Available online at www.sciencedirect.com

ScienceDirect

Nuclear Physics B 947 (2019) 114686

NUCLEAR PHYSICS B

www.elsevier.com/locate/nuclphysb

# Quantum variational principle and quantum multiform structure: The case of quadratic Lagrangians

S.D. King *, F.W. Nijhoff

*School of Mathematics, University of Leeds, Leeds, LS2 9JT, United Kingdom*

Received 31 December 2018; received in revised form 4 June 2019; accepted 28 June 2019
Available online 3 July 2019
Editor: Hubert Saleur

**Abstract**

A modern notion of integrability is that of multidimensional consistency (MDC), which classically implies the coexistence of (commuting) dynamical flows in several independent variables for one and the same dependent variable. This property holds for both continuous dynamical systems as well as for discrete ones defined in discrete space-time. Possibly the simplest example in the discrete case is that of a linear quadrilateral lattice equation, which can be viewed as a linearised version of the well-known lattice potential Korteweg-de Vries (KdV) equation. In spite of the linearity, the MDC property is non-trivial in terms of the parameters of the system. The Lagrangian aspects of such equations, and their nonlinear analogues, has led to the notion of Lagrangian multiform structures, where the Lagrangians are no longer scalar functions (or volume forms) but genuine $p$-forms in a multidimensional space of independent variables. The variational principle involves variations not only with respect to the field variables, but also with respect to the geometry in the space of independent variables. In this paper we consider a quantum analogue of this new variational principle by means of quantum propagators (or equivalently Feynman path integrals). In the case of quadratic Lagrangians these can be evaluated in terms of Gaussian integrals. We study also periodic reductions of the lattice leading to discrete multi-time dynamical commuting mappings, the simplest example of which is the discrete harmonic oscillator, which surprisingly reveals a rich integrable structure behind it. On the basis of this study we propose a new quantum variational principle in terms of multiform path integrals.



* Corresponding author.
*E-mail addresses:* steven.king@cantab.net (S.D. King), F.W.Nijhoff@leeds.ac.uk (F.W. Nijhoff).

https://doi.org/10.1016/j.nuclphysb.2019.114686

## 1. Introduction

Discrete integrable systems [1] have started to play an increasingly important role in deepening the understanding of integrability as a mathematical notion, thereby forging new perspectives in both analysis (e.g. the discovery of difference analogues of the Painlevé equations), geometry (the development of discrete differential geometry, [2]) and algebra (e.g. the development of cluster algebras through the so-called Laurent phenomenon). In physics, at the quantum level, discrete integrable systems appear in connection with random matrix theory and quantum spin models of statistical mechanics, and in aspects of relativistic many-body systems [3], but more directly in approaches to establish integrable quantum field theories on the space-time lattice [4].

Integrable systems are important not only because they can be treated by exact and rigorous methods, but also because they appear to be universal: they have a rare tendency of emerging in a large variety of contexts and physical situations, such as in correlations functions in scaling limits, random matrices and in energy level statistics of even chaotic systems. Furthermore, their intricate underlying structures gave rise to new mathematical theories, such as quantum groups and cluster algebras, revealing novel types of combinatorics. Thus, one could argue, letting these systems "speak for themselves" the stories they tell us will lead us to new principles and insights, even perhaps about the structure of Nature itself. One such story is about their variational description in terms of a least-action principle and its connection to one of the key integrability features, multi-dimensional consistency (MDC). The latter is the phenomenon that integrable equations do not come in isolation, but tend to come in combination with whole families of equations, all simultaneously imposable on one and the same field variable (the dependent variable of the equations). Such equations manifest themselves as higher or generalized symmetries, as *hierarchies* of equations or as compatible systems, their very compatibility being the signature of the integrability. In fact, it is this very feature that forms a powerful tool in the exact solvability of such equations through techniques such as the inverse scattering transform (a nonlinear analogue of the Fourier transform), Lax pairs and Bäcklund transformations.

This story about the variational description of integrable systems started with the paper [5], where the Lagrangian structure of a class of 2D quadrilateral lattice equations was studied, which are integrable in the sense of the MDC property. It was shown that for particularly well-chosen discrete Lagrangians for those equations, embedded through the MDC property in higher-dimensional space-time lattice, the Lagrangians obey a closure property, suggesting that these Lagrangians should be viewed as components of a discrete $p$-form that is closed on solutions of the quad equations. This remarkable property led to the formulation of a novel least-action principle in which the action is supposed to attain a critical point not only w.r.t. variations of the field variables, but also the action being stationary w.r.t. variations of the space-time surfaces in the higher-dimensional lattice of independent discrete variables on which the equations are defined. This allows one to derive from this extended variational principle not one single equation (in the conventional way on a fixed space-time surface) but the full set of compatible equations that possess the MDC property. Furthermore, this property was also shown to extend to corresponding integrable differential equations defined on smooth surfaces in a multidimensional space-time of independent continuous variables, as well as on systems of higher dimension and of higher rank, [6–8] as well as to many-body systems [9–11]. Further extensions and deepening understanding of these results were obtained in a number of papers, cf. [12–14].

A natural question is whether the Lagrangian multiform structure described above extends also to the quantum regime, since, after all, a canonical quantization formalism for reductions of quadrilateral lattice equations and higher-rank systems, using non-ultralocal $R$ matrix structures, was already established some while ago [15,16], as well as for a quantum lattice Hirota type system [17], cf. also [18]. However, the natural setting for a Lagrangian approach in the quantum case is obviously the Feynman path integral [19], which has remained curiously unexplored in the context of integrable systems theory where there has been a predilection for the Hamiltonian point of view. However, when dealing with discrete systems, e.g. systems evolving in discrete time, the Hamiltonian view point is no longer natural, and the Lagrangian point of view may become preferable. The further advantage is that in discrete time, path integrals are no longer marred by the infinite time-slicing limit which causes such objects to be notoriously ill-defined in general. Thus, first steps to set up a path integral approach for integrable quantum mappings,[1] i.e. integrable systems with discrete-time evolution, were undertaken in [21,22]. However, the main aim of the present paper is to arrive at an understanding of the Lagrangian multiform structure on the quantum level. In order to achieve that, and to avoid analytical complications arising from the nonlinearities, we restrict ourselves in this initial treatment to the case of quadratic Lagrangians, associated with linear multidimensionally consistent equations. Although this may seem restrictive, the quadratic case is surprisingly rich and exhibits most of the properties of the wider classes of nonlinear models when it comes to the MDC aspects. Those reveal themselves in the way the lattice parameters govern the compatible systems of equations, and it is there where even these linear equations exhibit quite non-trivial features. In fact, an interesting role reversal between discrete independent variables and continuous parameters allows the corresponding quantum propagators to be interpreted at the same time as discrete as well as continuous path integrals. The periodic reductions are particularly noteworthy, since they lead to propagators that can be readily computed, and it is here that the humble quantum harmonic oscillator makes its reappearance in quite a new context.

The outline of the paper is as follows. In section 2 we describe the classical quad equation, i.e. a 2-dimensional partial difference equation defined on elementary quadrilaterals, and its Lagrangian 2-form structure. In section 3, we consider its periodic reductions on the classical level, and construct commuting flows for the lowest period cases. The simplest (3-step) reduction leads to the harmonic oscillator, but even this case there is a non-trivial Lagrangian 1-form structure on the classical level. Next, in section 4 we consider the quantization of the reductions through discrete-time step path integrals which at the same time provides a natural discretization of the underlying continuous-time model in terms of the lattice parameters. The MDC property here is reflected in a path-independence property of the propagators. This leads us to suggest a quantum variational principle which we expect may extend to models beyond the quadratic case. In section 5 we return to the quad lattice case, which resembles a quantum field type of situation, and we establish surface-independence of the relevant propagators, suggestive of a quantum variational principle in the field theoretic case. Finally, in section 6 we discuss some possible ramifications of our findings, and how they connect to some ongoing questions regarding quantum mechanics and foundational aspects.

## 2. Linearised lattice KdV equation

Our starting point is a 2 dimensional quadrilateral lattice equation, whose dependent variable $u(n,m)$ is defined on lattice points labelled by discrete variables $(n,m)$, which are variables

[1] The notion of *quantum mapping* is essentially due to M.V. Berry et al., [20].

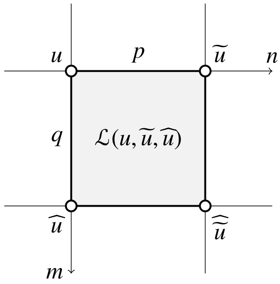


Fig. 1. An elementary plaquette in the lattice.

shifting by units, and with lattice parameters $p$ and $q$, each associated with the $n$ and $m$ directions on the lattice respectively. We adopt the shift notation by accents $\widetilde{\ }$ and $\widehat{\ }$, i.e. for $u := u(n,m)$, we have $\widetilde{u} := u(n+1,m)$, $\widehat{u} := u(n,m+1)$. The equation of interest in this paper is in the linear quadrilateral equation:

$$(p+q)(\widetilde{u}-\widehat{u}) = (p-q)(u-\widehat{\widetilde{u}}) \ . \tag{1}$$

This quadrilateral equation is supposed to hold on every elementary plaquette across a 2 dimensional lattice; the elementary plaquette is shown in Fig. 1. This is something of a "universal" linear quad equation, being the natural linearisation of nearly all the integrable quad equations of the ABS list [23]. Although for the sake of the present study it is not really relevant, we mention that (1) admits a *scalar inhomogeneous* Lax representation of the form[2]

$$\widetilde{\varphi} = u + \frac{p+k}{p-k}(\varphi-\widetilde{u}) \ , \quad \widehat{\varphi} = u + \frac{q+k}{q-k}(\varphi-\widehat{u}) \ , \tag{2}$$

where $\varphi$ is an auxiliary field variable and $k$ a spectral parameter. The compatibility condition $\widehat{\widetilde{\varphi}} = \widetilde{\widehat{\varphi}}$ leads to the linear equation (1) for the main field $u$. The latter equation can be derived via discrete Euler-Lagrange equations on the three-point Lagrangian

$$\mathcal{L}(u,\widetilde{u},\widehat{u}) = u(\widetilde{u}-\widehat{u}) - \frac{1}{2}\frac{p+q}{p-q}(\widetilde{u}-\widehat{u})^2 \ ; \quad \widehat{\widetilde{\left(\frac{\partial\mathcal{L}}{\partial u}\right)}} + \widehat{\left(\frac{\partial\mathcal{L}}{\partial\widetilde{u}}\right)} + \widetilde{\left(\frac{\partial\mathcal{L}}{\partial\widehat{u}}\right)} = 0, \tag{3}$$

where, for the action, we sum across every plaquette in the lattice:

$$\mathcal{S} = \sum_{(n,n)\in\mathbb{Z}^2} \mathcal{L}(u_{n,m}, u_{n+1,m}, u_{n,m+1}) \ . \tag{4}$$

Note that the Lagrangian (3) is also the natural linearisation of the Lagrangians for the non-linear quad equations of the ABS list from which (1) can be derived.

In fact, the standard variational principle on (3) produces two copies of (1). In order to regain precisely the linearised KdV equation, we must make use of the multiform variational principle introduced in [5,12]. (1) can be consistently embedded into a *multidimensional* lattice, with directions labelled by subscripts $i$, $j$, $k$. Across an elementary plaquette in the $i-j$ plane, (1) takes the form:

$$(p_i+p_j)(u_i-u_j) = (p_i-p_j)(u-u_{ij}) \ , \tag{5}$$

[2] Similar Lax representations in the continuous case of PDEs were considered in [24,25] to study initial boundary-value problems for linear PDEs.

where $u_i$ indicated $u$ shifted once in the $i$ direction on the lattice, and $p_i$ is now the lattice parameter associated to the $i$ direction. This equation has multidimensional consistency, which can be checked by establishing closure around the cube [26] - field variables at any point in the multi-dimensional lattice can be calculated via any route in a consistent manner.

The variational principle proposed in [5] was elaborated further in [12], where the system of generalised Euler-Lagrange equations was derived, cf. also [13]. The action being defined as the sum of Lagrangians on elementary plaquettes across a 2-dimensional surface $\sigma$, embedded in the multidimensional space, to derive the equations of motion, we demand the action be stationary not only under the variation of the field variables $u$, but also under the variation of the surface $\sigma$ itself. For this to hold, we require closure of the Lagrangian: if we consider the combination of oriented Lagrangians on the faces of a cube, we require that *on the equations of motion*, the Lagrangians sum to zero. In other words,

$$\begin{aligned}&\Delta_1\mathcal{L}_{23}(u)+\Delta_2\mathcal{L}_{31}(u)+\Delta_3\mathcal{L}_{12}(u)\\&\quad=\mathcal{L}_{23}(u_1)-\mathcal{L}_{23}(u)+\mathcal{L}_{31}(u_2)-\mathcal{L}_{31}(u)+\mathcal{L}_{12}(u_3)-\mathcal{L}_{12}(u)=0\,,\end{aligned}\tag{6}$$

where we have used the shorthand $\mathcal{L}_{ij}(u):=\mathcal{L}(u,u_i,u_j;p_i,p_j)$, and the final equality in (6) holds only when we apply (5). According to [12], such a system must be described by a Lagrangian of the form $\mathcal{L}(u,u_i,u_j;p_i,p_j)=A(u,u_i;p_i)-A(u,u_j;p_j)+C(u_i,u_j;p_i,p_j)$; where we require $C_{ij}$ to be antisymmetric under interchange of $i$ and $j$. Notice that the Lagrangian (3) is already in this form. By using the multidimensional consistency, a set of Euler-Lagrange equations are derived, which simplify on a single plaquette to:

$$\frac{\partial}{\partial u_i}\Big(A(u,u_i;p_i)-A(u_i,u_{ij};p_j)+C(u_i,u_j;p_i,p_j)\Big)=0\,.\tag{7}$$

This yields precisely the equation (1). This structure allows us to describe the multiple consistent equations (5) in a single Lagrangian framework - that of the 2-form. This is then the appropriate variational structure to describe multi-dimensionally consistent systems [5].

In fact, the Lagrangian (3) is the almost unique quadratic Lagrangian with a 2-form structure (i.e. exhibiting the closure property). Considering the general form for a three-point Lagrangian 2-form and equation of motion (7), we restrict our attention to quadratic Lagrangians and have the general form:

$$\begin{aligned}\mathcal{L}_{ij}(u,u_i,u_j)=&\left(\tfrac{1}{2}a_iu^2+c_iuu_i\right)-\left(\tfrac{1}{2}a_ju^2+c_juu_j\right)\\&+\left(\tfrac{1}{2}b_{ij}u_i^2-\tfrac{1}{2}b_{ji}u_j^2+\delta_{ij}u_iu_j\right)\,,\end{aligned}\tag{8}$$

where we require $\delta_{ji}=-\delta_{ij}$. Here, subscripts on coefficients indicate dependence on the lattice parameters $p_i$ and $p_j$. This Lagrangian yields the equation of motion: $c_iu-c_ju_{ij}=(a_j-b_{ij})u_i-\delta_{ij}u_j$. This is a quad equation, and as such we require it to be symmetric under the interchange of $i$ and $j$. This leads to the conditions $c_i=c_j=c$, constant, $a_j-b_{ij}=\delta_{ij}$.

Noting that the Lagrangian (8) already obeys the closure relation (6) on the equations of motion above, we use our freedom to multiply by an overall constant to let $c=1$, and hence the general Lagrangian is given by:

$$\mathcal{L}_{ij}(u,u_i,u_j)=u(u_i-u_j)-\tfrac{1}{2}\delta_{ij}(u_i-u_j)^2+\tfrac{1}{2}a_i(u^2-u_j^2)-\tfrac{1}{2}a_j(u^2-u_i^2)\,.\tag{9}$$

We can see this has the same form as (3), but with a more general dynamical, anti-symmetric parameter $\delta_{ij}$, and the free parameter $a_i$ that does not effect the equations of motion.

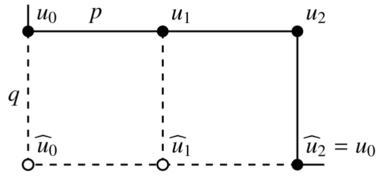


Fig. 2. Periodic initial value problem on the lattice equation.

## 3. One dimensional reduction: the discrete harmonic oscillator

### 3.1. Periodic reduction

Reductions of lattice equations to integrable symplectic mappings have been considered since the early 1990s [27–30]. Here, we are considering a linearised version of the lattice KdV equation as our starting point, and follow the same reduction procedure as has been considered previously for non-linear quad equations. The reduction is obtained by imposing a periodic initial value problem, where the evolution of the data progresses through the lattice according to a dynamical map, or equivalently a system of ordinary difference equations, which is constructed by implementing the lattice equation (1). We begin with initial data $u_0$, $u_1$ and $u_2$, and let $\widehat{u}_2 = u_0$, according to Fig. 2. This unit is then repeated periodically across an infinite staircase in the lattice. This is the simplest meaningful reduction we can perform on the lattice equation.

Applying the linear lattice equation (1) to each plaquette, we can write equations for the dynamical mapping $(u_0, u_1, u_2) \to (\widehat{u}_0, \widehat{u}_1, \widehat{u}_2)$:

$$\widehat{u}_0 = u_1 + s(\widehat{u}_1 - \widehat{u}_2) \;, \quad \widehat{u}_1 = u_2 + s(u_0 - u_1) \;, \quad \widehat{u}_2 = u_0 \;; \quad s := \frac{p-q}{p+q} \;. \tag{10}$$

This is a finite-dimensional discrete system. We introduce the reduced variables $x := u_1 - u_0$, $y := u_2 - u_1$ and, by eliminating $y$, write the second order difference equation:

$$\widehat{x} + 2bx + \underset{\frown}{x} = 0 \;, \quad b := 1 + 2s - s^2 \;, \tag{11}$$

where the underhat $\underset{\frown}{x}$ indicates a backwards step. This equation can be expressed by a Lagrangian-type generating function, with the equation arising from discrete Euler-Lagrange equations:

$$\mathcal{L}(x, \widehat{x}) = x\widehat{x} + bx^2 \;, \quad \widehat{\frac{\partial \mathcal{L}}{\partial x}} + \frac{\partial \mathcal{L}}{\partial \widehat{x}} = 0 \;, \tag{12}$$

and so is symplectic, $\mathrm{d}\widehat{x} \wedge \mathrm{d}\widehat{y} = \mathrm{d}x \wedge \mathrm{d}y$. The map also possesses an exact invariant[3]:

$$I_b(x, \widehat{x}) = x^2 + \widehat{x}^2 + 2bx\widehat{x} \;, \tag{13}$$

the invariance of which, i.e., $I_b(x, \widehat{x}) = I_b(\underset{\frown}{x}, x)$ can be readily checked by direct computation.

[3] Unlike in the nonlinear case of [27,29], in the linear case the invariants of the reduced system cannot be readily computed by using the Lax pair (2), which turns out to be ineffective for deriving integrals of the motion. Consequently, also a classical and quantum $R$-matrix formulation has not yet been found for the linearised lattice KdV system, while for the nonlinear case a quantum $R$ matrix structure was presented in [31]. The construction of the analogue of the latter for the linear case is under investigation by the authors.

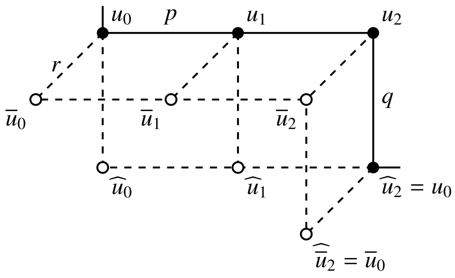


Fig. 3. The variables $\overline{u}_i$ extend from the plane in a third direction.

The equation (11) is a discrete harmonic oscillator. It is not difficult to see that the most general solution to (11) is given by

$$x_m = c_1 \sin(\mu m) + c_2 \cos(\mu m) \; ; \quad \cos\mu = -b \; , \tag{14}$$

where $m$ is the discrete variable. This has a clear relation to the solution for the continuous time harmonic oscillator. This solution can alternatively be written as $x_m = A\lambda^m + B\lambda^{-m}$, $\lambda = -b + \sqrt{b^2 - 1}$. By considering derivatives with respect to the *parameter* $b$, we can then derive the equations:

$$\frac{dx}{db} = \frac{m}{1-b^2}(bx + \widehat{x}) \; , \; \frac{dx}{db} = -\frac{m}{1-b^2}(bx + \underset{\frown}{x}) \; , \tag{15}$$

Eliminating $\widehat{x}$ yields the second order differential equation in $b$:

$$(1-b^2)\frac{d^2x}{db^2} - b\frac{dx}{db} + m^2 x = 0 \; . \tag{16}$$

A remarkable exchange has taken place: the parameter and independent variable of the discrete case, $b$ and $m$, have exchanged roles to become the independent variable and parameter of a continuous time model. Note that (16) can be simplified by taking $\mu := \cos^{-1}(-b)$ as the "time" variable, so that: $d^2x/d\mu^2 + m^2 x = 0$. This is the equation for the harmonic oscillator, with a quantised frequency $\omega = m$.

### 3.2. *Commuting discrete flow*

Recall that the linear lattice equation (5) can be embedded in a multidimensional lattice. From the periodic reduction in the plane (Fig. 2) we consider the embedding within a three dimensional lattice. The third lattice direction has lattice parameter $r$, and we introduce shifted variables $\overline{u}_i$, as shown in Fig. 3.

To derive the mapping, we now use the lattice equations (5):

$$(q+r)(\widehat{u} - \overline{u}) = (q-r)(u - \widehat{\overline{u}}) \; , \quad (r+p)(\overline{u} - \widetilde{u}) = (r-p)(u - \widetilde{\overline{u}}) \; ,$$

which, in terms of the $u_i$, yield

$$\begin{array}{rcl} \overline{u}_0 &=& u_1 + t(\overline{u}_1 - u_0) \; , \\ \overline{u}_1 &=& u_2 + t(\overline{u}_2 - u_1) \; , \\ \overline{u}_2 &=& u_0 + t'(\overline{u}_0 - u_2) \; . \end{array} \quad \text{where} \quad \begin{array}{l} t := \frac{p-r}{p+r} \; , \\ t' := \frac{q-r}{q+r} \; . \end{array} \tag{17}$$

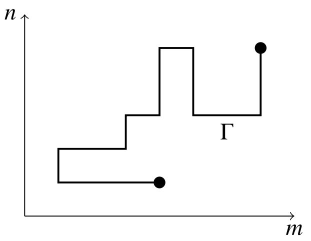


Fig. 4. A curve Γ in the discrete variables.

Again, we use reduction variables $(x, y)$, which yield the map $(x, y) \to (\overline{x}, \overline{y})$. This map can be written in a matrix form, from which it can be shown to be area preserving, $\mathrm{d}\overline{x} \wedge \mathrm{d}\overline{y} = \mathrm{d}x \wedge \mathrm{d}y$. Eliminating $y$ again produces a second order difference equation in $x$:

$$\overline{x} + 2ax + \underline{x} = 0\,, \quad \text{with} \quad 2a := \frac{(2t+1-t^2) - t'(2t-1+t^2)}{1-t^2t'}\,. \tag{18}$$

This equation has the same form as (11), that of a discrete harmonic oscillator, along with invariant $I_a(x, \overline{x}) = x^2 + \overline{x}^2 + 2ax\overline{x}$.

We can write both maps $(x, y) \to (\widehat{x}, \widehat{y})$ and $(x, y) \to (\overline{x}, \overline{y})$ in matrix form: $\widehat{\mathbf{x}} = \mathbf{S}\,\mathbf{x}$, $\overline{\mathbf{x}} = \mathbf{T}\,\mathbf{x}$, $\mathbf{x} := (x, y)^T$. It is then clear that the two maps commute, $(\widehat{\overline{x}}, \widehat{\overline{y}}) = (\overline{\widehat{x}}, \overline{\widehat{y}})$, since we have $[\mathbf{S}, \mathbf{T}] = 0$. This last relation relies on the parameter identity, $stt' = s - t + t'$, which is easily shown using the definitions for $s$, $t$ and $t'$.

Our equations are slightly simplified by introducing the parameters $P := p^2 + pq$, $Q := q^2$ and $R := r^2$, in terms of which $a = (P-R)/(P+R)$, $b = (P-Q)/(P+Q)$. By returning to earlier evolution equations in terms of $x$ and $y$ and eliminating $y$ in a different manner, we derive "corner equations" for the evolution, linking $x$, $\widehat{x}$ and $\overline{x}$; or $\widehat{x}$, $\overline{x}$ and $\widehat{\overline{x}}$ respectively. Thus:

$$\begin{aligned} \left(\frac{P-Q}{q} - \frac{P-R}{r}\right) x &= \frac{P+R}{r}\overline{x} - \frac{P+Q}{q}\widehat{x}\,, \\ \left(\frac{P-Q}{q} - \frac{P-R}{r}\right) \widehat{\overline{x}} &= \frac{P+R}{r}\widehat{x} - \frac{P+Q}{q}\overline{x}\,. \end{aligned} \tag{19}$$

Thus we have multiple equations of motion (11), (18), (19) all holding simultaneously on the same variable $x$.

### 3.3. *Lagrangian 1-form structure*

A recent development in understanding discrete integrable systems with commuting flows has been the Lagrangian multiform theory [12,5,9,10,32,11,14]. A system with two or more commuting, discrete flows can be described by a Lagrangian 1-form structure, which provides a way to obtain a simultaneous system of equations for a single dependent variable from a variational principle. Thus, the Lagrangians generating the flows $x \to \widehat{x}$ and $x \to \overline{x}$ should form the components of a *difference 1-form*, each associated with an oriented direction on a 2D lattice.

The action functional is then defined as a sum of elementary Lagrangian elements over an arbitrary discrete curve Γ in the 2D lattice, as shown in Fig. 4.

$$\mathcal{S}[x(\mathbf{n}); \Gamma] = \sum_{\gamma(\mathbf{n}) \in \Gamma} \mathcal{L}_i(x(\mathbf{n}), x(\mathbf{n} + \mathbf{e}_i))\,. \tag{20}$$

The usual variational principle demands that, on the equations of motion, the action $\mathcal{S}$ be stationary under the variation of the dynamical variables $x$. In addition, we also demand that $\mathcal{S}$ be stationary under variations of the curve $\Gamma$ itself. This principle leads to the compatibility of equations of motion and corner equations, under the condition of *closure* of the Lagrangians. That is, on the equations of the motion, the action should be locally invariant under changes to the curve $\Gamma$ and therefore:

$$\Box\mathcal{L} := \mathcal{L}_a(\widehat{x},\widehat{\overline{x}}) - \mathcal{L}_a(x,\overline{x}) - \mathcal{L}_b(\overline{x},\widehat{\overline{x}}) + \mathcal{L}_b(x,\widehat{x}) = 0\,, \tag{21}$$

where this last equality holds only on the equations of motion.

In the model we are considering, we already have compatible flows with consistent corner equations, and so it is natural for us to seek a Lagrangian form exhibiting closure. However, if we naively seek to satisfy the closure relation (21) with any simple Lagrangian yielding the equations of motion, we will find that this does not suffice - we must seek a more specific form. By considering the general form for the quadratic Lagrangians:

$$\mathcal{L}_a = \alpha\big(x\overline{x} + (a-a_0)x^2 + a_0\overline{x}^2\big)\,, \quad \mathcal{L}_b = \beta\big(x\widehat{x} + (b-b_0)x^2 + b_0\widehat{x}^2\big)\,, \tag{22}$$

we can apply the closure $\Box\mathcal{L} = 0$ as a condition. Recall that we require closure only on the solutions to the equations of motion, so we apply the corner equations (19) to $\Box\mathcal{L}$, and then compare coefficients of the remaining terms. Demanding that $\alpha, a_0$ and $\beta, b_0$ be independent of $Q$ and $R$ respectively, we find the conditions on the coefficients:

$$\begin{aligned}
&\alpha = \frac{P+R}{r}\gamma\,, \quad \beta = \frac{P+Q}{q}\gamma\,,\\
&a_0 = \frac{r}{P+R}f(P) + \frac{1}{2}a\,, \quad b_0 = \frac{q}{P+Q}f(P) + \frac{1}{2}b\,,
\end{aligned} \tag{23}$$

where $\gamma$ is some overall constant, and $f(P)$ is a free function of $P$. $f$ does not make any contribution to what follows, and so we ignore it: we let $a_0 = a/2$ and $b_0 = b/2$.

This yields the Lagrangians:

$$\begin{aligned}
\mathcal{L}_a(x,\overline{x}) &= \frac{1}{r}\left((P+R)\,x\overline{x} + \frac{1}{2}(P-R)\,(x^2+\overline{x}^2)\right),\\
\mathcal{L}_b(x,\widehat{x}) &= \frac{1}{q}\left((P+Q)\,x\widehat{x} + \frac{1}{2}(P-Q)\,(x^2+\widehat{x}^2)\right).
\end{aligned} \tag{24}$$

By construction, these obey the condition $\Box\mathcal{L} = 0$ on the equations of motion, and also yield the equations of motion (11) and (18) by the usual variational principle. This eliminates a great deal of the usual freedom in choosing our Lagrangian: the closure condition mandates a specific form of the Lagrangian.

In fact, not only the equations (11) and (18) arise from a variational principle on this action, but also the corner equations (19). We have four elementary curves in the space of two discrete variables, shown in Fig. 5.[4] Across each curve, we can define an action, and then a variation with respect to the middle point, which leads to an equation of motion.

The action and Euler Lagrance equation for curve 5(i) are

$$S = \mathcal{L}_a(x,\overline{x}) + \mathcal{L}_b(\overline{x},\widehat{\overline{x}})\,, \quad \frac{\partial S}{\partial \overline{x}} = 2\Big[\Big(\tfrac{P-R}{r} + \tfrac{P-Q}{q}\Big)\overline{x} + \tfrac{P+R}{r}x + \tfrac{P+Q}{q}\widehat{\overline{x}}\Big] = 0\,, \tag{25}$$

[4] Such elementary curves defining a complete set of discrete EL equations were first considered in [33].

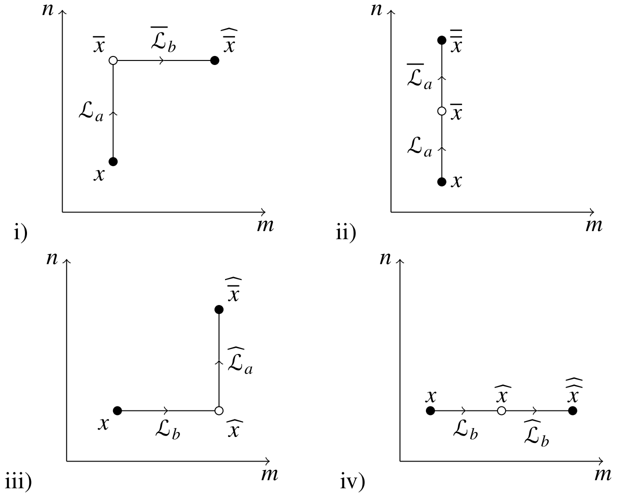


Fig. 5. Simple discrete curves for variables $m$ and $n$.

which is compatible with equations (19). Similarly, for curve 5(ii):

$$S = \mathcal{L}_a(x, \overline{x}) + \mathcal{L}_a(\overline{x}, \overline{\overline{x}}) \;, \qquad \frac{\partial S}{\partial \overline{x}} = 2\left[2\tfrac{P-R}{r}\overline{x} + \tfrac{P+R}{r}\left(x + \overline{\overline{x}}\right)\right] = 0 \;, \tag{26}$$

which is equation (18) (i.e. this is a "standard" Euler-Lagrange equation). Curves 5(iii) and (iv) yield similarly (11) and the other part of (19). We therefore have, *for the specific choice of Lagrangians described*, a consistent 1-form structure, yielding the equations of motion and corner equations, and obeying a Lagrangian closure relation. The discrete harmonic oscillator then, despite its simplicity, nonetheless has an underlying structure of a Lagrangian one-form expressing commuting flows: this is the simplest example yet discovered of such a structure.

Recall the invariants, it is straightforward to show using the equations of motion that both invariants are preserved under both evolutions, $\widehat{I_b} = \overline{I}_b = I_b$, $\overline{I}_a = \widehat{I_a} = I_a$. It is not clear, however, that these invariants are necessarily equal: $I_b$ has an apparent dependence on $Q$, and $I_a$ on $R$, that must be resolved. Taking our special choice of Lagrangians (24), we can then define canonical momenta, and rewrite our invariants in those terms. Writing $X_a$ as the momentum conjugate to $x$ in $\mathcal{L}_a$, and $X_b$ similarly for $\mathcal{L}_b$, we find:

$$\begin{aligned} X_a &= -\frac{\partial \mathcal{L}_a}{\partial x} = -\frac{P+R}{r}\overline{x} - \frac{P-R}{r}x \;, \\ X_b &= -\frac{\partial \mathcal{L}_b}{\partial x} = -\frac{P+Q}{q}\widehat{x} - \frac{P-Q}{q}x \;. \end{aligned} \tag{27}$$

As a direct consequence of the corner equation (19) we then have precisely that $X_a = X_b =: X$. In other words, we can define a common conjugate momentum for both evolutions. If we then write our invariants in terms of $x$ and $X$ we find after multiplication by a constant (which clearly does not change the nature of the invariants) that

$$I_a = I_b = \frac{1}{2}X^2 + 2Px^2 \;. \tag{28}$$

Note that in this form $I_a$, $I_b$ appear $Q$ and $R$ independent, and are nothing other than the Hamiltonian for the continuous harmonic oscillator, with angular frequency $\omega = 2\sqrt{P}$. This form is

Lagrangian dependent. A different choice of Lagrangian yields different conjugate momenta that are no longer equal, and where the equivalence of the invariants is no longer apparent. Requiring equality of the invariants turns out to be an equivalent condition to demanding Lagrangian closure.

The compatibility of the two discrete evolutions and their corner equations (guaranteed by the Lagrangian 1-form structure) allows us to consider a joint solution to the equations $x_{m,n}$. We allow $m$ to label the hat evolution, and $n$ to label the bar evolution, such that $x = x_{m,n}$, $\widehat{x} = x_{m+1,n}$, $\overline{x} = x_{m,n+1}$, and so on. Requiring $x_{m,n}$ to obey (11), (18) and (19), we have the joint solution for the evolutions:

$$x_{m,n} = c_1 \sin(\mu m + \nu n) + c_2 \cos(\mu m + \nu n)\ ; \quad b = -\cos\mu\ ,\ a = -\cos\nu\ . \tag{29}$$

In the same way as the parameter $b$ generates a continuous flow compatible with the discrete evolution (16), so we can find a continuous flow in the parameter $a$:

$$(1-a^2)\frac{d^2x}{da^2} - a\frac{dx}{da} + n^2 x = 0\ . \tag{30}$$

Now the joint solution (29) guarantees the compatibility of the $a$ and $b$ flows with the commuting discrete evolutions. The compatibility of the continuous flows can be further verified by checking the relation $\frac{d}{da}\frac{dx}{db} = \frac{d}{db}\frac{dx}{da}$ using (15) and similar equations for $a$. The continuous time-flows are generated by the usual Euler-Lagrange equations on continuous time Lagrangians of the form

$$\begin{aligned} \mathcal{L}_b(x, x_b) &= \frac{1}{2m}\sqrt{1-b^2}\left(\frac{\partial x}{\partial b}\right)^2 - \frac{m}{2\sqrt{1-b^2}}x^2\ ; \\ \mathcal{L}_a(x, x_a) &= \frac{1}{2n}\sqrt{1-a^2}\left(\frac{\partial x}{\partial a}\right)^2 - \frac{n}{2\sqrt{1-a^2}}x^2. \end{aligned} \tag{31}$$

Using the corner equations (19) these Lagrangians exhibit *continuous* multiform compatibility, obeying the relations

$$\frac{\partial \mathcal{L}_a}{\partial x_a} = \frac{\partial \mathcal{L}_b}{\partial x_b}\ , \quad \frac{\partial}{\partial a}\left(\frac{\partial \mathcal{L}_b}{\partial x}\right) = \frac{\partial}{\partial b}\left(\frac{\partial \mathcal{L}_a}{\partial x}\right)\ . \tag{32}$$

So, by considering the discrete parameters $a, b$ now as *continuous variables*, we find a continuous-time 1-form structure.

As in [34], the harmonic oscillator continues to display surprising new features. On the discrete level, we discover compatible flows that can be expressed through the structure of a Lagrangian form, even for this very simple case. This deeper structure then extends beyond the discrete case also into compatible continuous flows and we have an interplay between these discrete and continuous one-form structures. Having endowed the harmonic oscillator with these multi-dimensional structures, how are they revealed in the quantum harmonic oscillator case?

### 3.4. Higher periodicity

The periodic reduction defined in section 3.1 is part of a more general family of periodic staircase initial value problems [27,29,35]. In general, we define $2P$ initial conditions, $u_0, u_1, \ldots, u_{2P-1}$ such that $u_0 = \widehat{u}_{2P-1}$, along a staircase as shown in Fig. 6. The linearised KdV equation (1) defines a dynamical map $(u_0, u_1, \ldots, u_{2P-1}) \to (\widehat{u}_0, \widehat{u}_1, \ldots, \widehat{u}_{2P-1})$. As before, we introduce reduced variables $x_1, \ldots, x_{P-1}, y_1, \ldots, y_{P-1}$ and can eliminate the $y_i$ to give a $P-1$ dimensional system of second order difference equations in terms of the $x_i$ variables.

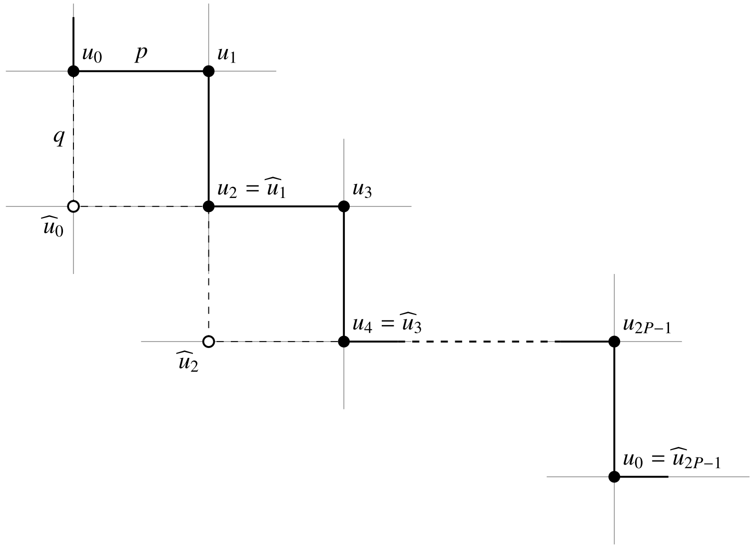


Fig. 6. The periodic staircase for period $P$.

The $P = 2$ case yields a 1 dimensional mapping that is entirely equivalent to the case we have considered in section 3.1, except the lattice parameters combine in a slightly different way to give the coefficient of the harmonic oscillator.

The $P = 3$ case is the next case of interest, as here we find a system of coupled harmonic oscillators in $x_1$ and $x_2$, with two commuting invariants and a similar commuting flow structure. In a similar manner to (11) we can derive equations for a discrete flow in variables $x_1$ and $x_2$:

$$\widehat{x}_1 + \widehat{x}_2 + \underset{\frown}{x}_1 + s(2x_1 + x_2) = 0 \,, \quad \widehat{x}_2 + \underset{\frown}{x}_1 + \underset{\frown}{x}_2 + s(x_1 + 2x_2) = 0 \,. \tag{33}$$

As in section 3.2, we can also derive a commuting flow for the evolution:

$$(1 + tt')(\overline{x}_1 + \underline{x}_1) + \overline{x}_2 + tt'\underline{x}_2 + (t + t')(2x_1 + x_2) = 0 \,, \tag{34}$$

$$(1 + tt')(\overline{x}_2 + \underline{x}_2) + tt'\overline{x}_1 + \underline{x}_1 + (t + t')(x_1 + 2x_2) = 0 \,. \tag{35}$$

Commutativity of these evolutions can be easily shown from the first order form (with $x$ and $y$ variables) by writing each evolution in matrix form; the resulting matrices commute. The evolution then also possesses corner equations, which can be derived using the eliminated $y$ variables. These allow us to write closed form Lagrangians, such that $\square\mathcal{L} = 0$ (21) on the equations of motion (33), (34), (35):

$$\mathcal{L}_1(x, \widehat{x}) = x_1(\widehat{x}_1 + \widehat{x}_2) + x_2\widehat{x}_2 + \tfrac{1}{2}s(x_1^2 + x_1x_2 + x_2^2 + \widehat{x}_1^2 + \widehat{x}_1\widehat{x}_2 + \widehat{x}_2^2) \,, \tag{36}$$

$$\begin{aligned}\mathcal{L}_2(x, \overline{x}) = {} & \frac{1 + tt'}{1 - tt'}(x_1\overline{x}_1 + + x_2\overline{x}_2) + \frac{1}{1 - tt'}(x_1\overline{x}_2 + tt'x_2\overline{x}_1) \\ & + \frac{1}{2}\frac{t + t'}{1 - tt'}(x_1^2 + x_1x_2 + x_2^2 + \overline{x}_1^2 + \overline{x}_1\overline{x}_2 + \overline{x}_2^2) \,,\end{aligned} \tag{37}$$

recalling the relation of $s, t, t'$. A Lagrangian 1-form structure as in section 3.3 follows. Note that $\mathcal{L}_2$ represents a Bäcklund transform with parameter $r$.

The Lagrangians (36), (37) allow us to define the momenta conjugate to $x_1, x_2$ writing $X_i = -\partial\mathcal{L}_1/\partial x_i$,

$$X_1 = -\left(\widehat{x}_1 + \widehat{x}_2 + \tfrac{1}{2}s(2x_1 + x_2)\right), \quad X_2 = -\left(\widehat{x}_2 + \tfrac{1}{2}s(x_1 + 2x_2)\right), \tag{38}$$

with respect to which we have the invariant Poisson structure $\{x_i, X_j\} = \delta_{ij}$, preserved under the mappings. We could also write expressions for $X_i$ using $\mathcal{L}_2$, with equality of these expressions producing the corner equations.

We can additionally derive two quadratic invariants of the mapping $I_1, I_2$, which are invariant under both maps. The canonical structure of (38) allows us to show the critical integrability property that the two invariants are in involution with each other, with respect to the canonical Poisson bracket: $\{I_1, I_2\} = 0$ where

$$\begin{aligned} I_1 &= x_1X_1 - 2x_1X_2 + 2x_2X_1 - x_2X_2 , \\ I_2 &= \left(1 - \tfrac{3}{4}s^2\right)(x_1^2 + x_1x_2 + x_2^2) + X_1^2 - X_1X_2 + X_2^2 . \end{aligned} \tag{39}$$

The invariance and involutivity of these can be shown by direct calculation. $I_1$ and $I_2$ will thus generate two commuting continuous flows to the mapping.

For both the hat and the bar evolutions (33), (34), (35) it is possible to write explicit solutions, and indeed we can find a joint solution to the discrete evolutions:

$$\begin{aligned} x_2(m,n) &= a\cos(\mu_+ m + \nu_+ n) + b\sin(\mu_+ m + \nu_+ n) \\ &\quad + c\cos(\mu_- m + \nu_- n) + b\sin(\mu_- m + \nu_- n) , \end{aligned} \tag{40}$$

where $\cos\mu_\pm = -3s/4 \pm \frac{1}{2}\sqrt{1 - 3s^2/4}$ and

$$\cos\nu_\pm = -\frac{3(t+t')(1+tt')}{4(1+tt'+t^2t'^2)} \pm \frac{1}{2}\left(\frac{1-tt'}{1+tt'+t^2t'^2}\right)^2 \sqrt{1+tt'+t^2t'^2 - \tfrac{3}{4}(t+t')^2} . \tag{41}$$

We find $x_1(m,n)$ similarly as a linear combination of shifts of $x_2$. By considering derivatives with respect to the parameters $s$ and $t$ (recalling $t'$ is not independent of $s, t$), we can therefore derive commuting continuous flows from the solution structure (40). We observe then again the interchange between continuous and discrete parameters and variables, as in the lower periodic case. We expect this will lead to a continuous Lagrangian 1-form structure, but defer further investigation to a later paper.

## 4. The quantum reduction

In section 3.3, the discrete harmonic oscillator model, arising as a special reduction from the linearised lattice KdV equation (1), albeit a simple linear model nonetheless displays commuting discrete flows. In the classical case, the Lagrangian 1-form structure captures these commuting flows in a variational principle. A natural question is: what is the quantum analogue for such a structure? Since the harmonic oscillator is well known and understood, it forms a good first toy model for investigating Lagrangian form structures at the quantum level.

Integrable quantum mappings, arising from the quantisation of mapping reductions from lattice equations, were constructed and studied within the framework of canonical quantization and (non-ultralocal) R-matrix structures in [15,20,36,31,37]. In a pioneering paper [38] Dirac took the position that the Lagrangian approach to Physics is the more *natural* one and proposed the first steps towards incorporating the Lagrangian into quantum mechanics, a route that was later pursued by Feynman leading to his concept of the path integral [39]. Concurring with Dirac's point of view, we seek here to understand the extended Lagrangian multiform variational principle on the quantum level, leading naturally to problem of finding a path integral version of that formalism in order to capture its natural quantum analogue. To make first steps in that direction the simple case of the quantum mappings derived in the previous section is a good starting point, exploiting the well-known formal techniques of path integrals, cf. e.g. [19,40,41]. As we will point out later there are some similarities with ideas developed by Rovelli in [42,43] who also uses the harmonic oscillator to develop ideas on reparametrisation invariant discretisations within the path integral framework, in particular the natural emergence of conservation of the energy of the continuous model within a time-slicing discretisation.

### *4.1. Feynman propagators*

Beginning from our Lagrangian $\mathcal{L}_b$ (24) we write the conjugate momenta $X := X_b$ (27) and $\widehat{X} = \partial\mathcal{L}_b/\partial\widehat{x}$. In canonical quantisation, position $x$ and momentum $X$ become *operators* $\mathbf{x}$ and $\mathbf{X}$, such that $[\mathbf{x}, \mathbf{X}] = i\hbar$. The momentum equations (27) become operator equations of motion:

$$\widehat{\mathbf{x}} - \mathbf{x} = \frac{q}{P-Q}\widehat{\mathbf{X}} - \frac{2P}{P-Q}\mathbf{x}\,, \quad \widehat{\mathbf{X}} - \mathbf{X} = -\frac{4Pq}{P-Q}\mathbf{x} + \frac{2P}{P-Q}\widehat{\mathbf{X}}\,. \tag{42}$$

To understand the discrete time evolution we wish to express the evolution $(\mathbf{x}, \mathbf{X}) \to (\widehat{\mathbf{x}}, \widehat{\mathbf{X}})$, in terms of a time-evolution operator $U_b$, such that $\mathbf{x} \to \widehat{\mathbf{x}} = U_b^{-1}\mathbf{x}U_b$, $\mathbf{X} \to \widehat{\mathbf{X}} = U_b^{-1}\mathbf{X}U_b$. This is a canonical approach to discrete quantisation, see for example [15]. Considering (42), it is not hard to see that an appropriate choice of $U_b$ is given by:

$$U_b = e^{iV(\mathbf{x})/2\hbar}e^{iT(\mathbf{X})/\hbar}e^{iV(\mathbf{x})/2\hbar} = \exp\left(\frac{iP\mathbf{x}^2}{\hbar q}\right)\exp\left(\frac{iq\mathbf{X}^2}{2\hbar(P+Q)}\right)\exp\left(\frac{iP\mathbf{x}^2}{\hbar q}\right)\,. \tag{43}$$

In other words, a separated form for $U_b$ exists, but it is required to have three terms. Note that (43) is not a unique form for $U_b$.

In discrete time, the one time-step propagator is then given by $K_b(x, n; \widehat{x}, n+1) = {}_{n+1}\langle\widehat{x}|x\rangle_n = \langle\widehat{x}|U_b|x\rangle$, where we have moved in the second equality from time-dependent, Heisenberg picture eigenstates to time-independent, Schrödinger picture eigenstates. Since we have an explicit form for $U_b$, we can calculate this expression by inserting a complete set of momentum eigenstates:

$$\begin{aligned}\langle\widehat{x}|U_b|x\rangle &= \int \mathrm{d}X e^{iV(\widehat{x})/2\hbar}\langle\widehat{x}|X\rangle e^{iT(X)/\hbar}\langle X|x\rangle e^{iV(x)/2\hbar}\,,\\ &= \left(\frac{i(P+Q)}{2\pi\hbar q}\right)^{1/2}\exp\left\{\frac{i}{\hbar q}\Big((P+Q)x\widehat{x} + \tfrac{1}{2}(P-Q)(x^2+\widehat{x}^2)\Big)\right\}\,,\\ &= \left(\frac{i(P+Q)}{2\pi\hbar q}\right)^{1/2}\exp\left[\frac{i}{\hbar}\mathcal{L}_b(x,\widehat{x})\right]\,.\end{aligned} \tag{44}$$

The second line results from a Gaussian integral: the linearity of our system justifies taking the integration region over the whole real line (we make some assumptions here on the Hilbert space). The final line recalls the Lagrangian (24). This is what might be expected for a "one-step" path integral (such as in [44,21]) noting that this approach also specifies the normalisation constant.

This is sufficient to define the discrete-time path integral. By iterating (44) over $N$ steps, we can write precisely the propagator for our discrete system:

$$K_b(x_0, 0; x_N, N) = \left(\frac{i(P+Q)}{2\pi\hbar q}\right)^{N/2} \prod_{n=1}^{N-1} \int_{-\infty}^{\infty} \mathrm{d}x_n \; e^{i\mathcal{S}[x(n)]/\hbar} \,,$$

$$\mathcal{S}[x(n)] = \sum_{n=0}^{N-1} \mathcal{L}_b(x_n, x_{n+1}) \,. \tag{45}$$

In this discrete case, equation (45) gives a precise definition to the path integral notation:

$$K_b(x_0, 0; x_N, N) = \int_{x(0)=x_0}^{x(N)=x_N} [\mathcal{D}x(n)] \; e^{i\mathcal{S}[x(n)]/\hbar} \,. \tag{46}$$

Notice in particular that the normalisation associated to the measure is here unambiguous. In our quadratic regime, we can now calculate this explicitly. Details are given in Appendix A, but we first expand our quantum variables around the classical path, where the classical action can be evaluated as $\mathcal{S}_{cl} = \sqrt{P}[2x_0x_N - (x_0^2 + x_N^2)\cos\mu N]/\sin\mu N$. Evaluating the discrete path integral as a series of $N$ Gaussian integrations, and recalling the normalisation constant in (45), we calculate the propagator:

$$K_b(x_0, 0; x_N, N) = \left(\frac{i\sqrt{P}}{\pi\hbar\sin(\mu N)}\right)^{1/2} \times \exp\left\{\frac{i\sqrt{P}}{\hbar\sin(\mu N)}\Big(2x_0x_N - (x_0^2 + x_N^2)\cos(\mu N)\Big)\right\} \,. \tag{47}$$

Note that this has the same form as the propagator for the continuous time harmonic oscillator. Dependence on the parameter $b$ is evident through $\cos\mu = -b$. We note, then, that the propagator is common to both the discrete flow and to the interpolating continuous time flow.

Using the operator equations of motion (42), it is easy to see that we have an operator invariant:

$$\mathbf{I}_b = \frac{1}{2}\mathbf{X}^2 + 2P\mathbf{x}^2 = \frac{1}{2}\left(-\hbar^2\frac{\partial^2}{\partial x^2} + 4Px^2\right) \,, \tag{48}$$

This is, of course, simply the operator version of the classical invariant (28), and is precisely the Hamiltonian for the continuous time harmonic oscillator, where $4P = \omega^2$. Note that $\mathbf{I}_b$ is $Q$ independent, and so it is clear that the same process applied to the bar evolution generated by $\mathcal{L}_a$ will give the same result. In other words, both discrete quantum evolutions share the same invariant, which is the harmonic oscillator. The invariant can also be considered from the perspective of path integrals and the unitary operator following the method of [21]; this is elaborated in Appendix B. We can relate $\mathbf{I}_b$ (48) to the evolution operator $U_b$ (43) in principle by a Campbell-Baker-Hausdorff expansion ([45,46]); an explicit form is given by algebraic manipulation:

$$U_b = \exp\left[\frac{1}{\hbar\sqrt{P}} \operatorname{arctanh}\left(\frac{i\sqrt{P}}{q}\right) \mathbf{I}_b\right] \,. \tag{49}$$

So we can see clearly how the discrete quantum evolution relates to a continuous time flow.

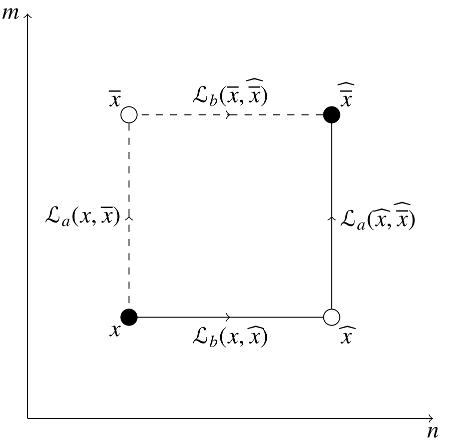


Fig. 7. The solid line shows path (i) for $K_{\lrcorner}$, and the dashed line path (ii) for $K_{\ulcorner}$. The white circles represent variables that are integrated over.

### 4.2. Path independence of the propagator

In equation (47) we have established the propagator for an evolution in one discrete time variable; but we have in the classical case two compatible discrete flows (24). The one-step propagator in the hat direction is given in (44), whilst in the bar direction it is easily deduced by the same method:

$$K_a(x,\overline{x};1)=\left(\frac{i(P+R)}{2\pi\hbar r}\right)^{1/2}\exp\left[\frac{i}{\hbar}\mathcal{L}_a(x,\overline{x})\right]. \tag{50}$$

We remark that, as we have here a second time direction, we might plausibly introduce a second $\hbar$ parameter. We ignore such considerations for the time being and allow $\hbar$ to be the same in both time directions. In general, if we begin at a time co-ordinate $(0,0)$ and evolve along integer time co-ordinates to a new time $(N,M)$, the propagator could depend not only on the endpoints, but also on the path $\Gamma$ taken through the time variables, see Fig. 4. We associate to the path an action $\mathcal{S}_\Gamma := \mathcal{S}[x(\boldsymbol{n});\Gamma]$ (20). We can then define a propagator for the evolution along the time-path $\Gamma$, made up of the one-step elements (44), (50):

$$K_\Gamma\big(x_b,(N,M);x_a,(0,0)\big):=\mathcal{N}_\Gamma\prod_{(n,m)\in\Gamma}\int \mathrm{d}x_{n,m}\,\exp\left[\frac{i}{\hbar}\mathcal{S}_\Gamma[x(\boldsymbol{n})]\right], \tag{51}$$

where we have integrated over all internal points $x_{n,m}$ on the curve $\Gamma$. Here $\mathcal{N}_\Gamma$ represents the product of normalisation factors from the relevant elements of (44), (50).

We begin by considering the simple case of an evolution of one step in each direction. There are two routes to achieve this, as shown in Fig. 7. Either we evolve first in the hat direction, followed by an evolution in the bar direction, or vice versa. In path (i), we evolve first according to the hat evolution $\mathcal{L}_b$, and then according to the bar evolution $\mathcal{L}_a$. We evaluate the propagator as:

$$K_{\lrcorner}(x,\widehat{\overline{x}})=\left(\frac{(P+Q)(P+R)}{(-2\pi i\hbar)^2qr}\right)^{1/2}\int_{-\infty}^{\infty}\mathrm{d}\widehat{x}\,\exp\left\{\frac{i}{\hbar}\left(\mathcal{L}_b(x,\widehat{x})+\mathcal{L}_a(\widehat{x},\widehat{\overline{x}})\right)\right\}. \tag{52}$$

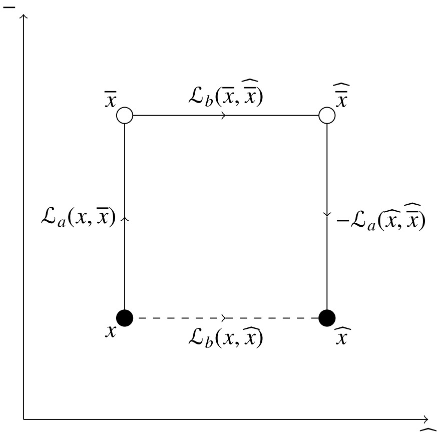


Fig. 8. The path for action $\mathcal{S}_{\sqcap}$. In the propagator, we integrate over the variables at the white circles. Note the minus sign on the backward step, $-\mathcal{L}_a(\widehat{x}, \widehat{\overline{x}})$.

For the alternative path (ii) we evolve first by the bar evolution $\mathcal{L}_a$, and then the hat evolution $\mathcal{L}_b$:

$$K_{\ulcorner}(x, \widehat{\overline{x}}) = \left( \frac{(P+Q)(P+R)}{(-2\pi i\hbar)^2 qr} \right)^{1/2} \int_{-\infty}^{\infty} \mathrm{d}\overline{x} \; \exp\left\{ \frac{i}{\hbar} \left( \mathcal{L}_a(x, \overline{x}) + \mathcal{L}_b(\overline{x}, \widehat{\overline{x}}) \right) \right\} . \tag{53}$$

These are both resolved by substituting Lagrangians (24) and evaluating the Gaussian integral. The result is totally symmetric under interchange of the parameters $q$ and $r$, as are (52) and (53); so that

$$K_{\ulcorner}(x, \widehat{\overline{x}}) = K_{\lrcorner}(x, \widehat{\overline{x}}) . \tag{54}$$

We find the same propagator for either path. It is an obvious corollary of this result that, so long as we take only forward steps in time, the propagator $K_{N,M}(x_a, x_b)$ is independent of the path taken in the time variables.

We could also consider a path in the time variables allowing backward time steps. As in the classical case, we can construct an action for such a trajectory, using an appropriate orientation for the Lagrangians. In the quantum case we perform a path integral over this action, integrating over all intermediate points. As $U_b$ generates a time-step in the $b$ direction (section 4.1), $U_b^{-1}$ generates the backward evolution.

Considering once more the simplest case, we imagine a trajectory around three sides of a square, shown in Fig. 8. Including the normalisation factors from (44) this is described by the propagator,

$$K_{\sqcap}(x, \widehat{x}) = \frac{(P+Q)^{1/2}(P+R)}{(2\pi\hbar)^{3/2}(-iq)^{1/2}r} \int_{-\infty}^{\infty} \mathrm{d}\overline{x} \int_{-\infty}^{\infty} \mathrm{d}\widehat{\overline{x}} \; \exp\left( \tfrac{i}{\hbar} \mathcal{L}_a(x, \overline{x}) + \mathcal{L}_b(\overline{x}, \widehat{\overline{x}}) - \mathcal{L}_a(\widehat{x}, \widehat{\overline{x}}) \right) . \tag{55}$$

This is easily calculated by Gaussian integrals, and yields:

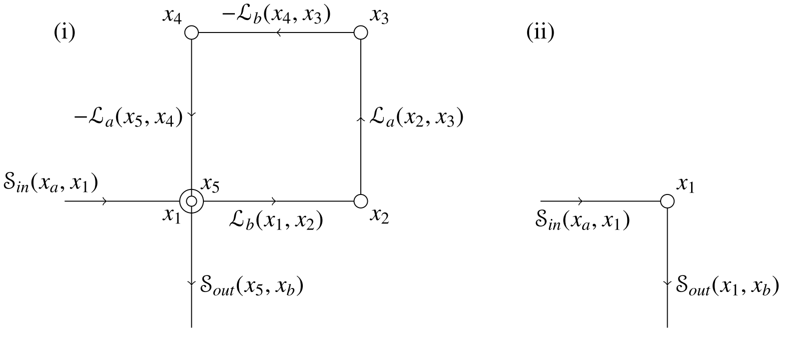


Fig. 9. (i) shows the loop in discrete variables. (ii) is what remains after collapse of the loop.

$$K_{\sqcap}(x, \widehat{x}) = \left(\frac{i(P+Q)}{2\pi\hbar q}\right)^{1/2} \exp\left(\frac{i}{\hbar}\mathcal{L}_b(x, \widehat{x})\right) = K_b(x, \widehat{x}; 1) . \tag{56}$$

So we regain exactly our one step propagator from (44). Remarkably, we again achieve Lagrangian closure, but now on the quantum level. Recall that classically Lagrangian closure depended upon the equations of motion: here we have left the equations of motion behind, and yet this key result still holds.

We could also consider the possibility of a *loop* in the discrete variables, illustrated in Fig. 9(i). We imagine some unspecified incoming and outgoing actions $\mathcal{S}_{in}(x_a, x_1)$ and $\mathcal{S}_{out}(x_5, x_b)$, a simple loop in discrete steps, and five integration variables $x_1, \ldots, x_5$. Note that we assign two integration variables to the same vertex, as it is visited twice by the path: the following calculation will justify this choice as the correct one.

We then consider the action for the loop, $\mathcal{S}_{loop} = \mathcal{L}_b(x_1, x_2) + \mathcal{L}_a(x_2, x_3) - \mathcal{L}_b(x_4, x_3) - \mathcal{L}_a(x_5, x_4)$, noting the orientations on the Lagrangians. With normalising factors from (44) and complex conjugations in the backward steps, we then have:

$$K_{loop}(x_a, x_b) = \frac{P+Q}{2\pi\hbar q}\frac{P+R}{2\pi\hbar r}\int \mathrm{d}x_1 \ldots \int \mathrm{d}x_5 \ \exp\frac{i}{\hbar}\left\{\mathcal{S}_{in} + \mathcal{S}_{loop} + \mathcal{S}_{out}\right\} . \tag{57}$$

The $x_2$ and $x_4$ integrals are evaluated as in (52) yielding,

$$\begin{aligned} K_{loop}(x_a, x_b) = \frac{(P+Q)(P+R)}{2\pi\hbar(P-qr)(q+r)} \iiint \mathrm{d}x_1\mathrm{d}x_3\mathrm{d}x_5 \ \exp\frac{i}{\hbar}\Bigg\{ &\mathcal{S}_{in}(x_a, x_1) + \mathcal{S}_{out}(x_5, x_b) \\ &- \frac{(P+Q)(P+R)}{(P-qr)(q+r)}(x_1 - x_5)x_3 \\ &+ \frac{1}{2}\left(\frac{P-qr}{q+r} - \frac{P(q+r)}{P-qr}\right)(x_1^2 - x_5^2)\Bigg\} . \end{aligned} \tag{58}$$

The quadratic term in the exponent in $x_3$ disappears, and so the integral $\mathrm{d}x_3$ yields a Dirac delta function: $\delta(x_1 - x_5)$. Combined with the integral over $x_5$ this forces $x_5 = x_1$ (as expected) and we finally conclude,

$$K_{loop}(x_a, x_b) = \int \mathrm{d}x_1 \ \exp\frac{i}{\hbar}\left\{\mathcal{S}_{in}(x_a, x_1) + \mathcal{S}_{out}(x_1, x_b)\right\} . \tag{59}$$

Diagrammatically, this is equivalent to the disappearance of the loop, shown in Fig. 9(ii). Loops in the discrete variables therefore "close" and do not effect the overall propagator.

**Proposition 1.** *For the special choice of Lagrangians (24), the propagator along the time path* $\Gamma$ *(51) is independent of the choice of* $\Gamma$*, depending only on the end points.*

**Proof.** Equations (54), (56) and (59) together show that the propagator is unchanged under elementary deformations of the curve $\Gamma$. Since we have a simple topology, a curve $\Gamma_1$ can be deformed into any other curve $\Gamma_2$ (with the same endpoints) by a series of elementary deformations. The proposition follows.

The proposition now allows us to calculate the general propagator for $N$ steps in the hat direction and $M$ steps in the bar direction, compare (51). We denote such a propagator from $x_a$ to $x_b$ by $K_{N,M}(x_a, x_b)$. As a consequence of the path independence, it is then clear that we can calculate this as $K_{N,M}(x_a, x_b) = \int \mathrm{d}x\, K_{N,0}(x_a, x) K_{0,M}(x, x_b)$. In other words, we can consider taking first all the hat-steps, followed by all the bar-steps. Taking our discrete propagator from (47), we can then carry out the integral as another Gaussian, but in fact the result follows immediately from the group property of the propagator, using its shared form with the continuous time case, so:

$$K_{N,M}(x_a, x_b) = \left( \frac{i\sqrt{P}}{\pi\hbar \sin(\mu N + \eta M)} \right)^{1/2} \times \exp\left\{ \frac{i\sqrt{P}}{\hbar \sin(\mu N + \eta M)} \left( 2x_a x_b - (x_a^2 + x_b^2)\cos(\mu N + \eta M) \right) \right\}, \quad (60)$$

which bears a clear relation to the continuous time case.

### *4.3. Uniqueness*

The time-path independence for the propagator of section 4.2 is a special property of our choice of Lagrangian (24) that does not hold in general. As classically the Lagrangian 1-form obeys the closure condition (21), so in the quantum case we have time-path independence of the propagators as a natural quantum analogue. Whilst classically this closure holds only on the equations of motion, in the quantum case the path-independence occurs as we perform the path integral over intermediate variables. It emerges that, for given oscillator parameters $a$ and $b$, there is a fairly unique choice of Lagrangians exhibiting time-path independence.

Consider the generalised oscillator Lagrangians of equation (22) and define propagators around two corners of a square, as in equations (52) and (53). Here we allow $a$ and $b$ to be free oscillator parameters.

$$K_{\lrcorner}(x, \widehat{\overline{x}}) = \mathcal{N}_{\lrcorner} \int_{-\infty}^{\infty} \mathrm{d}\widehat{x}\ \exp\left\{ \frac{i}{\hbar} \left( \mathcal{L}_b(x, \widehat{x}) + \mathcal{L}_a(\widehat{x}, \widehat{\overline{x}}) \right) \right\}, \quad (61)$$

$$K_{\ulcorner}(x, \widehat{\overline{x}}) = \mathcal{N}_{\ulcorner} \int_{-\infty}^{\infty} \mathrm{d}\overline{x}\ \exp\left\{ \frac{i}{\hbar} \left( \mathcal{L}_a(x, \overline{x}) + \mathcal{L}_b(\overline{x}, \widehat{\overline{x}}) \right) \right\}. \quad (62)$$

$\mathcal{N}_{\lrcorner}$ and $\mathcal{N}_{\ulcorner}$ are undetermined normalisation constants. These paths are illustrated in Fig. 7.

We demand equality of the exponents in these two expressions, once the integral has been carried out; in other words we demand $K_{\lrcorner}(x, \widehat{\overline{x}}) = K_{\ulcorner}(x, \widehat{\overline{x}})$, up to a normalisation. Calculating these propagators via a Gaussian integral, we then derive conditions for time-path-independence on our coefficients, which can be found in Appendix C. We find the necessary conditions on the coefficients:

$$a_0 = \frac{1}{2}a + \frac{f}{2\alpha}\,, \quad b_0 = \frac{1}{2}b + \frac{f}{2\beta}\,, \quad \alpha = \frac{\gamma}{\sqrt{a^2-1}}\,, \quad \beta = \frac{\gamma}{\sqrt{b^2-1}}\,. \tag{63}$$

As in (23) the constant $f$ makes no contribution and we ignore it. The general Lagrangians (22) are therefore restricted to a symmetric form, with a specified overall constant given by the oscillator parameters $a, b$. Note that taking $a = (P-R)/(P+R)$, $b = (P-Q)/(P+Q)$ leads us to *exactly* the conditions of (23) and the Lagrangians (24). In conclusion:

**Proposition 2.** *For given oscillator parameters a and b, the Lagrangians (24) are the unique Lagrangians, up to constants $\gamma$ and $f$ (23), such that the multi-time propagator is path independent.*

In other words, demanding time-path independence of the propagator is the natural quantum analogue of the closure relation on the Lagrangian.

### *4.4. Quantum variational principle: Lagrangian 1-form case*

Consider a quantum mechanical evolution from an initial time $(0, 0)$ to a new time $(N, M)$, along a time-path $\Gamma$: shown in Fig. 4. We can consider a propagator for the evolution $K_\Gamma(x_b; x_a)$ defined in (51). We have shown that, in the special case of Lagrangians (24), the propagator defined above is independent of the path $\Gamma$ (it depends only on the endpoints); but that this is not true in general. For a generic Lagrangian, $K_\Gamma$ will depend on the time-path chosen, as shown in section 4.3.

Classically, the system is defined as the critical point for the variation of the action over not only the dependent variables, but also over the independent variables, i.e., it is a critical point with respect to the variation of the time-path. This not only yields all the compatible equations of motion for the system, but also selects certain "permissible" Lagrangians which obey a closure relation (21). This then yields a system of extended EL equations of which the Lagrangian can be considered to be the solution, cf. [9].

In the quantum case, we consider the dependence of the propagator on all possible (discrete) time-paths $\Gamma$ between fixed initial and final times. In general, there are an infinite number of possible time paths from $(0, 0)$ to $(N, M)$, including shortest time-paths as well as those with long "diversions," or loops, as illustrated in Fig. 10. For a generic Lagrangian, as we vary the time path, each $\Gamma$ yields a different propagator (51) viewed as a functional of the path. In the special case of the Lagrangian (24), however, the propagator $K_\Gamma$ is *independent of the path taken through the time variables*, and so remains unchanged across the variation of the time-path $\Gamma$. This suggests that this path independence property is the natural quantum analogue of the Lagrangian closure condition (21).

Pushing this idea one step further: viewing the propagator as a functional of the Lagrange function, the Lagrangian itself can be thought of as representing a critical point (in a properly chosen function space of Lagrange functions) for the path-dependent propagator, with regard to variations of the time-path. We suppose we can vary the path in such a way that the critical point analysis *selects* the path independent Lagrangian from the space of possible Lagrangians (this was the point of view put forward in [12] in the classical case). In a quantum setting this principle would be represented by a "sum over all time-paths" scenario, i.e. by means of posing a new

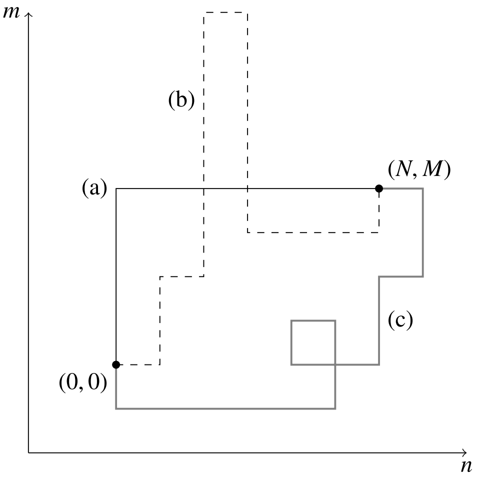


Fig. 10. Three possible paths in the time-variables. Path (a) is a direct path. Path (b) extends for some distance in the *m* direction before returning. Path (c) includes a loop in the time variables.

quantum object of the form as was proposed in the continuous time-case in [32]. As a functional of the Lagrangian such an object would have a singular point for those Lagrangians which possess the quantum closure condition, i.e., those where the contributions of the path-independent propagators over which one integrates all contribute the same amount. How to control the singular behaviour of such an object is a matter of ongoing investigation.

## 5. Quantisation of the lattice equation

In section 2 we introduced the linear lattice equation (5), and in the subsequent sections looked at a special type of reductions (associated with periodic initial value problems) leading to commuting dynamical maps and their quantum counterparts. This is obviously by far not the only type of reduction of the 2D lattice system to a finite number of degrees of freedom system: for example, another class is given by symmetry reductions leading to self-similar solutions.[5] Thus, the quantisation of the full lattice case of the MDC system associated with (5) is interesting in its own right, which would be the simplest example of the quantisation of a Lagrangian 2-form structure. This approach involving a path integral approach, would complement the quantisation of (nonlinear) lattice models that has been previously considered from the canonical (quantum inverse scattering method) perspective, cf. e.g. [4,48].

Classically, we suppose the equation (5) to hold on all plaquettes in the multidimensional lattice at the same time. The equation is generated by the oriented Lagrangian:

$$\mathcal{L}_{ij}(u,u_i,u_j;p_i,p_j)=u(u_i-u_j)-\frac{1}{2}s_{ij}(u_i-u_j)^2\,,\qquad s_{ij}=\frac{p_i+p_j}{p_i-p_j}\,. \tag{64}$$

[5] Such reductions are often connected to non-autonomous (i.e. explicitly time-dependent) equations of the motion, and the latter arise also in connection with random matrix ensembles. In particular, the linear case should be related to the case of Gaussian ensembles (see, e.g., Chapter 1 of [47]).

The Lagrangian itself is a critical point of the classical variational principle over surfaces: it obeys the closure property on the classical equations of motion, such that the surface can be allowed to freely vary under local moves. Indeed, it is also fairly unique, as seen in (9).

How might we proceed to quantise such a system? A canonical approach is to transform (5) into an operator equation of motion, but we are concerned here with a Lagrangian approach. The clear analogy is to quantum field theory: we have a discretised space-time and a Lagrangian in two dimensions over field variables $u(\boldsymbol{n})$ indexed by a discrete vector $\boldsymbol{n}$. We imagine some space-time boundary $\partial\sigma$ enclosing a multi-dimensional surface $\sigma$ made up of elementary plaquettes $\sigma_{ij}$. We can then construct an action by summing the directed Lagrangians over the surface, as we would classically:

$$\mathcal{S}_\sigma = \sum_{\sigma_{ij}\in\sigma} \mathcal{L}_{ij}(u, u_i, u_j) \ , \tag{65}$$

where we define the shorthand $\mathcal{L}_{ij}(u) := \mathcal{L}(u, u_i, u_j; p_i, p_j)$.

We then consider the propagator $K_\sigma(\partial\sigma)$, where all interior field variables on the surface are integrated over. The propagator depends, in principle, on the surface $\sigma$ and is a function of the field variables on the boundary $\partial\sigma$, which form some boundary value problem (see a similar point made in [43]):

$$K_\sigma(\partial\sigma) = \int [\mathcal{D}u(\boldsymbol{n})]_\sigma \ e^{i\mathcal{S}_\sigma[u(\boldsymbol{n})]/\hbar} = \mathcal{N}_\sigma \prod_{\boldsymbol{n}\in\sigma} \int \mathrm{d}u(\boldsymbol{n}) \ e^{i\mathcal{S}_\sigma[u(\boldsymbol{n})]/\hbar} \ . \tag{66}$$

We will see as we go on that this object is subject to infra-red divergences, as particular surface configurations produce integrations yielding volume factors. Since our main statements involves only the combinatorics of the exponential factors involving the action arising through Gaussian integrals, we tacitly assume $K_\sigma$ can be renormalised by an appropriate choice of normalisation factor $\mathcal{N}_\sigma$. $K_\sigma(\partial\sigma)$ describes a propagator in the sense of a surface gluing procedure: two propagators $K_{\sigma_1}$ and $K_{\sigma_2}$ are combined to a new propagator by multiplication and integration over all variables living on the shared boundary $\partial\sigma_1 \cap \partial\sigma_2$. Thus, the one-step surface gluing can be written symbolically as

$$\begin{aligned} K_{\sigma_1\cup\sigma_2} &= \int\limits_{\partial\sigma_1\cap\partial\sigma_2} K_{\sigma_1} * K_{\sigma_2} \\ &:= \mathcal{N}_{\partial\sigma_1\cap\partial\sigma_2} \left[ \prod_{\boldsymbol{n}\in\partial\sigma_1\cap\partial\sigma_2} \int \mathrm{d}u(\boldsymbol{n}) \right] K_{\sigma_1}(\partial\sigma_1).K_{\sigma_2}(\partial\sigma_2) \ , \end{aligned} \tag{67}$$

where the integral is over appropriately chosen coordinates of the joined boundary. Iterating the gluing formula is tantamount to setting up a "surface-slicing" procedure for the path integral.

### 5.1. Motivation: the pop-up cube

Classically, for a Lagrangian 2-form we vary the surface $\sigma$ so that the Lagrangian and equations of motion sit at a critical point: the action should be invariant under the variation of not only the dependent variables $u$, but also the variation of the surface itself. As we move to the quantum regime, we then naturally ask what happens to our propagator $K_\sigma(\partial\sigma)$ (66) under variation of the surface $\sigma$? We consider the effect of a simple variation of the surface: from a flat surface to a popped-up cube, see Fig. 11.

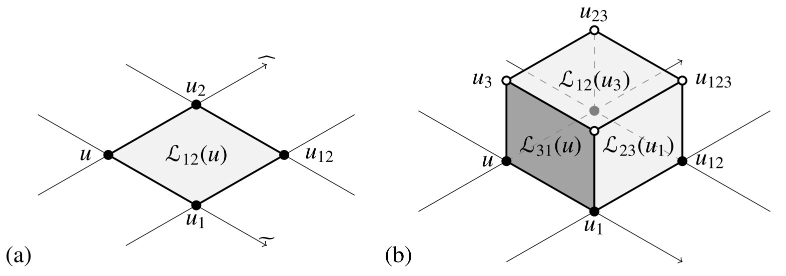


Fig. 11. A flat surface in (a), compared to a pop-up cube shown in (b).

The contribution to the action given by surface (a) is a single Lagrangian, $\mathcal{L}_{12}(u)$. In surface (b) we have five plaquettes, with a contribution to the action given by a sum of oriented Lagrangians: $\mathcal{S}_{pop}[u_{n,m}] = \mathcal{L}_{23}(u_1) + \mathcal{L}_{31}(u_2) + \mathcal{L}_{12}(u_3) - \mathcal{L}_{23}(u) - \mathcal{L}_{31}(u)$. Note that the orientations lead to the negative contributions. In our path integral perspective (66), in (b) we must also integrate over the "popped-up" variables $u_3, u_{23}, u_{31}, u_{123}$. The boundary variables on which the contributions depend are $u$, $u_1$, $u_2$ and $u_{12}$. So altogether, the contribution to the propagator for the pop-up cube is like this:

$$K_{pop} = \iiiint \mathrm{d}u_3 \mathrm{d}u_{31} \mathrm{d}u_{23} \mathrm{d}u_{123} \ \exp\left(\frac{i}{\hbar}\mathcal{S}_{pop}[u_{n,m}]\right) . \tag{68}$$

Now note that $\mathcal{S}_{pop}[u_{n,m}]$ contains no factor of $u_{123}$, so that the integral $\int \mathrm{d}u_{123}$ produces a volume factor $V$. Equation (68) can then be written in a matricial form:

$$\begin{aligned} K_{pop} = V \int \mathrm{d}^3\boldsymbol{u} \ \exp\frac{i}{\hbar}\Bigg(&\frac{1}{2}\boldsymbol{u}^T A\boldsymbol{u} + \boldsymbol{B}^t\boldsymbol{u} \\ &+ \frac{1}{2}\left[s_{31}(u_1^2 - u_{12}^2) + s_{23}(u_2^2 - u_{12}^2) + (u + u_{12})(u_1 - u_2)\right]\Bigg) , \end{aligned} \tag{69}$$

where $\boldsymbol{u}^T = (u_3, u_{31}, u_{23})$, $\boldsymbol{B}^T = (-s_{31}u_1 - s_{23}u_2, -u_1 + s_{23}u_{12}, u_2 + s_{31}u_{12})$, and

$$A = \begin{pmatrix} s_{23} + s_{31} & 1 & -1 \\ 1 & -(s_{12} + s_{23}) & s_{12} \\ -1 & s_{12} & -(s_{12} + s_{31}) \end{pmatrix} . \tag{70}$$

Now, in principle, equation (69) could be solved as a set of three Gaussian integrals, but matrix $A$ is in fact singular. The parameter identity for $s_{ij}$ (64):

$$s_{12}s_{23} + s_{23}s_{31} + s_{31}s_{12} + 1 = 0 , \tag{71}$$

leads to $\det A = 0$. We therefore resolve (69) by carrying out *two* Gaussian integrals, knowing for the third integration variable we shall be left with an exponent that is at most linear. Performing Gaussian integrations with respect to $u_3$ and $u_{31}$, we therefore have:

$$\begin{aligned} K_{pop} &= V\frac{2\pi\hbar}{s_{23}} \int \mathrm{d}u_{23} \exp\frac{i}{\hbar}\left(u(u_1 - u_2) - \tfrac{1}{2}s_{12}(u_1 - u_2)^2\right) \\ &= V^2\frac{2\pi\hbar}{s_{23}} \exp\left(\frac{i}{\hbar}\mathcal{L}_{12}(u, u_1, u_2)\right) , \end{aligned} \tag{72}$$

where in the first equality we note that all terms containing $u_{23}$ have vanished entirely. This is now *exactly* the exponent expected from the diagram (a) in Fig. 11. So, whilst it is clear that

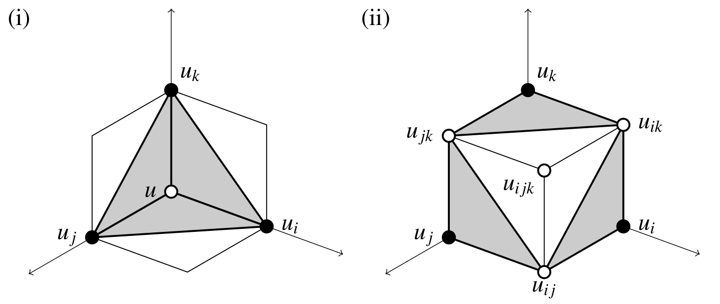


Fig. 12. Elementary move (a). We pass between (i) and (ii); white circles indicate variables to be integrated over in the move.

there are non-trivial issues to resolve with respect to volume factors and normalisation factors in (72),[6] in the critical issue of the contribution to the *action* in the exponent between diagrams 11(a) and 11(b), the two pictures make the *same* contribution. In other words, there is some sense in which the action is unchanged by the local move that transforms the surface $\sigma$ by the pop-up cube. Inspired by this discovery, we consider a more general situation.

### 5.2. Surface independence of the propagator

In the classical case, there are three *elementary configurations* of Lagrangians in three dimensions, that form the basis of all other possible configurations [12]. We can attach to these configurations three *elementary moves* in the quantum mechanical case that form the basis for deformations of the surface $\sigma$. These elementary moves are shown in Figs. 12, 13 and 14. Combined with the pop-up cube of Fig. 11 these give a full set of local moves for deforming the surface $\sigma$.

The first move is shown in Fig. 12. The action and contribution to the propagator (66) for Fig. 12(i) are given by:

$$\mathcal{S}_{(ai)} = \mathcal{L}_{ij}(u) + \mathcal{L}_{jk}(u) + \mathcal{L}_{ki}(u) , \quad K_{(ai)} = \mathcal{N}_{(ai)} \int \mathrm{d}u \exp\left[ i\mathcal{S}_{(ai)}/\hbar \right] . \tag{73}$$

In contrast, for Fig. 12(ii):

$$\begin{aligned} &\mathcal{S}_{(aii)} = \mathcal{L}_{ij}(u_k) + \mathcal{L}_{jk}(u_i) + \mathcal{L}_{ki}(u_j) , \\ &K_{(aii)} = \mathcal{N}_{(aii)} \iiiint \mathrm{d}u_{ij} \mathrm{d}u_{jk} \mathrm{d}u_{ki} \mathrm{d}u_{ijk} \exp\left[ i\mathcal{S}_{(aii)}/\hbar \right] . \end{aligned} \tag{74}$$

We have some issue in both of these cases with volume factors appearing in the evaluation; but we proceed under the assumption that these can be dealt with through some regularisation and normalisation. As shown in Appendix D, we then find that the *exponents* in $K_{(ai)}$ and $K_{(aii)}$ are the same. With the correct choice of normalisation and regularisation, we have identical contributions to the propagator.

We then consider elementary move (b), shown in Fig. 13. We have the action and propagator contribution for Fig. 13(i):

[6] The asymmetrical factor of $s_{23}$ in the prefactor is an indicator that renormalisation requires some careful thought.

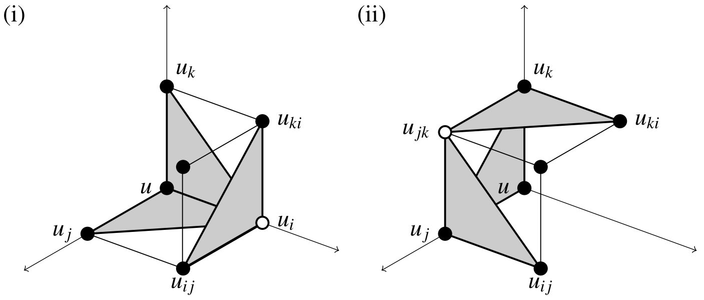


Fig. 13. Elementary move (b). White circles indicate integration variables.

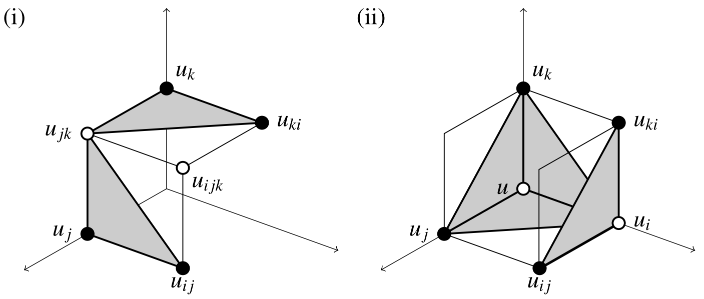


Fig. 14. Picture for elementary move (c).

$$\mathcal{S}_{(bi)} = \mathcal{L}_{ij}(u) + \mathcal{L}_{ki}(u) - \mathcal{L}_{jk}(u_i)\ , \quad K_{(bi)} = \mathcal{N}_{(bi)} \int \mathrm{d}u_i \exp\left[i\mathcal{S}_{(bi)}/\hbar\right]\ . \tag{75}$$

Similarly for Fig. 13(ii):

$$\mathcal{S}_{(bii)} = \mathcal{L}_{ij}(u_k) + \mathcal{L}_{ki}(u_j) - \mathcal{L}_{jk}(u)\ , \quad K_{(bii)} = \mathcal{N}_{(bii)} \int \mathrm{d}u_{jk} \exp\left[i\mathcal{S}_{(bii)}/\hbar\right]\ . \tag{76}$$

In this case, no volume factors appear and we find $K_{(bii)} = K_{(bi)}$. So the contributions to the propagator are directly identical here.

Lastly, consider elementary move (c) shown in Fig. 14. These bear a clear relation to Fig. 13: the element $\mathcal{L}_{jk}(u)$ has been shifted from one diagram to the other, inducing also a slight change in the integration variables. For 14(i):

$$\mathcal{S}_{(ci)} = \mathcal{L}_{ij}(u_k) + \mathcal{L}_{ki}(u_j)\ , \quad K_{(ci)} = \mathcal{N}_{(ci)} \iint \mathrm{d}u_{jk}\mathrm{d}u_{ijk} \exp\left[i\mathcal{S}_{(ci)}/\hbar\right]\ . \tag{77}$$

Similarly, 14(ii) is derived from 13(i) with an additional integral over $u$.

$$\begin{aligned} &\mathcal{S}_{(cii)} = \mathcal{L}_{ij}(u) + \mathcal{L}_{jk}(u) + \mathcal{L}_{ki}(u) - \mathcal{L}_{jk}(u_i)\ , \\ &K_{(cii)} = \mathcal{N}_{(cii)} \iint \mathrm{d}u\mathrm{d}u_i\ \exp\left[i\mathcal{S}_{(cii)}/\hbar\right]\ . \end{aligned} \tag{78}$$

Once more we find that $K_{(cii)} = K_{(ci)}$ (although this time a volume factor is involved on both sides) and the contributions to the propagator are the same.

**Proposition 3.** *The system characterised by Lagrangian (64) is* independent *of the choice of surface $\sigma$, up to the choice of normalising constants.*

**Proof.** The combination of elementary moves above, combined with the pop-up of Fig. 11, allows us to deform any surface $\sigma$ to another topologically equivalent surface $\sigma'$ by a series of elementary moves, without changing the exponent in the propagator. This free deformation gives us independence from the surface. □

An obvious consequence is that the propagator (66) depends only on the surface boundary $\partial\sigma$, and the field variables specified there - i.e. it is a function only of the boundary value problem. Note that since different topologies are specified by changes of the boundary, we have not considered these explicitly.

### 5.3. Uniqueness

The Lagrangian (64) has the property that it produces a propagator (66) which is independent of variations of the surface $\sigma$. In fact, it turns out that (64) is the unique quadratic Lagrangian 2-form such that this holds. Consider a general, 3-point, quadratic Lagrangian, imposing anti-symmetry under interchange of $i$ and $j$:

$$\mathcal{L}_{ij}(u,u_i,u_j) = \tfrac{1}{2}a_{ij}u^2 + \tfrac{1}{2}b_{ij}u_i^2 - \tfrac{1}{2}b_{ji}u_j^2 + c_{ij}uu_i - c_{ji}uu_j + d_{ij}u_iu_j \ . \tag{79}$$

For coefficients, a subscript $i$ indicates dependence on the lattice parameter $p_i$, with the ordering of subscripts important. The 2-form structure requires $a_{ji} = -a_{ij}$, $d_{ji} = -d_{ij}$ ($a_{ij}$ and $d_{ij}$ are anti-symmetric under interchange of the parameters). Our interest is in the subset of Lagrangians that display the surface independence property in the propagator. We therefore look for conditions on the Lagrangian such that elementary moves will leave the contribution to the action (i.e. the exponent in the propagator) unchanged. We assume that external factors and even volume factors can be resolved by renormalisation, so that we only consider that part of the propagator in the exponent.

Consider (79) under elementary move (a) - shown in Fig. 12. The contributions to the propagator, $K_{(ai)}$ and $K_{(aii)}$, are calculated according to (73) and (74). For surface independence, we require $K_{(ai)} = K_{(aii)}$.

$K_{(ai)}$ is calculated via an integral d$u$, as in (73). In general, the coefficient of $u$ in the exponent may be either quadratic, linear, or zero: yielding a Gaussian integral, Dirac delta function, or volume factor, respectively. However, a Dirac delta function would force linear dependence of field variables at different lattice points: since this is undesirable, we exclude this possibility. The remaining cases divide on the totally antisymmetric coefficient $\mathfrak{a}_{ijk} := a_{ij} + a_{jk} + a_{ki}$ (see Appendix E.1 for details). For $\mathfrak{a}_{ijk} \neq 0$ we have a Gaussian integral, and:

$$\begin{aligned} K_{(ai),G} = \left(\frac{2\pi i\hbar}{\mathfrak{a}_{ijk}}\right)^{1/2} \exp\frac{i}{\hbar}\Bigg[&\frac{1}{2}\Big(b_{ij} - b_{ik} - \frac{1}{\mathfrak{a}_{ijk}}(c_{ij}-c_{ik})^2\Big)u_i^2 + cyclic \\ &+ \Big(d_{ij} - \frac{1}{\mathfrak{a}_{ijk}}(c_{ij}-c_{ik})(c_{jk}-c_{ji})\Big)u_iu_j + cyclic\Bigg] . \end{aligned} \tag{80}$$

Conversely, for $\mathfrak{a}_{ijk} = 0$, we require the integral to reduce to a volume factor (linear coefficients of $u$ in the exponent must disappear) requiring the conditions

$$a_{ij} = a_i - a_j \ , \quad c_{ij} = c_i \ . \tag{81}$$

(the coefficient $a_{ij}$ must separate into a part depending on $p_i$ and a part depending on $p_j$ and $c_{ij}$ is a function of $p_i$ only). Under these conditions,

$$K_{(ai),V} = V \ \exp \frac{i}{\hbar} \left[ \tfrac{1}{2}(b_{ij} - b_{ik})u_i^2 + cyclic + d_{ij}u_iu_j + cyclic \right] . \tag{82}$$

This is a critical point of the variation - a volume factor appears uniquely for this special choice of Lagrangian, which can be written as:

$$\begin{aligned} \mathcal{L}_{ij}(u, u_i, u_j) &= \tfrac{1}{2}a_i u^2 + c_i u u_i - \tfrac{1}{2}a_j u^2 - c_j u u_j + \tfrac{1}{2}(b_{ij}u_i^2 - b_{ji}u_j^2) + d_{ij}u_iu_j \ , \\ &= A_i(u, u_i) - A_j(u, u_j) + C_{ij}(u_i, u_j) \ , \end{aligned} \tag{83}$$

with $C_{ij}(u_i, u_j)$ antisymmetric under interchange of $i$ and $j$. This is the most general classical Lagrangian 2-form (7) as found in [12], here specialised to the quadratic case. So we have two cases for $K_{(ai)}$: (82) when $\mathfrak{a}_{ijk} = 0$, and (80) when $\mathfrak{a}_{ijk} \neq 0$.

For $K_{(aii)}$, as in (74), we have four integrations $\mathrm{d}u_{ij}\mathrm{d}u_{jk}\mathrm{d}u_{ki}\mathrm{d}u_{ijk}$. The integral $\mathrm{d}u_{ijk}$ always produces a volume factor due to the three-point form of the Lagrangian. As for $K_{(ai)}$, we wish to avoid these integrals reducing to a Dirac delta function, and so we have 2 cases. The remaining integrals are either evaluated as three Gaussian integrations, or one integration reduces to a volume factor. This rests on the value of $\det A$ (see Appendix E.2 for details):

$$A = \begin{pmatrix} b_{jk} - b_{ik} & d_{ki} & d_{jk} \\ d_{ki} & b_{ki} - b_{ji} & d_{ij} \\ d_{jk} & d_{ij} & b_{ij} - b_{kj} \end{pmatrix} . \tag{84}$$

For $\det A \neq 0$ (equivalently $b_{ij} \neq -d_{ij}$) we have three Gaussian integrations, producing:

$$K_{(aii),G} = V \sqrt{\frac{(2\pi i \hbar)^3}{\det A}} \exp\left( -\frac{i}{2\hbar} \boldsymbol{B}^T A^{-1} \boldsymbol{B} \right) \exp \frac{i}{\hbar} \left( \frac{1}{2} a_{jk} u_i^2 + cyclic \right) , \tag{85}$$

where $\boldsymbol{B}^T = \big(c_{jk}u_i - c_{ik}u_j, \text{perm } (ijk), \text{perm } (kji)\big)$. Alternatively, when $\det A = 0$, evaluating $K_{(aii)}$ requires two Gaussian integrations. We then require linear terms in the third integrand to disappear in order to prohibit the appearance of a Dirac delta function (see Appendix E.2) hence we require the conditions

$$b_{ij} = -d_{ij} \ , \quad c_{ij} = c_{ji} \qquad \forall i, j \ . \tag{86}$$

So, $b_{ij}$ is also anti-symmetric, and $c_{ij}$ symmetric. We can then evaluate $K_{(aii)}$ as:

$$\begin{aligned} K_{(aii),V} = &\frac{2\pi\hbar}{(1 - \Lambda_{ijk})^{1/2}} V^2 \\ &\times \exp \frac{i}{\hbar} \left[ \frac{1}{2} a_{jk} u_i^2 + cyclic - \frac{1}{2} \frac{d_{ij}}{1 - \Lambda_{ijk}} (c_{jk}u_i - c_{ki}u_j)^2 + cyclic \right] , \end{aligned} \tag{87}$$

where we have introduced the totally symmetric parameter

$$\Lambda_{ijk} := d_{ij}d_{jk} + d_{jk}d_{ki} + d_{ki}d_{ij} + 1 \ . \tag{88}$$

Once more there are two cases. For $K_{(aii)}$, when $\det A = 0$, we find (87), and when $\det A \neq 0$ we have (85).

Comparing now the two configurations of the elementary move, we demand that the exponents from each configuration be the same; i.e. both make the same contribution to the propagator. More details of this comparison are given in Appendix E.3. We find a solution to the problem at the critical point of the system: where some of our integrals become singular. Allowing $\mathfrak{a}_{ijk} = 0$ and $\det A = 0$, we compare the exponent in (82) with (87). Recalling that at this critical point we have also the conditions (81), (86), we find that we require $c_{ij} = c$, constant, $\Lambda_{ijk} = 1 - c^2$, $a_{ij} = 0$. Finally, since our Lagrangian is defined only up to an overall multiple, we let $c = 1$. We therefore find the unique quadratic Lagrangian:

$$\mathcal{L}_{ij}(u, u_i, u_j) = u(u_i - u_j) - \tfrac{1}{2} d_{ij}(u_i - u_j)^2 , \tag{89}$$

along with the condition on $d_{ij}$ that $\Lambda_{ijk} = 0$. Comparing (88) with (71) we see that we require precisely $d_{ij} = s_{ij}$. But then (89) is uniquely the Lagrangian (64). We already know from section 5.2 that this Lagrangian also exhibits surface independence for the other elementary moves. This principle of surface independence is then sufficient to determine the admissible Lagrangian even more uniquely than in the classical case (9) as was treated in [12]. We mention also that in the classical case a classification of quadratic, so-called pluri-Lagrangian systems, was given in [49] which parallels the multiform variational approach.

**Proposition 4.** *The Lagrangian (64) is the unique quadratic Lagrangian 2-form yielding a surface independent propagator (66).*

**Proof.** (89), with the restriction $\Lambda_{ijk} = 0$ (88), gives us that this is the unique Lagrangian exhibiting surface independence for elementary move (a). We also have from Proposition 3 that Lagrangian (64) has surface independence under all other elementary moves.

### 5.4. *Quantum variational principle: Lagrangian 2-form case*

This result suggests a quantum variational principle in analogy to the one dimensional case of section 4.4. We consider the propagator over a discrete surface $\sigma$, $K_\sigma(\partial\sigma)$, defined in (66). We have shown that, for the special choice of Lagrangian (64), the propagator $K_\sigma(\partial\sigma)$ is *independent* of the surface $\sigma$. It depends only on the variables sitting on the boundary, $\partial\sigma$. Additionally, this is a very unique choice of Lagrangian: for a generic Lagrangian, $K_\sigma(\partial\sigma)$ will depend also on the surface $\sigma$ itself.

Recall that, classically, the Lagrangian 2-form structure arises from a variational principle *over surfaces* as in [12]. An extended set of Euler-Lagrange equations arise as we vary not only the dependent field variables $u_{\boldsymbol{n}}$, but also the surface $\sigma$. This restricts the class of admissible Lagrangians to those obeying the closure property (6): it is only for such Lagrangians and equations of motion that the classical action remains invariant under variations of the surface.

As we move to the quantisation, parallel to what we argued in the 1-form case, we consider the variation over all possible surfaces $\sigma$ with a fixed boundary $\partial\sigma$. For a generic Lagrangian, as we vary the surface $\sigma$ the propagator $K_\sigma(\partial\sigma)$ (66) changes. However, for the special "integrable" choice of Lagrangians (64) the propagator $K_\sigma(\partial\sigma)$ remains unchanged as we vary the surface. This therefore represents a critical (i.e., singular) point for a new quantum object which we conjecture to be a "sum over all surfaces" of which the surface-dependent propagator forms the summand,[7] viewed as a functional in a well-chosen space of Lagrange functions. Once again, controlling the singular behaviour of such an object, and arriving at mathematically concise definition is the subject of ongoing investigation. Nonetheless, we conjecture that critical/singular point analysis of such an object, leading to the selection of Lagrangians whose propagator are

[7] The sum over surfaces idea has also emerged in the theory of loop quantum gravity but with a different motivation, cf. [50,51].

surface-independent, would form a key ingredient for understanding the path integral quantisation of discrete field theories that are integrable in the sense of multidimensional consistency.

## 6. Discussion

In his seminal paper of 1933, [38], Paul Dirac expressed his *credo* that the Lagrangian formulation of classical dynamics, in comparison to the Hamiltonian one, was more fundamental, and he posed the question of a Lagrangian approach to quantum mechanics. In this important precursor to Feynman's development of the path integral [39] the analogy between classical and quantum mechanics was emphasized, cf. also [52]. In this context, the related question of what would constitute a variational point of view in quantum mechanics was partly, but not fully, answered by those approaches. In the present paper we have attempted to arrive to a more complete answer to these questions in the context of integrable systems in the sense of multidimensional consistency. This is pursued by setting up a quantum analogue of the Lagrangian multi-form approach.

We emphasize once more that so far, within the context of integrable systems theory, the Hamiltonian approach was largely favoured while the Lagrangian approach was largely ignored. The reason was obvious: the conventional Lagrangian approach seemed unsuitable to capture the fundamental aspect of multidimensional consistency. With the introduction of the concept of Lagrangian multiforms in [5] that drawback of the Lagrangian approach was overcome, and a variational theory of multidimensional consistency is now in full development. This poses then also the scientific imperative to develop alongside this new classical theory, a corresponding quantum theory, and the present paper forms the very first step in that direction. Although the theory seems to apply merely to the context of integrable systems, we believe that, since such models have turned out to be quite universal and fundamental, the development of a Lagrangian quantum theory of integrable systems will provide novel insights into the nature of quantum mechanics (e.g. using integrable models, it may be feasible to investigate more thoroughly the analytic aspects of the path integral measure for models beyond the quadratic ones in the momenta).

In the present study we focused on the simplest possible model exhibiting the MDC property, the linearised lattice KdV system, leading to a quadratic Lagrangian multiform structure for both the reduced case (Lagrangian 1-form) and the full lattice case (Lagrangian 2-form). This choice of model was primarily motivated by the ability to perform the corresponding Gaussian integrals, to assert the path- respectively surface independence of the corresponding Feynman propagators, which constitutes the quantum analogue of the closure property of the Lagrangian multiform structure in the classical case of [5]. This forms one of the main postulates of a quantum multiform theory. Further aspects of the theory, e.g. the derivation of multi-time Schrödinger equations for the propagators, will be pursued in subsequent work.

There are a number of further points to make in connection with the results obtained in this study.

First, although the results were obtained by restricting ourselves to only quadratic Lagrangians, the multidimensional consistency aspects do not essentially rely on the linearity of the equations. In fact, most of the combinatorics at the classical level carries through for all Lagrangians associated with nonlinear quad equations in the ABS list, cf. [8]. Due to the suspected close analogy between classical theory and quantum theory in the integrable case, it is therefore to be expected that some quantization procedure for those models would exist such that the results obtained here also carry through to the quantum level for those nonlinear models. This may, however, require non-conventional quantization prescriptions in terms of suitable integrals replacing the Gaussian integrals used in the quadratic case. Initial results along this direction were obtained in [22] and [21]. The choice of Hilbert space (in the canonical quantization picture), and of integration measure (in the path integral picture) may be driven by the integrable combinatorics of those models. An alternative approach would be to proceed along the route of [53,54] and to consider the introduction of stochasticity in the quadratic models in order to quantize the discrete-time models. However, although there are similarities in some of the discretization aspects with the present studies (effectively the role of Bäcklund transforms in the constructions), it seems that the point of view in those studies is quite different from the one developed in the present paper in particular w.r.t. the multi-time aspects. We also note that although we invoked at some points the standard connection with the canonical formalism in terms of operators on Hilbert space, the Feynman ideology in our view should enable one to move away from the operator approach altogether and compute the quantum objects purely in terms of the Lagrangian formalism. Our hope is that the multiform structure would eventually provide tools to perform such computations.

Second, another general feature of the models in question is the role-reversal interplay between parameters and independent variables and between the discrete and continuous models. Thus, the continuous models do not only appear as continuum limits, but more intrinsically as additional commuting flows: the classical equations hold simultaneously on a common set of solutions. On the quantum level this property extends in the fact that there is a common propagator of the underlying continuous and discrete quadratic models. If this feature is general enough to extend to the nonlinear case (which it does in the classical case) there is scope that this property can eventually be used to extract information on the time-sliced path integral from the discrete finite-step path integral.

Third, turning things around and imposing the path and surface independence of the propagator for a general parameter class of quadratic Lagrangians, we have shown that this quantum MDC property leads uniquely to the Lagrangians that arise from the integrable case, in the same spirit as in [12]. In fact, the point made in that paper is that the Lagrangians themselves should be viewed as solutions of an extended set of Euler-Lagrange equations, which incorporates the stationarity under variations with respect to both the field (i.e., dependent) variables as well as the geometry in the independent variables. This poses a new paradigm in variational calculus, as it signifies a departure from the conventional point of view of most physical theories, namely that Lagrangians have to be chosen based on tertiary considerations. In this new point of view, the Lagrangians are not necessarily given in advance, but follow from the variational principle itself.

We finish by making a few general remarks on further ramifications. In general it is not known how to derive a path integral formalism for non-conventional, i.e. non-Newtonian models, through a time-slicing procedure when Gaussian integrals no longer apply. Nonetheless, in integrable systems theories such non-Newtonian models do abundantly appear and often can also be readily quantized through the canonical formalism, e.g. the relativistic many-body systems of Ruijsenaars-Schneider type, [3]. This poses, in our view, a lacuna in the theory which is imperative to rectify as such integrable quantum systems cannot be simply discarded as potentially physical models. Thus, integrable systems can play a role of a litmus test for the completeness of a theory, which most reasonably should be applicable to those models for which in principle exact and rigorous computations can be performed. However, one may speculate that there is a deeper significance for those systems, since they have proved their merit in forming a fruitful breeding ground for new concepts and new understandings on a fundamental level. In fact, the ideas exposed in the present paper, based on simple toy problems, have some interesting resemblances to proposals that that in recent years have been put forward on the quantization of scaling invariant theories [42,43,55]. A particular parallel may be drawn between path and surface independence of propagators in our examples, and certain formulations of loop quantum gravity and "sum over surfaces", [50,51]. Furthermore, the interplay between discrete and continuous, which is prominent in our examples, may perhaps feed into views that G.'t Hooft has been promoting with regard to the quantum nature of the universe, cf. [56].

**Acknowledgements**

Steven King would like to thank EPSRC for funding this research. Frank Nijhoff was partly supported by EPSRC grant EP/I038683/1. FWN acknowledges early discussions with Sikarin Yoo-Kong and Chris Field on related topics.

## Appendix A. Calculating the discrete propagator

### *A.1. The classical action*

We wish to evaluate the classical action for the path beginning at $x_0$, and reaching $x_N$ after $N$ time steps. Recalling our discrete equation of motion (11) and classical solution we have the classical path

$$x_n = \frac{1}{\sin \mu N}\left(x_N \sin \mu N - x_0 \sin \mu (n-N)\right) . \tag{A.1}$$

The action along the classical path is then:

$$\begin{aligned} \mathcal{S}_{cl} &= \sum_{n=0}^{N-1}\left(\frac{P+Q}{q} x_n x_{n+1} + \frac{P-Q}{2q}(x_n^2 + x_{n+1}^2)\right) \\ &= \frac{\sqrt{P}}{\sin \mu N}\left[2x_0 x_N - (x_0^2 + x_N^2)\cos \mu N\right] , \end{aligned} \tag{A.2}$$

where we have used the identities:

$$\cos \mu = -b = -\frac{P-Q}{P+Q} , \quad \sin \mu = \frac{2q\sqrt{P}}{P+Q} . \tag{A.3}$$

We note two things about this result. First, there is no explicit $Q$ dependence: all $Q$ dependence is contained within the parameter $\mu$, which only appears as $\mu N$. Second, we can easily extend this result to the $\mathcal{L}_a$ (bar evolution) case, by a change of parameter. We replace $\mu$ by $\eta$, such that $\cos \eta = -a$.

### *A.2. The discrete propagator*

It is left for us to evaluate the discrete path integral:

$$\tilde{K}_N(0,0) = \int_{y(0)=0}^{y(N)=0} \mathcal{D}[y_n]\, e^{i\mathcal{S}[y_n]/\hbar} . \tag{A.4}$$

In the discrete case, we can consider this via a time slicing procedure without needing to worry about the problematic shrinking to zero. So we consider:

$$\tilde{K}_N(0,0) = \mathcal{N} \int \mathrm{d}y_1 \ldots \int \mathrm{d}y_{N-1} \times \exp\left\{ \frac{i}{\hbar q} \sum_{n=0}^{N-1} \left( (P+Q) y_n y_{n+1} + \frac{1}{2}(P-Q)(y_n^2 + y_{n+1}^2) \right) \right\} , \tag{A.5}$$

where $\mathcal{N}$ is the normalising factor appearing in (45) and with the boundary values $y_0 = y_N = 0$. This expression is quadratic in all $y_n$ variables, and so can be evaluated as $N-1$ Gaussian integrals. This is most easily achieved by writing the equation in a matrix form (as in [41], for example). We define $\mathbf{y}^T = (y_1, \ldots, y_{N-1})$, in order to write

$$\tilde{K}_N = \mathcal{N} \int \mathrm{d}^{N-1}\mathbf{y}\ \exp(-\mathbf{y}^T \sigma \mathbf{y}) = \frac{\pi^{(N-1)/2}}{\sqrt{\det \sigma}} , \tag{A.6}$$

with $\sigma$ the symmetric, tri-diagonal matrix:

$$\sigma = \frac{i(P+Q)}{\hbar q} \begin{pmatrix} -\frac{P-Q}{P+Q} & -1/2 & & \\ -1/2 & -\frac{P-Q}{P+Q} & \ddots & \\ & \ddots & \ddots & -1/2 \\ & & -1/2 & -\frac{P-Q}{P+Q} \end{pmatrix} . \tag{A.7}$$

In the case that all parameters $P$, $Q$ and $q$ are real valued, the result (A.6) (where the branch of the square root is understood to be taken in accordance with the sign of $(P-Q)/q$) is a consequence of standard Fresnel type integrals:

$$\int_{-\infty}^{\infty} e^{i\alpha x^2} \mathrm{d}x = \sqrt{\frac{\pi}{|\alpha|}}\, e^{\mathrm{sgn}(\alpha)\pi i/4} ,$$

cf. also [41] in the context of path integrals. (We note that the choice of these parameters allow us to move into imaginary time if required.) From (A.6) onward, it remains to calculate $\det \sigma$. The determinant for a tri-diagonal matrix can be found by forming a recursion relation on the size of the matrix, and solving as a discrete equation. Let

$$X_n = \begin{vmatrix} a & b & & \\ b & a & \ddots & \\ & \ddots & \ddots & b \\ & & b & a \end{vmatrix} \quad \text{of size } n . \tag{A.8}$$

Performing the cofactor expansion, we find

$$X_n = a X_{n-1} - b^2 X_{n-2} , \quad n \geq 2 , \tag{A.9}$$

with initial conditions $X_0 = 1$ and $X_1 = a$. The solution is thus given by

$$X_n = \frac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-} , \quad \text{where} \quad \lambda_\pm = \frac{1}{2}\left( a \pm \sqrt{a^2 - 4b^2} \right) , \tag{A.10}$$

are the roots of the characteristic equation of (A.9). Now, in the case of $\sigma$, recall that $a = -(P-Q)/(P+Q) = \cos\mu$ and $b = -1/2$, so that $\sqrt{a^2-4b^2} = i\sin\mu$: this leads to significant simplifications of the above expression. Working through these calculations, we then find:

$$\det\sigma = \left(\frac{i(P+Q)}{2\hbar q}\right)^{N-1}\frac{\sin\mu N}{\sin\mu}\ . \tag{A.11}$$

Putting this together, then,

$$\tilde{K}_N = \left(\frac{i(P+Q)}{2\pi\hbar q}\right)^{N/2}\left(\frac{2\pi\hbar q}{i(P+Q)}\right)^{(N-1)/2}\sqrt{\frac{\sin\mu}{\sin\mu N}}\ , \tag{A.12}$$

and therefore

$$K_N(x_0,x_N) = \left(\frac{i\sqrt{P}}{\pi\hbar\sin\mu N}\right)^{1/2} \times \exp\left\{\frac{i\sqrt{P}}{\hbar\sin\mu N}\left(2x_0x_N - (x_0^2+x_N^2)\cos\mu N\right)\right\}\ . \tag{A.13}$$

## Appendix B. Quantum invariants

In [21], the authors investigated quantum systems possessing invariants under a one time-step path integral evolution. Begin by considering the evolution in the hat direction, generated by $\mathcal{L}_b(x,\widehat{x})$ (24). A wavefunction $\psi_n(x)$ evolves under this transformation according to

$$\psi_{n+1}(\widehat{x}) = \mathcal{N}\int_C \exp\left(\frac{i}{\hbar}\mathcal{L}_b(x,\widehat{x})\right)\psi_n(x)\mathrm{d}x\ , \tag{B.1}$$

and to look for an invariant we desire $\psi_n$ and $\psi_{n+1}$ to be solutions of the same eigenvalue problem, with the same eigenvalue:

$$M_x\psi_n(x) = E\psi_n(x) \quad\Rightarrow\quad M_{\widehat{x}}\psi_{n+1}(\widehat{x}) = E\psi_{n+1}(\widehat{x})\ . \tag{B.2}$$

$M_x$ is a differential operator, and we restrict to considering the second order case:

$$M_x = p_0(x)\frac{\partial^2}{\partial x^2} + p_1(x)\frac{\partial}{\partial x} + p_2(x)\ . \tag{B.3}$$

Now,

$$\begin{aligned} E\psi_{n+1}(\widehat{x}) &= \mathcal{N}\int_C \exp\left(\frac{i}{\hbar}\mathcal{L}_b(x,\widehat{x})\right)(M_x\psi_n(x))\,\mathrm{d}x \\ &= \mathcal{N}\int_C \left(\overline{M}_x\exp\left(\frac{i}{\hbar}\mathcal{L}_b(x,\widehat{x})\right)\right)\psi_n(x)\mathrm{d}x\ + \mathcal{S}\ , \end{aligned} \tag{B.4}$$

where $\overline{M}_x$ is an adjoint to $M_x$ constructed under integrations by parts, and $\mathcal{S}$ is the resulting surface term. If we assume $\psi_n$ and $\psi_n'$ to vanish at infinity (a reasonable physical assumption) then the surface term $\mathcal{S}$ vanishes. We can also write,

$$E\psi_{n+1}(\widehat{x}) = M_{\widehat{x}}\psi_{n+1}(\widehat{x}) \; = \mathcal{N}\int_C \left( M_{\widehat{x}} \exp\left(\frac{i}{\hbar}\mathcal{L}_b(x,\widehat{x})\right)\right)\psi_n(x)\,\mathrm{d}x \; . \tag{B.5}$$

So the condition we require is for $\overline{M}_x \exp\left(\frac{i}{\hbar}\mathcal{L}_b(x,\widehat{x})\right) = M_{\widehat{x}} \exp\left(\frac{i}{\hbar}\mathcal{L}_b(x,\widehat{x})\right)$. Following the analysis in [21], and using the given Lagrangian, we find this can only hold under the restrictions:

$$p_0(x) = -\hbar^2 C_0 \; , \quad p_1(x) \equiv 0 \; , \quad p_2(x) = 4PC_0x^2 + C_2 \; , \tag{B.6}$$

so that

$$M_x = C_0\left(-\hbar^2\frac{\partial^2}{\partial x^2} + 4Px^2\right) + C_2 \; . \tag{B.7}$$

This is precisely the quantum invariant (48).

## Appendix C. Path independence for a general Lagrangian

We calculate the propagators (52) and (53) by a Gaussian integral:

$$K_\lrcorner(x,\widehat{\widehat{x}}) = \mathcal{N}_\lrcorner\left(\frac{\pi i\hbar}{\beta b_0 + \alpha(a-a_0)}\right)^{1/2} \exp\left\{\frac{i}{\hbar}\left[\left(\beta(b-b_0) - \frac{\beta^2}{4(\beta b_0 + \alpha(a-a_0))}\right)x^2 + \left(\alpha a_0 - \frac{\alpha^2}{4(\beta b_0+\alpha(a-a_0))}\widehat{\widehat{x}}^2\right) - \frac{\alpha\beta}{2(\beta b_0+\alpha(a-a_0))}x\widehat{\widehat{x}}\right]\right\} , \tag{C.1}$$

and,

$$K_\ulcorner(x,\widehat{\widehat{x}}) = \mathcal{N}_\ulcorner\left(\frac{\pi i\hbar}{\alpha a_0 + \beta(b-b_0)}\right)^{1/2} \exp\left\{\frac{i}{\hbar}\left[\left(\alpha(a-a_0) - \frac{\alpha^2}{4(\alpha a_0 + \beta(b-b_0))}\right)x^2 + \left(\beta b_0 - \frac{\beta^2}{4(\alpha a_0+\beta(b-b_0))}\widehat{\widehat{x}}^2\right) - \frac{\alpha\beta}{2(\alpha a_0+\beta(b-b_0))}x\widehat{\widehat{x}}\right]\right\} . \tag{C.2}$$

By comparing the coefficients of $x^2$, $\widehat{\widehat{x}}^2$ and $x\widehat{\widehat{x}}$ in the exponent, we derive conditions for time-path-independence on our coefficients:

$$\beta(b-b_0) - \frac{\beta^2}{4(\beta b_0+\alpha(a-a_0))} = \alpha(a-a_0) - \frac{\alpha^2}{4(\alpha a_0+\beta(b-b_0))} \; , \tag{C.3}$$

$$\alpha a_0 - \frac{\alpha^2}{4(\beta b_0+\alpha(a-a_0))} = \beta b_0 - \frac{\beta^2}{4(\alpha a_0+\beta(b-b_0))} \; , \tag{C.4}$$

$$\frac{\alpha\beta}{2(\beta b_0+\alpha(a-a_0))} = \frac{\alpha\beta}{2(\alpha a_0+\beta(b-b_0))} \; . \tag{C.5}$$

Note that an immediate consequence of (C.5) is that the multiplicative factors in (C.1) and (C.2) are the same. Analysis of these three conditions leads to (63).

## Appendix D. Elementary moves

We consider elementary move (a), shown in Fig. 12, in more detail as an illustrative case. The action and contributions to the propagator for Figs. 12(i) and (ii) are given in (73) and (74). We then have

$$
\begin{aligned}
K_{(ai)} &= \int \mathrm{d}u \exp \frac{i}{\hbar}\Big( u(u_i - u_j) - \tfrac{1}{2}s_{ij}(u_i-u_j)^2 \\
&\quad + u(u_j-u_k) - \tfrac{1}{2}s_{jk}(u_j-u_k)^2 + u(u_k-u_i) - \tfrac{1}{2}s_{ki}(u_k-u_i)^2\Big)\,, \\
&= V \exp \frac{-i}{2\hbar}\Big( s_{ij}(u_i-u_j)^2 + s_{jk}(u_j-u_k)^2 + s_{ki}(u_k-u_i)^2\Big)\,,
\end{aligned}
\tag{D.1}
$$

where we note that all the $u$ terms have cancelled out, leaving a volume factor. We compare this to

$$
\begin{aligned}
K_{(aii)} &= \iiiint \mathrm{d}u_{ij}\mathrm{d}u_{jk}\mathrm{d}u_{ki}\mathrm{d}_{ijk} \exp \frac{i}{\hbar}\Big( u_k(u_{ki} - u_{jk}) - \tfrac{1}{2}s_{ij}(u_{ki}-u_{jk})^2 \\
&\quad + u_i(u_{ij}-u_{ki}) - \tfrac{1}{2}s_{jk}(u_{ij}-u_{ki})^2 + u_j(u_{jk}-u_{ij}) - \tfrac{1}{2}s_{ki}(u_{jk}-u_{ij})^2\Big)\,, \\
&= V \int \mathrm{d}^3\boldsymbol{u}\ \exp \frac{i}{\hbar}\left( -\frac{1}{2}\boldsymbol{u}^T A\boldsymbol{u} + \boldsymbol{B}^t\boldsymbol{u}\right)\,,
\end{aligned}
\tag{D.2}
$$

where

$$
\begin{aligned}
\boldsymbol{u}^T &= (u_{ij}, u_{jk}, u_{ki})\,, \\
A &= \begin{pmatrix} s_{jk}+s_{ki} & -s_{ki} & -s_{jk} \\ -s_{ki} & s_{ki}+s_{ij} & -s_{ij} \\ -s_{jk} & -s_{ij} & s_{ij}+s_{jk}) \end{pmatrix}\,, \\
\boldsymbol{B}^T &= \big(u_i-u_j, u_j-u_k, u_k-u_i\big)\,.
\end{aligned}
\tag{D.3}
$$

Critically, we note that $\det A = 0$, so again we have a singular integral. Carrying out two integrals in turn, so that the third integration produces a volume factor, we therefore have:

$$
K_{(aii)} = V^2 2\pi\hbar\ \exp \frac{-i}{2\hbar}\Big( s_{ij}(u_i-u_j)^2 + s_{jk}(u_j-u_k)^2 + s_{ki}(u_k-u_i)^2\Big)\,. \tag{D.4}
$$

Thus, the *exponents* in $K_{(ai)}$ and $K_{(aii)}$ are the same. With the correct choice of normalisation and regularisation, we have identical contributions to the propagator.

## Appendix E. Uniqueness of the surface independent Lagrangian

### *E.1. Elementary move (a), configuration (i)*

For Lagrangian (79), the expression for $K_{(ai)}$ is shown in Fig. 12 and given by (73). We then have:

$$K_{(ai)} = \int \mathrm{d}u \exp \frac{i}{\hbar}\Big[\tfrac{1}{2}(a_{ij}+a_{jk}+a_{ki})u^2 + \big((c_{ij}-c_{ik})u_i + (c_{jk}-c_{ji})u_j + (c_{ki}-c_{kj})u_k\big)u\Big] \times \exp \frac{i}{\hbar}\Big[\tfrac{1}{2}(b_{ij}-b_{ik})u_i^2 + cyclic + d_{ij}u_iu_j + cyclic\Big]. \tag{E.1}$$

This integral is Gaussian providing the coefficient of $u^2$ does not vanish; i.e. $\mathfrak{a}_{ijk} \neq 0$. In that case the integral yields (80).

The other case occurs when $\mathfrak{a}_{ijk} = 0 \;\Rightarrow\; a_{ij} = a_i - a_j$. To avoid the integral producing a delta function (which would threaten the independence of our field variables) we then also require terms linear in $u$ to vanish, so that $c_{ij} - c_{ik} = 0\ \forall i, j, k \;\Rightarrow\; c_{ij} = c_i$. In other words, $c_{ij}$ must be a function of $p_i$ only. These are precisely the conditions (81). If these conditions hold, we are left with the contribution to the propagator (82).

### *E.2. Elementary move (a), configuration (ii)*

For $K_{(aii)}$ in Fig. 12(ii), we have a contribution to the propagator given by (74). For Lagrangian (79) this gives us:

$$K_{(aii)} = V \iiint \mathrm{d}^3\boldsymbol{u}\ \exp \frac{i}{\hbar}\left(\frac{1}{2}\boldsymbol{u}^T A\boldsymbol{u} + \boldsymbol{B}^t\boldsymbol{u}\right) \exp \frac{i}{\hbar}\left[\frac{1}{2}(a_{jk}u_i^2 + cyclic)\right], \tag{E.2}$$

with $A$ and $\boldsymbol{B}$ as in (84) and (85), and $\boldsymbol{u}^T = (u_{ij}, u_{jk}, u_{ki})$. Clearly, when $\det A \neq 0$ this can be evaluated as a trio of Gaussian integrals, giving (85). We must consider the critical point $\det A = 0$ separately. The condition $\det A = 0$ is a *functional equation* connecting the $b_{ij}$ with the $d_{ij}$. Considering the rows of $A$ in (84), it is clear that $\det A = 0$ if $b_{ij} = -d_{ij}$, our first condition of (86). In this case we must carry out the two remaining Gaussian integrals in turn. First, integrating over $\mathrm{d}u_{ij}$ in (E.2):

$$K_{(aii),V} = V\left(\frac{2\pi\hbar}{i(d_{jk}+d_{ki})}\right)^{1/2} \iint \mathrm{d}u_{jk}\mathrm{d}u_{ki}\ \exp\frac{i}{\hbar}\left[\frac{1}{2}\frac{1-\Lambda_{ijk}}{d_{jk}+d_{ki}}(u_{jk}-u_{ki})^2 + \left(c_{ki}u_j - c_{ji}u_k + \frac{d_{ki}}{d_{jk}+d_{ki}}(c_{jk}u_i - c_{ik}u_j)\right)(u_{jk}-u_{ki}) + \Big((c_{jk}-c_{kj})u_i + (c_{ki}-c_{ik})u_j + (c_{ij}-c_{ji})u_k\Big)u_{ki}\right] \exp\frac{i}{\hbar}\left[\frac{1}{2}\left(a_{jk}u_i^2 + cyclic + \frac{1}{d_{jk}+d_{ki}}(c_{jk}u_i - c_{ik}u_j)\right)\right], \tag{E.3}$$

with $\Lambda_{ijk}$ given in (88). Here it is clear that we can shift our integration by the substitution $v = u_{jk} - u_{ki}$. Thus, to avoid a delta function integral for $\mathrm{d}u_{ki}$ and gain the volume factor we desire, we require also all terms linear in $u_{ki}$ in the exponent to vanish. Hence we require: $c_{ij} - c_{ji} = 0\ \forall i, j$. This is the second condition of (86). Evaluation of the second Gaussian integral then gives us (87).

### *E.3. Elementary move (a): comparing results*

In the generic case ($\mathfrak{a}_{ijk} \neq 0$, $\det A \neq 0$) we compare equation (80) with (85). Comparing coefficients of $u_i^2$ and $u_iu_j$ in the exponent, this gives the functional equations:

$$b_{ij} - b_{ik} - \frac{1}{\mathfrak{a}_{ijk}}(c_{ij} - c_{ik})^2 = a_{jk} + \frac{1}{\det A}\Big\{\big(d_{ij}^2 - (b_{ki} - b_{ji})(b_{ij} - b_{kj})\big)c_{jk}^2 + \big(d_{ki}^2 - (b_{jk} - b_{ik})(b_{ki} - b_{ji})\big)c_{kj}^2 + 2\big(d_{ij}d_{ki} - d_{jk}(b_{ki} - b_{ji})\big)c_{jk}c_{kj}\Big\}\,, \tag{E.4}$$

$$d_{ij} - \frac{1}{\mathfrak{a}_{ijk}}(c_{ij} - c_{ik})(c_{jk} - c_{ji}) = \frac{1}{\det A}\Big[\big((b_{ki} - b_{ji})(b_{ij} - b_{kj}) - d_{ij}^2\big)c_{jk}c_{ik} - \big(d_{ij}d_{jk} - d_{ki}(b_{ij} - b_{kj})\big)c_{jk}c_{ki} + \big(d_{jk}d_{ki} - d_{ij}(b_{jk} - b_{ik})\big)c_{ki}c_{kj} - \big(d_{ij}d_{ki} - d_{jk}(b_{ki} - b_{ji})\big)c_{ik}c_{kj}\Big]\,. \tag{E.5}$$

It is not at all obvious that a solution to these equations, under the constraints, exists.

However, in the special case $\mathfrak{a}_{ijk} = 0$, $\det A = 0$ we compare the exponent in (82) with (87). This gives equations from the coefficients of $u_i^2$ and $u_iu_j$:

$$b_{ij} - b_{ik} = a_{jk} - \frac{1}{1 - \Lambda_{ijk}}(d_{ij} + d_{ki})c_{jk}^2\,, \quad d_{ij} = \frac{d_{ij}}{1 - \Lambda_{ijk}}c_{jk}c_{ki}\,. \tag{E.6}$$

Combined with the constraints (81), (86), this yields the Lagrangian (89).

## References


[1] J. Hietarinta, N. Joshi, F.W. Nijhoff, Discrete Systems and Integrability, Cambridge Texts in Applied Mathematics, vol. 54, Cambridge University Press, 2016.

[2] A.I. Bobenko, Yu.B. Suris, Discrete Differential Geometry, Graduate Studies in Mathematics, vol. 98, American Mathematical Society, 2008.

[3] S.N.M. Ruijsenaars, H. Schneider, A new class of integrable systems and its relation to solitons, Ann. Phys. 170 (2) (1986) 370–405.

[4] A.Yu. Volkov, L.D. Faddeev, Quantum inverse scattering method on a spacetime lattice, Theor. Math. Phys. 92 (2) (1992) 837–842.

[5] S.B. Lobb, F.W. Nijhoff, Lagrangian multiforms and multidimensional consistency, J. Phys. A, Math. Theor. 42 (45) (2009) 454013.

[6] S.B. Lobb, F.W. Nijhoff, G.R.W. Quispel, Lagrangian multiform structure for the lattice KP system, J. Phys. A, Math. Theor. 42 (47) (2009) 472002.

[7] S.B. Lobb, F.W. Nijhoff, Lagrangian multiform structure for the lattice Gel'fand–Dikii hierarchy, J. Phys. A, Math. Theor. 43 (7) (2010) 072003.

[8] P. Xenitidis, F.W. Nijhoff, S.B. Lobb, On the Lagrangian formulation of multidimensionally consistent systems, Proc. R. Soc. A, Math. Phys. Eng. Sci. 467 (2135) (2011) 3295–3317.

[9] S. Yoo-Kong, S.B. Lobb, F.W. Nijhoff, Discrete-time Calogero–Moser system and Lagrangian 1-form structure, J. Phys. A, Math. Theor. 44 (36) (2011) 365203.

[10] S. Yoo-Kong, F.W. Nijhoff, Discrete-time Ruijsenaars-Schneider system and Lagrangian 1-form structure, arXiv preprint, arXiv:1112.4576, 2011.

[11] Yu.B. Suris, Variational formulation of commuting Hamiltonian flows: multi-time Lagrangian 1-forms, J. Geom. Mech. 5 (3) (2013).

[12] S.B. Lobb, F.W. Nijhoff, A variational principle for discrete integrable systems, SIGMA 14 (041) (2018), 18 pp.

[13] R. Boll, M. Petrera, Yu.B. Suris, What is integrability of discrete variational systems?, Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci. 470 (2162) (2013).

[14] R. Boll, M. Petrera, Yu.B. Suris, Multi-time Lagrangian 1-forms for families of Bäcklund transformations: Toda-type systems, J. Phys. A, Math. Theor. 46 (27) (2013) 275204.

[15] F.W. Nijhoff, H.W. Capel, Quantization of integrable mappings, in: Geometric and Quantum Aspects of Integrable Systems, Springer, 1993, pp. 187–211.

[16] F.W. Nijhoff, H.W. Capel, Integrability and fusion algebra for quantum mappings, J. Phys. A, Math. Gen. 26 (22) (1993) 6385.
[17] L.D. Faddeev, A.Yu. Volkov, Algebraic Quantization of Integrable Models in Discrete Space-Time, Oxford Lecture Series in Mathematics and Its Applications, vol. 16, 1999, pp. 301–320.
[18] V.V. Bazhanov, V.V. Mangazeev, S.M. Sergeev, Exact solution of the Faddeev–Volkov model, Phys. Lett. A 372 (10) (2008) 1547–1550.
[19] R.P. Feynman, A.R. Hibbs, Quantum Mechanics and Path Integrals, McGraw-Hill, 1965.
[20] M.V. Berry, N.L. Balazs, M. Tabor, A. Voros, Quantum maps, Ann. Phys. 122 (1) (1979) 26–63.
[21] C.M. Field, F.W. Nijhoff, Time-sliced path integrals with stationary states, J. Phys. A, Math. Gen. 39 (20) (2006) L309.
[22] C.M. Field, On the Quantization of Integrable Discrete-Time Systems, PhD thesis, University of Leeds, March 2005.
[23] V.E. Adler, A.I. Bobenko, Yu.B. Suris, Classification of integrable equations on quad-graphs. The consistency approach, Commun. Math. Phys. 233 (3) (2003) 513–543.
[24] A.S. Fokas, A unified transform method for solving linear and certain nonlinear PDEs, Proc. R. Soc. Lond. A 453 (1962) (1997).
[25] A.S. Fokas, B. Pelloni, The solution of certain initial boundary-value problems for the linearized Korteweg—de Vries equation, Proc. R. Soc. Lond. A 454 (1970) (1998).
[26] F.W. Nijhoff, A.J. Walker, The discrete and continuous Painlevé VI hierarchy and the Garnier systems, Glasg. Math. J. 43 (A) (2001) 109–123.
[27] V.G. Papageorgiou, F.W. Nijhoff, H.W. Capel, Integrable mappings and nonlinear integrable lattice equations, Phys. Lett. A 147 (2–3) (1990) 106–114.
[28] M. Bruschi, O. Ragnisco, P.M. Santini, G.Z. Tu, Integrable symplectic maps, Phys. D: Nonlinear Phenom. 49 (3) (1991) 273–294.
[29] H.W. Capel, F.W. Nijhoff, V.G. Papageorgiou, Complete integrability of Lagrangian mappings and lattices of KdV type, Phys. Lett. A 155 (6) (1991) 377–387.
[30] G.R.W. Quispel, H.W. Capel, V.G. Papageorgiou, F.W. Nijhoff, Integrable mappings derived from soliton equations, Phys. A, Stat. Mech. Appl. 173 (1–2) (1991) 243–266.
[31] F.W. Nijhoff, H.W. Capel, V.G. Papageorgiou, Integrable quantum mappings, Phys. Rev. A 46 (1992) 2155–2158.
[32] F.W. Nijhoff, Lagrangian multiform theory and variational principle for integrable systems, in: Discrete Integrable Systems Follow-Up Meeting, Isaac Newton Institute, Cambridge, July 2013, https://www.newton.ac.uk/seminar/20130710093010001.
[33] S. Yoo-Kong, Calogero-Moser Type Systems, Associated KP Systems, and Lagrangian Structures, PhD thesis, University of Leeds, 2011.
[34] A. Degasperis, S.N.M. Ruijsenaars, Newton-equivalent Hamiltonians for the harmonic oscillator, Ann. Phys. 293 (1) (2001) 92–109.
[35] F.W. Nijhoff, V.G. Papageorgiou, H.W. Capel, G.R.W. Quispel, The lattice Gel'fand-Dikii hierarchy, Inverse Probl. 8 (4) (1992) 597.
[36] G.R.W. Quispel, F.W. Nijhoff, Integrable two-dimensional quantum mappings, Phys. Lett. A 161 (5) (1992) 419–422.
[37] C.M. Field, F.W. Nijhoff, H.W. Capel, Exact solutions of quantum mappings from the lattice KdV as multi-dimensional operator difference equations, J. Phys. A, Math. Gen. 38 (43) (2005) 9503.
[38] P.A.M. Dirac, The Lagrangian in quantum mechanics, Phys. Z. Sowjetunion 3 (1) (1933) 64–72.
[39] R.P. Feynman, Space-time approach to non-relativistic quantum mechanics, Rev. Mod. Phys. 20 (April 1948) 367–387.
[40] C. Grosche, F. Steiner, Handbook of Feynman Path Integrals, vol. 1, Springer, 1998.
[41] L.S. Schulman, Techniques and Applications of Path Integration, vol. 140, Dover, 2005.
[42] C. Rovelli, Discretizing parametrized systems: the magic of Ditt-invariance, arXiv preprint, arXiv:1107.2310, 2011.
[43] C. Rovelli, On the structure of a background independent quantum theory: Hamilton function, transition amplitudes, classical limit and continuous limit, arXiv preprint, arXiv:1108.0832, 2011.
[44] C.M. Field, On the Quantisation of Integrable Discrete-Time Systems, PhD thesis, The University of Leeds, 2005.
[45] J.A. Oteo, The Baker–Campbell–Hausdorff formula and nested commutator identities, J. Math. Phys. 32 (2) (1991) 419–424.
[46] V.S. Varadarajan, Lie Groups, Lie Algebras, and Their Representations, vol. 102, Springer Science & Business Media, 2013.
[47] P.J. Forrester, Log-Gases and Random Matrices, Princeton University Press, 2010.

[48] A.Yu. Volkov, Quantum lattice KdV equation, Lett. Math. Phys. 39 (4) (1997) 313–329.
[49] A.I. Bobenko, Yu.B. Suris, Discrete pluriharmonic functions as solutions of linear pluri-Lagrangian systems, Commun. Math. Phys. 336 (1) (2015) 199–215.
[50] M.P. Reisenberger, C. Rovelli, "Sum over surfaces" form of loop quantum gravity, Phys. Rev. D 56 (6) (1997) 3490.
[51] M.P. Reisenberger, A left-handed simplicial action for Euclidean general relativity, Class. Quantum Gravity 14 (7) (1997) 1753.
[52] P.A.M. Dirac, On the analogy between classical and quantum mechanics, Rev. Mod. Phys. 17 (April 1945) 195–199.
[53] N. O'Connell, Stochastic Bäcklund transformations, in: In Memoriam Marc Yor - Séminaire de Probabilités XLVII, Springer International Publishing, 2015, pp. 467–496.
[54] A. Doikou, S.J.A. Malham, A. Wiese, Stochastic analysis and discrete quantum systems, arXiv preprint, arXiv:1810.08095, 2018.
[55] B. Bahr, B. Dittrich, S. Steinhaus, Perfect discretization of reparametrization invariant path integrals, Phys. Rev. D 83 (10) (2011) 105026.
[56] G. t'Hooft, The cellular automaton interpretation of quantum mechanics. A view on the quantum nature of our universe, compulsory or impossible?, arXiv preprint, arXiv:1405.1548, 2014.