mathbb{R}^3, H) \to \mathbb{Z}_2$, we choose $s \in R(\mathbb{R}^3, H)$. Using the short exact sequence (6.11), presentation (6.2) of $R(\mathbb{R}^3, H_1, H_2)$, and relations (6.8), we have that $R(\mathbb{R}^3, H)$ admits the desired presentation (6.9). □

7. The ring group of a Hopf link with a ring

In this section we discuss the ring group of an H-trivial link of type $(1,1)$, *i.e.*, the split union of a Hopf link and a ring.

Let $H = H_1 \sqcup H_2$ be a Hopf link and C a ring with $H_1 = \{(x,y,0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$, $H_1 = \{(0,y,z) \in \mathbb{R}^3 \mid (y-1)^2 + z^2 = 1\}$ and $C = \{(x,y,0) \in \mathbb{R}^3 \mid x^2 + (y-5)^2 = 1\}$.

7.1. An exact sequence for $R(\mathbb{R}^3, H, C)$.

Lemma 7.1. *Let H and C be as above. Consider the composition of e and p_1:*

$$R(\mathbb{R}^3 \setminus H, C) \xrightarrow{\ e\ } R(\mathbb{R}^3, H, C) \xrightarrow{\ p_1\ } R(\mathbb{R}^3, H). \tag{7.1}$$

(1) The sequence (7.1) is exact.
(2) The map p_1 is surjective.

Proof. (1) By Proposition 4.1, we have that I*m* $e \subset$ K*er* p_1. We show that K*er* $p_1 \subset$ I*m* e. Let $[\{L_{1(t)}\}_{t\in[0,1]} \sqcup \{L_{2(t)}\}_{t\in[0,1]}]$ belong to K*er* p_1. Then $[\{L_{1(t)}\}_{t\in[0,1]}] = 1$ in $R(\mathbb{R}^3, H)$. Thus the ring motion $\{L_{1(t)}\}_{t\in[0,1]}$ is homotopic to the stationary motion $\{H\}_{t\in[0,1]}$ of H. We show that $\{L_{1(t)}\}_{t\in[0,1]} \sqcup \{L_{2(t)}\}_{t\in[0,1]}$ are homotopic as ring motions to the union of the stationary motion of H and a ring motion of C.

Step 1. First deform the ring motion $\{L_{1(t)}\}_{t\in[0,1]}$ of H and the motion $\{L_{2(t)}\}_{t\in[0,1]}$ of C in such a way that the restriction to H_1 becomes a stationary motion of H_1 keeping the condition that the new $\{L_{1(t)}\}_{t\in[0,1]}$ and $\{L_{2(t)}\}_{t\in[0,1]}$ are disjoint ring motions. This is done by the strategy used in the proof of Lemma 5.1 to deform the motion of H_1 to the stationary motion. (Recall the proof of Lemma 6.3.)

Now we may assume that the restriction of $\{L_{1(t)}\}_{t\in[0,1]}$ to H_1 is the stationary motion. The restriction of $\{L_{1(t)}\}_{t\in[0,1]}$ to H_2 is a ring motion of H_2 in $\mathbb{R}^3 \setminus H_1$.

Step 2. Secondly, deform the ring motion $\{L_{1(t)}\}_{t\in[0,1]}$ of H and the motion $\{L_{2(t)}\}_{t\in[0,1]}$ of C so that the restriction to H becomes the stationary motion of H keeping the condition that the new $\{L_{1(t)}\}_{t\in[0,1]}$ and $\{L_{2(t)}\}_{t\in[0,1]}$ are disjoint ring motions. This is done as follows: Since the restriction of $\{L_{1(t)}\}_{t\in[0,1]}$ to H_2 is a ring motion of H_2 in $\mathbb{R}^3 \setminus H_1$, it is homotopic to the power of the positive or negative standard rotation of H_2 along H_1 by the argument in the proof of Lemma 6.1. During the homotopy for $\{L_{1(t)}\}_{t\in[0,1]}$, we may deform $\{L_{2(t)}\}_{t\in[0,1]}$ keeping the condition that it is a ring motion.

Now, $\{L_{1(t)}\}_{t\in[0,1]} \sqcup \{L_{2(t)}\}_{t\in[0,1]}$ satisfies that $\{L_{1(t)}\}_{t\in[0,1]}$ is the stationary motion of H and $\{L_{2(t)}\}_{t\in[0,1]}$ is a ring motion of H_2 in $\mathbb{R}^3 \setminus H$. Thus it represents an element of the image of $e : R(\mathbb{R}^3 \setminus H, C) \to R(\mathbb{R}^3, H, C)$.

(2) By Lemma 6.6, the ring group $R(\mathbb{R}^3, H)$ is generated by τ_H and s.

Let $\tilde{\tau}_H$ (or $\tilde{s}$) be elements of $R(\mathbb{R}^3, H, C)$ which is the union of τ_H (or s) and the stationary motion on C. Then $p_1(\tilde{\tau}_H) = \tau_H$ and $p_1(\tilde{s}) = s$. Thus p_1 is surjective.

□

Later, in Lemma 7.6, we will see that sequence (7.1) induces a short exact sequence that will allow us to use once more the standard method to write presentations of group extensions.

7.2. **On the ring group** $R(\mathbb{R}^3 \setminus H, C)$**.** Let $H = H_1 \sqcup H_2$ be the Hopf link and C the ring disjoint from H as before. Let us choose a base point for the fundamental group $\pi_1(\mathbb{R}^3 \setminus (H \sqcup C))$ in such a way that the z-coordinate is sufficiently large. Let a, b and c be elements of $\pi_1(\mathbb{R}^3 \setminus (H \sqcup C))$ represented by meridian loops of H_1, H_2, and C, respectively. We assume that these meridian loops are oriented such that the linking number is $+1$ when we give H_1, H_2, and C orientations induced from the xy-plane and the yz-plane, see Figure 1. The fundamental group is the free product of the free abelian group of rank 2 generated by a and b and the infinite cyclic group generated by c:

$$\pi_1(\mathbb{R}^3 \setminus (H \sqcup C)) = \langle a, b, c \mid [a, b] = 1 \rangle \cong (\mathbb{Z} \oplus \mathbb{Z}) * \mathbb{Z}.$$

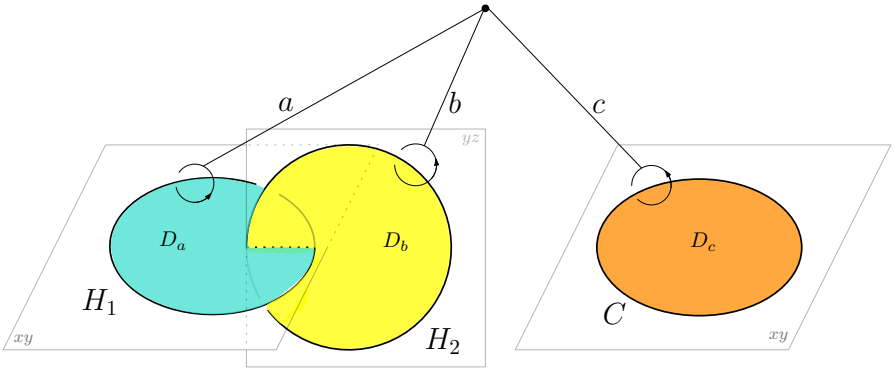

FIGURE 1. Generators of $\pi_1(\mathbb{R}^3 \setminus (H \sqcup C))$.

Let us introduce some ring motions:

- g_a: C pulls through H_1, see Figure 2;
- g_b: C pulls through H_2, see Figure 3;
- τ_C: C rotates by 180 degrees around the y-axis, as in Section 5;
- ε_C: C moves above H, slides downwards encircling H, and then moves back to its original position, see Figure 4.

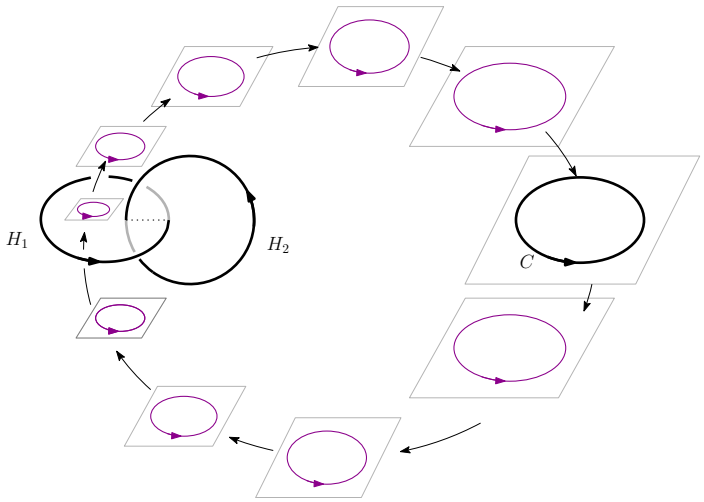

FIGURE 2. The ring motion g_a.

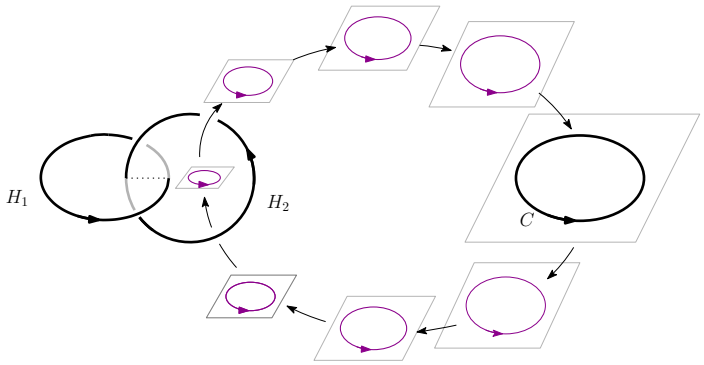

FIGURE 3. The ring motion g_b.

Lemma 7.2. *The ring group $R(\mathbb{R}^3 \setminus H, C)$ is generated by $g_a, g_b, \varepsilon_C, \tau_C$. The following relations are satisfied.*

$$[g_a, g_b] = 1, \tau_C^2 = 1, [g_a, \tau_C] = [g_b, \tau_C] = 1, \tau_C \varepsilon_C \tau_C = \varepsilon_C^{-1}. \tag{7.2}$$

Proof. First of all, remark that the motion group $R(\mathbb{R}^3 \setminus H, *)$, where $*$ is a point, is the fundamental group $\pi_1(\mathbb{R}^3 \setminus H) = \langle a, b \mid [a, b] = 1 \rangle \cong (\mathbb{Z} \oplus \mathbb{Z})$, and recall that $R(\mathbb{R}^3, C) = \langle \tau_C \mid \tau_C^2 = 1 \rangle \cong \mathbb{Z}_2$ (Section 5).

Consider D_a, D_b and D_c to be disks bounded by H_1, H_2 and C, flatly embedded in the planes where H_1, H_2 and C lie, as in Figure 1. Let $\{C_t\}_{t \in [0,1]}$ be a ring motion of C in $\mathbb{R}^3 \setminus H$, and D_{C_t} be the flat disk bounded by C_t, for $t \in [0, 1]$. Let us distinguish two cases.

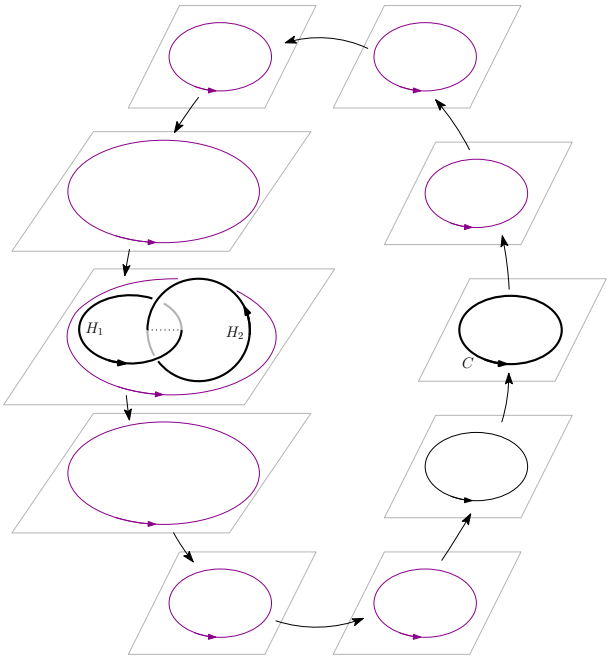

FIGURE 4. The ring motion ε_C.

1. Suppose that for all $t \in [0,1]$, $D_{C_t} \cap (D_a \cup D_b) = \emptyset$. After a deformation of $\{C_t\}_{t\in[0,1]}$ by a homotopy, we may assume that there exists a convex 3-ball B_C, disjoint from $(D_a \cup D_b)$, and such that C_t lies in B_C for all $t \in [0,1]$. Then $\{C_t\}_{t\in[0,1]}$ represents an element of $R(B_C, C) \cong R(\mathbb{R}^3, C) = \langle \tau_C \mid \tau_C^2 = 1\rangle$.
2. Suppose that for some value of t, $D_{C_t} \cap (D_a \cup D_b) \neq \emptyset$. Then let us consider these two subcases.
   2.1. The disks D_{C_t} intersects the interior of D_a and/or D_b for t in a finite number of intervals $[\tilde{t}-\varepsilon, \tilde{t}+\varepsilon]$, and $H_1 \cap \mathrm{int}(D_{C_t}) = H_2 \cap \mathrm{int}(D_{C_t}) = \emptyset$ for all $t \in [0,1]$. Then $\{C_t\}_{t\in[0,1]}$, modulo τ_C, represents an element of $R(\mathbb{R}^3 \setminus H, *) = \langle a, b \mid [a,b] = 1\rangle$.
   2.2. The interiors $\mathrm{int}(D_{C_t})$ intersects H_1 and/or H_2 for t in a finite number of intervals $[\tilde{t}-\varepsilon, \tilde{t}+\varepsilon]$, and $C_t \cap (\mathrm{int}(D_a) \cup \mathrm{int}(D_b)) = \emptyset$ for all $t \in [0,1]$. Then $\{C_t\}_{t\in[0,1]}$, modulo τ_C, represents an element of the subgroup of $R(\mathbb{R}^3 \setminus H, C)$ generated by the motion ε_C (Figure 4).

Every generic ring motion of C in $\mathbb{R}^3 \setminus H$ can be decomposed in a combination of motions that fall in the considered cases, thus τ_C, g_a, g_b and ε_C are a generating set for $R(\mathbb{R}^3 \setminus H, C)$.

The relations in the statement descend from relations of $R(\mathbb{R}^3 \setminus H, *)$ and $R(\mathbb{R}^3, C)$, with the exception of $\tau_C \varepsilon_C \tau_C = \varepsilon_C^{-1}$. This last relation can be seen from the sequence of Figures 5, 6, and 7. □

Let $R^+(\mathbb{R}^3 \setminus H, C)$ be the index 2 subgroup of $R(\mathbb{R}^3 \setminus H, C)$ consisting of equivalence classes of ring motions of C that preserve an orientation of C. This is the subgroup generated by g_a, g_b, ε_C.

Lemma 7.3. *The ring group $R^+(\mathbb{R}^3 \setminus H, C)$ admits the presentation*

$$\langle\, g_a, g_b, \varepsilon_C \mid [g_a, g_b] = 1 \,\rangle, \tag{7.3}$$

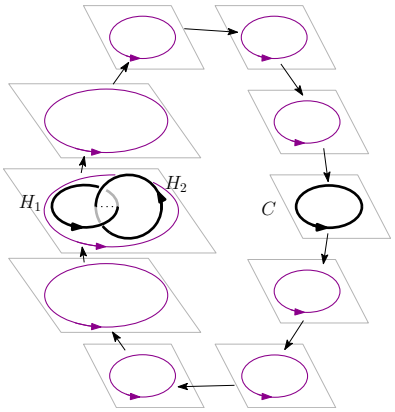

FIGURE 5. The ring motion ε_C^{-1}.

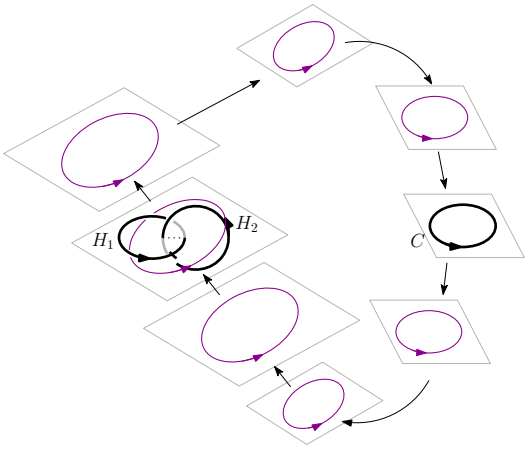

FIGURE 6. A deformation of ε_C^{-1}, where the plane where C is lying slightly tilts before encircling H.

and the Dahm homomorphism $D\colon R^+(\mathbb{R}^3 \setminus H, C) \to \mathrm{A}ut(\pi_1(\mathbb{R}^3 \setminus (H \sqcup C)))$ *is injective.*

Proof. The images of the elements g_a, g_b and ε_C under the Dahm homomorphism $D : R^+(\mathbb{R}^3 \setminus H, C) \to \mathrm{A}ut(\pi_1(\mathbb{R}^3 \setminus (H \sqcup C))) = \mathrm{A}ut\big(\langle a, b, c \mid [a, b] = 1\rangle\big)$ are the following automorphisms:

(7.4)

$$D(g_a)\colon \begin{cases} a \mapsto a \\ b \mapsto b \\ c \mapsto aca^{-1} \end{cases} \quad D(g_b)\colon \begin{cases} a \mapsto a \\ b \mapsto b \\ c \mapsto bcb^{-1} \end{cases} \quad D(\varepsilon_C)\colon \begin{cases} a \mapsto cac^{-1} \\ b \mapsto cbc^{-1} \\ c \mapsto c. \end{cases}$$

Let G_1 be the free abelian group generated by g_1 and g_2, let G_2 be the infinite cyclic group generated by ε_C, and let G be the free product of G_1 and G_2, *i.e.*, $G = \langle g_a, g_b, \varepsilon_C \mid [g_a, g_b] = 1\rangle$. We show that the natural epimorphism $\mu : G \to R^+(\mathbb{R}^3 \setminus H, C)$ is injective by showing that the homomorphism $D' = D \circ \mu : G \to \mathrm{A}ut(\pi_1(\mathbb{R}^3 \setminus (H \sqcup C)))$ is injective.

Let $W\colon G \to \langle a, b, c \mid [a, b] = 1\rangle$ be the isomorphism with $g_a \mapsto a, g_b \mapsto b, \varepsilon_C \mapsto c$. Note that for any $g \in G$, $D'(g)$ is the inner automorphism of $\langle a, b, c \mid [a, b] = 1\rangle$ by $W(g)$, *i.e.*, $D'(g)(x) = W(g)xW(g)^{-1}$. This implies that $D'(g) = 1$ if and only if $W(g) = 1$. Thus, D' is an isomorphism and we have the presentation (7.3). □

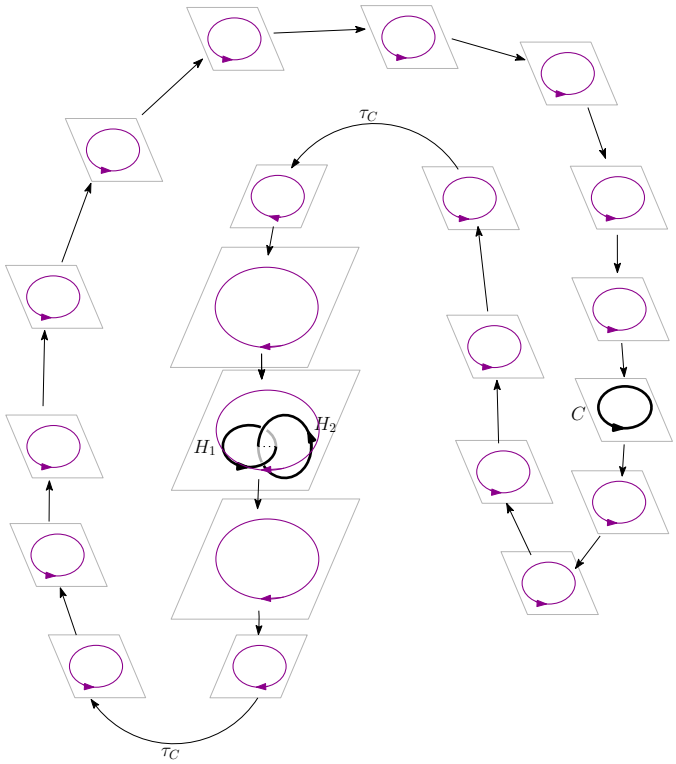

FIGURE 7. The plane where C is lying tilts by 180 degrees before encircling H. The ring motion ε_C^{-1} has been continuously deformed into the motion $\tau_C\varepsilon_C\tau_C$.

Remark 7.4. Remark that $\pi_1(\mathbb{R}^3 \setminus (H \sqcup C))$ is a right-angled Artin group, and that $\{D(g_a), D(g_b), D(\varepsilon_C)\}$ is the set of (partial) conjugations in $\mathrm{A}ut(\pi_1(\mathbb{R}^3 \setminus (H \sqcup C)))$. Then $\{D(g_a), D(g_b), D(\varepsilon_C)\}$ is a generating set for a particular case of *group of vertex-conjugating automorphisms of a right-angled Artin group*, for which Toinet gives a complete presentation in [Toi12]. In this paper he generalises a method used by McCool [McC86] to study *groups of basis-conjugating automorphisms of free groups*. We recall that these last ones are isomorphic to *pure untwisted* ring groups, and to *pure* loop braid groups [BH13, Dam17]

Lemma 7.5. *The ring group $R(\mathbb{R}^3 \setminus H, C)$ admits the presentation*

$$\langle\, g_a, g_b, \varepsilon_C, \tau_C \mid [g_a, g_b] = 1, \tau_C^2 = 1,\ [g_a, \tau_C] = [g_b, \tau_C] = 1, \tau_C\varepsilon_C\tau_C = \varepsilon_C^{-1} \,\rangle, \tag{7.5}$$

and the Dahm homomorhism $D\colon R(\mathbb{R}^3 \setminus H, C) \to \mathrm{A}ut(\pi_1(\mathbb{R}^3 \setminus (H \sqcup C)))$ is injective.

Proof. Presentation (7.5) is obtained from presentation (7.3) and Lemma 7.2 by using the short exact sequence

$$1 \longrightarrow R^+(\mathbb{R}^3 \setminus H, C) \xrightarrow{\ \iota\ } R(\mathbb{R}^3 \setminus H, C) \longrightarrow \mathbb{Z}_2 \longrightarrow 1. \tag{7.6}$$

Let $g \in R(\mathbb{R}^3 \setminus H, C)$ be an element of the kernel of D. In Lemma 7.3 we have seen that D is injective on the subgroup $R^+(\mathbb{R}^3 \setminus H, C)$. Suppose $g \in R(\mathbb{R}^3 \setminus H, C) \setminus R^+(\mathbb{R}^3 \setminus H, C)$. Then $g = g_0\tau_C$ for some $g_0 \in R^+(\mathbb{R}^3 \setminus H, C)$. Since

$$D(\tau_C)\colon \begin{cases} a & \mapsto & a \\ b & \mapsto & b \\ c & \mapsto & c^{-1}, \end{cases} \tag{7.7}$$

$D(\tau_C)$ is never an inner automorphism of $\pi_1(\mathbb{R}^3 \setminus (H \sqcup C)))$. This contradicts to that $D(g_0)$ is an inner automorphism. Thus, D is injection on $R(\mathbb{R}^3 \setminus H, C)$. □

Lemma 7.6. *The sequence involving e and p_1 in Lemma 7.1 induces the short exact sequence*

$$1 \longrightarrow R(\mathbb{R}^3 \setminus H, C) \xrightarrow{\ e\ } R(\mathbb{R}^3, H, C) \xrightarrow{\ p_1\ } R(\mathbb{R}^3, H) \longrightarrow 1. \tag{7.8}$$

Proof. By Lemma 7.1, it is sufficient to show that e is injective. This follows from the injectivity of the Dahm homomorphism $D\colon R(\mathbb{R}^3 \setminus H, C) \to \pi_1(\mathbb{R}^3 \setminus (H \sqcup C)))$. □

7.3. The ring group $R(\mathbb{R}^3, H \sqcup C)$.

Theorem 7.7. *The ring group $R(\mathbb{R}^3, H \sqcup C)$ $(= R(\mathbb{R}^3, H, C))$ admits the following presentation: Generators:*

$$g_a, g_b, \varepsilon_C, \tau_C, \tau_H, s. \tag{7.9}$$

Relations:

$$[g_a, g_b] = 1,\ \tau_C^2 = 1,\ [g_a, \tau_C] = [g_b, \tau_C] = 1,\ \tau_C \varepsilon_C \tau_C = \varepsilon_C, \tag{7.10}$$

$$\tau_H^4 = 1,\ s^2 = \tau_H^2,\ s\tau_H s^{-1} = \tau_H^{-1}, \tag{7.11}$$

$$\tau_H g_a \tau_H^{-1} = g_a^{-1},\ \tau_H g_b \tau_H^{-1} = g_b^{-1},\ \tau_H \varepsilon_C \tau_H^{-1} = \varepsilon_C,\ \tau_H \tau_C \tau_H^{-1} = \tau_C, \tag{7.12}$$

$$s g_a s^{-1} = g_a,\ s g_b s^{-1} = g_b,\ s \varepsilon_C s^{-1} = \varepsilon_C,\ s \tau_C s^{-1} = \tau_C. \tag{7.13}$$

Proof. Consider the short exact sequence (7.6). Let $\tilde{\tau}_H$ (or $\tilde{s}$) be elements of $R(\mathbb{R}^3, H \sqcup C)$ which is the union of τ_H (or s) and the stationary motion on C. Then $p_1(\tilde{\tau}_H) = \tau_H$ and $p_1(\tilde{s}) = s$. We have a section $R(\mathbb{R}^3, H) \to R(\mathbb{R}^3, H \sqcup C)$ sending τ_H to $\tilde{\tau}_H$ and s to $\tilde{s}$. Thus, the short exact sequence (7.6) is split. We may denote the elements $\tilde{\tau}_H$ and $\tilde{s}$ by τ_H and s for simplicity. Using the presentation (7.5) of $R(\mathbb{R}^3 \setminus H, C)$, and the presentation (6.9) of $R(\mathbb{R}^3, H)$, we have the generators (7.9) and relations (7.10) and (7.11). The actions of τ_H and s yield relations (7.12) and (7.13). □

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References

[BH13] T. E. Brendle and A. Hatcher. Configuration spaces of rings and wickets. *Comment. Math. Helv.*, 88(1):131–162, 2013.

[CKSS02] J. S. Carter, S. Kamada, M. Saito, and S. Satoh. Bordism of unoriented surfaces in 4-space. *Michigan Math. J.*, 50(3):575–591, 2002.

[Dah62] D. M. Dahm. *A Generalisation of Braid Theory*. PhD thesis, Princeton University, 1962.

[Dam17] C. Damiani. A journey through loop braid groups. *Expo. Math.*, 35(3):252–285, 2017.

[Gol81] D. L. Goldsmith. The theory of motion groups. *The Michigan Mathematical Journal*, 28(1):3–17, 1981.

[Gol82] D. L. Goldsmith. Motion of links in the 3-sphere. *Mathematica Scandinavica*, 50(2):167–205, 1982.

[Joh97] D. L. Johnson. *Presentations of groups*, volume 15 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, second edition, 1997.

[KK17] S. Kamada and K. Kawamura. Ribbon-clasp surface-links and normal forms of immersed surface-links. *Topology Appl.*, 230:181–193, 2017.

[KKKL17] S. Kamada, A. Kawauchi, J. Kim, and S. Y. Lee. Presentation of immersed surface-links by marked graph diagrams. *ArXiv e-prints*, July 2017.

[McC86] J. McCool. On basis-conjugating automorphisms of free groups. *Canad. J. Math.*, 38(6):1525–1529, 1986.

[Sav96] A. G. Savushkina. On the group of conjugating automorphisms of a free group. *Mathematical Notes*, 60(1):68–80, 1996.

[SW17] A. I Suciu and H. Wang. The pure braid groups and their relatives. In *Perspectives in Lie theory, Springer INdAM series*, volume 19. Springer, 2017.

[Toi12] E. Toinet. A finitely presented subgroup of the automorphism group of a right-angled Artin group. *J. Group Theory*, 15(6):811–822, 2012.

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