AUSLANDER-REITEN THEORY IN QUASI-ABELIAN AND KRULL-SCHMIDT CATEGORIES

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Abstract. We generalise some of the theory developed for abelian categories in papers of Auslander and Reiten to semi-abelian and quasi-abelian categories. In addition, we generalise some Auslander-Reiten theory results of S. Liu for Krull-Schmidt categories by removing the Hom-finite and indecomposability restrictions. Finally, we give equivalent characterisations of Auslander-Reiten sequences in a quasi-abelian, Krull-Schmidt category.

1. Introduction

As is well-known, the work of Auslander and Reiten on *almost split sequences* (which later also became known as *Auslander-Reiten sequences*), introduced in [5], has played a large role in comprehending the representation theory of artin algebras. In trying to understand these sequences, it became apparent that two types of morphisms would also play a fundamental role (see [6]). Irreducible morphisms and minimal left/right almost split morphisms (see Definitions 3.6 and 3.13, respectively) were defined in [6], and the relationship between these morphisms and Auslander-Reiten sequences was studied. In fact, many of the abstract results of Auslander and Reiten were proven for an arbitrary abelian category, not just a module category, and in this article we show that much of this Auslander-Reiten theory also holds in a more general context—namely in that of a quasi-abelian category.

A *quasi-abelian* category is an additive category that has kernels and cokernels, and in which kernels are stable under pushout and cokernels are stable under pullback. Classical examples of such categories include: any abelian category; the category of filtered modules over a filtered ring; and the category of topological abelian groups. A modern example has recently arisen from cluster theory as we recall now. Let $\mathcal{C}$ denote the cluster category (see [13], [17]) associated to a finite-dimensional hereditary k-algebra, where k is a field. Fix a basic rigid object R of $\mathcal{C}$ and consider the *partial cluster-tilted* algebra $\Lambda_R := (\operatorname{End}_{\mathcal{C}} R)^{\mathrm{op}}$. In the study of the category $\Lambda_R-\mathsf{mod}$ of finitely generated left Λ_R-modules, the additive quotient $\mathcal{C}/[\mathcal{X}_R]$ has been useful, where $[\mathcal{X}_R]$ is the ideal of morphisms factoring through objects of $\mathcal{X}_R = \operatorname{Ker}(\operatorname{Hom}_{\mathcal{C}}(R,-))$, because a certain Gabriel-Zisman localisation (see [21]) of it is equivalent to $\Lambda_R-\mathsf{mod}$ (see [14, Thm. 5.7]). We showed in [42] that $\mathcal{C}/[\mathcal{X}_R]$ is a quasi-abelian category, and hence the generalisations of the results of Auslander and

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Reiten that we prove in §3 of this article can be used to more fully understand this category.

Furthermore, $\mathcal{C}$, and hence $\mathcal{C}/[\mathcal{X}_R]$, is a Hom-finite, Krull-Schmidt category, and so one can utilise techniques from a different perspective. Liu initiated the study of Auslander-Reiten theory in Hom-finite, Krull-Schmidt categories in [28] and, in particular, introduced the notion of an admissible ideal (see Definition 4.10). It was shown in [28, §1] that if $\mathcal{A}$ is a Hom-finite, Krull-Schmidt category and $\mathcal{I}$ is an admissible ideal of $\mathcal{A}$, then, under suitable assumptions, irreducible morphisms (between indecomposables) and minimal left/right almost split morphisms behave well under the quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{I}$. We extend these results of Liu in the following ways: first, we are able to remove the Hom-finite assumption for all the results in [28, §1]; and second, we are able to remove the indecomposability assumption in [28, Lem. 1.7 (1)]. We also prove new observations in this setting, inspired by work of Auslander and Reiten. Moreover, in case $\mathcal{A}$ is quasi-abelian and Krull-Schmidt we are able to provide equivalent criteria (see Theorem 4.19) for when a short exact sequence in $\mathcal{A}$ is an *Auslander-Reiten sequence* (in the sense of Definition 4.6). In particular, this characterisation applies to the category $\mathcal{C}/[\mathcal{X}_R]$.

This article is structured as follows. In §2 we recall some background material on preabelian categories, and in particular we remind the reader on how one may define the first extension group for such categories. In §3 we develop Auslander-Reiten theory for semi-abelian and quasi-abelian categories. At the beginning of §4, we switch focus to Auslander-Reiten theory in Krull-Schmidt categories, and then present a characterisation theorem for Auslander-Reiten sequences in a Krull-Schmidt, quasi-abelian category. At the end of §4 we present an example of a Hom-infinite, Krull-Schmidt category communicated to the author by P.-G. Plamondon. Lastly, in §5 we explore an example coming from the cluster category as discussed above, which demonstrates the theory we have developed in the earlier sections.

2. Preliminaries

2.1. **Preabelian categories.** The categories we study in §3 are semi-abelian and quasi-abelian categories (see Definitions 3.1 and 3.2, respectively). A category of either kind is a preabelian category (see Definition 2.1) with some additional structure. In this section, we recall the notion of a preabelian category, and provide some basic results that will be helpful later. For more details, we refer the reader to [39].

Definition 2.1. [15, §5.4] A *preabelian* category is an additive category in which every morphism has a kernel and a cokernel.

Remark 2.2. By [11, Prop. 6.5.4], any preabelian category $\mathcal{A}$ *has split idempotents*, or is *idempotent complete*, (see [11, Def. 6.5.3]) because every morphism in $\mathcal{A}$ admits a kernel, so in particular every idempotent does. See also [4, p. 188] and [16, §6].

The following lemma is standard but often useful.

Lemma 2.3. *Suppose $\mathcal{A}$ is an additive category with split idempotents. Let $f\colon X \to Y$ be a morphism in $\mathcal{A}$.*

(i) *Suppose X is indecomposable and $Y \neq 0$. If f is a retraction, then f is an isomorphism.*
(ii) *Suppose Y is indecomposable and $X \neq 0$. If f is a section, then f is an isomorphism.*

Proof. We only prove (i); the proof for (ii) is dual. Suppose X is indecomposable, $Y \neq 0$ and that $f\colon X \to Y$ is a retraction. Then, by [16, Rem. 7.4], $X \cong Y \oplus Y'$ with f corresponding to the canonical projection $Y \oplus Y' \twoheadrightarrow Y$. However, X is indecomposable and $Y \neq 0$ implies $Y' = 0$, so f is an isomorphism. ■

The next lemma may be found as an exercise in [32], and follows from [1, Lem. IX.1.8]: the proof in [1] is for the corresponding result in an abelian category, but is sufficient for Lemma 2.4 since only the existence of (co)kernels is needed.

Lemma 2.4. [32, Exer. 7.13] *In a preabelian category:*

(i) *every kernel is the kernel of its cokernel; and*
(ii) *every cokernel is the cokernel of its kernel.*

For a morphism $f\colon X \to Y$ in an additive category, we denote the kernel (respectively, cokernel) of f, if it exists, by $\ker f\colon \operatorname{Ker} f \to X$ (respectively, $\operatorname{coker} f\colon Y \to \operatorname{Coker} f$).

Lemma 2.5. *Let $f\colon X \to Y$ be a morphism in a preabelian category. If f is an epimorphism and a kernel (respectively, a monomorphism and a cokernel), then f is an isomorphism.*

Proof. We prove the statement for when f is an epic kernel. The other statement is dual.

Suppose $f\colon X \to Y$ is an epimorphism and a kernel. Note that f is monic by [1, Lem. IX.1.4], and $Y \xrightarrow{\operatorname{coker} f} 0$ is a cokernel of f by [1, Lem. IX.1.5]. Now consider the identity morphism $1_Y\colon Y \to Y$, and notice that since $(\operatorname{coker} f) \circ 1_Y = 0 \circ 1_Y = 0$ we have that 1_Y factors through $\ker(\operatorname{coker} f) = f$ (using Lemma 2.4). Thus, there exists $g\colon Y \to X$ such that $fg = 1_Y$. Thus, f is a monic retraction, and so an isomorphism by [30, Thm. I.1.5]. ■

Definition 2.6. [35, p. 23] Given a morphism $f\colon A \to B$ in an additive category, the *coimage* $\operatorname{coim} f\colon A \to \operatorname{Coim} f$, if it exists, is the cokernel $\operatorname{coker}(\ker f)$ of the kernel of f. Dually, the *image* $\operatorname{im} f\colon \operatorname{Im} f \to B$ is the kernel $\ker(\operatorname{coker} f)$ of the cokernel of f.

It is then immediate from Definition 2.6 that any image morphism is a monomorphism and any coimage is an epimorphism. Furthermore, in a preabelian category $\mathcal{A}$, any morphism f admits a factorisation as follows:

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle \operatorname{coim} f}\downarrow & \circ & \uparrow{\scriptstyle \operatorname{im} f} \\ \operatorname{Coim} f & \xrightarrow[\quad \widetilde{f} \quad]{} & \operatorname{Im} f \end{array}$$

where $\widetilde{f}$ is known as the *parallel of* f. This parallel morphism is always an isomorphism in an abelian category, but this may not be the case in general.

Recall that a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of morphisms in an additive category is called *short exact* if $f = \ker g$ and $g = \operatorname{coker} f$. We state a version of the well-known Splitting Lemma, which is normally stated in the context of an abelian category (see, for example, [29, Prop. I.4.3] or [12, Prop. 1.8.7]), for an additive category. Additionally, we do not assume initially that the sequence of morphisms is short exact, since this is a consequence of the equivalent conditions. We omit the proof because the one in [12] works essentially unchanged.

Proposition 2.7 (Splitting Lemma). *Let $\mathcal{A}$ be an additive category with a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of morphisms. Then the following are equivalent.*

(i) *There is an isomorphism $Y \cong X \oplus Z$, where f corresponds to the canonical inclusion $X \hookrightarrow X \oplus Z$ and g to the canonical projection $X \oplus Z \twoheadrightarrow Z$.*
(ii) *The morphism f is a section and $g = \operatorname{coker} f$.*
(iii) *The morphism g is a retraction and $f = \ker g$.*

In this case, $X \xrightarrow{f} Y \xrightarrow{g} Z$ is short exact.

Definition 2.8. A short exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in an additive category $\mathcal{A}$ is called *split* if it satisfies any of the equivalent conditions of Proposition 2.7. Otherwise, the sequence is said to be *non-split*.

In a non-split short exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have that f is not a section and g is not a retraction by Proposition 2.7. However, more can be said as we see now.

Lemma 2.9. *Let $\mathcal{A}$ be an additive category, and suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a non-split short exact sequence in $\mathcal{A}$. Then both f and g are neither sections nor retractions.*

Proof. As noted above, we need only show that f is not a retraction and that g is not a section. Assume, for contradiction, that f is a retraction. Then f is an epimorphism and so $Z \cong \operatorname{Coker} f \cong 0$ by [1, Lem. IX.1.5]. However, this implies that g is a retraction which is a contradiction. Therefore, f cannot be a retraction. Showing g is not a section is dual. ■

2.2. Ext **in a preabelian category.** In order to avoid some Hom-finiteness restrictions in later arguments, we recall in this section how a first extension group (see Definition 2.15) may be defined in a preabelian category in such a way that it is a bimodule (see Theorem 2.19). Although we follow the development in [38], there is an error in their Theorem 4 ([38, p. 523]) that is corrected in [18]. However, we also believe there should be more (set-theoretic) assumptions in place to ensure that the first extension group is indeed a group (see Remark 2.16).

Throughout this section, $\mathcal{A}$ denotes a preabelian category and we suppose X, Z are objects of $\mathcal{A}$. We note for the next definition that in a preabelian category pullbacks and pushouts always exist as we have the existence of kernels and cokernels.

Definition 2.10. [38, p. 523] Let $\xi\colon X \overset{f}{\to} Y \overset{g}{\to} Z$ be a *complex* (i.e. $g \circ f = 0$) in $\mathcal{A}$. Let $a\colon X \to X'$ be any morphism in $\mathcal{A}$. We define a new complex $a\xi$ as follows. First, we form the pushout $X' \amalg_X Y$ of a along f with morphisms $f'\colon X' \to X' \amalg_X Y$ and $a'\colon Y \to X' \amalg_X Y$. Then we obtain a unique morphism $g'\colon X' \amalg_X Y \to Z$ using the universal property of the pushout with the morphisms $0\colon X' \to Z$ and $g\colon Y \to Z$ as in the following commutative diagram.

$$\begin{CD} X @>f>> Y @>g>> Z \\ @VaVV @Va'VV @| \\ X' @>f'>> X' \amalg_X Y @>\exists! g'>> Z \end{CD} \qquad (0\colon X' \to Z)$$

The complex $a\xi$ is then defined to be $X' \overset{f'}{\to} X' \amalg_X Y \overset{g'}{\to} Z$.

Dually, we define ξb for a morphism $b\colon Z' \to Z$. The commutative diagram

$$\begin{CD} X @>\exists! f'>> Y \amalg_Z Z' @>g'>> Z' \\ @| @Vb'VV @VbVV \\ X @>f>> Y @>g>> Z \end{CD} \qquad (0\colon X \to Z')$$

summarises the construction and ξb is the complex $X \overset{f'}{\to} Y \amalg_Z Z' \overset{g'}{\to} Z'$.

The next definition is standard terminology.

Definition 2.11. Suppose

$$\begin{CD} A @>a>> B \\ @VbVV @VVcV \\ C @>>d> D \end{CD}$$

is a commutative diagram in a category $\mathcal{A}$. Let $\mathcal{P}$ be a class of morphisms in $\mathcal{A}$ (e.g. the class of all kernels in $\mathcal{A}$). We say that $\mathcal{P}$ is *stable under pullback* (respectively, *stable under pushout*) if, in any diagram above that is a pullback (respectively, pushout), d is in $\mathcal{P}$ implies a is in $\mathcal{P}$ (respectively, a is in $\mathcal{P}$ implies d is in $\mathcal{P}$).

In a preabelian category, kernels are stable under pullback (see [38, Thm. 1]), but they may not be stable under pushout. Dually for cokernels. Thus, Richman and Walker make the following definition.

Definition 2.12. [38, p. 524] Let $\xi\colon X \overset{f}{\to} Y \overset{g}{\to} Z$ be a short exact sequence in $\mathcal{A}$. We say that ξ is *stable*, if $a\xi$ and ξb are short exact for all $a\colon X \to X', b\colon Z' \to Z$. In this case, we call $f = \ker g$ a *stable kernel* and $g = \operatorname{coker} f$ a *stable cokernel*. We will sometimes also call ξ *stable exact* in this case to emphasise the exactness of ξ.

Suppose $\nu\colon A \xrightarrow{a} B \xrightarrow{b} C$ and $\xi\colon X \xrightarrow{f} Y \xrightarrow{g} Z$ are two short exact sequences in $\mathcal{A}$. Recall that a *morphism* $(u,v,w)\colon \nu \to \xi$ *of short exact sequences* is a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{a} & B & \xrightarrow{b} & C \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

in $\mathcal{A}$. If $A = X$ and $C = Z$, then a morphism of the form $(1_X, v, 1_Z)$ in which $v\colon B \to Y$ is an isomorphism is called an *isomorphism of short exact sequences with the same end-terms*, and we denote this by $\nu \;{}_X{\cong}_Z\; \xi$. This is clearly an equivalence relation on the class of short exact sequences of the form $X \to - \to Z$.

Theorem 2.13. [18, Thm. 2] *Suppose $\xi\colon X \xrightarrow{f} Y \xrightarrow{g} Z$ is a stable exact sequence in $\mathcal{A}$. Then there is an isomorphism $a(\xi b) \;{}_{X'}{\cong}_{Z'}\; (a\xi)b$ for all $a\colon X \to X', b\colon Z' \to Z$ in $\mathcal{A}$.*

Remark 2.14. It can readily be seen that a statement like Theorem 2.13 is needed to give a definition of an extension group $\operatorname{Ext}^1_{\mathcal{A}}(Z,X)$ that is also an $(\operatorname{End}_{\mathcal{A}} X, \operatorname{End}_{\mathcal{A}} Z)$-bimodule. Theorem 2.13 was claimed to hold for all sequences in [38] (see [38, Thm. 4]), but a counterexample was given in [18]. Cooper presents the corrected statement as [18, Thm. 2], presenting one half of the argument and suggesting a diagram to use for the dual argument. However, the suggested dual diagram is not the right one to consider.

First, let us recall the setup in [18, Thm. 2]. Let $E\colon A \xrightarrow{f} B \xrightarrow{g} C$ be a stable exact sequence, and suppose we have morphisms $\alpha\colon A \to A'$ and $\beta\colon C' \to C$. Then, as obtained in [18, p. 266], there is a commutative diagram

$$\begin{array}{cccccc} \alpha(E\beta)\colon & A' & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C' \\ {\scriptstyle (1_{A'},\varphi_2,1_{C'})}\downarrow & \| & & \downarrow{\scriptstyle \varphi_2} & & \| \\ (\alpha E)\beta\colon & A' & \xrightarrow{f_3} & B_3 & \xrightarrow{g_3} & C' \end{array}$$

It is shown in detail that $g_3 = \operatorname{coker} f_3$ and φ_2 is an epimorphism. It is then suggested that the diagram

$$\alpha(E\beta) \to (\alpha E)\beta \to \alpha E \to f_3\alpha E$$

with a dual proof strategy will yield $f_2 = \ker g_2$ and φ_2 is a monomorphism. However, this diagram should be replaced by

$$\alpha(E\beta) \to (\alpha E)\beta \to \alpha E \to f_2\alpha E.$$

Furthermore, we note that it is straightforward to find a morphism $\alpha(E\beta) \to (\alpha E)\beta$ of the form $(1_{A'}, \varphi_2, 1_{C'})$, but to then show that φ_2 is an isomorphism requires the stability of E (see [38, Cor. 7]). In an abelian category, the fact that φ_2 is an isomorphism would follow quickly, for example, from the Five Lemma.

Following [29], we introduce some notation to help the reading of the sequel. Let A, B, C, D be objects in $\mathcal{A}$. We denote by ∇_A the *codiagonal* morphism $(\,1_A\;\;1_A\,)\colon A \oplus A \to A$, and denote by Δ_A the *diagonal* morphism $\left(\begin{smallmatrix} 1_A \\ 1_A \end{smallmatrix}\right)\colon A \to A \oplus A$. For two morphisms

$a\colon A \to C$ and $b\colon B \to D$, we let $a \oplus b$ denote the morphism $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)\colon A \oplus B \to C \oplus D$. We are now in a position to define $\operatorname{Ext}^1_{\mathcal{A}}(Z, X)$.

Definition 2.15. [38, §4] Let $\mathcal{A}$ be a preabelian category. Define $\operatorname{Ext}^1_{\mathcal{A}}(Z, X)$ to be the class of equivalence classes under ${}_X{\cong}_Z$ of stable short exact sequences of the form $X \to - \to Z$ in $\mathcal{A}$.

By abuse of terminology/notation, by an element ξ of $\operatorname{Ext}^1_{\mathcal{A}}(Z, X)$ we will really mean the equivalence class $[\xi]_{{}_X{\cong}_Z}$ of ξ in $\operatorname{Ext}^1_{\mathcal{A}}(Z, X)$. If $\xi\colon X \xrightarrow{f} Y \xrightarrow{g} Z$, $\xi'\colon X \xrightarrow{f'} Y' \xrightarrow{g'} Z$ are elements of $\operatorname{Ext}^1_{\mathcal{A}}(Z, X)$, then we define the *Baer sum* of ξ and ξ' to be (the equivalence class of)

$$\xi + \xi' \coloneqq \nabla_X(\xi \oplus \xi')\Delta_Z.$$

Note that by [38, Thm. 8] and [18, Thm. 2], $\xi + \xi'$ is stable exact and the Baer sum $+$ is a closed binary operation on $\operatorname{Ext}^1_{\mathcal{A}}(Z, X)$.

Remark 2.16. It is observed in [38] that one may then follow [29, pp. 70–71] in order to show that $\operatorname{Ext}^1_{\mathcal{A}}(Z, X)$ is an abelian group. However, $\operatorname{Ext}^1_{\mathcal{A}}(Z, X)$ may not form a (small) set and hence may not be a group. A similar issue arises in [18].

Note, however, that if $\mathcal{A}$ is skeletally small, then $\operatorname{Ext}^1_{\mathcal{A}}(Z, X)$ will be a set. Indeed, for objects Y, Y' in $\mathcal{A}$, the sets $\{\xi\colon X \to Y \to Z \mid \xi \text{ is short exact}\}$ and $\{\xi'\colon X \to Y' \to Z \mid \xi' \text{ is short exact}\}$ are in bijection whenever Y is isomorphic to Y'. So, up to equivalence with respect to ${}_X{\cong}_Z$, the collection of all short exact sequences of the form $X \xrightarrow{f} Y \xrightarrow{g} Z$ is determined only by the isomorphism class of Y and the morphisms f, g since the end-terms X and Z are fixed. Therefore, the collection of all ${}_X{\cong}_Z$-equivalence classes will form a set, and hence restricting our attention to the classes of stable exact sequences will also yield a set.

These set-theoretic considerations lead us to the next theorem.

Theorem 2.17. [38, §4] *Suppose $\mathcal{A}$ is a preabelian category with objects X, Z, and suppose $\operatorname{Ext}^1_{\mathcal{A}}(Z, X)$ is a set. Then $\operatorname{Ext}^1_{\mathcal{A}}(Z, X)$ is an abelian group with the group operation given by the Baer sum defined in* Definition 2.15*. The class of the split extension $\xi_0\colon X \to X \oplus Z \to Z$ is the identity element, and the inverse of $\xi \in \operatorname{Ext}^1_{\mathcal{A}}(Z, X)$ is $(-1_X)\xi$.*

Therefore, if $\mathcal{A}$ is preabelian, $\operatorname{Ext}^1_{\mathcal{A}}(Z, X)$ is known as a *first extension group.* We state without proof one corresponding result from [29] that is needed for the last theorem of this section.

Lemma 2.18. [29, p. 71] *Let ξ_0 denote the split short exact sequence $X \to X \oplus Z \to Z$ and let $\xi\colon X \xrightarrow{f} Y \xrightarrow{g} Z$ be an arbitrary stable exact sequence. Let 0_X (respectively, 0_Z) denote the zero morphism in $\operatorname{End}_{\mathcal{A}} X$ (respectively, $\operatorname{End}_{\mathcal{A}} Z$). Then $0_X\xi \; {}_X{\cong}_Z \; \xi_0$ and $1_X\xi = \xi$, and $\xi 0_Z \; {}_X{\cong}_Z \; \xi_0$ and $\xi 1_Z = \xi$.*

Theorem 2.19. *Let $\mathcal{A}$ be a preabelian category with objects X, Z, and suppose $\operatorname{Ext}^1_{\mathcal{A}}(Z, X)$ is a set. Then $\operatorname{Ext}^1_{\mathcal{A}}(Z, X)$ is an $(\operatorname{End}_{\mathcal{A}} X, \operatorname{End}_{\mathcal{A}} Z)$-bimodule.*

Proof. This follows from Theorem 2.17, [38, Thm. 4], Lemma 2.18 and Theorem 2.13. ■

3. Auslander-Reiten theory in quasi-abelian categories

In this section, after recalling the definitions of a semi-abelian and a quasi-abelian category, we will explore some Auslander-Reiten theory type results in connection with these categories. We remark here that quasi-abelian categories carry a canonical exact structure: a quasi-abelian category $\mathcal{A}$ endowed with the class of all short exact sequences forms an exact category in the sense of Quillen [37] (see [41, Rem. 1.1.11]). Some Auslander-Reiten theory for exact categories was developed in [20], but our results are different in nature: we explore properties of the morphisms involved in Auslander-Reiten sequences, whereas [20] focuses more on the existence and construction of such sequences. See [23] also.

Definition 3.1. [39, p. 167] Let $\mathcal{A}$ be a preabelian category. We call $\mathcal{A}$ *left semi-abelian* if each morphism $f\colon A \to B$ factorises as $f = ip$ for some monomorphism i and cokernel p. Similarly, $\mathcal{A}$ is said to be *right semi-abelian* if instead each morphism f decomposes as $f = ip$ with i a kernel and p some epimorphism. If $\mathcal{A}$ is both left and right semi-abelian, then it is called *semi-abelian*.

Note that a preabelian category is semi-abelian if and only if, for every morphism f, the parallel $\widetilde{f}$ (see §2.1) of f is *regular*, i.e. both monic and epic (see [39, pp. 167–168]).

Definition 3.2. [39, p. 168] Let $\mathcal{A}$ be a preabelian category. We call $\mathcal{A}$ *left quasi-abelian* if cokernels are stable under pullback (see Definition 2.11) in $\mathcal{A}$. If kernels are stable under pushout in $\mathcal{A}$, then we call $\mathcal{A}$ *right quasi-abelian*. If $\mathcal{A}$ is left and right quasi-abelian, then $\mathcal{A}$ is simply called *quasi-abelian*.

Example 3.3. Any abelian category is quasi-abelian.

Example 3.4. The category of *Banach spaces*, i.e. complete normed vector spaces, over $\mathbb{R}$ or $\mathbb{C}$ is quasi-abelian, but not abelian; see [39, p. 214].

Quasi-abelian categories, as we define them here, were called 'almost abelian' categories in [39], but the terminology we adopt is the more widely accepted one. See the 'Historical remark' in [40] for more details.

Remark 3.5. Rump shows that every left (respectively, right) quasi-abelian category is left (respectively, right) semi-abelian (see [39, p.129, Cor. 1]). Furthermore, in a left (respectively, right) semi-abelian category, if a morphism f factorises as $f = ip$ with i monic and p a cokernel (respectively, i a kernel and p epic), then $p = \operatorname{coim} f$ (respectively, $i = \operatorname{im} f$) up to unique isomorphism.

Auslander and Reiten showed that irreducible and (minimal) left/right almost split morphisms (introduced in [6]) play a large role in the study of almost split sequences (defined in [5]) in abelian categories. The same is true in the generality we consider in this article, and we begin by recalling the definition of an irreducible morphism.

Definition 3.6. [6, §2] A morphism $f\colon X \to Y$ of an arbitrary category is *irreducible* if the following conditions are satisfied:

(i) f is not a section;
(ii) f is not a retraction; and
(iii) if $f = hg$, for some $g\colon X \to Z$ and $h\colon Z \to Y$, then either h is a retraction or g is a section.

For the results presented here that are analogues of those in known work, we omit the proofs that carry over or that are easy generalisations. Instead, we focus on those arguments that need significant modification or that have been omitted in previous work. Furthermore, many of the results in the remainder of the article have duals, which we state but do not prove.

The next proposition is a version of [6, Prop. 2.6 (a)] for the semi-abelian setting. Recall that a monomorphism (respectively, epimorphism) that is not an isomorphism is called a *proper monomorphism* (respectively, *proper epimorphism*).

Proposition 3.7. *Suppose a category $\mathcal{A}$ is left or right semi-abelian. If $f\colon X \to Y$ is irreducible in $\mathcal{A}$, then it is a proper monomorphism or a proper epimorphism.*

Proof. Suppose $f\colon X \to Y$ is an irreducible morphism with coimage $\operatorname{coim} f\colon X \twoheadrightarrow M$. Note that f cannot be an isomorphism since it is not, for example, a section.

Suppose now that $\mathcal{A}$ is left semi-abelian. Then we have a factorisation $f = i \circ \operatorname{coim} f$ where i is monic (see Remark 3.5). If f is a proper monomorphism then we are done, so suppose not. Then $f = i \circ \operatorname{coim} f$ is irreducible implies i is a retraction or $\operatorname{coim} f$ is a section. The latter implies $\operatorname{coim} f$ is monic and this in turn yields that f is monic, which is contrary to our assumption that f is not a proper monomorphism. Thus, i must be a retraction and hence an epimorphism. Then f is the composition of two epimorphisms and is thus epic itself, i.e. f is a proper epimorphism.

The case when $\mathcal{A}$ is right semi-abelian is proved similarly. ■

In an abelian category, we have that a morphism f is an isomorphism if and only if it is regular. Therefore, in an abelian setting an irreducible morphism is either a proper monomorphism or a proper epimorphism, but never both simultaneously. However, this need not hold in an arbitrary category. In particular, we will see in Example 5.1 irreducible morphisms (so non-isomorphisms) that are regular.

The following two results give a version of [6, Prop. 2.7] in a more general setting. For the first proposition, we only really need that the category $\mathcal{A}$ is right semi-abelian and left quasi-abelian, but by [39, Prop. 3] this is equivalent to $\mathcal{A}$ being (left and right) quasi-abelian because left quasi-abelian implies left semi-abelian. Dually, for the second result we only require that $\mathcal{A}$ is left semi-abelian and right quasi-abelian. The proof we give is inspired by that of Auslander and Reiten; however, since regular morphisms may not be isomorphisms or, for example, monomorphisms may not be kernels in the categories we are dealing with, we must consider some different short exact sequences in the proof.

Proposition 3.8. *Suppose $\mathcal{A}$ is a quasi-abelian category and that $f\colon X \to Y$ is a morphism in $\mathcal{A}$ with cokernel $c\colon Y \to C$. If f is irreducible, then for all $v\colon V \to C$ either there exists $v_1\colon V \to Y$ such that $cv_1 = v$ or there exists $v_2\colon Y \to V$ such that $c = vv_2$.*

Furthermore, if $X \xrightarrow{f} Y \xrightarrow{c} C$ *is a non-split short exact sequence, then the converse also holds.*

Proof. First, suppose that $f\colon X \to Y$ is irreducible and that $v\colon V \to C$ is arbitrary. Since $\mathcal{A}$ is quasi-abelian we may consider the following commutative diagram

$$\begin{array}{ccccc} L & \xrightarrow{\ker a} & E & \xrightarrow{a} & V \\ \| & & \downarrow{\scriptstyle u} & \square & \downarrow{\scriptstyle v} \\ L & \xrightarrow{\operatorname{im} f} & Y & \xrightarrow{c} & C \\ {\scriptstyle p}\uparrow & \nearrow{\scriptstyle f} & & & \\ X & & & & \end{array}$$

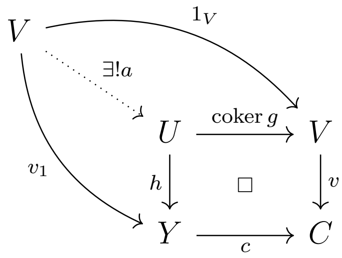
where E is the pullback of c along v, $f = (\operatorname{im} f) \circ p$ with p epic, and a is a cokernel since cokernels are stable under pullback in (left) quasi-abelian categories. Then $f = (\operatorname{im} f) \circ p = u \circ ((\ker a)p)$, so either $(\ker a)p$ is a section or u is a retraction as f is irreducible.

If $(\ker a)p$ is a section then there exists $v_1'\colon E \to X$ such that $v_1'(\ker a)p = 1_X$. Then $pv_1'(\ker a)p = p = 1_L p$, so $pv_1' \ker a = 1_L$ as p is epic. Since a is a cokernel, we have $a = \operatorname{coker}(\ker a)$ by Lemma 2.4. Therefore, $\ker a$ is a section implies a is a retraction by the Splitting Lemma (Proposition 2.7). That is, there exists $v_1''\colon V \to E$ such that $av_1'' = 1_V$. Now define $v_1 := uv_1''$ and note that $cv_1 = cuv_1'' = vav_1'' = v$. Otherwise, u is a retraction and so there exists $v_2'\colon Y \to E$ with $uv_2' = 1_Y$. Setting $v_2 := av_2'$ we see that $vv_2 = vav_2' = cuv_2' = c$. This concludes the proof of the first statement.

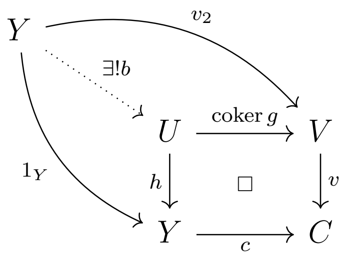
For the converse, we assume that $X \xrightarrow{f} Y \xrightarrow{c} C$ is a non-split short exact sequence and, further, that for all $v\colon V \to C$ either there exists $v_1\colon V \to Y$ such that $cv_1 = v$ or there exists $v_2\colon Y \to V$ such that $c = vv_2$. Then f is not a section or a retraction, by Lemma 2.9, as $X \xrightarrow{f} Y \xrightarrow{c} C$ is non-split. It remains to show part (iii) of Definition 3.6. To this end, suppose $f = hg$ for some $g\colon X \to U$ and $h\colon U \to Y$. Since $hg = f = \ker c$ is a kernel, g is also a kernel by [39, Prop. 2] as $\mathcal{A}$ is quasi-abelian and so, in particular, right semi-abelian. Thus, $g = \ker(\operatorname{coker} g)$ (by Lemma 2.4) and $X \xrightarrow{g} U \xrightarrow{\operatorname{coker} g} V$ is short exact. Consider the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{g} & U & \xrightarrow{\operatorname{coker} g} & V \\ \| & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle v} \\ X & \xrightarrow{f} & Y & \xrightarrow{c} & C \end{array}$$

where v exists since $(ch)g = cf = 0$. Since 1_X is an isomorphism, $\operatorname{coker} g$ and c are cokernels, and $\mathcal{A}$ is quasi-abelian, by [39, Prop. 5] we have that the right square is *exact*, i.e. simultaneously a pullback and a pushout. Then, by assumption, either there exists $v_1\colon V \to Y$ such that $cv_1 = v$ or there exists $v_2\colon Y \to V$ such that $c = vv_2$. In the first case, we have the following situation



since $cv_1 = v = v1_V$, and hence there exists (a unique) $a\colon V \to U$ such that $(\operatorname{coker} g) \circ a = 1_V$ (and $v_1 = ha$). Thus, $\operatorname{coker} g$ is a retraction and by the Splitting Lemma (Proposition 2.7) we have that g is a section. Otherwise, in the case where v_2 exists, we have that there is (a unique) $b\colon Y \to U$ such that $hb = 1_Y$ (and $(\operatorname{coker} g) \circ b = v_2$), in which case h is seen to be a retraction. The following diagram summarises this case.



Therefore, f is irreducible and the proof is complete. ■

The dual statement is as follows.

Proposition 3.9. *Suppose $\mathcal{A}$ is a quasi-abelian category. Suppose $f\colon X \to Y$ is a morphism in $\mathcal{A}$ with kernel* $\ker f\colon \operatorname{Ker} f \to X$. *If f is irreducible, then for all $u\colon \operatorname{Ker} f \to U$ either there exists $u_1\colon X \to U$ such that $u_1 \ker f = u$ or there exists $u_2\colon U \to X$ such that* $\ker f = u_2 u$. *Furthermore, if* $\operatorname{Ker} f \xrightarrow{\ker f} X \xrightarrow{f} Y$ *is a non-split short exact sequence, then the converse also holds.*

Let k be a commutative (unital) ring and suppose $\mathcal{A}$ is a *k-category*, i.e. an additive category in which the set of morphisms between any two objects is a k-module and composition of morphisms is k-bilinear. The *radical* $\operatorname{rad}_{\mathcal{A}}(-,-)$ of a k-category is the (two-sided) ideal of $\mathcal{A}$ defined by

$$\operatorname{rad}_{\mathcal{A}}(X,Y) \coloneqq \{ f \in \operatorname{Hom}_{\mathcal{A}}(X,Y) \mid 1_X - gf \text{ is invertible for all } g\colon Y \to X \}$$

for any two objects $X, Y \in \mathcal{A}$. By a *radical* morphism $f\colon X \to Y$, we mean an element of $\operatorname{rad}_{\mathcal{A}}(X,Y)$. Furthermore, $\operatorname{rad}_{\mathcal{A}}(X,X) \subseteq \operatorname{End}_{\mathcal{A}} X$ coincides with the Jacobson radical $J(\operatorname{End}_{\mathcal{A}} X)$ of the ring $\operatorname{End}_{\mathcal{A}} X$. For $n \in \mathbb{Z}_{>0}$, $\operatorname{rad}^n_{\mathcal{A}}(X,Y)$ denotes the k-submodule of $\operatorname{Hom}_{\mathcal{A}}(X,Y)$ generated by morphisms that are a composition of n radical morphisms. See [26] or [27, §2] for more details.

The next two propositions together give a combined version of [8, Lem. 3.8] and [3, Lem. IV.1.9] valid for any k-category. We provide a full proof as the proof of the corresponding result for module categories is omitted in [8]. We also note that the equivalence of (i) and

(iii) in each statement below has already appeared in the proof of [6, Thm. 2.4] in the setting of an additive category with split idempotents.

Proposition 3.10. *Let $f\colon X \to Y$ be a morphism in a k-category $\mathcal{A}$, where $\operatorname{End}_{\mathcal{A}} X$ is a local ring. Then the following are equivalent:*

(i) *f is not a section;*
(ii) *$f \in \operatorname{rad}_{\mathcal{A}}(X, Y)$;*
(iii) *$\operatorname{Im}(\operatorname{Hom}_{\mathcal{A}}(f, X)) \subseteq J(\operatorname{End}_{\mathcal{A}} X) = \operatorname{rad}_{\mathcal{A}}(X, X)$; and*
(iv) *for all $Z \in \mathcal{A}$, the image $\operatorname{Im}(\operatorname{Hom}_{\mathcal{A}}(f, Z))$ of the map $\operatorname{Hom}_{\mathcal{A}}(f, Z)\colon \operatorname{Hom}_{\mathcal{A}}(Y, Z) \to \operatorname{Hom}_{\mathcal{A}}(X, Z)$ is contained in $\operatorname{rad}_{\mathcal{A}}(X, Z)$.*

Proof. First, assume $f = 0$. If f were a section then we would have $1_X = gf = 0$ for some $g\colon Y \to X$, but this is impossible as $\operatorname{End}_{\mathcal{A}} X$ is local so $X \neq 0$. Thus, f is not a section and (i) holds true. Furthermore, for any $Z \in \mathcal{A}$, we have $\operatorname{Im}(\operatorname{Hom}_{\mathcal{A}}(f, Z)) = 0$ if $f = 0$, and so is contained in $\operatorname{rad}_{\mathcal{A}}(X, Z)$. That is, (iii) and (iv) are satisfied. Since $\operatorname{rad}_{\mathcal{A}}$ is an ideal of $\mathcal{A}$, $f = 0 \in \operatorname{rad}_{\mathcal{A}}(X, Y)$ and (ii) also holds in this case.

Therefore, we may now assume $f \neq 0$. It is clear that (iv) implies (iii).

(iii) $\Rightarrow$ (i). If f is a section, then there exists $g\colon Y \to X$ with $\operatorname{Hom}_{\mathcal{A}}(f, X)(g) = gf = 1_X \in \operatorname{Im}(\operatorname{Hom}_{\mathcal{A}}(f, X)) \backslash J(\operatorname{End}_{\mathcal{A}} X)$, so $\operatorname{Im}(\operatorname{Hom}_{\mathcal{A}}(f, X)) \nsubseteq J(\operatorname{End}_{\mathcal{A}} X)$.

(i) $\Rightarrow$ (ii). Suppose f is not a section. Since $\operatorname{End}_{\mathcal{A}} X$ is local, $\operatorname{rad}_{\mathcal{A}}(X, X) = J(\operatorname{End}_{\mathcal{A}} X)$ is the set of all non-left invertible elements of $\operatorname{End}_{\mathcal{A}} X$. Let $g\colon Y \to X$ be arbitrary and consider $gf\colon X \to X$. Notice that gf cannot have a left inverse because we are assuming f is not a section. Therefore, $gf \in \operatorname{rad}_{\mathcal{A}}(X, X)$ and $1_X - gf$ is invertible. This is precisely the requirement for f to be radical.

(ii) $\Rightarrow$ (iv). Suppose $f \in \operatorname{rad}_{\mathcal{A}}(X, Y)$, and let $Z \in \mathcal{A}$ and $g\colon Y \to Z$ be arbitrary. Since $\operatorname{rad}_{\mathcal{A}}$ is an ideal of $\mathcal{A}$, we immediately see that $\operatorname{Hom}_{\mathcal{A}}(f, Z)(g) = gf \in \operatorname{rad}_{\mathcal{A}}(X, Z)$ and so $\operatorname{Im}(\operatorname{Hom}_{\mathcal{A}}(f, Z)) \subseteq \operatorname{rad}_{\mathcal{A}}(X, Z)$. ■

Proposition 3.11. *Let $f\colon X \to Y$ be a morphism in a k-category $\mathcal{A}$, where $\operatorname{End}_{\mathcal{A}} Y$ is a local ring. Then the following are equivalent:*

(i) *f is not a retraction;*
(ii) *$f \in \operatorname{rad}_{\mathcal{A}}(X, Y)$;*
(iii) *$\operatorname{Im}(\operatorname{Hom}_{\mathcal{A}}(Y, f)) \subseteq J(\operatorname{End}_{\mathcal{A}} Y) = \operatorname{rad}_{\mathcal{A}}(Y, Y)$; and*
(iv) *for all $Z \in \mathcal{A}$, the image $\operatorname{Im}(\operatorname{Hom}_{\mathcal{A}}(Z, f))$ of the map $\operatorname{Hom}_{\mathcal{A}}(Z, f)\colon \operatorname{Hom}_{\mathcal{A}}(Z, X) \to \operatorname{Hom}_{\mathcal{A}}(Z, Y)$ is contained in $\operatorname{rad}_{\mathcal{A}}(Z, Y)$.*

Immediately from the above two results, we have

Corollary 3.12. *Let $\mathcal{A}$ be a k-category and $f\colon X \to Y$ a morphism in $\mathcal{A}$, and suppose $\operatorname{End}_{\mathcal{A}} X$ is local or $\operatorname{End}_{\mathcal{A}} Y$ is local. If f is neither a section nor a retraction, then $f \in \operatorname{rad}_{\mathcal{A}}(X, Y)$.*

So far we have only studied irreducible morphisms and, as mentioned earlier, we will also be concerned with (minimal) left/right almost split morphisms.

Definition 3.13. [6, §2] Let $f\colon X \to Y$ be a morphism in an arbitrary category. We call f *right almost split* if

(i) f is not a retraction; and
(ii) for any non-retraction $u\colon U \to Y$ there exists $\hat{u}\colon U \to X$ such that $f\hat{u} = u$.

If $fg = f$ implies g is an automorphism for any $g\colon X \to X$, then f is said to be *right minimal.* If f is both right minimal and right almost split, then f is called *minimal right almost split.*

Dually, one can define the notions of *left almost split*, *left minimal* and *minimal left almost split.*

We recall that in an additive category if $f\colon X \to Y$ is right almost split (respectively, left almost split), then Y (respectively, X) has local endomorphism ring; see [6, Lem. 2.3].

Proposition 3.14. *Let $\mathcal{A}$ be an additive category with split idempotents.*

(i) *If $f\colon X \to Y$ is minimal left almost split and $Y \neq 0$, then f is irreducible.*
(ii) *If $f\colon X \to Y$ is minimal right almost split and $X \neq 0$, then f is irreducible.*

Proof. For (i), notice that f satisfies the criterion in [6, Thm. 2.4 (b)]. Statement (ii) is dual. ■

The next proposition is an observation that we may generalise [10, Prop. 2.18] to a category with split idempotents that is not necessarily Krull-Schmidt, e.g. the category of all left R-modules for a ring R, or the category of all Banach spaces (over $\mathbb{R}$, for example). This result generalises [6, Cor. 2.5] since an irreducible morphism with a domain or codomain that has local endomorphism ring is radical by Corollary 3.12. We omit the proof as the one given in [10] holds in our generality using [16, Rem. 7.4]. See [9, Prop. 3.2] also.

Proposition 3.15. *Let $\mathcal{A}$ be an additive category with split idempotents. Suppose $f\colon X \to Y$ is a radical irreducible morphism in $\mathcal{A}$.*

(i) *If $0 \neq g\colon W \to X$ is a section, then $fg\colon W \to Y$ is irreducible.*
(ii) *If $0 \neq h\colon Y \to Z$ is a retraction, then $hf\colon X \to Z$ is irreducible.*

The following is a version of [6, Prop. 2.10] for the quasi-abelian setting. We will see that the idea behind the proof is the same, but we have to negotiate around the fact that the class of kernels (respectively, cokernels) does not necessarily coincide with the class of monomorphisms (respectively, epimorphisms) in the category.

Proposition 3.16. *Suppose $\mathcal{A}$ is a quasi-abelian category.*

(i) *If $f\colon X \to Y$ is an irreducible monomorphism, Y is indecomposable and $v\colon V \to \operatorname{Coker} f$ is any irreducible morphism, then v is epic.*
(ii) *If $f\colon X \to Y$ is an irreducible epimorphism, X is indecomposable and $u\colon \operatorname{Ker} f \to X$ is any irreducible morphism, then u is monic.*

Proof. We only prove (i) as (ii) is dual. Suppose $f\colon X \to Y$ is an irreducible monomorphism, and that Y is an indecomposable object. First, if $C := \operatorname{Coker} f = 0$ then any morphism $v\colon V \to C$ is trivially epic as $\mathcal{A}$ is additive, so we may suppose $C \neq 0$.

Consider the (not necessarily short exact) sequence $X \xrightarrow{f} Y \xrightarrow{c:=\operatorname{coker} f} C$. Let $v\colon V \to C$ be an irreducible morphism in $\mathcal{A}$. By Proposition 3.8 either there exists $v_1\colon V \to Y$ such that $cv_1 = v$ or there exists $v_2\colon Y \to V$ such that $c = vv_2$. In the latter case, as c is epic, v would also be epic and we would be done. Thus, suppose no such v_2 exists. Then there exists $v_1\colon V \to Y$ with $cv_1 = v$. But now v is irreducible, and so either c is a retraction or v_1 is a section. If $c\colon Y \to C$ is a retraction, then by Lemma 2.3 we have that c is an isomorphism since Y is indecomposable and $C \neq 0$. However, this implies $f = c^{-1}cf = 0$ and in turn yields $1_X = 0$, since $f \circ 1_X = f = 0$ and f is monic. Thus, we would have $X = 0$ and f is in fact a section, which contradicts that f is irreducible. Hence, c cannot be a retraction and so $v_1\colon V \to Y$ must be a section.

If $V = 0$ then $v\colon 0 = V \to C$ is a section, which is impossible as v is assumed to be irreducible. Therefore, $V \neq 0$ and hence, by Lemma 2.3 again, $v_1\colon V \to Y$ must be an isomorphism and, in particular, an epimorphism. Finally, we observe that $v = cv_1$ is the composition of two epimorphisms and hence an epimorphism itself. ■

Definition 3.17. [38, p. 522] Let $\mathcal{A}$ be an additive category. A kernel (respectively, cokernel) is called *semi-stable* if every pushout (respectively, pullback) of it is again a kernel (respectively, a cokernel).

Example 3.18. Consider a stable exact sequence $\xi\colon X \overset{f}{\to} Y \overset{g}{\to} Z$. Let $a\colon X \to X'$ be a morphism, and form the sequence $a\xi$ as in Definition 2.10. Since ξ is stable, the sequence $a\xi$ is short exact and hence the pushout of f along a is again a kernel. Thus, f is a semi-stable kernel. Dually, g is a semi-stable cokernel.

Remark 3.19. All kernels are semi-stable in a right quasi-abelian category and all cokernels are semi-stable in a left quasi-abelian category. In particular, all short exact sequences are stable in a quasi-abelian category.

Lemma 3.20. *Let $\mathcal{A}$ be a preabelian category. Suppose $\xi\colon X \overset{f}{\to} Y \overset{g}{\to} Z$ is short exact. If f or g is semi-stable, then any morphism $(1_X, v, 1_Z)\colon \xi \to \xi$ is an isomorphism.*

Proof. This follows from [38, Thm. 6], noting that the dual of this result by Richman and Walker holds when the morphism $(1_X, v, 1_Z)$ of short exact sequences is an endomorphism. ■

Our main theorem in §4 generalises [3, Thm. IV.1.13], and part of the proof uses tools to detect when an endomorphism (u, v, w) of a short exact sequence is in fact an *isomorphism*, i.e. u, v, w are all isomorphisms. We present generalisations of these tools now, and we will see the work of §2.2 used below. We will assume for simplicity that a preabelian category $\mathcal{A}$ is skeletally small whenever our proofs require the use of an extension group. However, we only really need that the first extension group is a set in the relevant arguments (see Remark 2.16).

Definition 3.21. [27, p. 547] An additive category $\mathcal{A}$ is called Hom-*finite* if $\mathcal{A}$ is a k-category, for some commutative ring k, and $\operatorname{Hom}_{\mathcal{A}}(X,Y)$ is a finite length k-module for any $X, Y \in \mathcal{A}$.

Proposition 3.22. *Let $\mathcal{A}$ be a* Hom-*finite category. Suppose we have a commutative diagram*

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z \end{array}$$

in $\mathcal{A}$ with non-split short exact rows. If $\operatorname{End}_{\mathcal{A}} X$ (respectively, $\operatorname{End}_{\mathcal{A}} Z$) is local and w (respectively, u) is an automorphism, then u (respectively, w) is an automorphism.

Further, if $\mathcal{A}$ is also preabelian and if f or g is semi-stable, then v is also an automorphism in this case.

Proof. Suppose that $\operatorname{End}_{\mathcal{A}} X$ is local and w is an automorphism of Z. Showing that u is an automorphism in this case is the same as in [3, Lem. IV.1.12].

Now assume $\mathcal{A}$ is also preabelian, and that f is a semi-stable kernel or g is a semi-stable cokernel. Consider the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{fu^{-1}} & Y & \xrightarrow{wg} & Z \\ \| & & \downarrow{\scriptstyle v} & & \| \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z \end{array}$$

that has short exact rows. Then v is an automorphism by Lemma 3.20. ■

The following corollary is a generalisation of [3, Lem. IV.1.12] to a quasi-abelian Hom-finite setting, and it follows quickly from Proposition 3.22 in light of Remark 3.19.

Corollary 3.23. *Let $\mathcal{A}$ be a* Hom-*finite category, which is left or right quasi-abelian. Suppose we have a commutative diagram*

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z \end{array}$$

in $\mathcal{A}$ with non-split short exact rows. If $\operatorname{End}_{\mathcal{A}} X$ (respectively, $\operatorname{End}_{\mathcal{A}} Z$) is local and w (respectively, u) is an automorphism, then u (respectively, w) is an automorphism. Furthermore, v is also an automorphism.

The next proposition is a generalisation of [6, Lem. 2.13] for preabelian categories. However, note that we need to assume the short exact sequence in question is stable.

Proposition 3.24. *Let $\mathcal{A}$ be a skeletally small, preabelian category. Suppose we have a commutative diagram*

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \big\downarrow{\scriptstyle u} & & \big\downarrow{\scriptstyle v} & & \big\downarrow{\scriptstyle w} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z \end{array}$$

in $\mathcal{A}$, where $\xi \colon X \overset{f}{\to} Y \overset{g}{\to} Z$ is a non-split stable exact sequence. If $\operatorname{End}_{\mathcal{A}} X$ (respectively, $\operatorname{End}_{\mathcal{A}} Z$) is local and w (respectively, u) is an automorphism, then u (respectively, w) and hence v are automorphisms.

Proof. The proof is that of [6] with the following adjustments. In order to show u is an automorphism, one needs that $u\xi \cong \xi 1_Z = \xi$ and that $\operatorname{Ext}^1_{\mathcal{A}}(Z, X)$ is a left $\operatorname{End}_{\mathcal{A}} X$-module, which follow from [38, Cor. 7] and Theorem 2.19, respectively. Lastly, an application of Lemma 3.20 yields that v is also an automorphism. ■

Since all short exact sequences are stable in a quasi-abelian category (see Remark 3.19), we obtain a direct generalisation of [6, Lem. 2.13] as follows.

Corollary 3.25. *Let $\mathcal{A}$ be a skeletally small, quasi-abelian category. Suppose we have a commutative diagram*

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \big\downarrow{\scriptstyle u} & & \big\downarrow{\scriptstyle v} & & \big\downarrow{\scriptstyle w} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z \end{array}$$

in $\mathcal{A}$, where $X \overset{f}{\to} Y \overset{g}{\to} Z$ is a non-split short exact sequence. If $\operatorname{End}_{\mathcal{A}} X$ (respectively, $\operatorname{End}_{\mathcal{A}} Z$) is local and w (respectively, u) is an automorphism, then u (respectively, w) and hence v are automorphisms.

The following definition has been given by Auslander for abelian categories (see [4, p. 292]), but we may make the same definition for additive categories and are able to derive some of the same consequences.

Definition 3.26. [4, p. 292] Let $\mathcal{A}$ be an additive category with an object X. Suppose $\mathscr{F} \colon \mathcal{A} \to \mathsf{Ab}$ is a covariant additive functor to the category Ab of all abelian groups. An element $x \in \mathscr{F}(X)$ is said to be *minimal* if $x \neq 0$ (where 0 is the identity element of the abelian group $\mathscr{F}(X)$), and if for all proper epimorphisms $f \colon X \to Y$ in $\mathcal{A}$, we have that $\mathscr{F}(f) \colon \mathscr{F}(X) \to \mathscr{F}(Y)$ satisfies $\mathscr{F}(f)(x) = 0$.

A definition of minimal can be made for a contravariant functor $\mathscr{G} \colon \mathcal{A} \to \mathsf{Ab}$ by considering $\mathscr{G}$ as a covariant functor $\mathcal{A}^{\mathrm{op}} \to \mathsf{Ab}$.

An immediate result is a version of [4, p. 292, Lem. 3.2 (a)] for additive categories:

Proposition 3.27. *Let $\mathcal{A}$ be an additive category with an object X. Suppose $\mathscr{F} \colon \mathcal{A} \to \mathsf{Ab}$ is a covariant additive functor. If $\mathscr{F}(X)$ has a minimal element, then X is indecomposable in $\mathcal{A}$.*

Proof. Assume $x \in \mathscr{F}(X)$ is minimal, and that $X = X_1 \oplus X_2$ with X_1, X_2 both non-zero. Let $\iota_i\colon X_i \hookrightarrow X$ and $\pi_i\colon X \twoheadrightarrow X_i$ be the canonical inclusion and projection morphisms, respectively, for $i = 1, 2$. Note that π_i is a proper epimorphism for $i = 1, 2$ since X_1 and X_2 are non-zero. Therefore, $\mathscr{F}(\pi_i)(x) = 0$ for $i = 1, 2$ as x is minimal. However, this implies

$$\begin{array}{rcll} x & = & 1_{\mathscr{F}(X)}(x) = \mathscr{F}(1_X)(x) & \text{as } \mathscr{F} \text{ is a functor} \\ & = & \mathscr{F}(\iota_1\pi_1 + \iota_2\pi_2)(x) & \text{as } X = X_1 \oplus X_2 \\ & = & \mathscr{F}(\iota_1\pi_1)(x) + \mathscr{F}(\iota_2\pi_2)(x) & \text{as } \mathscr{F} \text{ is additive} \\ & = & \mathscr{F}(\iota_1)(\mathscr{F}(\pi_1)(x)) + \mathscr{F}(\iota_2)(\mathscr{F}(\pi_2)(x)) & \text{as } \mathscr{F} \text{ is covariant} \\ & = & 0 & \text{since } \mathscr{F}(\pi_i)(x) = 0. \end{array}$$

This is a contradiction because $x \neq 0$ since it is minimal. Hence, X must be indecomposable. ■

The next proposition generalises [6, Prop. 2.6 (b)] to a semi-abelian setting. The strategy in the proof is the same, but we need a technical result from [39] in order to work in a category with less structure. Note that if $\mathcal{A}$ skeletally small, then $\operatorname{Ext}^1_{\mathcal{A}}(-,-)$ is an additive bifunctor (see [38, §4], or [18, p. 267]).

Proposition 3.28. *Let $\mathcal{A}$ be a skeletally small, semi-abelian category with objects X, Z. Consider the covariant additive functor $\operatorname{Ext}^1_{\mathcal{A}}(Z,-)\colon \mathcal{A} \to \mathsf{Ab}$ and the contravariant additive functor $\operatorname{Ext}^1_{\mathcal{A}}(-,X)\colon \mathcal{A} \to \mathsf{Ab}$. Suppose $\xi\colon X \xrightarrow{f} Y \xrightarrow{g} Z$ is an element of $\operatorname{Ext}^1_{\mathcal{A}}(Z,X)$.*

(i) *If f is irreducible, then $\xi \in \operatorname{Ext}^1_{\mathcal{A}}(-,X)(Z)$ is minimal, and hence Z is indecomposable.*

(ii) *If g is irreducible, then $\xi \in \operatorname{Ext}^1_{\mathcal{A}}(Z,-)(X)$ is minimal, and hence X is indecomposable.*

Proof. We prove (ii); the proof for (i) is similar. Suppose g is irreducible in the stable exact sequence $\xi\colon X \xrightarrow{f} Y \xrightarrow{g} Z$. Since g is not a retraction, by the Splitting Lemma (Proposition 2.7) we know ξ is not split and hence $\xi \neq 0$ in $\operatorname{Ext}^1_{\mathcal{A}}(Z,X)$. Suppose $a\colon X \to X_1$ is a proper epimorphism. We will show that $\operatorname{Ext}^1_{\mathcal{A}}(Z,a)(\xi) = a\xi = 0$, i.e. the short exact sequence $a\xi$ is split. By definition, $a\xi$ comes with some commutative diagram

$$\begin{array}{rccccc} \xi\colon & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ & \downarrow{\scriptstyle a} & \square & \downarrow{\scriptstyle b} & & \| \\ a\xi\colon & X_1 & \xrightarrow{f_1} & Y_1 & \xrightarrow{g_1} & Z \end{array}$$

where the left square is a pushout square. Thus, $g = g_1 b$ and so g_1 is a retraction or b is a section as g is assumed to be irreducible.

Assume, for contradiction, that b is a section. Then there exists $r\colon Y_1 \to Y$ such that $rb = 1_Y$. This yields $(rf_1)a = rbf = f = \ker g$. As $\mathcal{A}$ is (right) semi-abelian, we have that a is also a kernel by [39, Prop. 2]. Therefore, a is an epic kernel and hence an isomorphism by Lemma 2.5, which contradicts that a is a proper epimorphism. Hence, b cannot be a section.

Thus, g_1 must be a retraction, whence $a\xi\colon X_1 \xrightarrow{f_1} Y_1 \xrightarrow{g_1} Z$ is split (Proposition 2.7 again) and $a\xi = 0$ in $\operatorname{Ext}^1_{\mathcal{A}}(Z,-)(X_1)$. Since $a\colon X \to X_1$ was an arbitrary proper epimorphism, we see that ξ is a minimal element of $\operatorname{Ext}^1_{\mathcal{A}}(Z,-)(X)$ and that X is indecomposable by Proposition 3.27. ■

We also observe that [6, Prop. 2.11] remains valid in a more general situation:

Proposition 3.29. *Let $\mathcal{A}$ be a skeletally small, preabelian category. Suppose $f\colon X \to Y$ is an irreducible morphism in $\mathcal{A}$ and let Z be an object of $\mathcal{A}$.*

(i) *If* $\operatorname{Hom}_{\mathcal{A}}(Y,Z) = 0$*, then* $0 \longrightarrow \operatorname{Ext}^1_{\mathcal{A}}(Z,X) \xrightarrow{\operatorname{Ext}^1_{\mathcal{A}}(Z,f)} \operatorname{Ext}^1_{\mathcal{A}}(Z,Y)$ *is exact.*

(ii) *If* $\operatorname{Hom}_{\mathcal{A}}(Z,X) = 0$*, then* $0 \longrightarrow \operatorname{Ext}^1_{\mathcal{A}}(Y,Z) \xrightarrow{\operatorname{Ext}^1_{\mathcal{A}}(f,Z)} \operatorname{Ext}^1_{\mathcal{A}}(X,Z)$ *is exact.*

Proof. This is an arrow-theoretic translation of the proof from [6]. ■

The next result is an analogue of [6, Prop. 2.8] for which we need the theory of subobjects in an abelian category. Recall that two monomorphisms $i_1\colon X_1 \to X$ and $i_2\colon X_2 \to X$ in an abelian category are said to be *equivalent* if there is an isomorphism $f\colon X_1 \xrightarrow{\cong} X_2$ such that $i_1 = i_2 \circ f$. Then a *subobject* of an object X in an abelian category is an equivalence class of monomorphisms into X. Furthermore, if $i\colon V \to X$ and $j\colon W \to X$ are representatives of subobjects of X, then we write $V \subseteq W$ if there exists a morphism $g\colon V \to W$ such that $i = j \circ g$. See [27] or [19] for more details.

Proposition 3.30. *Let $\mathcal{A}$ be a skeletally small, quasi-abelian category and suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a non-split short exact sequence in $\mathcal{A}$.*

(i) *The morphism f is irreducible if and only if, for any subobject $\mathscr{F}$ of* $\operatorname{Hom}_{\mathcal{A}}(-,Z)$*, we have either $\mathscr{F}$ contains or is contained in* $\operatorname{Im}(\operatorname{Hom}_{\mathcal{A}}(-,g))$*, the image of the natural transformation* $\operatorname{Hom}_{\mathcal{A}}(-,g)\colon \operatorname{Hom}_{\mathcal{A}}(-,Y) \to \operatorname{Hom}_{\mathcal{A}}(-,Z)$.
(ii) *The morphism g is irreducible if and only if, for any subobject $\mathscr{F}$ of* $\operatorname{Hom}_{\mathcal{A}}(X,-)$*, we have either $\mathscr{F}$ contains or is contained in* $\operatorname{Im}(\operatorname{Hom}_{\mathcal{A}}(f,-))$*, the image of the natural transformation* $\operatorname{Hom}_{\mathcal{A}}(f,-)\colon \operatorname{Hom}_{\mathcal{A}}(Y,-) \to \operatorname{Hom}_{\mathcal{A}}(X,-)$.

Proof. Note that the category of functors $\mathcal{A} \to \mathsf{Ab}$ is abelian as $\mathcal{A}$ is skeletally small (see [36, Thm. 10.1.3]). The proof is identical to that for [6, Prop. 2.8], noting that we may use [36, Prop. 10.1.13], and Propositions 3.8 and 3.9. ■

Proposition 3.31 and its dual, Proposition 3.32, below give analogues of one direction of parts (a) and (b) of [6, Cor. 2.9] for quasi-abelian categories. There is an obvious method to prove (ii) in light of Proposition 3.8, but the details are omitted in [6] so we include them here for completeness.

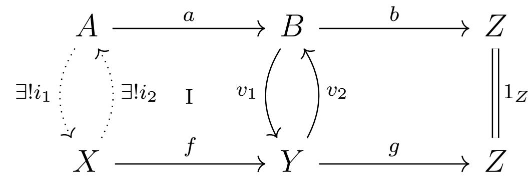
Proposition 3.31. *Let $\mathcal{A}$ be a quasi-abelian category and suppose $\xi\colon X \xrightarrow{f} Y \xrightarrow{g} Z$ is a non-split short exact sequence in $\mathcal{A}$. Suppose f is irreducible. Then the following statements hold.*

(i) *For any proper subobject $\iota\colon Y' \hookrightarrow Y$ such that $\operatorname{Im} f \subseteq Y'$ given by a monomorphism $s\colon \operatorname{Im} f \hookrightarrow Y'$, we have that s is a section.*

(ii) *For any short exact sequence* $\xi' \colon A \xrightarrow{a} B \xrightarrow{b} Z$, *either there exists* $j \colon A \to X$ *such that* $j\xi' = \xi$ *or there exists* $i \colon X \to A$ *such that* $\xi i = \xi'$.

Proof. Suppose f is irreducible. To show (i) holds, we assume $\iota \colon Y' \to Y$ is a proper monomorphism and $s \colon \operatorname{Im} f \to Y'$ is a monomorphism such that $\operatorname{im} f = \iota s$. Since $\mathcal{A}$ is preabelian and ξ is short exact, we have that $f = \ker g = \ker(\operatorname{coker} f) = \operatorname{im} f$ by Lemma 2.4. Therefore, $f = \iota s$ and hence either ι is a retraction or s is a section. If ι is a retraction then it would be a monic retraction, and hence an isomorphism by [30, Thm. I.1.5]. But this contradicts our assumption on ι, so we must have that s is a section.

For (ii), if $A \xrightarrow{a} B \xrightarrow{b} Z$ is a short exact sequence, then by Proposition 3.8 either there exists $v_1 \colon B \to Y$ with $gv_1 = b$ or there exists $v_2 \colon Y \to B$ with $g = bv_2$. This will yield one of the two morphisms of short exact sequences indicated in the following diagram.



Therefore, we need only show that the left square I is a pushout in either case. However, this follows immediately from the dual of [39, Prop. 5] since a, f are kernels and 1_Z is an isomorphism. ■

Proposition 3.32. *Let* $\mathcal{A}$ *be a quasi-abelian category and suppose* $\xi \colon X \xrightarrow{f} Y \xrightarrow{g} Z$ *is a non-split short exact sequence in* $\mathcal{A}$. *Suppose* g *is irreducible. Then the following statements hold.*

(i) *For any non-zero subobject* $X' \overset{\iota}{\hookrightarrow} X$, *the induced morphism* $r \colon Y/X' = \operatorname{Coker}(f\iota) \to Z$ *is a retraction.*

(ii) *For any short exact sequence* $\xi' \colon X \xrightarrow{b} B \xrightarrow{c} C$, *either there exists* $j \colon Z \to C$ *such that* $\xi' j = \xi$ *or there exists* $i \colon C \to Z$ *such that* $i\xi = \xi'$.

4. Auslander-Reiten theory in Krull-Schmidt categories

Let $\mathcal{A}$ denote a k-category, for some commutative ring k, and let $\mathcal{I}$ denote a (two-sided) ideal of $\mathcal{A}$. For a morphism $f \colon X \to Y$ in $\mathcal{A}$, we will denote by $\overline{f}$ the morphism $f + \mathcal{I}(X, Y)$ in the additive quotient k-category $\mathcal{A}/\mathcal{I}$. For most of this section we will study Krull-Schmidt categories that are not necessarily Hom-finite. There are very interesting examples of Hom-infinite generalised cluster categories (see [2], [33], [25]) coming from quivers with potential. In these examples, a certain Krull-Schmidt category has been used in [34] to show the existence of cluster characters for Hom-infinite cluster categories. We provide an example of this kind at the end of this section (see Example 4.20).

Before we begin our study of Auslander-Reiten theory in Krull-Schmidt categories, we present a series of lemmas inspired by [7, Lem. 1.1]. The proofs are omitted since they are easy generalisations of those in [7].

Lemma 4.1. *Suppose* $X, Y \in \mathcal{A}$ *and* $f \in \mathcal{I}(X, Y)$. *If* $1_X \notin \mathcal{I}(X, X)$ *or* $1_Y \notin \mathcal{I}(Y, Y)$, *then* $f \colon X \to Y$ *is not an isomorphism.*

Lemma 4.2. *Suppose* $X = \bigoplus_{i=1}^{n} X_i$ *in* $\mathcal{A}$*, with* $\operatorname{End}_{\mathcal{A}} X_i$ *local and* $1_{X_i} \notin \mathcal{I}(X_i, X_i)$ *for each* $i = 1, \ldots, n$*. For an endomorphism* $f\colon X \to X$*, if* $f \in \mathcal{I}(X, X)$ *then* $f \in \operatorname{rad}_{\mathcal{A}}(X, X)$*.*

Lemma 4.3. *Suppose* $X = \bigoplus_{i=1}^{n} X_i$ *and* $Y = \bigoplus_{j=1}^{m} Y_j$ *in* $\mathcal{A}$*, with* $\operatorname{End}_{\mathcal{A}} X_i$ *and* $\operatorname{End}_{\mathcal{A}} Y_j$ *local for all* i, j*. Let* $f\colon X \to Y$ *be a morphism in* $\mathcal{A}$*.*

(i) *If* $1_{X_i} \notin \mathcal{I}(X_i, X_i)\ \forall 1 \leqslant i \leqslant n$*, then* f *is a section in* $\mathcal{A} \iff \overline{f}$ *is a section in* $\mathcal{A}/\mathcal{I}$*.*
(ii) *If* $1_{Y_j} \notin \mathcal{I}(Y_j, Y_j)\ \forall 1 \leqslant j \leqslant m$*, then* f *is a retraction in* $\mathcal{A} \iff \overline{f}$ *is a retraction in* $\mathcal{A}/\mathcal{I}$*.*
(iii) *If* $1_{X_i} \notin \mathcal{I}(X_i, X_i)$ *and* $1_{Y_j} \notin \mathcal{I}(Y_j, Y_j)$ *for all* i, j*, then* f *is an isomorphism in* $\mathcal{A} \iff \overline{f}$ *is an isomorphism in* $\mathcal{A}/\mathcal{I}$*.*

The forward direction of the next lemma can be found in [28] just above [28, Def. 1.6], but it is a short argument so we include it here for completeness.

Lemma 4.4. *Suppose* $X = \bigoplus_{i=1}^{n} X_i$ *in* $\mathcal{A}$*, with* $\operatorname{End}_{\mathcal{A}} X_i$ *local and* $1_{X_i} \notin \mathcal{I}(X_i, X_i)$ *for each* $i = 1, \ldots, n$*. Then* $\operatorname{End}_{\mathcal{A}} X$ *is local if and only if* $\operatorname{End}_{\mathcal{A}/\mathcal{I}} X$ *is local.*

Proof. ($\Rightarrow$) If $\operatorname{End}_{\mathcal{A}} X$ is local, then $\mathcal{I}(X, X)$ is contained in the Jacobson radical $J(\operatorname{End}_{\mathcal{A}} X)$, which is the unique maximal ideal of $\operatorname{End}_{\mathcal{A}} X$, since $1_X \notin \mathcal{I}(X, X)$. Then $\operatorname{End}_{\mathcal{A}/\mathcal{I}} X$ has unique maximal ideal $J(\operatorname{End}_{\mathcal{A}} X)/\mathcal{I}(X, X)$.

($\Leftarrow$) Conversely, assume $\operatorname{End}_{\mathcal{A}/\mathcal{I}} X$ is local and let $u\colon X \to X$ be a non-unit in $\operatorname{End}_{\mathcal{A}} X$. We will show that $1_X - u$ is a unit in $\operatorname{End}_{\mathcal{A}} X$. If $\overline{u}$ is a unit in $\operatorname{End}_{\mathcal{A}/\mathcal{I}} X$ then $\overline{1_X} = \overline{uv}$ for some $v\colon X \to X$. Then $1_X - uv \in \mathcal{I}(X, X)$, so $1_X - uv$ is radical by Lemma 4.2. Then $uv = 1_X - (1_X - uv)$ is a unit, so that u has a right inverse. A similar argument also shows u has a left inverse and so u is a unit, contrary to our assumption on u. Thus, $\overline{u}$ is not invertible and, as $\operatorname{End}_{\mathcal{A}/\mathcal{I}} X$ is local, $\overline{u}$ must be radical. Therefore, $\overline{1_X} - \overline{u}$ is a unit in $\operatorname{End}_{\mathcal{A}/\mathcal{I}} X$. So for some $w\colon X \to X$ we have $\overline{w}(\overline{1_X} - \overline{u}) = \overline{1_X}$. This shows that $1_X - w(1_X - u) \in \mathcal{I}(X, X)$ must be radical using Lemma 4.2 as before, so $w(1_X - u) = 1_X - (1_X - w(1_X - u))$ is invertible and $1_X - u$ has a left inverse. Again, a similar argument shows $1_X - u$ has a right inverse, and hence a two-sided inverse. ■

Definition 4.5. Suppose $f\colon X \to Y$ is a morphism in an additive category $\mathcal{A}$. A *weak kernel* of f is a morphism $w\colon W \to X$ such that $f \circ w = 0$, with the following property: for every morphism $g\colon V \to X$ with $fg = 0$, there exists $\widehat{g}\colon V \to W$ such that $w\widehat{g} = g$. A *weak cokernel* is defined dually. We call a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ *short weak exact* if f is a weak kernel of g and g is a weak cokernel of f.

It is easy to show that a morphism is a pseudo-(co)kernel, in the sense of [28], if and only if it is a weak (co)kernel. We adopt the terminology 'weak' as it seems more widely used.

Definition 4.6. We call a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$ an *Auslander-Reiten sequence (in an additive category)* if the following conditions are satisfied.

(i) The sequence is short weak exact.

(ii) The morphism f is minimal left almost split.
(iii) The morphism g is minimal right almost split.

In an almost identical way, Liu defines an Auslander-Reiten sequence for a Hom-finite, Krull-Schmidt category in [28]. However, we do not impose the condition that the middle term be non-zero, because Auslander-Reiten sequences of the form $X \to 0 \to Z$ do appear, for example, in the bounded derived category $\mathsf{D}^b(k\mathbb{A}_1 - \mathsf{mod})$ of the path algebra $k\mathbb{A}_1$, where k is an algebraically closed field, and $\mathbb{A}_1$ is the quiver with one vertex and no arrows. As we will see now, the results of [28, §1] can be generalised to the not necessarily Hom-finite setting. First, we note that [28, Lem. 1.1] is still valid for an arbitrary additive category.

Next, we give a more general version of [28, Prop. 1.5]:

Proposition 4.7. *Suppose $\mathcal{A}$ is a preabelian category, and let $\xi\colon X \xrightarrow{f} Y \xrightarrow{g} Z$ be an Auslander-Reiten sequence in $\mathcal{A}$ with $Y \neq 0$. Then ξ is short exact.*

Recall that an additive category $\mathcal{A}$ is *Krull-Schmidt* if, for any object X of $\mathcal{A}$, there exists a finite direct sum decomposition $X = X_1 \oplus \cdots \oplus X_n$ where $\operatorname{End}_{\mathcal{A}}(X_i)$ is a local ring for $i = 1, \ldots, n$ (see [27, p. 544]).

Throughout the remainder of this section, we further assume that $\mathcal{A}$ is a Krull-Schmidt category unless otherwise stated. In particular, $\mathcal{A}$ has split idempotents (see Remark 2.2), and an object $X \in \mathcal{A}$ is indecomposable if and only if $\operatorname{End}_{\mathcal{A}} X$ is local. We still assume $\mathcal{I}$ is an ideal of $\mathcal{A}$.

The following lemma is a generalisation of [28, Lem. 1.2] that will be needed to prove a uniqueness result about Auslander-Reiten sequences in a Krull-Schmidt category (see Theorem 4.9). Part of the proof in [28] uses heavily that the category is Hom-finite, so the corresponding part of the proof below is quite different in nature.

Lemma 4.8. *Suppose*

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ Y & \xrightarrow{g} & Z \end{array}$$

is a commutative diagram in $\mathcal{A}$, with f, g both non-zero.

(i) *If f, g are minimal right almost split, then $u \in \operatorname{Aut}_{\mathcal{A}} Y \iff v \in \operatorname{Aut}_{\mathcal{A}} Z$.*
(ii) *If f, g are minimal left almost split, then $u \in \operatorname{Aut}_{\mathcal{A}} Y \iff v \in \operatorname{Aut}_{\mathcal{A}} Z$.*

Proof. We prove only (i) as the proof for (ii) is dual. Assume f, g are non-zero, minimal right almost split morphisms with $vf = gu$. Note that the argument in [28] that u is an automorphism of Y whenever v is an automorphism of Z works here as well, so we only show the converse. We observe for later use that Y, Z are both non-zero since there exists a non-zero morphism between them.

Therefore, suppose $u \in \operatorname{Aut}_{\mathcal{A}} Y$ with inverse u^{-1}. Since f is right almost split, we have that $\operatorname{End}_{\mathcal{A}} Z$ is local, so Z is indecomposable. Assume, for contradiction, that v is not a retraction. Then v factors through the right almost split morphism g as, say, $v = ga$

for some $a\colon Z \to Y$. In particular, we see that $g = guu^{-1} = vfu^{-1} = gafu^{-1}$ and hence afu^{-1} is an automorphism of Y as g is right minimal. This means that af is also an automorphism of Y and that a is a retraction. Then, by Lemma 2.3, we have that a is an isomorphism because Z is indecomposable and $Y \neq 0$. However, this yields that $f = a^{-1}af$ is an isomorphism, and hence a retraction, which contradicts that f is right almost split. Hence, v must be a retraction, and thus also an isomorphism by Lemma 2.3. ■

Now we generalise [28, Thm. 1.4] to a not necessarily Hom-finite (but still Krull-Schmidt) setting.

Theorem 4.9. *Let $\mathcal{A}$ be a Krull-Schmidt category, and suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ is an Auslander-Reiten sequence in $\mathcal{A}$ with $Y \neq 0$.*

(i) *Up to isomorphism, $X \xrightarrow{f} Y \xrightarrow{g} Z$ is the unique Auslander-Reiten sequence starting at X and the unique one ending at Z.*
(ii) *Any irreducible morphism $f_1\colon X \to Y_1$ or $g_1\colon Y_1 \to Z$ fits into an Auslander-Reiten sequence $X \xrightarrow{\left(\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right)} Y_1 \oplus Y_2 \xrightarrow{(\, g_1 \; g_2 \,)} Z$.*

Proof. Follow the proof in [28], replacing the use of [28, Lem. 1.2] with Lemma 4.8. ■

For a Hom-finite, Krull-Schmidt category, Liu identifies a nice class of ideals—admissible ideals. It is observed in [28] that, for such an ideal $\mathcal{I}$ of a Hom-finite, Krull-Schmidt category $\mathcal{A}$, irreducible morphisms (between indecomposables) and minimal left/right almost split morphisms remain, respectively, so under the quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{I}$. We adopt the same definition but without the Hom-finite restriction.

Definition 4.10. [28, Def. 1.6] Suppose $\mathcal{A}$ is a Krull-Schmidt k-category. An ideal $\mathcal{I}$ of $\mathcal{A}$ is called *admissible* if it satisfies the following.

(i) Whenever $X, Y \in \mathcal{A}$ are indecomposable such that $1_X \notin \mathcal{I}(X,X)$ and $1_Y \notin \mathcal{I}(Y,Y)$, then $\mathcal{I}(X,Y) \subseteq \operatorname{rad}^2_{\mathcal{A}}(X,Y)$.
(ii) If $f\colon X \to Y$ is minimal left almost split, where $1_X \notin \mathcal{I}(X,X)$, and $g \in \mathcal{I}(X,M)$, then we can express $g = hf$ for some $h \in \mathcal{I}(Y,M)$.
(iii) If $f\colon X \to Y$ is minimal right almost split, where $1_Y \notin \mathcal{I}(Y,Y)$, and $g \in \mathcal{I}(M,Y)$, then we can express $g = fh$ for some $h \in \mathcal{I}(M,X)$.

Example 4.11. Suppose $\mathcal{B} \subseteq \mathcal{A}$ is a full subcategory closed under direct sums and direct summands. Then the ideal $[\mathcal{B}]$ of morphisms factoring through objects of $\mathcal{B}$ is admissible. See [28, Prop. 1.9].

The next result follows quickly from the definition of an admissible ideal.

Lemma 4.12. *Suppose $\mathcal{I}$ is an admissible ideal of $\mathcal{A}$. Suppose $X = \bigoplus_{i=1}^{n} X_i$ and $Y = \bigoplus_{j=1}^{m} Y_j$ are decompositions into indecomposables in $\mathcal{A}$ with $1_{X_i} \notin \mathcal{I}(X_i, X_i), 1_{Y_j} \notin \mathcal{I}(Y_j, Y_j)$ for all i, j. Then $\mathcal{I}(X,Y) \subseteq \operatorname{rad}^2_{\mathcal{A}}(X,Y)$.*

Proof. Let $f \in \mathcal{I}(X,Y)$ be arbitrary and write

$$f = \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{m1} & \cdots & f_{mn} \end{pmatrix}$$

where $f_{ji}\colon X_i \to Y_j$. Then for each i,j we have $f_{ji} = \pi_j f \iota_i \in \mathcal{I}(X_i, Y_j)$, where $\pi_j\colon Y \to Y_j$ is the natural projection and $\iota_i\colon X_i \to X$ is the natural inclusion. Since $\mathcal{I}$ is admissible and $1_{X_i} \notin \mathcal{I}(X_i,X_i), 1_{Y_j} \notin \mathcal{I}(Y_j,Y_j)$, we have that $f_{ji} \in \mathcal{I}(X_i,Y_j) \subseteq \operatorname{rad}^2_{\mathcal{A}}(X_i,Y_j)$ for each i,j. Therefore, f is a sum of morphisms in $\operatorname{rad}^2_{\mathcal{A}}(X,Y)$ and hence $f \in \operatorname{rad}^2_{\mathcal{A}}(X,Y)$ as desired. ■

The next lemma generalises [28, Lem. 1.7 (1)]. The proof in [28] makes use of a specific characterisation of irreducible morphisms between indecomposables (see [9, Prop. 2.4]), which we cannot use since we make no indecomposability assumptions on the domain and codomain of the morphism. See also [7, Prop. 1.2].

Proposition 4.13. *Suppose $\mathcal{I}$ is an admissible ideal of $\mathcal{A}$. Suppose $X = \bigoplus_{i=1}^n X_i$ and $Y = \bigoplus_{j=1}^m Y_j$ are decompositions into indecomposables in $\mathcal{A}$ with $1_{X_i} \notin \mathcal{I}(X_i,X_i), 1_{Y_j} \notin \mathcal{I}(Y_j,Y_j)$ for all i,j. Then $f\colon X \to Y$ is irreducible in $\mathcal{A}$ if and only if $\overline{f} = f + \mathcal{I}(X,Y)$ is irreducible in $\mathcal{A}/\mathcal{I}$.*

Proof. ($\Rightarrow$) Assume $f\colon X \to Y$ is an irreducible morphism in $\mathcal{A}$. By Lemma 4.3, $\overline{f}$ is neither a section nor a retraction in $\mathcal{A}/\mathcal{I}$. Now suppose $\overline{f} = \overline{hg}$ in $\mathcal{A}/\mathcal{I}$ for some morphisms $g\colon X \to Z$, $h\colon Z \to Y$ of $\mathcal{A}$. Then $f - hg \in \mathcal{I}(X,Y) \subseteq \operatorname{rad}^2_{\mathcal{A}}(X,Y)$ by Lemma 4.12. Therefore, there is an object $W \in \mathcal{A}$ and morphisms $a \in \operatorname{rad}_{\mathcal{A}}(X,W), b \in \operatorname{rad}_{\mathcal{A}}(W,Y)$ such that $f - hg = ba$. This yields $f = hg + ba = (\,h\ \ b\,)\binom{g}{a}$, so that either $(\,h\ \ b\,)$ is a retraction or $\binom{g}{a}$ is a section because f is irreducible. First, assume $(\,h\ \ b\,)$ is a retraction. Then there is a morphism $\binom{s}{t}\colon Y \to Z \oplus W$ such that $1_Y = (\,h\ \ b\,)\binom{s}{t} = hs + bt$. Now $b \in \operatorname{rad}_{\mathcal{A}}(W,Y)$, so $bt \in \operatorname{rad}_{\mathcal{A}}(Y,Y)$ as $\operatorname{rad}_{\mathcal{A}}$ is an ideal of $\mathcal{A}$. Then $hs = 1_Y - bt$ is invertible, so h is a retraction and hence $\overline{h}$ is also a retraction. In the other case, we find that $\overline{g}$ is a section in a similar fashion. Thus, $\overline{f}$ is an irreducible morphism.

($\Leftarrow$) Conversely, suppose $\overline{f}\colon X \to Y$ is irreducible in $\mathcal{A}/\mathcal{I}$. By Lemma 4.3, f cannot be a section or a retraction. Assume $f = hg$ for some $g\colon X \to Z$, $h\colon Z \to Y$ of $\mathcal{A}$. Then in $\mathcal{A}/\mathcal{I}$ we have $\overline{f} = \overline{hg}$ and so either $\overline{h}$ is a retraction or $\overline{g}$ is a section, since $\overline{f}$ is irreducible. Therefore, h is a retraction or g is a section, respectively, by Lemma 4.3 again. Hence, f is irreducible. ■

In a not necessarily Hom-finite, Krull-Schmidt category, the results [28, Lem. 1.7 (2), (3)], [28, Prop. 1.8] and [28, Lem. 1.9 (2)] all hold using the same proofs that Liu provides. This concludes our work on generalisations of results of Liu. We now recall some last definitions from [28] and prove some new results.

Definition 4.14. [28, Def. 2.2] An object $X \in \mathcal{A}$ is called *pseudo-projective* (respectively, *pseudo-injective*) if there exists a minimal right almost split monomorphism $W \to X$ (respectively, minimal left almost split epimorphism $X \to Y$).

Definition 4.15. [28, Def. 2.6] Suppose $\mathcal{A}$ is a Krull-Schmidt k-category. We call $\mathcal{A}$ a *left Auslander-Reiten* category if, for every indecomposable $Z \in \mathcal{A}$, either Z is pseudo-projective or it is the last term of an Auslander-Reiten sequence in $\mathcal{A}$. Dually, $\mathcal{A}$ is a *right Auslander-Reiten* category if, for every indecomposable $X \in \mathcal{A}$, either X is pseudo-injective or it is the first term of an Auslander-Reiten sequence. If $\mathcal{A}$ is both a left and right Auslander-Reiten category, then we simply call $\mathcal{A}$ an *Auslander-Reiten* category.

Remark 4.16. Let $\mathcal{C}$ be a triangulated category with suspension functor Σ. Then $\mathcal{C}$ is said to *have Auslander-Reiten triangles* if for every indecomposable Z there is an Auslander-Reiten triangle ending at Z (see [22, p. 31]). That is, for each indecomposable Z there is a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ with f minimal left almost split and g minimal right almost split. Therefore, a Krull-Schmidt, Hom-finite, triangulated k-category that has Auslander-Reiten triangles is immediately seen to be a left Auslander-Reiten category in light of a result of Liu: [28, Lem. 6.1] shows that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is an Auslander-Reiten triangle if and only if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is an Auslander-Reiten sequence as in Definition 4.6. The Hom-finite assumption may be removed by noting that one can use Theorem 4.9 in the proof of [28, Lem. 6.1].

The following two propositions generalise [7, Prop. 1.2], and the last theorem of this section is an analogue of [3, Thm. IV.1.13] (see also [6, Thm. 2.14]). For the most part, the proofs are straightforward generalisations of those for the abelian case, using the more general results from this article and [28] as appropriate. Thus, we only outline the proofs indicating the required generalised results where it is clear what needs to be done, and provide more details otherwise.

Proposition 4.17. *Suppose $\mathcal{A}$ is a left Auslander-Reiten category. Let $f\colon X \to Y$ be a morphism in $\mathcal{A}$ and let $\mathcal{I}$ be an admissible ideal of $\mathcal{A}$. Suppose $X = \bigoplus_{i=1}^{n} X_i$ and $Y = \bigoplus_{j=1}^{m} Y_j$ are decompositions into indecomposables in $\mathcal{A}$ with $1_{X_i} \notin \mathcal{I}(X_i, X_i), 1_{Y_j} \notin \mathcal{I}(Y_j, Y_j)$ for all i, j. Then $\overline{f} = f + \mathcal{I}(X, Y)\colon X \to Y$ is irreducible and right almost split in $\mathcal{A}/\mathcal{I}$, if and only if there exists $g\colon X' \to Y$ in $\mathcal{A}$ with $1_{X'} \in \mathcal{I}(X', X')$ such that $(\, f \;\; g\,)\colon X \oplus X' \to Y$ is minimal right almost split in $\mathcal{A}$.*

Proof. ($\Rightarrow$) Use: Proposition 4.13 instead of [7, Prop. 1.2 (a)] to show f is irreducible; [6, Lem. 2.3] and Lemma 4.4 to show $\operatorname{End}_{\mathcal{A}} Y$ is local; and [6, Thm. 2.4] and that $\mathcal{A}$ is a left Auslander-Reiten category to obtain a minimal right almost split morphism $(\, f \;\; g\,)\colon X \oplus X' \to Y$.

By [28, Lem. 1.7], the morphism $(\,\overline{f} \;\; \overline{g}\,)\colon X \oplus X' \to Y$ is minimal right almost split, and hence a non-retraction, in $\mathcal{A}/\mathcal{I}$. Since $\overline{f}\colon X \to Y$ is right almost split, there exists $(\,\overline{a} \;\; \overline{b}\,)\colon X \oplus X' \to X$ such that $(\,\overline{fa} \;\; \overline{fb}\,) = \overline{f} \circ (\,\overline{a} \;\; \overline{b}\,) = (\,\overline{f} \;\; \overline{g}\,)$. We now deviate from the proof given in [7]. This implies

$$(\,\overline{f} \;\; \overline{g}\,)\begin{pmatrix}\overline{a} & \overline{b}\\ 0 & 0\end{pmatrix} = (\,\overline{fa} \;\; \overline{fb}\,) = (\,\overline{f} \;\; \overline{g}\,),$$

so $\left(\begin{smallmatrix}\overline{a} & \overline{b}\\ 0 & 0\end{smallmatrix}\right)$ is an automorphism of $X \oplus X'$ in $\mathcal{A}/\mathcal{I}$ as $(\,\overline{f}\;\;\overline{g}\,)$ is right minimal. Hence, there is $\left(\begin{smallmatrix}\overline{r} & \overline{s}\\ \overline{t} & \overline{u}\end{smallmatrix}\right) \in \operatorname{End}_{\mathcal{A}/\mathcal{I}}(X \oplus X')$ such that

$$\begin{pmatrix}\overline{r} & \overline{s}\\ \overline{t} & \overline{u}\end{pmatrix}\begin{pmatrix}\overline{a} & \overline{b}\\ 0 & 0\end{pmatrix} = \begin{pmatrix}1_X & 0\\ 0 & 1_{X'}\end{pmatrix} = \begin{pmatrix}\overline{a} & \overline{b}\\ 0 & 0\end{pmatrix}\begin{pmatrix}\overline{r} & \overline{s}\\ \overline{t} & \overline{u}\end{pmatrix} = \begin{pmatrix}\overline{ar}+\overline{bt} & \overline{as}+\overline{bu}\\ 0 & 0\end{pmatrix}.$$

Therefore, $\overline{1_{X'}} = 0$ and hence $1_{X'} \in \mathcal{I}(X', X')$.

($\Leftarrow$) Use [28, Lem. 1.7] to get that $(\,\overline{f}\;\;\overline{g}\,)$ is minimal right almost split in $\mathcal{A}/\mathcal{I}$, and that $\overline{\iota_X}\colon X \hookrightarrow X \oplus X'$ is an isomorphism in the factor category as $1_{X'} \in \mathcal{I}(X', X')$. ■

Dually, the following is also true.

Proposition 4.18. *Suppose $\mathcal{A}$ is a right Auslander-Reiten category. Let $f\colon X \to Y$ be a morphism in $\mathcal{A}$ and let $\mathcal{I}$ be an admissible ideal of $\mathcal{A}$. Suppose $X = \bigoplus_{i=1}^{n} X_i$ and $Y = \bigoplus_{j=1}^{m} Y_j$ are decompositions into indecomposables in $\mathcal{A}$ with $1_{X_i} \notin \mathcal{I}(X_i, X_i), 1_{Y_j} \notin \mathcal{I}(Y_j, Y_j)$ for all i, j. Then $f + \mathcal{I}(X, Y)\colon X \to Y$ is irreducible and left almost split in $\mathcal{A}/\mathcal{I}$, if and only if there exists $g\colon X \to Y'$ in $\mathcal{A}$ with $1_{Y'} \in \mathcal{I}(Y', Y')$ such that $\left(\begin{smallmatrix} f\\ g\end{smallmatrix}\right)\colon X \to Y \oplus Y'$ is minimal left almost split in $\mathcal{A}$.*

Our main result of this section is the following characterisation of Auslander-Reiten sequences, which is a more general version of [3, Thm. IV.1.13]. Furthermore, statement (f) in [3] has stronger assumptions than the corresponding statement (vi) below: more precisely, in (vi) we do not assume any indecomposability assumptions on the first and last term of the short exact sequence.

Theorem 4.19. *Let $\mathcal{A}$ be a skeletally small, preabelian category. Let $\xi\colon X \xrightarrow{f} Y \xrightarrow{g} Z$ be a stable exact sequence in $\mathcal{A}$, i.e. $\xi \in \operatorname{Ext}^1_{\mathcal{A}}(Z, X)$. Then statements* (i)–(iii) *are equivalent.*

(i) *ξ is an Auslander-Reiten sequence.*
(ii) $\operatorname{End}_{\mathcal{A}}(X)$ *is local and g is right almost split.*
(iii) $\operatorname{End}_{\mathcal{A}}(Z)$ *is local and f is left almost split.*

Suppose further that $\mathcal{A}$ is quasi-abelian and Krull-Schmidt. Then (i)–(vi) *are equivalent.*

(iv) *f is minimal left almost split.*
(v) *g is minimal right almost split.*
(vi) *f* and *g are irreducible.*

Proof. From Definition 4.6 and [6, Lem. 2.3], (ii) and (iii) follow from (i). To show (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i), use Proposition 3.24 instead of [3, Lem. IV.1.12]. And (iii) $\Rightarrow$ (ii) is dual to (ii) $\Rightarrow$ (iii), so this establishes the equivalence of (i)–(iii).

Now suppose further that $\mathcal{A}$ is quasi-abelian and Krull-Schmidt. Statements (iv) and (v) follow from (i) by definition, and (iv) $\Rightarrow$ (iii) is dual to (v) $\Rightarrow$ (ii).

First, we claim that if g is right almost split then Y is non-zero. Indeed, if $Y = 0$ then $1_Z \circ g = g = 0$, which implies $1_Z = 0$ as $g = \operatorname{coker} f$ is an epimorphism (since ξ is short exact). However, if g is right almost split, then $\operatorname{End}_{\mathcal{A}} Z$ is local by [6, Lem. 2.3] and hence 1_Z cannot be the zero morphism.

(v) $\Rightarrow$ (ii). Since g is right almost split, we may use our claim above to conclude that g is irreducible by Proposition 3.14 (ii). Then X is indecomposable by Proposition 3.28, which is equivalent to $\operatorname{End}_{\mathcal{A}} X$ being local as $\mathcal{A}$ is Krull-Schmidt.

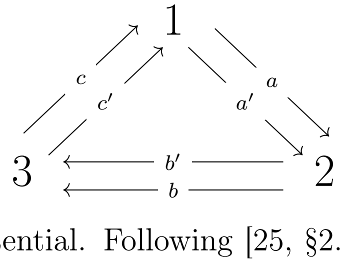
For (i) implies (vi), use Proposition 3.14 (noting again that Y is non-zero if g is right almost split).

(vi) $\Rightarrow$ (ii). Suppose that f, g are irreducible. First we show that g is right almost split. Note that g is not a retraction as it is irreducible by assumption. Thus, let $h\colon M \to Z$ be a non-retraction. Since $\mathcal{A}$ is Krull-Schmidt we may write $M = \bigoplus_{i=1}^{n} M_i$, for some indecomposable objects M_i, and $h = (\, h_1 \quad \cdots \quad h_n \,)$ where $h_i\colon M_i \to Z$. Since h is not a retraction, it follows that no h_i may be a retraction either. Fix $i \in \{1, \ldots, n\}$. As f is irreducible, the criterion from Proposition 3.8 tells us that either there exists $v_{i,1}\colon M_i \to Y$ such that $gv_{i,1} = h_i$ or there exists $v_{i,2}\colon Y \to M_i$ such that $g = h_i v_{i,2}$. Suppose we are in the latter case and that $g = h_i v_{i,2}$ for some $v_{i,2}\colon Y \to M_i$. Then, as g is irreducible and h_i is not a retraction, we have that $v_{i,2}$ is section. But M_i is indecomposable and $Y \neq 0$, so $v_{i,2}$ is in fact an isomorphism by Lemma 2.3. In this case, we then get $h_i = g \circ {v_{i,2}}^{-1}$. Therefore, for all $1 \leqslant i \leqslant n$ we have that $h_i = g \circ w_i$ for some $w_i\colon M_i \to Y$. Hence, $h = (\, h_1 \quad \cdots \quad h_n \,) = g \circ (\, w_1 \quad \cdots \quad w_n \,)$ and g is seen to be right almost split. Dually, we have that f is left almost split and hence $\operatorname{End}_{\mathcal{A}} X$ is local by [6, Lem. 2.3].

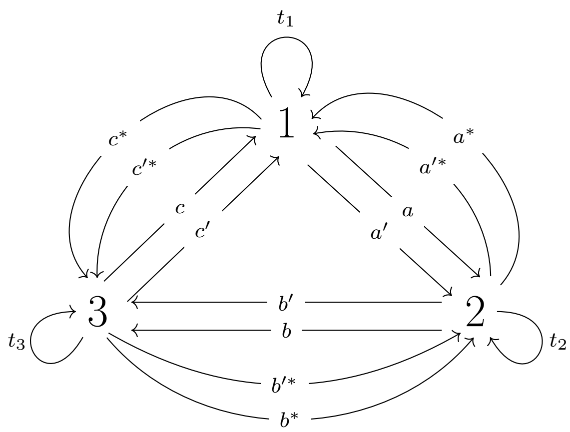
This shows (i)–(vi) are equivalent and finishes the proof. ■

We conclude this section with an example of a Hom-infinite, Krull-Schmidt category. The author is grateful to P.-G. Plamondon for communicating the following example and answering several questions.

Example 4.20. Let k be a field. Consider the quiver with potential (Q, W) where Q is the quiver



and $W = cba + c'b'a'$ is the potential. Following [25, §2.6], we recall the construction of the *complete Ginzburg dg algebra* $G := \widehat{\Gamma}(Q, W)$ associated to Q. From Q, consider the quiver $\widetilde{Q}$:



The quiver $\widetilde{Q}$ is given the following grading: arrows x, x' have degree 0 and arrows x^*, x'^* have degree -1 for $x \in \{a, b, c\}$, and the loop t_i has degree -2 for $1 \leqslant i \leqslant 3$. Then G has underlying graded algebra given by the completion of the graded path algebra $k\widetilde{Q}$ with respect to the ideal generated by the arrows of $\widetilde{Q}$ in the category of graded k-vector spaces. Furthermore, G is a dg algebra, equipped with a differential of degree $+1$.

Let $\mathsf{mod}-G$, $\mathsf{K}(G)$ and $\mathsf{D}(G)$ denote the category of right dg G-modules, the homotopy category of right dg G-modules and the corresponding derived category, respectively. The *perfect derived category* $\mathsf{per}\, G$ is the smallest, full, subcategory of $\mathsf{D}(G)$ that contains G, and is closed under shifts, extensions and direct summands. Let $J(Q, W)$ denote the *Jacobian algebra* associated to (Q, W). Then $J(Q, W)$ is the complete path algebra $\widehat{kQ}$ modulo the closure of the ideal generated by $\partial_x(W)$ and $\partial_{x'}(W)$ for $x \in \{a, b, c\}$, where

$$\partial_x(W) := \sum_{W=yxz} zy,$$

where the sum is over all decompositions of W with y, z (possibly trivial) paths. The sum $\partial_{x'}(W)$ is defined similarly. It is easy to check that $J(Q, W)$ is infinite-dimensional over k.

The category $\mathsf{per}\, G$ is Krull-Schmidt by [25, Lem. 2.17]. Furthermore, we have

$$\begin{aligned}
\mathrm{End}_{\mathsf{per}\, G}\, G &= \mathrm{End}_{\mathsf{D}(G)}\, G && \text{since } \mathsf{per}\, G \text{ is a full subcategory} \\
&\cong \mathrm{Hom}_{\mathsf{K}(G)}(G, G) && \text{since } G \text{ is cofibrant (see [25, pp. 2126–2127])} \\
&= H^0(\mathcal{H}om_{\mathsf{mod}-G}(G, G)) && \\
&\cong H^0(G) && \\
&= J(Q, W) && \text{by [25, Lem. 2.8].}
\end{aligned}$$

It follows that $\mathsf{per}\, G$ is a Hom-infinite k-category and is also Krull-Schmidt. We remark that, by [34, Lem. 2.9], the corresponding cluster category is also Hom-infinite k-category in this case.

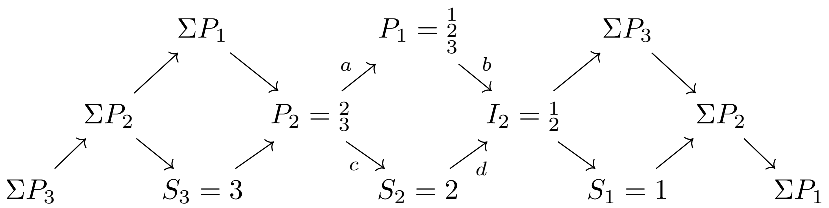
5. An example from cluster theory

We now present an example coming from cluster theory that encapsulates some of the theory we have explored.

Example 5.1. Let k be a field. Consider the cluster category $\mathcal{C} \coloneqq \mathcal{C}_{kQ}$ (as defined in [13]) associated to the linearly oriented Dynkin quiver

$$Q: \quad 1 \to 2 \to 3.$$

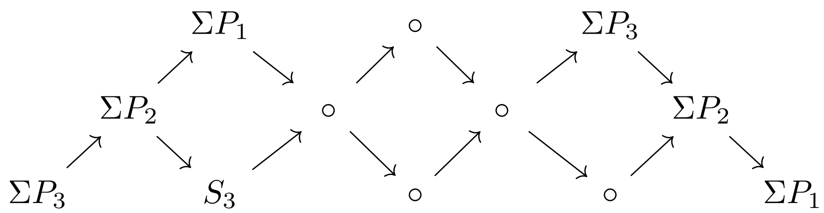
It is shown in [13] that $\mathcal{C}$ is Krull-Schmidt and it is triangulated by a result of Keller [24]. Let Σ denote the suspension functor of $\mathcal{C}$. Its Auslander-Reiten quiver, with the meshes omitted, is



where the lefthand copy of ΣP_i is identified with the righthand copy (for $i = 1, 2, 3$). We set $R \coloneqq P_1 \oplus P_2$, which is a basic, rigid object of $\mathcal{C}$. By $\mathsf{add}\,\Sigma R$ we denote the full subcategory of $\mathcal{C}$ consisting of objects that are isomorphic to direct summands of finite direct sums of copies of ΣR. The full subcategory $\mathcal{X}_R$ consists of objects X for which $\operatorname{Hom}_{\mathcal{C}}(R, X) = 0$. Then the pair $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})) = ((\mathsf{add}\,\Sigma R, \mathcal{X}_R), (\mathcal{X}_R, \mathsf{add}\,\Sigma R))$ is a twin cotorsion pair on $\mathcal{C}$ with heart $\overline{\mathcal{H}} = \mathcal{C}/[\mathcal{X}_R]$ (see [42, Lem. 5.4] and [42, Cor. 5.9], or [31, Exa. 2.10], for more details), where $[\mathcal{X}_R]$ is the ideal of morphisms factoring through objects of $\mathcal{X}_R$. Note that $[\mathcal{X}_R]$ is an admissible ideal by Example 4.11 as $\mathcal{X}_R$ is closed under direct summands.

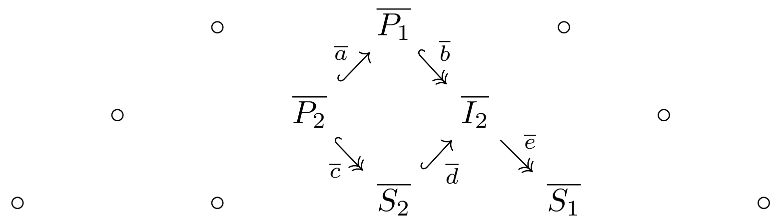
The subcategory $\mathcal{X}_R$ is described pictorially below, where "$\circ$" denotes that the corresponding object does not belong to the subcategory.

$$\mathcal{T} = \mathcal{X}_R = \mathcal{U}$$



The heart $\overline{\mathcal{H}} = \mathcal{C}/[\mathcal{X}_R]$ for this twin cotorsion pair is quasi-abelian by [42, Thm. 5.5], and, by [28, Prop. 2.9], has the following Auslander-Reiten quiver (ignoring the objects denoted by a "∘" that lie in $\mathcal{X}_R$)

$$\overline{\mathcal{H}} = \mathcal{C}/[\mathcal{X}_R]$$



where one may define the Auslander-Reiten quiver for a Krull-Schmidt category as in [28]. Again we have omitted the meshes. Furthermore, we have denoted by $\overline{X}$ the image in $\mathcal{C}/[\mathcal{X}_R]$ of the object X of $\mathcal{C}$, monomorphisms by "$\hookrightarrow$" and epimorphisms by "$\twoheadrightarrow$". In this example, we notice that there are precisely two irreducible morphisms (up to a scalar) that are regular (monic and epic simultaneously)—namely, $\overline{b}$ and $\overline{c}$.

Consider the Auslander-Reiten triangle $P_2 \xrightarrow{\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)} P_1 \oplus S_2 \xrightarrow{(b\ d)} I_2 \longrightarrow \Sigma P_2$ in $\mathcal{C}$, and note that the minimal left almost split morphism $\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)$ is irreducible by Proposition 3.14. Therefore, by Proposition 4.13, $\left(\begin{smallmatrix} \overline{a} \\ \overline{c} \end{smallmatrix}\right) \colon \overline{P_2} \to \overline{P_1} \oplus \overline{S_2}$ is also irreducible. Similarly, $(\overline{b}\ \overline{d}) \colon \overline{P_1} \oplus \overline{S_2} \to \overline{I_2}$ is irreducible in $\mathcal{C}/[\mathcal{X}_R]$. We remark that one cannot use [28, Lem. 1.7 (1)] since the morphisms are not between indecomposable objects.

One can check that $\left(\begin{smallmatrix} \overline{a} \\ \overline{c} \end{smallmatrix}\right) = \ker(\overline{b}\ \overline{d})$ and $(\overline{b}\ \overline{d}) = \operatorname{coker}\left(\begin{smallmatrix} \overline{a} \\ \overline{c} \end{smallmatrix}\right)$ by, for example, using the construction of (co)kernels as in [14, Lem. 3.4]. So, we have that

$$\overline{P_2} \xrightarrow{\left(\begin{smallmatrix} \overline{a} \\ \overline{c} \end{smallmatrix}\right)} \overline{P_1} \oplus \overline{S_2} \xrightarrow{(\overline{b}\ \overline{d})} \overline{I_2}$$

is a short exact sequence in the quasi-abelian, Krull-Schmidt category $\mathcal{C}/[\mathcal{X}_R]$. Hence, by Theorem 4.19, the sequence is an Auslander-Reiten sequence because it satisfies statement (vi) in the Theorem. Note that we could also have established this fact using [28, Prop. 1.8] and Proposition 4.7.

Furthermore, this example also shows that the indecomposability conditions in Proposition 3.16 cannot be removed. The morphism $\left(\begin{smallmatrix} \overline{a} \\ \overline{c} \end{smallmatrix}\right)$ is an irreducible monomorphism, but has decomposable target, and the morphism $\overline{d}$ is an irreducible morphism with codomain the cokernel of $\left(\begin{smallmatrix} \overline{a} \\ \overline{c} \end{smallmatrix}\right)$ that is not epic. Indeed, $\overline{e}\overline{d} = 0$ but $\overline{e} \neq 0$ so $\overline{d}$ cannot be an epimorphism.

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References

[1] P. Aluffi. *Algebra: chapter 0*, volume 104 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2009.

[2] C. Amiot. *Cluster categories for algebras of global dimension 2 and quivers with potential. Ann. Inst. Fourier (Grenoble)*, **59**(6):2525–2590, 2009.

[3] I. Assem, D. Simson, and A. Skowroński. *Elements of the representation theory of associative algebras. Vol. 1*, volume 65 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2006. Techniques of representation theory.

[4] M. Auslander. *Representation theory of Artin algebras. I, II. Comm. Algebra*, **1**:177–268; ibid. 1 (1974), 269–310, 1974.

[5] M. Auslander and I. Reiten. *Representation theory of Artin algebras. III. Almost split sequences. Comm. Algebra*, **3**:239–294, 1975.

[6] M. Auslander and I. Reiten. *Representation theory of Artin algebras. IV. Invariants given by almost split sequences. Comm. Algebra*, **5**(5):443–518, 1977.

[7] M. Auslander and I. Reiten. *Representation theory of Artin algebras. V. Methods for computing almost split sequences and irreducible morphisms. Comm. Algebra*, **5**(5):519–554, 1977.

[8] M. Auslander and S. O. Smalø. *Preprojective modules over Artin algebras. J. Algebra*, **66**(1):61–122, 1980.

[9] R. Bautista. *Irreducible morphisms and the radical of a category. An. Inst. Mat. Univ. Nac. Autónoma México*, **22**:83–135 (1983), 1982.

[10] R. Bautista and M. J. Souto Salorio. *Irreducible morphisms in the bounded derived category. J. Pure Appl. Algebra*, **215**(5):866–884, 2011.

[11] F. Borceux. *Handbook of categorical algebra. 1*, volume 50 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994. Basic category theory.

[12] F. Borceux. *Handbook of categorical algebra. 2*, volume 51 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994. Categories and structures.

[13] A. B. Buan, R. Marsh, M. Reineke, I. Reiten, and G. Todorov. *Tilting theory and cluster combinatorics. Adv. Math.*, **204**(2):572–618, 2006.

[14] A. B. Buan and R. J. Marsh. *From triangulated categories to module categories via localization II: calculus of fractions. J. Lond. Math. Soc. (2)*, **86**(1):152–170, 2012.

[15] I. Bucur and A. Deleanu. *Introduction to the theory of categories and functors*. With the collaboration of Peter J. Hilton and Nicolae Popescu. Pure and Applied Mathematics, Vol. XIX. Interscience Publication John Wiley & Sons, Ltd., London-New York-Sydney, 1968.

[16] T. Bühler. *Exact categories. Expo. Math.*, **28**(1):1–69, 2010.

[17] P. Caldero, F. Chapoton, and R. Schiffler. *Quivers with relations arising from clusters (A_n case). Trans. Amer. Math. Soc.*, **358**(3):1347–1364, 2006.

[18] Y. S. Cooper. *Stable sequences in pre-abelian categories. Pacific J. Math.*, **91**(2):263–267, 1980.

[19] P. J. Freyd. *Abelian categories. Repr. Theory Appl. Categ.*, **3**:1–190, 2003.

[20] P. Gabriel and A. V. Roiter. *Representations of finite-dimensional algebras*. Springer-Verlag, Berlin, 1997. Translated from the Russian, With a chapter by B. Keller, Reprint of the 1992 English translation.

[21] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35. Springer-Verlag New York, Inc., New York, 1967.

[22] D. Happel. *Triangulated categories in the representation theory of finite-dimensional algebras*, volume 119 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1988.

[23] O. Iyama, H. Nakaoka, and Y. Palu. *Auslander–Reiten theory in extriangulated categories*. Preprint, 2018.

[24] B. Keller. *On triangulated orbit categories. Doc. Math.*, **10**:551–581, 2005.

[25] B. Keller and D. Yang. *Derived equivalences from mutations of quivers with potential. Adv. Math.*, **226**(3):2118–2168, 2011.

[26] G. M. Kelly. *On the radical of a category. J. Austral. Math. Soc.*, **4**:299–307, 1964.

[27] H. Krause. *Krull-Schmidt categories and projective covers. Expo. Math.*, **33**(4):535–549, 2015.

[28] S. Liu. *Auslander-Reiten theory in a Krull-Schmidt category. São Paulo J. Math. Sci.*, **4**(3):425–472, 2010.

[29] S. Mac Lane. *Homology.* Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1975 edition.

[30] C. McLarty. *Elementary categories, elementary toposes*, volume 21 of *Oxford Logic Guides*. The Clarendon Press, Oxford University Press, New York, 1992. Oxford Science Publications.

[31] H. Nakaoka. *General heart construction for twin torsion pairs on triangulated categories. J. Algebra*, **374**:195–215, 2013.

[32] M. S. Osborne. *Basic homological algebra*, volume 196 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[33] P.-G. Plamondon. *Cluster algebras via cluster categories with infinite-dimensional morphism spaces. Compos. Math.*, **147**(6):1921–1954, 2011.

[34] P.-G. Plamondon. *Cluster characters for cluster categories with infinite-dimensional morphism spaces. Adv. Math.*, **227**(1):1–39, 2011.

[35] N. Popescu. *Abelian categories with applications to rings and modules.* Academic Press, London-New York, 1973. London Mathematical Society Monographs, No. 3.

[36] M. Prest. *Purity, spectra and localisation*, volume 121 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2009.

[37] D. Quillen. *Higher algebraic K-theory. I.* In H. Bass, editor, *Algebraic K-Theory I. Proceedings of the Conference Held at the Seattle Research Center of Battelle Memorial Institute, August 28 – September 8, 1972*, pages 85–147. Springer, Berlin, 1973.

[38] F. Richman and E. A. Walker. *Ext in pre-Abelian categories. Pacific J. Math.*, **71**(2):521–535, 1977.

[39] W. Rump. *Almost abelian categories. Cahiers Topologie Géom. Différentielle Catég.*, **42**(3):163–225, 2001.

[40] W. Rump. *A counterexample to Raikov's conjecture. Bull. Lond. Math. Soc.*, **40**(6):985–994, 2008.

[41] J.-P. Schneiders. *Quasi-abelian categories and sheaves. Mém. Soc. Math. Fr. (N.S.)*, **76**:vi+134, 1999.

[42] A. Shah. *Quasi-abelian hearts of twin cotorsion pairs on triangulated categories.* Preprint, 2018.

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