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17th CIRP Conference on Modelling of Machining Operations

# An analytical study of wheel regeneration in surface grinding

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## Abstract

Machining vibrations have been a topic of active research since the beginning of the 20<sup>th</sup> century, because of their harmful potential to deteriorate the surface finish of the workpiece, reduce the lifetime of the machine tool and limit the overall productivity of manufacturing operations. Grinding, often being a finishing operation responsible for the final quality of the workpiece, can be especially affected by unwanted process vibrations. Machining vibrations can arise from a number of sources, but they are usually classified as forced and self-excited vibrations. Forced vibrations, such as those resulting from runout or unbalance, are generally easier to avoid. However, self-excited vibrations, also known as chatter, originate in the machining process itself and are typically more difficult to predict and suppress. When it comes to grinding operations, chatter can occur as a result of uneven surface regeneration on both the wheel and the workpiece. In this paper we perform an analytical study of grinding chatter, to explore the intricate nature of wheel regeneration (i.e. chatter arising from uneven wear on the wheel) in surface grinding. Whilst there has been a great deal of previous research into grinding chatter, the present study explores a contrarian approach whereby a circumferential variation of the specific grinding energy occurs during the onset of instability. The specific energy is a fundamental quantity in grinding that relates the necessary grinding power to the prescribed material removal rate. It is also related to wheel wear and thus to grinding forces, since a worn wheel requires more grinding power and produces greater grinding forces to sustain the same material removal rate than a sharp one. Therefore, by characterising the specific energy variation around the circumference of the grinding wheel as a function of wheel vibration, we derive a stability model and discuss its practical potential.

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*Keywords:* Grinding chatter; wheel regeneration; specific energy; stability

## 1. Introduction

Grinding is the oldest machining operation invented by mankind [1] and has been used for thousands of years in many different ways to make parts of desired shapes and forms. It is an abrasive process relying on a tool (a grinding wheel) for material removal, which consists of tiny hard particles called grains or grits and a softer bonding agent to hold them all together. Grinding is a large and diverse area of manufacturing due to its intrinsic ability to produce fine surface finishes and accurate dimensions, cut difficult-to-machine materials, and achieve high material removal rates. However, besides the aforementioned advantages, there are

some challenges as well, especially when it comes to modelling grinding processes. The cutting edges of abrasive tools have inherently random and undefined geometries, which makes wheel modelling particularly complicated. Grinding wheels also wear significantly more than conventional cutting tools such as those used in milling and turning, which means that wheel wear cannot be neglected.

Similarly to other manufacturing operations, self-excited tool vibration or chatter poses a serious problem in grinding, negatively influencing the surface quality and dimensional accuracy of the workpiece, reducing the lifetime of the grinding wheel and limiting productivity. Therefore, predicting and avoiding chatter is of crucial importance in

efficient grinding practice. Due to the complexities mentioned above, the understanding of grinding chatter is considered to have lagged behind that of conventional processes [2]. Therefore, the overall aim of this paper is to further our grasp of grinding dynamics by investigating chatter vibrations from an angle that is different from the most common approaches.

## 2. Literature review

Much research has been done since the 1950s in order to gain a deeper understanding of grinding chatter, considering the challenges mentioned earlier [3–5]. The geometric uncertainty of the cutting edges was addressed by a number of authors [6–10] with the aim of developing more accurate force expressions, whereas grinding wheel wear was considered by many as an additional source of surface regeneration in the system [11–13] and termed by some “the doubly regenerative effect” [14–19]. The vast majority of the relevant papers deal with regeneration on the surface of the grinding wheel only in terms of an angle-dependent radius that varies around the circumference of the grinding wheel. In other words, the physical shape of the grinding wheel will differ from a perfect circle during the onset of instability. The work of Li and Shin [20], however, proposed an alternative approach, where unstable wheel regeneration can occur not only as a result of a varying wheel radius, but also as a result of a varying specific energy around the circumference of the grinding wheel. Their novel idea helped to shed light on a number of experimental observations which previous grinding dynamics models could not explain. The new approach is based on the simple fact that a blunt or worn grain produces a greater grinding force than a sharp one. The degree of wear on a single grain can be related to the specific energy increase corresponding to that grain which measures the amount of energy required to remove a unit volume of material. It is important to note that the specific energy consists of chip formation, ploughing and sliding components, where the latter two do not contribute to material removal [21]. Therefore, by characterising grit wear as a function of material removal (which itself depends on the vibration of the grinding wheel), a mathematical model can be established that governs the stability of the grinding system. Although Li and Shin presented the original idea, their approach was based on a finite element method, which provides less generality and insight than an analytical model. Nevertheless, their finite element analysis was able to accurately capture the phenomenon of wheel regeneration and explain some previously unclear experimental observations reported in the literature.

## 3. Objectives

In this paper a new analytical model is developed to address the problem and solution proposed by Li and Shin [20]. A variable specific energy distribution is considered around the circumference of the grinding wheel in order to assess the stability of the system in a rigorous fashion. The advantage of an analytical model as opposed to a finite element approach is closed-form solutions which provide great insight, clarity and generality about the problem in question.

## 4. Model description

The scenario considered is single-pass surface grinding in order to investigate wheel regeneration without having to model surface regeneration on the workpiece. The doubly regenerative effect has been the topic of a significant amount of research, so it is disregarded in this paper with the intention of analysing wheel regeneration more thoroughly. Single-pass surface grinding often takes place in a creep-feed grinding configuration, therefore, high depths of cut and low feed rates will be modelled, but dressing is not included in the analysis.

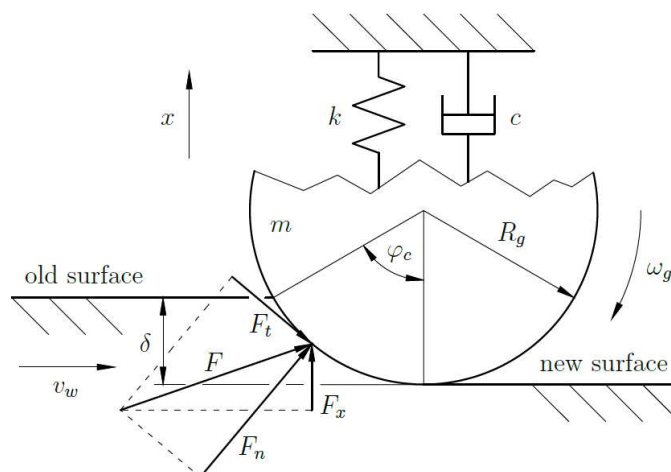


Fig. 1: Mechanical model of surface grinding

The grinding dynamics model under consideration is presented in Fig. 1. It can be seen that only one degree of freedom is taken into account which is described by the general coordinate  $x$ , and the grinding wheel is allowed to oscillate perpendicular to the workpiece, in the direction of the depth of cut  $\delta$ . The modal mass, damping and stiffness of the grinding wheel are denoted by  $m$ ,  $c$  and  $k$ , respectively. The rotational speed of the wheel is  $\omega_g$  and the feed rate is  $v_w$ . The different grinding force components are denoted by  $F_n$  (normal),  $F_t$  (tangential) and  $F_x$ , while the resultant of the real distributed force system that occurs in the grinding zone is  $F$ .

## 5. Chatter theory

According to the work of Li and Shin [20], grinding chatter will be investigated in terms of a varying specific energy distribution around the circumference of the grinding wheel which will be characterised by a single physical radius throughout the entire grinding process.

As an initial approximation, grinding wheel vibration is assumed to affect the depth of cut without influencing the chip thickness. That is to say, the vibration of the grinding wheel is equivalent to the oscillation of the old, not-yet-ground workpiece surface (see Fig. 1). Therefore, it is only the length of contact between the wheel and the workpiece that changes in such a way that the entering angle of the grinding zone remains constant and the exiting angle changes according to the vibration of the grinding wheel. This is a significant simplification in the model, and therefore presents an opportunity of high priority and great potential to improve the model in the future.

The applied wheel vibration model allows for regenerative chatter to occur, since wheel vibrations disturb the depth of cut, which causes the material removal to be different for each grit, leading to uneven wheel wear quantified by an uneven specific energy distribution around the circumference of the grinding wheel, which in turn produces varying grinding forces, resulting in more wheel vibrations. Therefore, the loop is closed – wheel vibrations generate more wheel vibrations. Depending on the phase difference between them, they can be either self-attenuating (stable) or self-amplifying (unstable).

## 6. Equation of motion

The mathematical description of the problem consists of the five aforementioned relationships:

- Wheel vibration and depth of cut,
- Depth of cut and material removed,
- Material removed and specific energy,
- Specific energy and grinding force,
- Grinding force and wheel vibration.

The relationship between the wheel vibration  $x$  and the depth of cut  $\delta$  is straightforward: the displacement of the grinding wheel changes the depth of cut in such a way that  $x = 0$  corresponds to a stable, steady-state operation characterised by the constant, nominal depth of cut  $\delta_0$ . Therefore, the instantaneous depth of cut can be written as:

$$\delta(t) = \delta_0 - x(t). \quad (1)$$

The relationship between the depth of cut and the amount of material removed by a single grain over one grinding wheel revolution can be determined by multiplying the feed rate  $v_w$ , the grit-passing period  $\tau_g$  and the depth of cut  $\delta$ :

$$V'_w(t) = v_w \tau_g \delta(t), \quad (2)$$

where  $V'_w$  is the specific material removed (material volume per unit grinding width).

The relationship between the specific material removed  $V'_w$  and the specific energy  $u$  can be described by the evolution of the specific energy as a result of wheel wear characterised by the coefficient of dulling  $C_d$ . The specific energy at time  $t$  corresponds to the angular location on the grinding wheel where the grains leave the grinding zone. This way of formulating the specific energy distribution around the circumference of the grinding wheel allows for a one-variable description of the specific energy without having to introduce a spatial coordinate in addition to the temporal one. Thus the evolution of the specific energy can be expressed as:

$$u(t) = u(t - T_g) + C_d V'_w(t). \quad (3)$$

The relationship between the specific energy and the grinding force  $F_x$  can be obtained by averaging the specific energy distribution in the grinding zone and then applying a simple grinding force expression derived by Malkin and Guo [21]:

$$F_x(t) = \frac{\mu_x \delta_0 w v_w}{\tau_{c,0} v_g} \int_0^{\tau_{c,0}} u(t - T_g + \tau_{c,0} - \tau) d\tau C. \quad (4)$$

The relationship between the grinding force and the wheel vibration is the classical second-order differential equation defining the connection between force and motion:

$$\ddot{x}(t) + 2\zeta\omega_n \dot{x}(t) + \omega_n^2 x(t) = \frac{1}{m} F_x(t). \quad (5)$$

The notations in Eqs. (1-5) are summarised in the Table of Nomenclature.

## Nomenclature

$\delta$	[m]	instantaneous depth of cut
$\delta_0$	[m]	nominal depth of cut
$\zeta$	[1]	damping ratio
$\mu_x$	[1]	grinding force ratio between $F_x$ and $F_t$
$\tau$	[s]	local time coordinate in the grinding zone
$\tau_{c,0}$	[s]	grit contact time
$\tau_g$	[s]	grit-passing period
$\omega_n$	[rad/s]	angular natural frequency
$C$	[N]	force constant
$C_d$	[J/m <sup>3</sup> /m <sup>2</sup> ]	coefficient of dulling
$F_x$	[N]	grinding force in the $x$ direction
$m$	[kg]	modal mass
$T_g$	[s]	grinding wheel period
$t$	[s]	time
$u$	[J/m <sup>3</sup> ]	specific energy
$V_w$	[m <sup>3</sup> ]	material volume removed
$V'_w$	[m <sup>3</sup> /m]	specific material removed
$v_g$	[m/s]	wheel circumferential speed
$v_w$	[m/s]	feed rate
$w$	[m]	grinding width
$x$	[m]	grinding wheel displacement

Combining Eqs. (1-5) together by eliminating  $x$ ,  $\delta$ ,  $V'_w$  and  $F_x$ , and keeping the specific energy  $u$  as the general coordinate of the problem described, the equation of motion reads:

$$\ddot{u}(t) + 2\zeta\omega_n \dot{u}(t) + \omega_n^2 u(t) = \ddot{u}(t - T_g) + 2\zeta\omega_n \dot{u}(t - T_g) + \omega_n^2 u(t - T_g) - \frac{\mu_x \delta_0 w C_d v_w^2 \tau_g}{m \tau_{c,0} v_g} \int_0^{\tau_{c,0}} u(t - T_g + \tau_{c,0} - \tau) d\tau, \quad (6)$$

where the force constant  $C$  has been neglected for the stability analysis of the process, since it only offsets and does not alter the dynamics of a linear system. Due to the fact that the equation of motion is written for the specific energy function instead of the grinding wheel displacement, the derivatives of the delayed term appear in the equation as well.

## 7. Analytical solution

The dynamical properties of Eq. (6) can be investigated by the Nyquist criterion which assesses the stability of a system based on the contour plot of its open-loop transfer function. A dynamical system is unstable if the Nyquist plot of the open-loop transfer function corresponding to the Nyquist contour, which encloses the entire right half side of the complex plane in a clockwise direction, encircles the  $(-1,0)$  point at least once in a clockwise direction (further details of the Nyquist stability criterion are given in [22], pages 624-650).

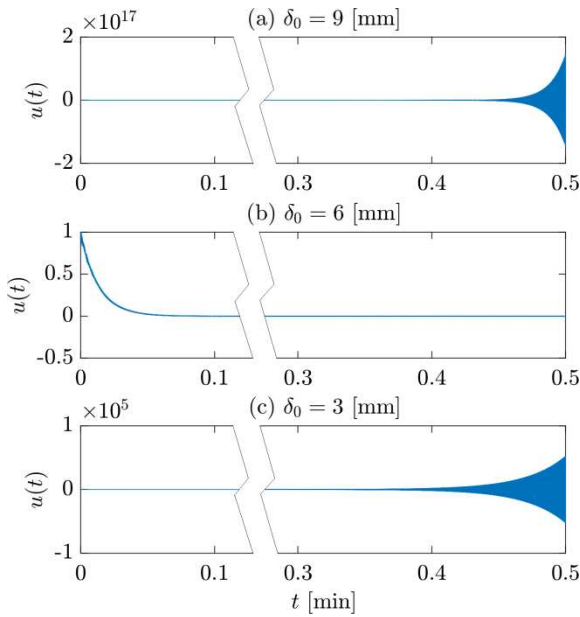


Fig. 2: Simulation results for  $\omega_n = 2000$  [rpm] and different nominal depths of cut  $\delta_0$

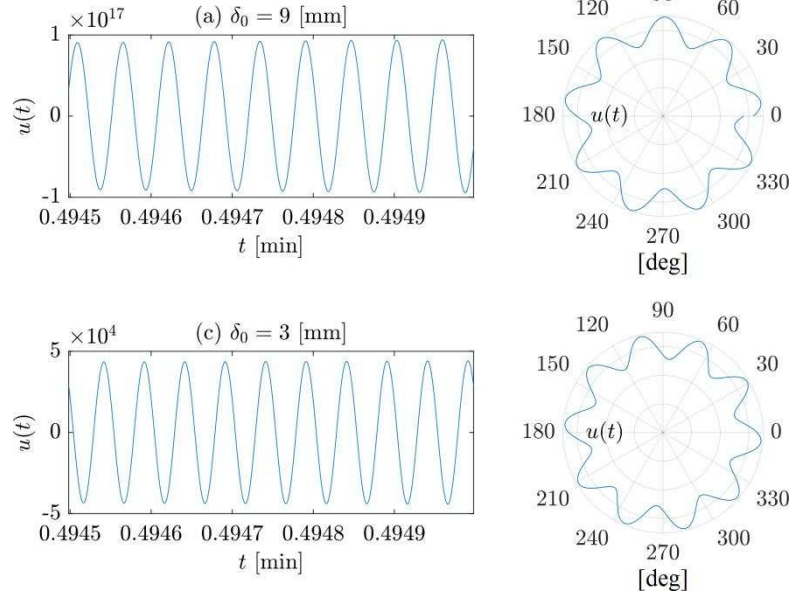


Fig. 3: Linear (left) and polar (right) specific energy distributions over one grinding wheel period for the two unstable scenarios presented in Fig. 2

Therefore, taking the Laplace transforms of Eq. (1-5) and establishing local transfer functions between the system variables described in Sec. 6, the block diagram of the grinding system can be constructed and the open-loop transfer function  $T_o(s)$  can be derived:

$$T_o(s) = \frac{C_d v_w^2 \tau_g \mu_x \delta_0 w e^{-T_g s} (e^{\tau_{c,0} s} - 1)}{(1 - e^{-T_g s}) \tau_{c,0} v_g s m (s^2 + 2\zeta \omega_n s + \omega_n^2)}, \quad (7)$$

where  $s$  is the complex Laplace variable. It can be seen that there are two time delays in the system: a long delay corresponding to the time period  $T_g$  of the grinding wheel and a short delay corresponding to the grit contact time  $\tau_{c,0}$ . It is possible to assess the stability of the system based on Eq. (7), but it is challenging because of the complex shape of the contour plot of  $T_o(s)$ . Therefore, a numerical solution of Eq. (6) is presented in this article in order to show that the analytical model under discussion is capable of capturing the behaviour of wheel regenerative chatter in surface grinding. Time domain simulations also give valuable insight into the physical behaviour of the system, i.e., what actually happens at the onset of instability. Plotting the specific energy variation around the circumference of the grinding wheel provides a graphical way of analysing the results, making it easier to visualise and understand the physics behind the equation of motion Eq. (6).

### 8. Numerical simulation

Applying the central difference scheme, Eq. (6) can be solved for a given set of grinding parameters. Figure 2 presents three different grinding scenarios. It can be seen that the process is unstable, then stable and then unstable again as the depth of cut increases for a certain spindle speed. A similar phenomenon can be observed if the depth of cut is fixed and the spindle speed increases.

Figure 3 zooms in on the two unstable cases after about half a minute of grinding time, revealing the behaviour of the specific energy distribution around the circumference of the grinding wheel. It can be seen that an approximately integer number of waves are formed, however, there is a phase difference between the beginning and the end of the specific energy distribution over one grinding wheel rotation. This is the result of a phenomenon called ‘precession’ which is well-known in the literature [23,24]. It means that surface waves do not only grow in time but also move slowly around the circumference of the grinding wheel. However it should be reiterated that in the present study, the precessional waves represent circumferential variations in the specific energy, rather than variations in wheel radius. According to Fig. 3, the higher the depth of cut, the faster the precession rate. The phase difference is about  $43.5^\circ$  and  $3.6^\circ$  for  $\delta_0 = 9$  [mm] and  $\delta_0 = 3$  [mm], respectively. In other words, considering the number of specific energy waves in each case, it takes approximately 74.5 and 1000 grinding wheel revolutions (or 2.2 and 30 seconds) for the surface waves to travel once around the circumference of the wheel. The phase difference is more easily detectable when the specific energy distribution is plotted for two revolutions in different colours for each one (see Fig. 4). Figure 4 also reveals that the direction of precession is different in the two cases. This is likely owing to the fact that scenarios (a) and (c) are separated by a stable region denoted by (b) in Fig. 2.

### 9. Discussion

This work has demonstrated that the proposed analytical model is capable of describing wheel regenerative chatter in surface grinding. Although an analytical stability analysis could not be presented in much detail due to its complexity and length, a numerical simulation of the grinding process gave valuable insight into the dynamical behaviour of the

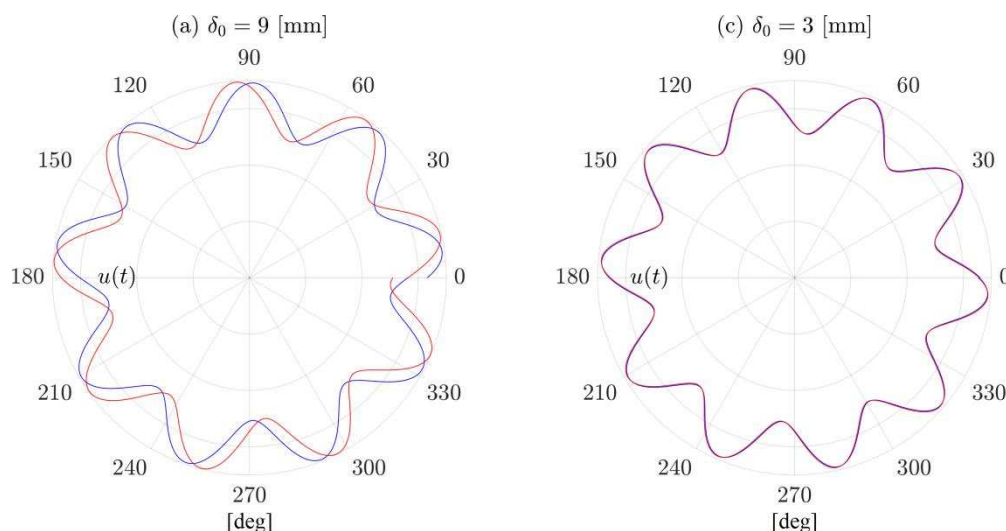


Fig. 4: Precession of specific energy waves around the grinding wheel for two different nominal depths of cut  $\delta_0$  (energy units are irrelevant in this case)

system. Both stability and instability have been predicted for certain grinding parameters, furthermore, the vibration frequency at the onset of chatter has been estimated as well. It is close to an integer multiple of the spindle speed, where the difference is responsible for the well-known phenomenon of ‘wheel lobe precession’ [23,24].

For the experimental validation of the proposed chatter model – apart from the obvious comparison where the outcome is merely a ‘stable’ or ‘unstable’ process – it is possible to go further in validation and compare the measured vibration frequencies (for unstable scenarios) with the ones predicted by the theory. For example, in case (c), the dominant vibration frequency is expected to be ten times the spindle speed, because ten specific energy waves have been predicted around the circumference of the wheel for that specific scenario. Therefore, setting up the same grinding parameters and measuring a grinding force with ten periods within one wheel revolution would be a very good indication that the proposed chatter theory is accurate.

Such experiments are currently being developed and implemented by the authors. In the meantime, an initial comparison to Thompson’s work [24,25] is given to ground the presented theory in measurements reported in the literature. Similarly to the proposed model, Thompson also considered surface grinding with a flexible wheel and a rigid workpiece. However, he quantified uneven wheel wear by an uneven wheel radius distribution instead of an uneven specific energy distribution around the circumference of the grinding wheel. Thompson’s calculated and measured chatter frequencies are summarised in Tab. 1 along with the predictions provided by the presented analytical model.

Frequency unit: [Hz]	Case #1	Case #2	Case #3
Thompson’s prediction	223.66	276.61	333.95
Thompson’s experiment	221.19	277.36	334.57
Authors’ prediction	206.92	259.40	311.87

Tab. 1: Comparison between chatter frequencies for three different test cases

It can be seen that the differences between the proposed theory and Thompson’s results are well within 10%.

In terms of the authors’ own machining trials, theoretical and practical results are not expected to align perfectly. However, in case of a significant mismatch between the two, certain modelling assumptions need to be reconsidered. One of them is that the grinding process can be described with a single degree of freedom. While this is a reasonable assumption, it can also introduce considerable inaccuracies into the model, depending on the structural properties of the actual grinding machine (e.g. symmetry) and the applied grinding parameters (e.g. depth of cut). Also, the current grinding force expression is relatively simple, so using or developing a more sophisticated force model is another potential way of achieving higher accuracy (e.g. considering sliding, ploughing and chip formation mechanisms separately [21]). Another simplification in the current form of the proposed theory is the neglecting of the so-called self-sharpening effect of the grinding wheel by which new cutting edges are exposed as a natural result of wheel wear. This is not of primary importance when it comes to explaining a potential mismatch between theory and practice, but could still be responsible for some inaccuracies. Revisiting these assumptions provides an opportunity to reconsider and reformulate them and thus make the new wheel chatter model even more realistic.

Further work could potentially include extending the current model to two degrees of freedom and, similarly to Li and Shin’s approach [20], considering not only a varying specific energy distribution but also a varying wheel radius around the circumference of the grinding wheel.

## 10. Conclusions

In this paper a new analytical chatter theory was proposed that is able to capture the intricate nature of wheel regeneration in surface grinding. The model considered wheel wear as a varying specific energy distribution around the circumference of the grinding wheel and relied on that variation alone to induce instability in the process. An analytical approach based on the Nyquist criterion was introduced to determine the stability properties of the system,

however, due to its complexity and length, it was not discussed in detail, rather, a numerical simulation was presented to shed some light on process stability. Although experiments have not been performed yet to validate the proposed theory, the findings reported in this paper already agree with well-established aspects of sound grinding practice published in the literature: our simulation results indicate that the chatter frequencies arising at the onset of instability are nearly integer multiples of the spindle speed (also reported in [20,25]), and the small differences from them being exact integer multiples are responsible for the well-known phenomenon of ‘wheel lobe precession’ [23–25]. Unstable grinding operations were analysed with regard to chatter growth as well, and it was demonstrated that deeper cuts are responsible for faster chatter growth (also reported in [5]). Therefore, based on this new analytical model, optimal grinding parameters can be chosen in order to prevent chatter vibrations, or – if unstable operating conditions cannot be avoided – to suppress them as much as possible.

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