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DOUBLE YANGIAN AND THE UNIVERSAL R-MATRIX

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Abstract. We describe the double Yangian of the general linear Lie algebra $\mathfrak{gl}_N$ by following a general scheme of Drinfeld. This description is based on the construction of the universal R-matrix for the Yangian. To make exposition self contained, we include the proofs of all necessary facts about the Yangian itself. In particular, we describe the centre of the Yangian by using its Hopf algebra structure, and provide a proof of the analogue of the Poincaré–Birkhoff–Witt theorem for the Yangian based on its representation theory. This proof extends to the double Yangian, thus giving a description of its underlying vector space.

Contents

Introduction

The main subject of this article is a Hopf algebra that appeared in the framework of quantum inverse scattering method introduced by L. D. Faddeev, E. K. Sklyanin and their collaborators, see for instance [9, 15, 26, 27, 28, 30]. This algebra then became a part of a family of examples in the theory of quantum groups created by V. G. Drinfeld [3, 4, 5]. He gave to this family the name *Yangians* in honour of C. N. Yang, the author of a seminal work [32]. The Yangian that we consider here corresponds to the general linear Lie algebra $\mathfrak{gl}_N$. It is a canonical deformation of the universal enveloping algebra of the polynomial current Lie algebra $\mathfrak{gl}_N[z]$.

The general notion of a quantum double was also introduced in [5]. However the Yangians were not discussed there in the context of this notion. Here we define the double Yangian of the Lie algebra $\mathfrak{gl}_N$ similarly to [10]. Yet many details and proofs are also missing in the latter work. In the present article we fill these gaps.

We denote by $\mathrm{Y}(\mathfrak{gl}_N)$ the Yangian of $\mathfrak{gl}_N$, and by $\mathrm{DY}(\mathfrak{gl}_N)$ its quantum double. There are several equivalent definitions of the Hopf algebra $\mathrm{Y}(\mathfrak{gl}_N)$ available [20]. In this article we use the definition that appeared first, see for instance [16, 17, 31]. Details of this definition are given in our Sections 1,2 and 4 by closely following [21]. Sections 3,5 and 6 describe basic properties of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ that we will use.

We will also use an analogue of the classical Poincaré–Birkhoff–Witt theorem [2] for the algebra $\mathrm{Y}(\mathfrak{gl}_N)$. The first proof of this analogue was given by V. G. Drinfeld but not published. Other proofs were given later in [18, 24]. In Section 8 we give yet another proof of this analogue by using the representation theory of current Lie algebras. The fact from the theory that we use is established in Section 7. It is this proof that will be extended to the double Yangian $\mathrm{DY}(\mathfrak{gl}_N)$ in the present article. This method was used in [23] to prove analogues of the Poincaré–Birkhoff–Witt theorem for the Yangian of the queer Lie superalgebra $\mathfrak{q}_N$ and its quantum double. For the algebra dual to the coalgebra $\mathrm{Y}(\mathfrak{gl}_N)$ the same method was used in [7].

The structure of a Hopf algebra includes a canonical anti-automorphism relative to both multiplication and comultiplication, called the antipodal map. In general this map is not involutive. In Section 3 we also compute the square of this map for the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$, by following [22] where the Yangian of the general linear Lie superalgebra $\mathfrak{gl}_{M|N}$ was considered. This yields a family of central elements of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$, see also [6]. In Section 9 we prove that these elements generate the whole centre. Our proof uses another general fact from the theory of current Lie algebras, which we establish in the beginning of the section. The idea of reducing the proof to that fact belongs to V. G. Drinfeld, as acknowledged in [21].

In Section 10 we introduce the bialgebra $\mathrm{Y}^*(\mathfrak{gl}_N)$ dual to $\mathrm{Y}(\mathfrak{gl}_N)$. First we define it in terms generators and relations similarly to $\mathrm{Y}(\mathfrak{gl}_N)$. However $\mathrm{Y}^*(\mathfrak{gl}_N)$ is not a Hopf algebra. The antipodal map is defined only on a certain completion $\mathrm{Y}^\circ(\mathfrak{gl}_N)$ of $\mathrm{Y}^*(\mathfrak{gl}_N)$ described at the end of that section. In Section 11 we define a bialgebra pairing between $\mathrm{Y}(\mathfrak{gl}_N)$ and $\mathrm{Y}^*(\mathfrak{gl}_N)$. This definition goes back to [25] where the quantized universal enveloping algebras of simple Lie algebras were considered. In Section 12 we prove that this pairing is non-degenerate. Details of this proof first appeared in [23] where instead of $\mathfrak{gl}_N$, the Lie superalgebra $\mathfrak{q}_N$ was considered.

In Section 13 we define the universal R-matrix for $\mathrm{Y}(\mathfrak{gl}_N)$. This is the canonical element of a suitable completion of the tensor product $\mathrm{Y}^*(\mathfrak{gl}_N) \otimes \mathrm{Y}(\mathfrak{gl}_N)$, which corresponds to the bialgebra pairing. There we also describe the basic properties of this element relative to the Hopf algebra structures on both $\mathrm{Y}(\mathfrak{gl}_N)$ and $\mathrm{Y}^\circ(\mathfrak{gl}_N)$.

In Section 14 we define the double Yangian $\mathrm{DY}(\mathfrak{gl}_N)$ as a bialgebra generated by $\mathrm{Y}(\mathfrak{gl}_N)$ and $\mathrm{Y}^*(\mathfrak{gl}_N)$. Following [5, 25] the cross relations between the elements of $\mathrm{Y}(\mathfrak{gl}_N)$ and $\mathrm{Y}^*(\mathfrak{gl}_N)$ are introduced by means of the universal R-matrix. Then we provide a more explicit description of the algebra $\mathrm{DY}(\mathfrak{gl}_N)$. Using this description one can define a central extension of $\mathrm{DY}(\mathfrak{gl}_N)$, see for instance [8, 11].

Finally, in Section 15 we introduce a filtration on the algebra $\mathrm{DY}(\mathfrak{gl}_N)$ and show that the corresponding graded algebra is isomorphic to the universal enveloping algebra of the current Lie algebra $\mathfrak{gl}_N[z, z^{-1}]$. This implies our analogue of the Poincaré–Birkhoff–Witt theorem for $\mathrm{DY}(\mathfrak{gl}_N)$. This also implies that the defining homomorphisms of the algebras $\mathrm{Y}(\mathfrak{gl}_N)$ and $\mathrm{Y}^*(\mathfrak{gl}_N)$ to $\mathrm{DY}(\mathfrak{gl}_N)$ are embeddings.

The purpose of the present article is to provide the basic facts about the double Yangian $\mathrm{DY}(\mathfrak{gl}_N)$ with their detailed proofs. We do not not aim to review all works which involve this remarkable object. Still let us mention here the pioneering works [1, 19, 29] where the double Yangian of the special linear Lie algebra $\mathfrak{sl}_2$ was studied. The double Yangians of all simple Lie algebras were studied in [13, 14] by using the definition of the underlying Yangians from [4]. This approach to double Yangians is different from ours. Recently some of the results on $\mathrm{DY}(\mathfrak{gl}_N)$ presented here have been extended to the double Yangians of the other classical Lie algebras [12].

1. Definition of the Yangian

The *Yangian* of the general linear Lie algebra $\mathfrak{gl}_N$ is a unital associative algebra $\mathrm{Y}(\mathfrak{gl}_N)$ over the complex field $\mathbb{C}$ with countably many generators

$$T_{ij}^{(1)},\ T_{ij}^{(2)},\dots \quad \text{where} \quad i,j=1,\dots,N\,.$$

The defining relations of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$ are

$$[\,T_{ij}^{(r+1)},T_{kl}^{(s)}\,]-[\,T_{ij}^{(r)},T_{kl}^{(s+1)}\,]=T_{kj}^{(r)}T_{il}^{(s)}-T_{kj}^{(s)}T_{il}^{(r)} \tag{1.1}$$

where $r,s=0,1,\dots$ and $T_{ij}^{(0)}=\delta_{ij}$. By introducing the formal generating series

$$T_{ij}(u)=\delta_{ij}+T_{ij}^{(1)}u^{-1}+T_{ij}^{(2)}u^{-2}+\ldots\in\mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]] \tag{1.2}$$

we can write (1.1) in the form

$$(u-v)\,[\,T_{ij}(u),T_{kl}(v)\,]=T_{kj}(u)\,T_{il}(v)-T_{kj}(v)\,T_{il}(u)\,. \tag{1.3}$$

Here the indeterminates u and v are considered to be commuting with each other and with the elements of the Yangian. The following is an equivalent form of (1.1).

Proposition 1.1. *The system of relations* (1.1) *is equivalent to the system*

$$[\,T_{ij}^{(r)},T_{kl}^{(s)}\,]=\sum_{a=1}^{\min(r,s)}\Big(T_{kj}^{(a-1)}T_{il}^{(r+s-a)}-T_{kj}^{(r+s-a)}T_{il}^{(a-1)}\Big)\,. \tag{1.4}$$

Proof. Observe that the multiplication of both sides of (1.3) by the formal series $\sum_{p=0}^{\infty}u^{-p-1}v^p$ yields an equivalent relation

$$[\,T_{ij}(u),T_{kl}(v)\,]=\Big(T_{kj}(u)T_{il}(v)-T_{kj}(v)T_{il}(u)\Big)\sum_{p=0}^{\infty}u^{-p-1}v^{p}.$$

Taking the coefficients of $u^{-r}v^{-s}$ on both sides gives

$$[\,T_{ij}^{(r)},T_{kl}^{(s)}\,]=\sum_{a=1}^{r}\Big(T_{kj}^{(a-1)}T_{il}^{(r+s-a)}-T_{kj}^{(r+s-a)}T_{il}^{(a-1)}\Big)\,.$$

This agrees with (1.4) in the case $r\leqslant s$. Finally, if $r>s$ observe that

$$\sum_{a=s+1}^{r}\Big(T_{kj}^{(a-1)}T_{il}^{(r+s-a)}-T_{kj}^{(r+s-a)}T_{il}^{(a-1)}\Big)=0\,. \qquad \square$$

We shall be often using formal series to define or describe maps between various algebras. If $A(u)$ and $B(u)$ are formal series in u with coefficients in certain algebras then assignments of the type $A(u)\mapsto B(u)$ are understood in the sense that every coefficient of $A(u)$ is mapped to the corresponding coefficient of $B(u)$.

Many applications of $\mathrm{Y}(\mathfrak{gl}_N)$ are based on the following observation. Let E_{ij} be the standard generators of the Lie algebra $\mathfrak{gl}_N$ so that

$$[\,E_{ij}, E_{kl}\,] = \delta_{jk}\, E_{il} - \delta_{li}\, E_{kj}\,. \tag{1.5}$$

Proposition 1.2. *The assignment*

$$T_{ij}(u) \mapsto \delta_{ij} + E_{ij}u^{-1} \tag{1.6}$$

defines a surjective homomorphism $\mathrm{Y}(\mathfrak{gl}_N) \to \mathrm{U}(\mathfrak{gl}_N)\,.$ *The assignment*

$$E_{ij} \mapsto T_{ij}^{(1)} \tag{1.7}$$

defines an embedding $\mathrm{U}(\mathfrak{gl}_N) \to \mathrm{Y}(\mathfrak{gl}_N)$.

Proof. By the definition (1.3) we need to verify the equality

$$(u-v)\,[\,E_{ij}, E_{kl}\,]\,u^{-1}v^{-1} =$$
$$(\delta_{kj} + E_{kj}u^{-1})(\delta_{il} + E_{il}v^{-1}) - (\delta_{kj} + E_{kj}v^{-1})(\delta_{il} + E_{il}u^{-1})\,.$$

But this clearly holds by the commutation relations (1.5) in $\mathfrak{gl}_N$, which proves the first part of the proposition. In order to prove the second part, put $r = s = 1$ in (1.4). This gives

$$[\,T_{ij}^{(1)}, T_{kl}^{(1)}\,] = \delta_{kj}T_{il}^{(1)} - \delta_{il}T_{kj}^{(1)}.$$

Thus (1.7) is an algebra homomorphism. Its injectivity follows from the observation that by applying (1.7) and then (1.6), we get the identity map on $\mathrm{U}(\mathfrak{gl}_N)\,.$ □

The homomorphism (1.6) is called the *evaluation homomorphism.* By its virtue any representation of the Lie algebra $\mathfrak{gl}_N$ can be regarded as representation of the $\mathrm{Y}(\mathfrak{gl}_N)\,.$ Any irreducible representation of $\mathfrak{gl}_N$ remains irreducible over $\mathrm{Y}(\mathfrak{gl}_N)$ due to surjectivity of this homomorphism. We will also be using its composition with the automorphism $E_{ij} \mapsto -E_{ji}$ of the algebra $\mathrm{U}(\mathfrak{gl}_N)\,.$ The composition maps

$$T_{ij}(u) \mapsto \delta_{ij} - E_{ji}u^{-1}\,. \tag{1.8}$$

The reason for using it rather than (1.6) will be explained in Section 6.

2. Matrix form of the definition

Introduce the $N \times N$ matrix $T(u)$ whose ij-th entry is the series $T_{ij}(u)\,.$ One can regard $T(u)$ as an element of the algebra $\mathrm{End}\,\mathbb{C}^N \otimes \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]]\,.$ Then

$$T(u) = \sum_{i,j=1}^{N} e_{ij} \otimes T_{ij}(u) \tag{2.1}$$

where $e_{ij} \in \mathrm{End}\,\mathbb{C}^N$ are the standard matrix units. If $e_1, \ldots, e_N$ are the standard basis vectors of $\mathbb{C}^N$, then $T(u)\,e_j$ is interpreted as the linear combination

$$T(u)\,e_j = \sum_{i=1}^{N} e_i \otimes T_{ij}(u) \in \mathbb{C}^N \otimes \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]]\,.$$

For any positive integer m we shall be using algebras of the form

$$(\mathrm{End}\,\mathbb{C}^N)^{\otimes m} \otimes \mathrm{Y}(\mathfrak{gl}_N)\,. \tag{2.2}$$

For any $a = 1, \ldots, m$ we denote by $T_a(u)$ the matrix $T(u)$ which corresponds to the a-th copy of the algebra $\operatorname{End}\mathbb{C}^N$ in the tensor product (2.2). That is, $T_a(u)$ is a formal power series in u^{-1} with the coefficients from the algebra (2.2),

$$T_a(u) = \sum_{i,j=1}^{N} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(m-a)} \otimes T_{ij}(u)$$

where e_{ij} belongs to the a-th copy of $\operatorname{End}\mathbb{C}^N$ and 1 is the identity matrix. If C is an element of the tensor square $(\operatorname{End}\mathbb{C}^N)^{\otimes 2}$ then for $a, b = 1, \ldots, m$ with $a < b$ we will denote by C_{ab} the image of C under this embedding $(\operatorname{End}\mathbb{C}^N)^{\otimes 2} \to (\operatorname{End}\mathbb{C}^N)^{\otimes m}$:

$$e_{ij} \otimes e_{kl} \mapsto 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{kl} \otimes 1^{\otimes(m-b)}.$$

Here the tensor factors e_{ij} and e_{kl} belong to the a-th and b-th copies of $\operatorname{End}\mathbb{C}^N$ respectively. The element C_{ab} can be identified with the element $C_{ab} \otimes 1$ of (2.2). If

$$t : \operatorname{End}\mathbb{C}^N \to \operatorname{End}\mathbb{C}^N : e_{ij} \mapsto e_{ji}$$

is the matrix transposition, then for any $a = 1, \ldots, m$ we shall denote by t_a the corresponding partial transposition on the algebra (2.2). It acts as t on the a-th copy of $\operatorname{End}\mathbb{C}^N$ and as the identity map on all the other tensor factors.

Consider now the permutation operator

$$P = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ji} \in \operatorname{End}\mathbb{C}^N \otimes \operatorname{End}\mathbb{C}^N. \tag{2.3}$$

The rational function

$$R(u) = 1 - P\, u^{-1} \tag{2.4}$$

with values in $\operatorname{End}\mathbb{C}^N \otimes \operatorname{End}\mathbb{C}^N$ is called the *Yang R-matrix*. Here and below we write 1 instead of $1 \otimes 1$, for brevity. We will be frequently using the identity

$$R(u)\, R(-u) = 1 - u^{-2}.$$

We will also work with the rational function

$$R^{\,t}(u) = 1 - Q\, u^{-1}$$

where

$$Q = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ij} = P^{\,t_1} = P^{\,t_2}.$$

We should write either $R^{\,t_1}(u)$ or $R^{\,t_2}(u)$ instead of $R^{\,t}(u)$ but we will *not* do so. Note that Q is a one-dimensional operator on $\mathbb{C}^N \otimes \mathbb{C}^N$ such that $Q^2 = N\, Q$. Hence

$$R^{\,t}(u)^{-1} = 1 + Q\,(u - N)^{-1}. \tag{2.5}$$

Proposition 2.1. *In the algebra* $(\operatorname{End}\mathbb{C}^N)^{\otimes 3}(u, v)$ *we have the identity*

$$R_{12}(u)\, R_{13}(u+v)\, R_{23}(v) = R_{23}(v)\, R_{13}(u+v)\, R_{12}(u). \tag{2.6}$$

Proof. Multiplying both sides of the relation (2.6) by $uv(u+v)$ we come to verify

$$(u + P_{12})(u + v + P_{13})(v + P_{23}) = (v + P_{23})(u + v + P_{13})(u + P_{12}). \tag{2.7}$$

Each operator P_{ij} is the image of the corresponding transposition $(ij) \in \mathfrak{S}_3$ under the natural action of the symmetric group $\mathfrak{S}_3$ on $(\mathbb{C}^N)^{\otimes 3}$ by permutations of the tensor factors. So (2.7) follows from the relations in the group algebra $\mathbb{C}[\mathfrak{S}_3]$. $\square$

The relation (2.6) is known as the *Yang–Baxter equation.* The Yang R-matrix is its simplest nontrivial solution. Below we regard $T_1(u)$ and $T_2(v)$ as formal power series with the coefficients from the algebra (2.2) where $m = 2$. We also identify $R(u-v)$ with the rational function $R(u-v) \otimes 1$ taking values in this algebra.

Proposition 2.2. *The defining relations of the algebra* $\mathrm{Y}(\mathfrak{gl}_N)$ *can be written as*

$$R(u-v)\,T_1(u)\,T_2(v) = T_2(v)\,T_1(u)\,R(u-v). \tag{2.8}$$

Proof. Let us apply both sides of (2.8) to an any basis vector $e_j \otimes e_l \in \mathbb{C}^N \otimes \mathbb{C}^N$ as explained in the beginning of this section. For the left hand side we get

$$\sum_{i,k} T_{ij}(u)\,T_{kl}(v) \otimes e_i \otimes e_k - \frac{1}{u-v} \sum_{i,k} T_{ij}(u)\,T_{kl}(v) \otimes e_k \otimes e_i\,,$$

while the right hand side gives

$$\sum_{i,k} T_{kl}(v)\,T_{ij}(u) \otimes e_i \otimes e_k - \frac{1}{u-v} \sum_{i,k} T_{kj}(v)\,T_{il}(u) \otimes e_i \otimes e_k\,.$$

Multiplying by $u-v$ and equating the coefficients of $e_i \otimes e_k$ we recover (1.3). $\square$

3. Automorphisms and anti-automorphisms

In this section, we will use the $N \times N$ matrix $T(u)$ to define several distinguished automorphisms and anti-automorphisms of the associative unital algebra $\mathrm{Y}(\mathfrak{gl}_N)$. For each of them, we will describe the $N \times N$ matrix whose ij-entry is the formal power series in u^{-1} with the coefficients being the images of the corresponding coefficients of the series $T_{ij}(u)$. For example, the assignment (3.2) below means that for all indices $r = 1, 2, \ldots$ and $i, j = 1, \ldots, N$

$$T_{ij}^{(r)} \mapsto (-1)^r\, T_{ij}^{(r)}\,.$$

Proposition 3.1. *For any* $c \in \mathbb{C}$ *an automorphism of* $\mathrm{Y}(\mathfrak{gl}_N)$ *can be defined by*

$$T(u) \mapsto T(u-c)\,. \tag{3.1}$$

Proof. The image of $T(u)$ relative to (3.1) clearly satisfies the defining relation (2.8). Further, the mapping (3.1) is obviously invertible which completes the proof. $\square$

We may regard the element $T(u)$ defined by (2.1) as a formal power series in u^{-1} whose coefficients are matrices with the entries from the algebra $\mathrm{Y}(\mathfrak{gl}_N)$. Since the leading term of this series is the identity matrix, the element $T(u)$ is invertible. We denote by $T^{-1}(u)$ the inverse element. Further, denote by $T^{\,t}(u)$ the transposed matrix for $T(u)$. Then

$$T^{\,t}(u) = \sum_{i,j=1}^{N} e_{ij} \otimes T_{ji}(u)\,.$$

Proposition 3.2. *Each of the assignments*

$$T(u) \mapsto T(-u)\,, \tag{3.2}$$

$$T(u) \mapsto T^{\,t}(u)\,, \tag{3.3}$$

$$\mathrm{S} : T(u) \mapsto T^{-1}(u) \tag{3.4}$$

defines an anti-automorphism of $\mathrm{Y}(\mathfrak{gl}_N)$.

Proof. The images $T'_{ij}(u)$ of the series $T_{ij}(u)$ under any anti-automorphism of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$ must satisfy the relations (1.3) with the opposite multiplication:

$$(u-v)\,[\,T'_{ij}(u),T'_{kl}(v)\,]=T'_{il}(u)\,T'_{kj}(v)-T'_{il}(v)\,T'_{kj}(u).$$

Exactly as in the proof of Proposition 2.2, one can show that these relations can be equivalently written in the following matrix form

$$R(u-v)\,T'_2(v)\,T'_1(u)=T'_1(u)\,T'_2(v)\,R(u-v)$$

where $T'(u)$ is the $N\times N$ matrix whose ij-th entry is $T'_{ij}(u)$. But the relation

$$R(u-v)\,T_2(-v)\,T_1(-u)=T_1(-u)\,T_2(-v)\,R(u-v)$$

follows from (2.8) if we conjugate both sides by P and replace (u,v) by $(-v,-u)$. This shows that (3.2) defines an anti-homomorphism. Furthermore, the application of the partial transposition t_1 to both sides of the relation (2.8) yields

$$T^{\,t}_1(u)\,R^{\,t}(u-v)\,T_2(v)=T_2(v)\,R^{\,t}(u-v)\,T^{\,t}_1(u). \tag{3.5}$$

Since $R(u-v)$ is fixed by the composition of t_1 with t_2, applying t_2 to (3.5) yields

$$T^{\,t}_1(u)\,T^{\,t}_2(v)\,R(u-v)=R(u-v)\,T^{\,t}_2(v)\,T^{\,t}_1(u).$$

Hence (3.3) is an anti-homomorphism. Finally, for (3.4) observe that the relation

$$R(u-v)\,T^{-1}_2(v)\,T^{-1}_1(u)=T^{-1}_1(u)\,T^{-1}_2(v)\,R(u-v)$$

is equivalent to (2.8). Note now that the mappings (3.2) and (3.3) are involutive and so these two anti-homomorphisms are bijective.

The bijectivity of the anti-homomorphism S of $\mathrm{Y}(\mathfrak{gl}_N)$ defined by (3.4) follows from the bijectivity of its square S^2 which is computed at the end of this section. $\square$

The anti-automorphisms (3.2) and (3.3) are involutive and commute with each other. Their composition is an involutive automorphism of $\mathrm{Y}(\mathfrak{gl}_N)$ such that

$$T(u)\mapsto T^{\,t}(-u)\,. \tag{3.6}$$

This automorphism of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$ will play an important role in Section 6. However, the anti-automorphism (3.4) is *not* involutive unless $N=1$. This is the antipodal map S of the Hopf algebra $\mathrm{Y}(\mathfrak{gl}_N)$, see Section 4 below.

To compute the square of the anti-homomorphism (3.4) consider $N\times N$ matrix obtained from $T^{-1}(u)$ by transposition. Let us denote this new matrix by $T^{\,\sharp}(u)$. Accordingly, the ij-th entry of this matrix will be denoted by $T^{\sharp}_{ij}(u)$. This entry is a formal power series in u^{-1} with coefficients from the algebra $\mathrm{Y}(\mathfrak{gl}_N)$. By definition,

$$\mathrm{S}:T_{ij}(u)\mapsto T^{\sharp}_{ji}(u)\,. \tag{3.7}$$

Our computation of the image of $T_{ij}(u)$ relative to S^2 is based on the next lemma.

Lemma 3.3. *There is a formal power series $Z(u)$ in u^{-1} with the coefficients from the centre of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$ and with the leading term 1 such that for all i and j*

$$\sum_{k=1}^{N} T_{ki}(u+N)\,T^{\sharp}_{kj}(u)=\delta_{ij}\,Z(u)\,. \tag{3.8}$$

Proof. Let us multiply both sides of the relation (2.8) by $T_2^{-1}(v)$ on the left and right and then apply transposition relative to the second copy of $\operatorname{End}\mathbb{C}^N$. We get

$$R^t(u-v)\,T_2^\sharp(v)\,T_1(u)=T_1(u)\,T_2^\sharp(v)\,R^t(u-v)\,.$$

Multiplying both sides of this result on the left and right $R^t(u-v)^{-1}$ we get

$$R^t(u-v)^{-1}\,T_1(u)\,T_2^\sharp(v)=T_2^\sharp(v)\,T_1(u)\,R^t(u-v)^{-1}\,. \tag{3.9}$$

Multiplying the latter equality by $u-v-N$ and then setting $u=v+N$ we get

$$Q\,T_1(v+N)\,T_2^\sharp(v)=T_2^\sharp(v)\,T_1(v+N)\,Q\,, \tag{3.10}$$

see (2.5). Because the operator Q is one-dimensional, either side of (3.10) must be equal to Q times a certain power series in v^{-1} with the coefficients from $\mathrm{Y}(\mathfrak{gl}_N)$. Denote this series by $Z(v)$. By applying the left hand side of (3.10) to the basis vector $e_i\otimes e_j$ we obtain the required equality (3.8).

It is immediate from (1.2) and (3.8) that the leading term of series $Z(v)$ is 1. Let us prove that all the coefficients of this series are central in $\mathrm{Y}(\mathfrak{gl}_N)$. We will work with the algebra (2.2) where $m=3$. By using the relations (2.8) and (3.9),

$$\begin{aligned}
&R^t_{13}(u-v)^{-1}\,R_{12}(u-v-N)\,T_1(u)\,T_2(v+N)\,T_3^\sharp(v)=\\
&R^t_{13}(u-v)^{-1}\,T_2(v+N)\,T_1(u)\,T_3^\sharp(v)\,R_{12}(u-v-N)=\\
&T_2(v+N)\,T_3^\sharp(v)\,T_1(u)\,R^t_{13}(u-v)^{-1}\,R_{12}(u-v-N)\,.
\end{aligned}$$

Note that by using the expressions (2.4) and (2.5) we obtain the equality

$$Q_{23}\,R^t_{13}(u-v)^{-1}\,R_{12}(u-v-N)=Q_{23}\,(\,1-(u-v-N)^{-2}\,).$$

So multiplying the first and third lines of previous display by Q_{23} on the left gives

$$(\,1-(u-v-N)^{-2}\,)\,T_1(u)\,Z(v)\,Q_{23}\;=\;Q_{23}\,Z(v)\,T_1(u)\,(\,1-(u-v-N)^{-2}\,)$$

where we also used (3.10). The last display shows that any generator $T_{ij}^{(r)}$ commutes with every coefficient of the series $Z(v)$. $\square$

It follows from (1.2) and (3.8) that the coefficient of the series $Z(u)$ at u^{-1} is zero. In Section 9 we will show that the coefficients of $Z(u)$ at $u^{-2},u^{-3},\ldots$ are free generators of the centre of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$. Hence we will again use Lemma 3.3.

Proposition 3.4. *The square of the map* S *is the automorphism of* $\mathrm{Y}(\mathfrak{gl}_N)$ *given by*

$$\mathrm{S}^2: T(u)\mapsto Z(u)^{-1}\,T(u+N)\,.$$

Proof. Let us apply the anti-homomorphism S to both sides of the identity

$$\sum_{k=1}^N T_{jk}(u)\,T_{ik}^\sharp(u)=\delta_{ij}\,.$$

Using (3.7) we get

$$\sum_{k=1}^N \mathrm{S}^2(T_{ki}(u))\,T_{kj}^\sharp(u)=\delta_{ij}\,.$$

Comparing this with (3.8) we conclude that $\mathrm{S}^2(T_{ki}(u))=Z(u)^{-1}\,T_{ki}(u+N)\,.$ $\square$

4. Hopf algebra structure

A *coalgebra* over the field $\mathbb{C}$ is a complex vector space A equipped with a linear map $\Delta : \mathrm{A} \to \mathrm{A} \otimes \mathrm{A}$ called the *comultiplication*, and another linear map $\varepsilon : \mathrm{A} \to \mathbb{C}$ called the *counit*, such that the following three diagrams are commutative:

$$\begin{tikzcd}
\mathrm{A} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathrm{A} \otimes \mathrm{A} \arrow[d, "\Delta\otimes\mathrm{id}"] \\
\mathrm{A} \otimes \mathrm{A} \arrow[r, "\mathrm{id}\otimes\Delta"'] & \mathrm{A} \otimes \mathrm{A} \otimes \mathrm{A}
\end{tikzcd}$$

which gives the *coassociativity* axiom of the comultiplication Δ , and

$$\begin{tikzcd}
\mathrm{A} \arrow[r, "\Delta"] \arrow[d, "\mathrm{id}"'] & \mathrm{A} \otimes \mathrm{A} \arrow[d, "\varepsilon\otimes\mathrm{id}"] \\
\mathrm{A} \arrow[r, "\cong"'] & \mathbb{C} \otimes \mathrm{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{A} \arrow[r, "\Delta"] \arrow[d, "\mathrm{id}"'] & \mathrm{A} \otimes \mathrm{A} \arrow[d, "\mathrm{id}\otimes\varepsilon"] \\
\mathrm{A} \arrow[r, "\cong"'] & \mathrm{A} \otimes \mathbb{C}
\end{tikzcd}$$

A *bialgebra* over $\mathbb{C}$ is a complex associative unital algebra A equipped with a coalgebra structure, such that Δ and ε are algebra homomorphisms. In particular, then $\Delta(1) = 1 \otimes 1$ and $\varepsilon(1) = 1$. A bialgebra A is called a *Hopf algebra*, if it is also equiped with an anti-automorphism $\mathrm{S} : \mathrm{A} \to \mathrm{A}$ called the *antipode*, such that another two diagrams are commutative:

$$\begin{tikzcd}
\mathrm{A} \arrow[r, "\delta\,\varepsilon"] \arrow[d, "\Delta"'] & \mathrm{A} \\
\mathrm{A} \otimes \mathrm{A} \arrow[r, "\mathrm{S}\otimes\mathrm{id}"'] & \mathrm{A} \otimes \mathrm{A} \arrow[u, "\mu"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{A} \arrow[r, "\delta\,\varepsilon"] \arrow[d, "\Delta"'] & \mathrm{A} \\
\mathrm{A} \otimes \mathrm{A} \arrow[r, "\mathrm{id}\otimes\mathrm{S}"'] & \mathrm{A} \otimes \mathrm{A} \arrow[u, "\mu"']
\end{tikzcd}$$

Here $\mu : \mathrm{A} \otimes \mathrm{A} \to \mathrm{A}$ is the algebra multiplication and $\delta : \mathbb{C} \to \mathrm{A}$ is the unit map of the algebra A, that is $\delta(c) = c \cdot 1$ for any $c \in \mathbb{C}$.

Proposition 4.1. *The Yangian* $\mathrm{Y}(\mathfrak{gl}_N)$ *is a Hopf algebra with comultiplication*

$$\Delta : T_{ij}(u) \mapsto \sum_{k=1}^{N} T_{ik}(u) \otimes T_{kj}(u), \tag{4.1}$$

the antipode (3.4) *and the counit* $\varepsilon : T(u) \mapsto 1$.

Proof. We start by verifying the axiom that $\Delta : \mathrm{Y}(\mathfrak{gl}_N) \to \mathrm{Y}(\mathfrak{gl}_N) \otimes \mathrm{Y}(\mathfrak{gl}_N)$ is an algebra homomorphism. We shall slightly generalize the notation used in Section 2. Let m and n be positive integers. Introduce the algebra

$$(\mathrm{End}\,\mathbb{C}^N)^{\otimes m} \otimes \mathrm{Y}(\mathfrak{gl}_N)^{\otimes n}. \tag{4.2}$$

For all $a \in \{1, \ldots, m\}$ and $b \in \{1, \ldots, n\}$ consider the formal power series in u^{-1} with the coefficients in this algebra,

$$T_{a[b]}(u) = \sum_{i,j=1}^{N} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(m-a)} \otimes 1^{\otimes(b-1)} \otimes T_{ij}(u) \otimes 1^{\otimes(n-b)}.$$

The definition of Δ can now be written in a matrix form,

$$\Delta : T(u) \mapsto T_{[1]}(u)\, T_{[2]}(u) \tag{4.3}$$

where $T_{[b]}(u)$ is an abbreviation for the series $T_{1[b]}(u)$ with the coefficients from the algebra (4.2) where $m=1$ and $n=2$. We need to show that $\Delta(T(u))$ obeys (2.8):

$$R(u-v)\, T_{1[1]}(u)\, T_{1[2]}(u)\, T_{2[1]}(v)\, T_{2[2]}(v) \;=\; T_{2[1]}(v)\, T_{2[2]}(v)\, T_{1[1]}(u)\, T_{1[2]}(u)\, R(u-v).$$

Here $m=n=2$, and $R(u-v)$ is identified with $R(u-v)\otimes 1\otimes 1$. But this relation is implied by the relation (2.8), and by the observation that the elements $T_{1[2]}(u)$ and $T_{2[1]}(v)$ commute, as well as the elements $T_{1[1]}(u)$ and $T_{2[2]}(v)$ do.

Our S is an anti-automorphism relative to multiplication due to Proposition 3.2. Since Δ is a homomorphism of algebras, the definition (4.3) implies that

$$\Delta : T^{-1}(u) \mapsto T^{-1}_{[2]}(u)\, T^{-1}_{[1]}(u)\,.$$

Therefore S is also an anti-automorphism relative to comultiplication. The other two axioms involving S are readily verified since

$$(\mathrm{S}\otimes\mathrm{id})\,\Delta : T(u) \mapsto T^{-1}_{[1]}(u)\, T_{[2]}(u)$$

and

$$(\mathrm{id}\otimes\mathrm{S})\,\Delta : T(u) \mapsto T_{[1]}(u)\, T^{-1}_{[2]}(u)$$

so that subsequent application of μ yields the identity matrix in both the cases. $\square$

We have $\varepsilon\left(T_{ij}^{(r)}\right)=0$ for $r\geqslant 1$. By expanding the formal power series in u^{-1} in (4.1) we obtain a more explicit definition of the comultiplication Δ on $\mathrm{Y}(\mathfrak{gl}_N)$,

$$\Delta\big(T_{ij}^{(r)}\big) = T_{ij}^{(r)}\otimes 1 + 1\otimes T_{ij}^{(r)} + \sum_{k=1}^{N}\sum_{s=1}^{r-1} T_{ik}^{(s)}\otimes T_{kj}^{(r-s)}\,. \tag{4.4}$$

Hence this comultiplication is not cocommutative unless $N=1$.

Proposition 4.2. *For the series $Z(u)$ defined above we have*

$$\Delta : Z(u) \mapsto Z(u)\otimes Z(u)\,.$$

Proof. The square S^2 of of the antipodal map is a coalgebra automorphism. Hence the images of $T(u)$ relative to the compositions $\Delta\,\mathrm{S}^2$ and $(\mathrm{S}^2\otimes\mathrm{S}^2)\,\Delta$ are the same. By Proposition 3.4 these images are respectively equal to

$$\Delta\,(\,Z(u)^{-1}\,T(u+N)) = \Delta\,(\,Z(u)^{-1})\,(\,T(u+N)\otimes T(u+N))$$

and

$$(\mathrm{S}^2\otimes\mathrm{S}^2)(T(u)\otimes T(u)) = (\,Z(u)^{-1}\,T(u+N))\otimes(\,Z(u)^{-1}\,T(u+N))\,.$$

Here we identify $Z(u)^{-1}$ with the series $1\otimes Z(u)^{-1}$ which takes its coefficients from $\mathrm{End}\,\mathbb{C}^N\otimes\mathrm{Y}(\mathfrak{gl}_N)$ and use the homomorphism property of Δ. By dividing the right hand sides of above two equalities by $T(u+N)\otimes T(u+N)$ and equating the results

$$\Delta : Z(u)^{-1} \mapsto Z(u)^{-1}\otimes Z(u)^{-1}\,.$$ $\square$

5. Two filtrations on the Yangian

There are two natural ascending filtrations on the associative algebra $\mathrm{Y}(\mathfrak{gl}_N)$. The first one is defined by

$$\deg T_{ij}^{(r)} = r\,.$$

For any $r \geqslant 1$ we will denote by $\widehat{T}_{ij}^{(r)}$ the image of the generator $T_{ij}^{(r)}$ in the degree r component of the corresponding graded algebra $\mathrm{gr}\,\mathrm{Y}(\mathfrak{gl}_N)$. It is immediate from the defining relations (1.4) that all these images pairwise commute. In Section 8 we will prove that these images are also algebraically independent.

Now introduce another filtration on $\mathrm{Y}(\mathfrak{gl}_N)$ by setting for $r \geqslant 1$

$$\deg' T_{ij}^{(r)} = r-1\,. \tag{5.1}$$

Let $\mathrm{gr}'\,\mathrm{Y}(\mathfrak{gl}_N)$ be the corresponding graded algebra. Let $\widetilde{T}_{ij}^{(r)}$ be the image of $T_{ij}^{(r)}$ in the component of $\mathrm{gr}'\,\mathrm{Y}(\mathfrak{gl}_N)$ of the degree $r-1$.

The graded algebra $\mathrm{gr}'\,\mathrm{Y}(\mathfrak{gl}_N)$ inherits from $\mathrm{Y}(\mathfrak{gl}_N)$ the Hopf algebra structure. Namely, by using (4.4) for any $r \geqslant 1$ we get

$$\Delta\big(\,\widetilde{T}_{ij}^{(r)}\big) = \widetilde{T}_{ij}^{(r)} \otimes 1 + 1 \otimes \widetilde{T}_{ij}^{(r)}, \tag{5.2}$$

$$\varepsilon\big(\,\widetilde{T}_{ij}^{(r)}\big) = 0 \quad \text{and} \quad \mathrm{S}\big(\,\widetilde{T}_{ij}^{(r)}\big) = -\widetilde{T}_{ij}^{(r)}. \tag{5.3}$$

For any Lie algebra $\mathfrak{g}$ over the field $\mathbb{C}$ consider the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$. There is a natural Hopf algebra structure on $\mathrm{U}(\mathfrak{g})$. The comultiplication Δ, the counit ε and the antipode S on $\mathrm{U}(\mathfrak{g})$ are defined by setting for $X \in \mathfrak{g}$

$$\Delta(X) = X \otimes 1 + 1 \otimes X\,, \tag{5.4}$$

$$\varepsilon(X) = 0 \quad \text{and} \quad \mathrm{S}(X) = -X\,. \tag{5.5}$$

In the next proposition $\mathfrak{g}$ is the polynomial current Lie algebra $\mathfrak{gl}_N[z] \cong \mathfrak{gl}_N \otimes \mathbb{C}[z]$. The latter Lie algebra is naturally graded by degrees of the indeterminate z.

Proposition 5.1. *The graded Hopf algebra* $\mathrm{gr}'\,\mathrm{Y}(\mathfrak{gl}_N)$ *is isomorphic to* $\mathrm{U}(\mathfrak{gl}_N[z])$.

Proof. Using the defining relations (1.4) we get

$$[\,\widetilde{T}_{ij}^{(r)}, \widetilde{T}_{kl}^{(s)}] = \delta_{kj}\,\widetilde{T}_{il}^{(r+s-1)} - \delta_{il}\,\widetilde{T}_{kj}^{(r+s-1)}.$$

Hence the assignments

$$E_{ij}\,z^{r-1} \mapsto \widetilde{T}_{ij}^{(r)} \quad \text{for} \quad r \geqslant 1 \tag{5.6}$$

define a surjective homomorphism

$$\mathrm{U}(\mathfrak{gl}_N[z]) \to \mathrm{gr}'\,\mathrm{Y}(\mathfrak{gl}_N) \tag{5.7}$$

of graded associative algebras. At the end of Section 8 we will show that the kernel of this homomorphism is trivial. Hence comparing the definitions (5.2),(5.3) with the general definitions (5.4),(5.5) completes the proof of Proposition 5.1. □

6. Vector and covector representations

We shall often use the matrix $T(u)$ to describe homomorphisms from $\mathrm{Y}(\mathfrak{gl}_N)$ to other algebras. Namely, let A be any unital associative algebra over the field $\mathbb{C}$. Let $X(u)$ be the $N \times N$ matrix whose ij-entry is any formal power series $X_{ij}(u)$ in u^{-1} with the leading term δ_{ij} and all coefficients from the algebra A. If $\alpha : \mathrm{Y}(\mathfrak{gl}_N) \to \mathrm{A}$ is any homomorphism, then the assignment

$$\alpha : T(u) \mapsto X(u) \tag{6.1}$$

means that every coefficient of the series $T_{ij}(u)$ gets mapped to the corresponding coefficient of the series $X_{ij}(u)$ for all indices $i, j = 1, \ldots, N$. If we regard $T(u)$ as a series in u with the coefficients from the algebra $\mathrm{End}\,\mathbb{C}^N \otimes \mathrm{Y}(\mathfrak{gl}_N)$ then, more formally, we may write

$$\mathrm{id} \otimes \alpha : T(u) \mapsto X(u)$$

instead of (6.1). Here

$$X(u) = \sum_{i,j=1}^{N} e_{ij} \otimes X_{ij}(u),$$

is regarded as a series in u with coefficients from the algebra $\mathrm{End}\,\mathbb{C}^N \otimes \mathrm{A}$; cf. (2.1).

Setting $\mathrm{A} = \mathrm{End}\,\mathbb{C}^N$ and $X(u) = R(u)$ above, we can define a homomorphism $\mathrm{Y}(\mathfrak{gl}_N) \to \mathrm{End}\,\mathbb{C}^N$ by the assignment $T(u) \mapsto R(u)$. To prove the homomorphism property by using the matrix form (2.8) of the defining relations of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$, we have to check the equality of rational functions in u and v with values in the algebra $(\mathrm{End}\,\mathbb{C}^N)^{\otimes 3}$,

$$R_{12}(u-v)\, R_{13}(u)\, R_{23}(v) = R_{23}(v)\, R_{13}(u)\, R_{12}(u-v)\,.$$

But this equality is just another form of (2.6). In other words, the assignment $T(u) \mapsto R(u)$ defines a representation of $\mathrm{Y}(\mathfrak{gl}_N)$ on the vector space $\mathbb{C}^N$. Here

$$T_{ij}(u) \mapsto \delta_{ij} - e_{ji}\, u^{-1}$$

by (2.3) and (2.4). Note that this representation of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$ can also be obtained by pulling the defining representation $E_{ij} \mapsto e_{ij}$ of the Lie algebra $\mathfrak{gl}_N$ back through the homomorphism (1.8). This remark justifies the definition (1.8).

By pulling the defining representation $E_{ij} \mapsto e_{ij}$ of the Lie algebra $\mathfrak{gl}_N$ back through the homomorphism (1.6), we get the representation of $\mathrm{Y}(\mathfrak{gl}_N)$ such that

$$T_{ij}(u) \mapsto \delta_{ij} + e_{ij}\, u^{-1}\,.$$

Hence this representation can be described by the assignment $T(u) \mapsto R^{\,t}(-u)$. Observe that the representations $T(u) \mapsto R(u)$ and $T(u) \mapsto R^{\,t}(-u)$ differ by the involutive automorphism (3.6) of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$.

By pulling the representation $T(u) \mapsto R(u)$ back through the automorphism (3.1) of $\mathrm{Y}(\mathfrak{gl}_N)$ for any $c \in \mathbb{C}$, we get the representation of $\mathrm{Y}(\mathfrak{gl}_N)$ on the vector space $\mathbb{C}^N$, such that $T(u) \mapsto R(u-c)$. It is called a *vector representation* of $\mathrm{Y}(\mathfrak{gl}_N)$, and is denoted by ρ_c. Thus

$$\rho_c : T_{ij}(u) \mapsto \delta_{ij} - e_{ji}\,(u-c)^{-1}$$

or equivalently,

$$\rho_c : T_{ij}^{(r)} \mapsto -\,c^{\,r-1}\, e_{ji} \quad \text{for any} \quad r \geqslant 1\,. \tag{6.2}$$

By pulling the representation $T(u) \mapsto R^t(-u)$ back through the automorphism (3.1), we get the representation of $\mathrm{Y}(\mathfrak{gl}_N)$ on $\mathbb{C}^N$, such that $T(u) \mapsto R^t(c-u)$. It is called a *covector representation* of $\mathrm{Y}(\mathfrak{gl}_N)$, and is denoted by σ_c. Thus

$$\sigma_c : T_{ij}(u) \mapsto \delta_{ij} + e_{ij}\,(u-c)^{-1}$$

or equvalently,

$$\sigma_c : T_{ij}^{(r)} \mapsto c^{\,r-1} e_{ij} \quad \text{for any} \quad r \geqslant 1\,. \tag{6.3}$$

In Section 5 we introduced an ascending filtration on algebra $\mathrm{Y}(\mathfrak{gl}_N)$ such that any generator $T_{ij}^{(r)}$ of $\mathrm{Y}(\mathfrak{gl}_N)$ has the degree $r-1$. We denoted the corresponding graded algebra by $\mathrm{gr}'\,\mathrm{Y}(\mathfrak{gl}_N)$ and defined a surjective homomorphism (5.7) by (5.6).

Under this homomorphism the element $T_{ij}^{(r)}$ of $\mathrm{Y}(\mathfrak{gl}_N)$, or rather its image $\widetilde{T}_{ij}^{(r)}$ in $\mathrm{gr}'\,\mathrm{Y}(\mathfrak{gl}_N)$, corresponds to the generator $E_{ij}\,z^{\,r-1}$ of $\mathrm{U}(\mathfrak{gl}_N[z])$. One can define a representation $\widetilde{\sigma}_c$ of the algebra $\mathrm{U}(\mathfrak{gl}_N[z])$ on the vector space $\mathbb{C}^N$ by

$$\widetilde{\sigma}_c : E_{ij}\,z^{\,r-1} \mapsto c^{\,r-1} e_{ij} \quad \text{for any} \quad r \geqslant 1\,, \tag{6.4}$$

so that

$$\widetilde{\sigma}_c(\,E_{ij}\,z^{\,r-1}\,) = \sigma_c(\,T_{ij}^{(r)}\,)\,.$$

The representation $\widetilde{\sigma}_c$ is an example of an evaluation representation of $\mathrm{U}(\mathfrak{gl}_N[z])$, see the general definition in Section 7 below.

7. Evaluation representations

For any Lie algebra $\mathfrak{a}$ over $\mathbb{C}$ consider the corresponding polynomial current Lie algebra $\mathfrak{a}[z] = \mathfrak{a} \otimes \mathbb{C}[z]$. Let θ be any representation of $\mathfrak{a}$ on the vector space $\mathbb{C}^N$, and c be any complex number. Then one can define a representation of $\mathfrak{a}[z]$ by

$$X\,z^s \mapsto c^{\,s}\theta(X) \quad \text{for any} \quad s \geqslant 0\,.$$

This is the *evaluation representation* of the Lie algebra $\mathfrak{a}[z]$, corresponding to θ at the point $z=c$ of the complex plane $\mathbb{C}$. When $\mathfrak{a} = \mathfrak{gl}_N$ and θ is the defining representation of the Lie algebra $\mathfrak{gl}_N$ on $\mathbb{C}^N$, we obtain $\widetilde{\sigma}_c$ in this way.

We will need the following general property of evaluation representations. For any $c_1,\dots,c_n \in \mathbb{C}$ let us denote by $\theta_{c_1\dots c_n}$ the tensor product of the evaluation representations of the Lie algebra $\mathfrak{a}[z]$ corresponding to θ at the points $c_1,\dots,c_n$. We extend the representation $\theta_{c_1\dots c_n}$ to the universal enveloping algebra $\mathrm{U}(\mathfrak{a}[z])$.

Lemma 7.1. *Suppose that the Lie algebra $\mathfrak{a}$ is finite-dimensional, and θ is its faithful representation. Let the parameters $c_1,\dots,c_n$ and integer $n \geqslant 0$ vary. Then the intersection in $\mathrm{U}(\mathfrak{a}[z])$ of the kernels of all representations $\theta_{c_1\dots c_n}$ is trivial.*

Proof. Using the faithful representation θ of the Lie algebra $\mathfrak{a}$, we can identify $\mathfrak{a}[z]$ with a subalgebra of the Lie algebra $\mathfrak{gl}_N[z]$. Hence it suffices to consider the case when $\mathfrak{a}$ is the Lie algebra $\mathfrak{gl}_N$, and $\theta : \mathfrak{gl}_N \to \mathrm{End}\,\mathbb{C}^N$ is the defining representation.

Choose any basis $X_1,\dots,X_{N^2}$ in $\mathfrak{gl}_N$ such that one of the basis vectors is

$$I = E_{11} + \dots + E_{NN}\,.$$

To distinguish between the algebras $\mathrm{U}(\mathfrak{gl}_N)$ and $\mathrm{End}\,\mathbb{C}^N$, the operators on $\mathbb{C}^N$ corresponding to the elements $X_1,\dots,X_{N^2} \in \mathfrak{gl}_N$ will be denoted by $x_1,\dots,x_{N^2}$ respectively. Note that one of these operators is the identity operator 1.

Take any finite non-zero linear combination C of the products

$$(X_{a_1}z^{s_1})\dots(X_{a_m}z^{s_m}) \in \mathrm{U}(\mathfrak{gl}_N[z]) \tag{7.1}$$

where the indices $a_1,\ldots,a_m$ and $s_1,\ldots,s_m \geqslant 0$ may vary. The number m of factors in (7.1) may also vary. Let $\mathfrak{S}_m$ be the symmetric group acting on the set $\{1,\ldots,m\}$. For each fixed $m \geqslant 0$, we will suppose that the elements

$$\sum_{q\in\mathfrak{S}_m} (\,X_{a_{q(1)}} z^{s_{q(1)}}) \otimes \ldots \otimes (\,X_{a_{q(m)}} z^{s_{q(1)}}) \in (\,\mathfrak{gl}_N[z]\,)^{\otimes m} \tag{7.2}$$

corresponding to the products (7.1) which have exactly m factors and appear in C with non-zero coefficients, are linearly independent. We may suppose so without loss of generality, due to the commutation relations in the algebra $\mathrm{U}(\mathfrak{gl}_N[z])$. Using the natural identification of vector spaces

$$(\,\mathfrak{gl}_N[z]\,)^{\otimes m} = \mathfrak{gl}_N^{\otimes m}[z_1,\ldots,z_m]\,,$$

the sum (7.2) may be also regarded as a polynomial function in m independent complex variables $z_1,\ldots,z_m$. This function takes values in the vector space $\mathfrak{gl}_N^{\otimes m}$.

For every element (7.1) appearing in the linear combination C, suppose that

$$X_{a_1},\ldots,X_{a_l} \neq I \quad \text{and} \quad X_{a_{l+1}} = \cdots = X_{a_m} = I$$

for a certain number $l \geqslant 0$. Then $x_{a_1},\ldots,x_{a_l} \neq 1$ and $x_{a_{l+1}} = \ldots = x_{a_m} = 1$. We may suppose so without loss of generality, as the elements $I, Iz, Iz^2, \ldots \in \mathfrak{gl}_N[z]$ are central. Further, suppose that $s_{l+1} \geqslant \cdots \geqslant s_m$. Of course, the number l here may depend on the given element (7.1); let l_0 be the maximum of these numbers.

Let us consider two cases. First suppose that $l_0 = 0$, so that $l = 0$ for every element (7.1) appearing in the linear combination C. The image of (7.1) under the representation $\theta_{c_1\ldots c_n}$ of $\mathrm{U}(\mathfrak{gl}_N[z])$ is then the operator of multiplication by

$$\prod_{k=1}^{m} (\,c_1^{s_k} + \cdots + c_n^{s_k}) \in \mathbb{C}. \tag{7.3}$$

Since $s_1 \geqslant \cdots \geqslant s_m$ here, any non-trivial linear combination of the scalars (7.3) corresponding to different sequences $s_1,\ldots,s_m$ cannot vanish identically for all $n \geqslant 0$ and $c_1,\ldots,c_n \in \mathbb{C}$. This proves the claim when $l_0 = 0$.

Second, suppose that $l_0 > 0$. For any element (7.1) appearing in the linear combination C, take its image under the representation $\theta_{c_1\ldots c_n}$ with $n \geqslant l$. This image belongs to the algebra $\mathrm{End}\,(\mathbb{C}^N)^{\otimes n}$, which we will identify with $(\mathrm{End}\,\mathbb{C}^N)^{\otimes n}$. Let V_l be the subspace in $(\mathrm{End}\,\mathbb{C}^N)^{\otimes n}$ spanned by the elements $x_{b_1}\otimes\ldots\otimes x_{b_n}$ where at least one of the first l tensor factors $x_{b_1},\ldots,x_{b_l}$ is 1. The indices $b_1,\ldots,b_n$ here range over $1,\ldots,N^2$. Modulo V_l the image of (7.1) in $(\mathrm{End}\,\mathbb{C}^N)^{\otimes n}$ equals the sum

$$\sum_{p\in\mathfrak{S}_l} (\,c_1^{s_{p(1)}} x_{a_{p(1)}}) \otimes \ldots \otimes (\,c_l^{s_{p(l)}} x_{a_{p(l)}}) \otimes 1^{\,\otimes(n-l)} \tag{7.4}$$

multiplied by

$$\prod_{k=l+1}^{m} (\,c_1^{s_k} + \cdots + c_n^{s_k})\,. \tag{7.5}$$

Notice that the sum (7.4) does not belong to V_l unless this sum is zero or $l = 0$.

In the linear combination C, take the terms where $l = l_0$. Let $D \in (\mathrm{End}\,\mathbb{C}^N)^{\otimes n}$ be the image of the sum of these terms under the representation $\theta_{c_1\ldots c_n}$. We assume that here $n \geqslant l_0$. The images of the terms with $l < l_0$ under the representation $\theta_{c_1\ldots c_n}$ belong to the subspace V_{l_0}. We will show that $D \neq 0$ for some $n \geqslant l_0$ and $c_1,\ldots,c_n \in \mathbb{C}$. Then $D \notin V_{l_0}$, and Lemma 7.1 will follow.

Observe that the sum (7.4) does not depend on the parameters $c_{l+1},\dots,c_n$ while the product (7.5) can depend on these parameters. Due to this observation, we may now assume that for all terms in the linear combination C with $l=l_0$, the sequences $s_{l_0+1},\dots,s_m$ are the same; see the case $l_0=0$ considered above. Moreover, we may assume that $m=l_0$ for all these terms. But under the latter assumption, the inequality $D\neq 0$ for some $c_1,\dots,c_{l_0}\in\mathbb{C}$ follows from the linear independence of the elements (7.2) with $m=l_0$. □

8. Poincaré–Birkhoff–Witt theorem

Let us now make use of the bialgebra structure on $\mathrm{Y}(\mathfrak{gl}_N)$. For any $c_1,\dots,c_n\in\mathbb{C}$ take the tensor product of the vector representations $\rho_{c_1},\dots,\rho_{c_n}$ of $\mathrm{Y}(\mathfrak{gl}_N)$. We get a representation

$$\rho_{c_1\dots c_n}:\mathrm{Y}(\mathfrak{gl}_N)\to(\mathrm{End}\,\mathbb{C}^N)^{\otimes n}.$$

If $n=0$, the representation $\rho_{c_1\dots c_n}$ is understood as the counit homomorphism $\varepsilon:\mathrm{Y}(\mathfrak{gl}_N)\to\mathbb{C}$. Using the matrix form (4.3) of the definition of the comultiplication on $\mathrm{Y}(\mathfrak{gl}_N)$, we see that

$$\mathrm{id}\otimes\rho_{c_1\dots c_n}:T(u)\mapsto R_{12}(u-c_1)\dots R_{1,n+1}(u-c_n).$$

Here we apply the convention made in the beginning of Section 6 to the algebra $\mathrm{A}=(\mathrm{End}\,\mathbb{C}^N)^{\otimes n}$ and to the homomorphism $\alpha=\rho_{c_1\dots c_n}$.

The tensor product of the covector representations $\sigma_{c_1},\dots,\sigma_{c_n}$ will be denoted by $\sigma_{c_1\dots c_n}$. By using the matrix form (4.3) of the definition of the comultiplication on $\mathrm{Y}(\mathfrak{gl}_N)$ again, we see that

$$\mathrm{id}\otimes\sigma_{c_1\dots c_n}:T(u)\mapsto R^t_{12}(c_1-u)\dots R^t_{1,n+1}(c_n-u).$$

By using Lemma 7.1, we will now prove the following proposition.

Proposition 8.1. *Let the parameters $c_1,\dots,c_n\in\mathbb{C}$ and the integer $n\geqslant 0$ vary. Then the intersection of the kernels of all representations $\sigma_{c_1\dots c_n}$ is trivial.*

Proof. Take any finite linear combination A of the products

$$T^{(r_1)}_{i_1j_1}\dots T^{(r_m)}_{i_mj_m}\in\mathrm{Y}(\mathfrak{gl}_N)$$

with certain coefficients

$$A^{\,r_1\dots r_m}_{\,i_1j_1\dots i_mj_m}\in\mathbb{C}$$

where the indices $r_1,\dots,r_m\geqslant 1$ and the number $m\geqslant 0$ may vary, as well as the indices $i_1,j_1,\dots,i_m,j_m$. Suppose that $A\neq 0$ as an element of $\mathrm{Y}(\mathfrak{gl}_N)$.

The algebra $\mathrm{Y}(\mathfrak{gl}_N)$ comes with an ascending filtration such that $T^{(r)}_{ij}$ has the degree $r-1$. Let d be the degree of A rtelative to this filtration. Let B be the image of A in the degree d component of the graded algebra $\mathrm{gr}'\,\mathrm{Y}(\mathfrak{gl}_N)$. Then $B\neq 0$.

We can also assume that

$$A^{\,r_1\dots r_m}_{\,i_1j_1\dots i_mj_m}=0\quad\text{if}\quad r_1+\dots+r_m>d+m.$$

Let C be the sum of the elements of the algebra $\mathrm{U}(\mathfrak{gl}_N[z])$,

$$\sum_{r_1+\dots+r_m=d+m}A^{\,r_1\dots r_m}_{\,i_1j_1\dots i_mj_m}\,(E_{i_1j_1}z^{\,r_1-1})\dots(E_{i_mj_m}z^{\,r_m-1}).$$

The image of C under the homomorphism (5.7) equals B. In particular, $C\neq 0$.

Consider the image of A under the representation $\sigma_{c_1\ldots c_n}$. This image depends on $c_1,\ldots,c_n$ polynomially. The degree of this polynomial does not exceed d by the definition (6.3). Let D be the sum of the terms of degree d of this polynomial.

Now equip the tensor product $\mathrm{Y}(\mathfrak{gl}_N)^{\otimes n}$ with the ascending filtration where the degree is the sum of the degrees on the tensor factors. Then under the n-fold comultiplication $\mathrm{Y}(\mathfrak{gl}_N)\to\mathrm{Y}(\mathfrak{gl}_N)^{\otimes n}$

$$T_{ij}^{(r)}\mapsto\sum_{b=1}^{n}1^{\otimes(b-1)}\otimes T_{ij}^{(r)}\otimes 1^{\otimes(n-b)}\quad\text{plus terms of degree less than}\quad r-1\,,$$

see (4.4). But under the n-fold comultiplication $\mathrm{U}(\mathfrak{gl}_N[z])\to\mathrm{U}(\mathfrak{gl}_N[z])^{\otimes n}$,

$$E_{ij}\,z^{r-1}\mapsto\sum_{b=1}^{n}1^{\otimes(b-1)}\otimes(E_{ij}\,z^{r-1})\otimes 1^{\otimes(n-b)}\,.$$

The definitions (6.3) and (6.4) now imply that the sum $D\in(\operatorname{End}\mathbb{C}^N)^{\otimes n}$ coincides with the image of the sum $C\in\mathrm{U}(\mathfrak{gl}_N[z])$ under the tensor product of the evaluation representations $\widetilde{\sigma}_{c_1},\ldots,\widetilde{\sigma}_{c_n}$. Since $C\neq 0$, using Lemma 7.1 we can choose n and $c_1,\ldots,c_n$ so that $D\neq 0$. Then $\sigma_{c_1\ldots c_n}(A)\neq 0$ by the definition of D. $\square$

Proposition 8.2. *Let the parameters $c_1,\ldots,c_n\in\mathbb{C}$ and the integer $n\geqslant 0$ vary. Then the intersection of the kernels of all representations $\rho_{c_1\ldots c_n}$ is trivial.*

The proof of Proposition 8.2 is similar to that of Proposition 8.1 and is omitted. We will now prove the injectivity of homomorphism (5.7) by modifying the logic of our proof of Proposition 8.1. Take any finite linear combination C of the products

$$(E_{i_1j_1}\,z^{r_1-1})\ldots(E_{i_mj_m}\,z^{r_m-1})\in\mathrm{U}(\mathfrak{gl}_N[z])$$

with certain coefficients

$$C^{\,r_1\ldots r_m}_{\,i_1j_1\ldots i_mj_m}\in\mathbb{C}$$

where the indices $r_1,\ldots,r_m\geqslant 1$ and the number $m\geqslant 0$ may vary, as well as the indices $i_1,j_1,\ldots,i_m,j_m$. Suppose that $C\neq 0$ as an element of $\mathrm{U}(\mathfrak{gl}_N[z])$.

The algebra $\mathrm{U}(\mathfrak{gl}_N[z])$ is graded so that for any integer $s\geqslant 0$, the generator $E_{ij}\,z^s$ has the degree s. The homomorphism (5.7) preserves the degree. Without loss of generality suppose that the element C is homogeneous of degree d, that is

$$C^{\,r_1\ldots r_m}_{\,i_1j_1\ldots i_mj_m}=0\quad\text{ if }\quad r_1+\cdots+r_m\neq d+m\,.$$

Now define the element $A\in\mathrm{Y}(\mathfrak{gl}_N)$ as the sum

$$\sum_{r_1+\cdots+r_m=d+m}C^{\,r_1\ldots r_m}_{\,i_1j_1\ldots i_mj_m}\,T^{(r_1)}_{i_1j_1}\ldots T^{(r_m)}_{i_mj_m}\,.$$

Let B be the image of A in the d-th component of the graded algebra $\mathrm{gr}'\,\mathrm{Y}(\mathfrak{gl}_N)$. The element B coincides with the image of C under the homomorhism (5.7).

Now let $D\in(\operatorname{End}\mathbb{C}^N)^{\otimes n}$ be the image of C under the tensor product of the evaluation representations $\widetilde{\sigma}_{c_1},\ldots,\widetilde{\sigma}_{c_n}$. The image of A under the representation $\sigma_{c_1\ldots c_n}$ depends on $c_1,\ldots,c_n$ polynomially. The degree of this polynomial does not exceed d by (6.3). The sum of the terms of degree d of this polynomial equals D, see the proof of Proposition 8.1. Since $C\neq 0$, using Lemma 7.1 we can choose n and $c_1,\ldots c_n$ so that $D\neq 0$. Then $\deg' A=d$. Indeed, if $\deg' A<d$ then the degree of the polynomial $\sigma_{c_1,\ldots,c_n}(A)$ would be also less then d. This would contradict to the non-vanishing of D. By the definition of the element $B\in\mathrm{gr}'\,\mathrm{Y}(\mathfrak{gl}_N)$, the equality $\deg' A=d$ means that $B\neq 0$. So the homomorphism (5.7) is injective.

Let us now invoke the classical Poincaré–Birkhoff–Witt theorem for the universal enveloping algebras of Lie algebras [2, Section 2.1]. By applying this theorem to the Lie algebra $\mathfrak{gl}_N[z]$ we now obtain its analogue for the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$.

Theorem 8.3. *Given an arbitrary linear ordering of the set of generators $T_{ij}^{(r)}$ with $r \geqslant 1$, any element of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$ can be uniquely written as a linear combination of ordered monomials in these generators.*

Corollary 8.4. *The graded algebra $\mathrm{gr}\,\mathrm{Y}(\mathfrak{gl}_N)$ is the algebra of polynomials in the generators $\widehat{T}_{ij}^{(r)}$ with $r \geqslant 1$.*

9. Centre of the Yangian

Let $\mathfrak{a}$ be any Lie algebra over the field $\mathbb{C}$. Consider the corresponding polynomial current Lie algebra $\mathfrak{a}[z]$. In the proof of Theorem 9.3 we will use a general property of the universal enveloping algebra $\mathrm{U}(\mathfrak{a}[z])$. It is stated as the lemma below.

Lemma 9.1. *Suppose that the Lie algebra $\mathfrak{a}$ is finite-dimensional and has the trivial centre. Then the centre of the algebra $\mathrm{U}(\mathfrak{a}[z])$ is also trivial, that is equal to $\mathbb{C}$.*

Proof. Consider adjoint action of the Lie algebra $\mathfrak{a}[z]$ on its symmetric algebra. It suffices to prove that the space of invariants of this action is trivial.

Let A be any element of the symmetric algebra of $\mathfrak{a}[z]$ invariant under the adjoint action. Let $M = \dim \mathfrak{a}$. Choose any basis $X_1, \ldots, X_M$ of $\mathfrak{a}$ and let

$$[\, X_p \,, X_q \,] = \sum_{r=1}^{M} c_{pq}^{\,r} \, X_r$$

where $c_{pq}^{\,r} \in \mathbb{C}$. Let L be the minimal non-negative integer such that

$$A \;=\; \sum_{d_1,\ldots,d_M} A_{\,d_1 \ldots\, d_M} \, (X_1 \, z^{\,L})^{d_1} \ldots (X_M \, z^{\,L})^{d_M}$$

where $d_1, \ldots, d_M$ range over non-negative integers and $A_{\,d_1 \ldots d_M}$ is a polynomial in the basis elements $X_p \, z^{\,s}$ of $\mathfrak{a}[z]$ with $1 \leqslant p \leqslant M$ and $0 \leqslant s < L$ only. We have

$$\mathrm{ad}\,(X_p \, z)(A) = 0$$

for every index $p = 1, \ldots, M$. The component of the left hand side of this equation that involves the basis elements of $\mathfrak{a}[z]$ of the form $X_r \, z^{\,L+1}$ must be zero. Thus

$$\sum_{d_1,\ldots,d_M} A_{\,d_1 \ldots\, d_M} \sum_{q,r=1}^{M} c_{pq}^{\,r} \, d_q \, (X_1 \, z^{\,L})^{d_1} \ldots (X_q \, z^{\,L})^{d_q - 1} \ldots (X_M \, z^{\,L})^{d_M} \, X_r \, z^{\,L+1} = 0 \,.$$

Taking here the coefficient of $X_r \, z^{\,L+1}$ we obtain that

$$\sum_{d_1,\ldots,d_M} A_{\,d_1 \ldots\, d_M} \sum_{q=1}^{M} c_{pq}^{\,k} \, d_q \, (X_1 \, z^{\,L})^{d_1} \ldots (X_q \, z^{\,L})^{d_q - 1} \ldots (X_M \, z^{\,L})^{d_M} = 0 \,.$$

If follows that for any non-negative integers $d_1^{\,\prime}, \ldots, d_M^{\,\prime}$ we have

$$\sum_{q=1}^{M} A_{\,d_1^{\prime} \ldots\, d_q^{\prime}+1 \ldots\, d_M^{\prime}} \, c_{pq}^{\,r} \, (d_q^{\,\prime} + 1) = 0 \quad \text{where} \quad p, r = 1, \ldots, M \,. \tag{9.1}$$

Let us now fix $d'_1,\ldots,d'_M$ and observe that the elements $X'_q=(d'_q+1)\,X_q$ with $q=1,\ldots,M$ also form a basis of $\mathfrak{a}$. Since the centre of $\mathfrak{a}$ is trivial, the system

$$[\,X_p\,,\sum_{q=1}^{M} a_q\,X'_q\,]=0\quad\text{where}\quad p=1,\ldots,M$$

of linear equations on $a_1,\ldots,a_M\in\mathbb{C}$ has only trivial solution. It can be written as

$$\sum_{q=1}^{M} a_q\,c_{pq}^{\,r}\,(d'_q+1)=0\quad\text{where}\quad p,r=1,\ldots,M\,.$$

Hence by comparing (9.1) with the latter system we obtain that $A_{\,d'_1\ldots\,d'_q+1\ldots\,d'_M}=0$ for every $q=1,\ldots,M$. It now follows that $A\in\mathbb{C}$, and $L=0$ in particular. $\square$

Now consider the series $Z(u)$ defined by (3.8). For any $r\geqslant 1$ let $Z^{(r)}$ be the coefficient of this series at u^{-r}. Just before stating Proposition 3.4 we noted that $Z^{(1)}=0$. Hence

$$Z(u)=1+Z^{(2)}u^{-2}+Z^{(3)}u^{-3}+\ldots\,.$$

Proposition 9.2. *For any $r\geqslant 2$ the element $Z^{(r)}\in\mathrm{Y}(\mathfrak{gl}_N)$ has the degree $r-2$ relative to the filtration* (5.1). *Its image in the graded algebra* $\mathrm{gr}'\,\mathrm{Y}(\mathfrak{gl}_N)$ *is equal to*

$$(\,1-r\,)\sum_{i=1}^{N}\widetilde{T}_{ii}^{\,(r-1)}\,.$$

Proof. Let us expand the factor $T_{ki}(u+N)$ appearing in the definition (3.8) as a formal power series in u^{-1}. The result has the form

$$T_{ki}(u)+N\,\dot{T}_{ki}(u)+X_{ki}(u)$$

where

$$\dot{T}_{ki}(u)=-\,T_{ki}^{(1)}u^{-2}-2\,T_{ki}^{(2)}u^{-3}-\ldots$$

is the formal derivative of the series $T_{ki}(u)$ and

$$X_{ki}(u)=X_{ki}^{(3)}u^{-3}+X_{ki}^{(4)}u^{-4}+\ldots$$

is a series with coefficients $X_{ki}^{(r)}\in\mathrm{Y}(\mathfrak{gl}_N)$ such that $\deg'X_{ki}^{(r)}=r-3$ for $r\geqslant 3$. By setting $i=j$ in (3.8) and summing over $i=1,\ldots,N$ we now get the equality

$$N+\sum_{i,k=1}^{N}\big(\,N\,\dot{T}_{ki}(u)+X_{ki}(u)\big)\,T_{ki}^{\sharp}(u)=N\,Z(u)\,. \tag{9.2}$$

Here we used the definition of the matrix $T^{\,\sharp}(u)$ as the transposed inverse of $T(u)$.

The leading term of the series $T_{ki}^{\sharp}(u)$ is δ_{ik} while for any $r\geqslant 1$ the coefficient of this series at u^{-r} has the degree $r-1$ relative to (5.1). It follows that modulo lower degree elements, for any $r\geqslant 1$ the coefficient at u^{-r} of the series at the left hand side of (9.2) equals

$$N\sum_{i,k=1}^{N}(1-r)\,T_{ki}^{(r-1)}\,\delta_{ik}=N\sum_{i=1}^{N}(1-r)\,T_{ii}^{(r-1)}\,.$$

Hence Proposition 9.2 follows from (9.2). Also we see once again that $Z^{(1)}=0$. $\square$

Theorem 9.3. *The coefficients $Z^{(2)},Z^{(3)},\ldots$ of the series $Z(u)$ are free generators of the centre of the associative algebra* $\mathrm{Y}(\mathfrak{gl}_N)$.

Proof. Let us apply Lemma 9.1 to the special linear Lie algebra $\mathfrak{a} = \mathfrak{sl}_N$. Since the centre of the universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_N[z])$ is trivial, the decomposition

$$\mathfrak{gl}_N[z] = \mathfrak{sl}_N[z] \oplus \mathbb{C}[z] \sum_{i=1}^{N} E_{ii}$$

of Lie algebras implies that the centre of $\mathrm{U}(\mathfrak{gl}_N[z])$ is generated by the elements

$$\sum_{i=1}^{N} E_{ii}\, z^{r-1} \tag{9.3}$$

where $r \geqslant 1$. Moreover these generators are free due to the Poincaré–Birkhoff–Witt theorem [2, Section 2.1] applied to the Lie algebra $\mathfrak{gl}_N[z]$.

Under the isomorphism (5.7), the elements (9.3) go respectively to the elements

$$\sum_{i=1}^{N} \widetilde{T}_{ii}^{(r)}$$

where again $r \geqslant 1$. Therefore the latter elements are free generators of the centre of the algebra $\mathrm{gr}'\,\mathrm{Y}(\mathfrak{gl}_N)$, see Proposition 5.1. On the other hand, we have already proved that the elements $Z^{(2)}, Z^{(3)}, \ldots$ of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$ belong to its centre, see Lemma 3.3. Hence Theorem 9.3 follows from Proposition 9.2. $\square$

10. Dual Yangian

The *dual Yangian* for the Lie algebra $\mathfrak{gl}_N$, denoted by $\mathrm{Y}^*(\mathfrak{gl}_N)$, is an associative unital algebra over the field $\mathbb{C}$ with a countable set of generators

$$T_{ij}^{(-1)},\ T_{ij}^{(-2)}, \ldots \quad \text{where} \quad i, j = 1, \ldots, N\,.$$

To write down the defining relations for these generators, introduce the series

$$T_{ij}^*(v) = \delta_{ij} + T_{ij}^{(-1)} + T_{ij}^{(-2)}\, v + T_{ij}^{(-3)} v^2 + \ldots \in \mathrm{Y}^*(\mathfrak{gl}_N)[[v]]\,. \tag{10.1}$$

The reason for separating the term δ_{ij} in (10.1) will become apparent in the next section. Now combine all the series (10.1) into the single element

$$T^*(v) = \sum_{i,j=1}^{N} T_{ij}^*(v) \otimes e_{ij} \in \mathrm{Y}^*(\mathfrak{gl}_N)[[v]] \otimes \mathrm{End}\,\mathbb{C}^N\,. \tag{10.2}$$

We will write the defining relations of the algebra $\mathrm{Y}^*(\mathfrak{gl}_N)$ first in their matrix form, to be compared with (2.8). For any positive integer n, consider the algebra

$$\mathrm{Y}^*(\mathfrak{gl}_N) \otimes (\mathrm{End}\,\mathbb{C}^N)^{\otimes n}. \tag{10.3}$$

For any index $b \in \{1, \ldots, n\}$ introduce the formal power series in the variable v with the coefficients from the algebra (10.3),

$$T_b^*(v) = \sum_{i,j=1}^{N} T_{ij}^*(v) \otimes 1^{\otimes(b-1)} \otimes e_{ij} \otimes 1^{(n-b)}. \tag{10.4}$$

Here the belongs to the b-th copy of $\mathrm{End}\,\mathbb{C}^N$. Setting $n = 2$ and identifying $R(u-v)$ with $1 \otimes R(u-v)$, the defining relations of $\mathrm{Y}^*(\mathfrak{gl}_N)$ can be written as

$$T_1^*(u)\, T_2^*(v)\, R(u-v) = R(u-v)\, T_2^*(v)\, T_1^*(u)\,. \tag{10.5}$$

The relation (10.5) is equivalent to the collection of relations

$$(u-v)\,[T^*_{ij}(u),T^*_{kl}(v)]=T^*_{il}(u)\,T^*_{kj}(v)-T^*_{il}(v)\,T^*_{kj}(u)$$

for all $i,j,k,l=1,\dots,N$. We omit the proof of the equivalence, because it is similar to the proof of Proposition 2.2. The last displayed relation can be rewritten as

$$[T^*_{ij}(u),T^*_{kl}(v)]=\sum_{p=0}^{\infty}u^{-p-1}v^p\,\Big(T^*_{il}(u)\,T^*_{kj}(v)-T^*_{il}(v)\,T^*_{kj}(u)\Big).$$

Expanding here the series in u,v and equating the coefficients at $u^{r-1}v^{s-1}$ we get

$$[T_{ij}^{(-r)},T_{kl}^{(-s)}]\,=\,\delta_{kj}\,T_{il}^{(-r-s)}-\delta_{il}\,T_{kj}^{(-r-s)}+ \sum_{b=1}^{s}\Big(T_{il}^{(b-r-s-1)}\,T_{kj}^{(-b)}-T_{il}^{(-b)}\,T_{kj}^{(b-r-s-1)}\Big). \tag{10.6}$$

The proof of next proposition is similar to that of Proposition 4.1 and is omitted.

Proposition 10.1. *The dual Yangian* $\mathrm{Y}^*(\mathfrak{gl}_N)$ *is a bialgebra over the field* $\mathbb{C}$ *with the counit defined* $\varepsilon:T^*(v)\mapsto 1$ *and the comultiplication defined by*

$$\Delta:T^*_{ij}(v)\mapsto\sum_{k=1}^{N}T^*_{ik}(v)\otimes T^*_{kj}(v)\,. \tag{10.7}$$

Expanding the power series in v in (10.7) and using the axiom $\Delta(1)=1\otimes 1$, we get a more explicit definition of the comultiplication on the dual Yangian $\mathrm{Y}^*(\mathfrak{gl}_N)$,

$$\Delta\big(T_{ij}^{(-r)}\big)=T_{ij}^{(-r)}\otimes 1+1\otimes T_{ij}^{(-r)}+\sum_{k=1}^{N}\sum_{s=1}^{r}T_{ik}^{(-s)}\otimes T_{kj}^{(s-r-1)} \tag{10.8}$$

for $r\geqslant 1$; cf. (4.4). Since $\varepsilon(1)=1$, for every $r\geqslant 1$ we get $\varepsilon\big(T_{ij}^{(-r)}\big)=0\,$.

The dual Yangian $\mathrm{Y}^*(\mathfrak{gl}_N)$ is a bialgebra but *not* a Hopf algebra. The antipodal map S is defined only for a completion $\mathrm{Y}^\circ(\mathfrak{gl}_N)$ of $\mathrm{Y}^*(\mathfrak{gl}_N)$ such that the element

$$T^*(0)\in\mathrm{Y}^\circ(\mathfrak{gl}_N)\otimes\operatorname{End}\mathbb{C}^N$$

is invertible. Then

$$T^*(v)\in\mathrm{Y}^\circ(\mathfrak{gl}_N)[[v]]\otimes\operatorname{End}\mathbb{C}^N$$

is also invertible, and the antipode S is defined by mapping $T^*(v)$ to its inverse. This inverse will be denoted by $T^\natural(v)$. It will be used again in the end of Section 15.

In order to construct such a completion, let us equip the algebra $\mathrm{Y}^*(\mathfrak{gl}_N)$ with a descending filtration, defined by assigning to the generator $T_{ij}^{(-r)}$ the degree r for any $r\geqslant 1$. Then $\mathrm{Y}^\circ(\mathfrak{gl}_N)$ is defined as the formal completion of $\mathrm{Y}^*(\mathfrak{gl}_N)$ relative to this descending filtration. The algebra $\mathrm{Y}^\circ(\mathfrak{gl}_N)\otimes\operatorname{End}\mathbb{C}^N$ contains the inverse of

$$T^*(0)=1\otimes 1+\sum_{i,j=1}^{N}T_{ij}^{(-1)}\otimes e_{ij}\,.$$

We extend the comultiplication Δ on $\mathrm{Y}^*(\mathfrak{gl}_N)$ to $\mathrm{Y}^\circ(\mathfrak{gl}_N)$, and also denote this extension by Δ. The image $\Delta(\mathrm{Y}^\circ(\mathfrak{gl}_N))$ lies in the formal completion of the algebra $\mathrm{Y}^*(\mathfrak{gl}_N)\otimes\mathrm{Y}^*(\mathfrak{gl}_N)$ with respect to the descending filtration, defined by assigning to the element $T_{ij}^{(-r)}\otimes T_{kl}^{(-s)}$ the degree $r+s$. Indeed, the image $\Delta\big(T_{ij}^{(-r)}\big)$ in $\mathrm{Y}^*(\mathfrak{gl}_N)\otimes\mathrm{Y}^*(\mathfrak{gl}_N)$ is a sum of elements of degrees r and $r+1$ by (10.8).

The kernel of the counit homomorphism $\varepsilon : \mathrm{Y}^*(\mathfrak{gl}_N) \to \mathbb{C}$ consists of all the elements which of positive degree relative to the filtration, see Proposition 10.1. Therefore ε extends to the algebra $\mathrm{Y}^\circ(\mathfrak{gl}_N)$. This extension is the counit map for the Hopf algebra $\mathrm{Y}^\circ(\mathfrak{gl}_N)$, it will be also denoted by ε.

For any $c \in \mathbb{C}$ the assignment $T^*(v) \mapsto T^*(v+c)$ determines an automorphism of the algebra $\mathrm{Y}^\circ(\mathfrak{gl}_N)$. This follows from the relations (10.5), cf. Proposition 3.1. But for $c \neq 0$ this automorphism does not preserve the subset $\mathrm{Y}^*(\mathfrak{gl}_N) \subset \mathrm{Y}^\circ(\mathfrak{gl}_N)$, and therefore does not determine an automorphism of $\mathrm{Y}^*(\mathfrak{gl}_N)$.

To find the square of the antipodal map S of the Hopf algebra $\mathrm{Y}^\circ(\mathfrak{gl}_N)$ let $T^\flat(v)$ be the result of applying to the inverse of (10.2) the transposition in $\operatorname{End}\mathbb{C}^N$. Write

$$T^\flat(v) = \sum_{i,j=1}^{N} T^\flat_{ij}(v) \otimes e_{ij} \in \mathrm{Y}^\circ(\mathfrak{gl}_N)[[v]] \otimes \operatorname{End}\mathbb{C}^N$$

so that

$$\mathrm{S} : T^*_{ij}(v) \mapsto T^\flat_{ji}(v)\,.$$

The proof of the next lemma is similar to that of Lemma 3.3 and is omitted here.

Lemma 10.2. *There is a formal power series $Z^\circ(v)$ in v with coefficients from the centre of the algebra $\mathrm{Y}^\circ(\mathfrak{gl}_N)$ such that for all indices i and j*

$$\sum_{k=1}^{N} T^*_{ki}(v-N)\, T^\flat_{kj}(v) = \delta_{ij}\, Z^\circ(v)\,.$$

In general, the coefficients of the series $Z^\circ(v)$ do not belong to the dual Yangian $\mathrm{Y}^*(\mathfrak{gl}_N)$. However, the proposition below can be derived from Lemma 10.2 just as Proposition 3.4 was derived from Lemma 3.3. Hence we again omit the proof.

Proposition 10.3. *The square of the map* S *is the automorphism of* $\mathrm{Y}^\circ(\mathfrak{gl}_N)$

$$\mathrm{S}^2 : T^*(v) \mapsto Z^\circ(v)^{-1}\, T^*(v-N)\,.$$

The latter result follows just as Proposition 4.2 followed from Proposition 3.4.

Proposition 10.4. *For the series $Z^\circ(v)$ defined above we have*

$$\Delta : Z^\circ(v) \mapsto Z^\circ(v) \otimes Z^\circ(v)\,.$$

The completion $\mathrm{Y}^\circ(\mathfrak{gl}_N)$ of the filtered algebra $\mathrm{Y}^*(\mathfrak{gl}_N)$ can be described more explicitly. At the end of Section 12 we will show that the vector space $\mathrm{Y}^*(\mathfrak{gl}_N)$ has a basis parameterized by all multisets of triples $(r_1, i_1, j_1), \ldots, (r_m, i_m, j_m)$ where

$$r_1, \ldots, r_m \in \{1, 2, \ldots\} \quad \text{and} \quad i_1, j_1, \ldots, i_m, j_m \in \{1, \ldots, N\}$$

while $m = 0, 1, 2, \ldots$. The corresponding basis vector in $\mathrm{Y}^*(\mathfrak{gl}_N)$ is the monomial

$$T^{(-r_1)}_{i_1 j_1} \ldots T^{(-r_m)}_{i_m j_m}. \tag{10.9}$$

The ordering of the factors in this monomial can be chosen arbitrarily. Choose any linear ordering of the basis monomials. For any positive integer r, there is only a finite number of the basis monomials (10.9) such that $r_1 + \cdots + r_m \leqslant r$. This means that when the index of the basis monomial (10.9) in any chosen linear ordering increases, then the filtration degree (10.9)

$$r_1 + \cdots + r_m \to \infty\,.$$

Therefore the vector space $\mathrm{Y}^\circ(\mathfrak{gl}_N)$ consists of all infinite linear combinations of the basis monomials (10.9), with the coefficients from the field $\mathbb{C}$.

11. Canonical pairing

There is a canonical bilinear pairing

$$\langle\ ,\ \rangle : \mathrm{Y}(\mathfrak{gl}_N) \times \mathrm{Y}^*(\mathfrak{gl}_N) \to \mathbb{C}. \tag{11.1}$$

We shall describe the corresponding linear map $\beta : \mathrm{Y}(\mathfrak{gl}_N) \otimes \mathrm{Y}^*(\mathfrak{gl}_N) \to \mathbb{C}$. It will be defined so that for all integers $m, n \geqslant 0$ the linear map

$$(\mathrm{End}\,\mathbb{C}^N)^{\otimes m} \otimes \mathrm{Y}(\mathfrak{gl}_N) \otimes \mathrm{Y}^*(\mathfrak{gl}_N) \otimes (\mathrm{End}\,\mathbb{C}^N)^{\otimes n} \to (\mathrm{End}\,\mathbb{C}^N)^{\otimes(m+n)}$$

given by $\mathrm{id} \otimes \beta \otimes \mathrm{id}$, will send

$$T_1(u_1) \dots T_m(u_m) \otimes T_1^*(v_1) \dots T_n^*(v_n) \;\mapsto \prod_{1 \leqslant a \leqslant m}^{\rightarrow} \; \prod_{1 \leqslant b \leqslant n}^{\rightarrow} R_{a,b+m}(u_a - v_b). \tag{11.2}$$

Here $u_1, \dots, u_m, v_1, \dots, v_n$ are independent variables. The coefficients of the series

$$T_1(u_1), \dots, T_m(u_m) \quad \text{and} \quad T_1^*(v_1), \dots, T_n^*(v_n) \tag{11.3}$$

belong to the algebras (2.2) and (10.3), respectively.

Note that the series in $u_1, \dots, u_m$ and $v_1, \dots, v_n$ at the left hand side of (11.2) satisfy certain relations, implied by the defining relations of the algebras $\mathrm{Y}(\mathfrak{gl}_N)$ and $\mathrm{Y}^*(\mathfrak{gl}_N)$. The following proposition guarantees that the pairing is well-defined.

Proposition 11.1. *The assignment* (11.2) *agrees with relations* (2.8) *and* (10.5).

Proof. This follows from the Yang-Baxter equation (2.6). For instance, let us consider the case when $m = 2$ and $n = 1$. Here we have to check that the series

$$\big(\, R(u_1 - u_2)\, T_1(u_1)\, T_2(u_2) \big) \otimes T^*(v)$$

and

$$\big(\, T_2(u_2)\, T_1(u_1)\, R(u_1 - u_2) \big) \otimes T^*(v)$$

with the coefficients in the algebra

$$(\mathrm{End}\,\mathbb{C}^N)^{\otimes 2} \otimes \mathrm{Y}(\mathfrak{gl}_N) \otimes \mathrm{Y}^*(\mathfrak{gl}_N) \otimes \mathrm{End}\,\mathbb{C}^N,$$

have the same images in under the map $\mathrm{id} \otimes \beta \otimes \mathrm{id}$. These images are series with the coefficients in $(\mathrm{End}\,\mathbb{C}^N)^{\otimes 3}$. Note that the second element can be rewritten as

$$\big(\, P\, T_1(u_2)\, T_2(u_1)\, P\, R(u_1 - u_2) \big) \otimes T^*(v)$$

By the definition (11.2), the images of the two elements are respectively

$$R_{12}(u_1 - u_2)\, R_{13}(u_1 - v)\, R_{23}(u_2 - v)$$

and

$$\begin{aligned} &P_{12}\, R_{13}(u_2 - v)\, R_{23}(u_1 - v)\, P_{12}\, R_{12}(u_1 - u_2) \\ &= \; R_{23}(u_2 - v)\, R_{13}(u_1 - v)\, R_{12}(u_1 - u_2). \end{aligned}$$

The equality of two images is now evident due to (2.6). Using (2.6) repeatedly, one can prove Proposition 11.1 for any $m, n \geqslant 0$. □

Let us show that the assignments (11.2) for all $m, n = 0, 1, 2, \dots$ determine the values of the bilinear pairing (11.1) uniquely. When $m = n = 0$, we get from (11.2) the equality $\langle\, 1\,, 1\, \rangle = 1$. By choosing $m = 1$ and $n = 0$, we obtain from (11.2) that $\langle\, T_{ij}^{(r)}, 1\, \rangle = 0$ for any $r \geqslant 1$. When $m = 0$ and $n = 1$, we obtain that $\langle\, 1, T_{ij}^{(-s)}\, \rangle = 0$ for any $s \geqslant 1$. In both cases, we had to use the equality $\langle\, 1\,, 1\, \rangle = 1$ obtained above.

Now suppose that $m, n \geqslant 1$. To determine the pairing values

$$\langle\, T^{(r_1)}_{i_1 j_1} \ldots T^{(r_m)}_{i_m j_m} \,,\, T^{(-s_1)}_{k_1 l_1} \ldots T^{(-s_n)}_{k_n l_n} \rangle \tag{11.4}$$

for any indices

$$r_1, \ldots, r_m, s_1, \ldots, s_n \in \{1, 2, \ldots\}$$

and

$$i_1, j_1, \ldots, i_m, j_m, k_1, l_1, \ldots, k_n, l_n \in \{1, \ldots, N\},$$

the product of the rational functions $R_{a,b+m}(u_a - v_b)$ on the right hand side of (11.2) should be expanded as power series in the variables $u_1^{-1}, \ldots, u_m^{-1}, v_1, \ldots, v_n$. The series (11.3) should be then also expanded.

Note that although the coefficient of v^0 in the series (10.1) is a sum of two terms, δ_{ij} and $T^{(-1)}_{ij}$, the pairing value (11.4) can be still determined by (11.2) for any indices $s_1, \ldots, s_n \geqslant 1$ by using the induction on n. Namely, if some of the indices $s_1, \ldots, s_n$ are equal to 1, the value (11.4) can be determined by (11.2), using the values (11.4) with n replaced by $0, \ldots, n-1$.

Consider the case $m = n = 1$ in more detail. Then the map $\mathrm{id} \otimes \beta \otimes \mathrm{id}$ maps

$$T(u) \otimes T^*(v) = \sum_{i,j,k,l=1}^{N} e_{ij} \otimes \Bigl(\sum_{r=0}^{\infty} T^{(r)}_{ij}\, u^{-r} \Bigr) \otimes \Bigl(\delta_{kl} + \sum_{s=1}^{\infty} T^{(-s)}_{kl} v^{s-1} \Bigr) \otimes e_{kl}$$

to the series

$$R(u - v) = 1 \otimes 1 - \sum_{i,j=1}^{N} \sum_{r=1}^{\infty} u^{-r} v^{r-1}\, e_{ij} \otimes e_{ji}\,; \tag{11.5}$$

see (2.4) and (10.1). Using the equality $\langle\, T^{(r)}_{ij}, 1 \,\rangle = \delta_{0r}$ for $r \geqslant 0$, we get

$$\langle\, T^{(r)}_{ij}, T^{(-s)}_{kl} \rangle = -\,\delta_{rs}\, \delta_{il}\, \delta_{jk} \quad \text{for} \quad r, s \geqslant 1.$$

More explicitly the value (11.4) will be determined in the course of the proof of the next lemma. This lemma describes a basic property of the bilinear pairing (11.1). It is valid for any integers $m, n \geqslant 0$.

Lemma 11.2. *If* $r_1 + \cdots + r_m < s_1 + \cdots + s_n$ *then the value* (11.4) *is zero.*

Proof. First suppose that $s_1, \ldots, s_n \geqslant 2$. Then by the definition of the pairing (11.2), the value (11.4) is the coefficient of

$$e_{i_1 j_1} \otimes \ldots \otimes e_{i_m j_m} \otimes e_{k_1 l_1} \otimes \ldots \otimes e_{k_n l_n} \cdot u_1^{-r_1} \ldots u_m^{-r_m}\, v_1^{s_1 - 1} \ldots v_n^{s_n - 1} \tag{11.6}$$

in the expansion of the product in $(\mathrm{End}\, \mathbb{C}^N)^{\otimes (m+n)}[[u_1^{-1}, \ldots, u_m^{-1}, v_1, \ldots, v_n]]$

$$\prod_{1 \leqslant a \leqslant m}^{\rightarrow} \; \prod_{1 \leqslant b \leqslant n}^{\rightarrow} R_{a,b+m}(u_a - v_b) = \prod_{1 \leqslant a \leqslant m}^{\rightarrow} \; \prod_{1 \leqslant b \leqslant n}^{\rightarrow} \bigl(1 - \sum_{r=1}^{\infty} u_a^{-r} v_b^{r-1} P_{a,b+m} \bigr)\,.$$

If the coefficient of (11.6) is non-zero in this expansion then clearly we have the inequality $r_1 + \cdots + r_m \geqslant s_1 + \cdots + s_n$.

Now suppose that some of the numbers $s_1, \ldots, s_n$ are equal to 1. Without loss of generality we will assume that $s_1, \ldots, s_d \geqslant 2$ and $s_{d+1}, \ldots, s_n = 1$ for some $d < n$. Rewrite the product at the right hand side of the definition (11.2) as

$$\prod_{1 \leqslant b \leqslant d}^{\rightarrow} \; \prod_{1 \leqslant a \leqslant m}^{\rightarrow} R_{a,b+m}(u_a - v_b) \;\cdot\; \prod_{d < b \leqslant n}^{\rightarrow} \; \prod_{1 \leqslant a \leqslant m}^{\rightarrow} R_{a,b+m}(u_a - v_b)\,.$$

By definition, the coefficient of (11.6) in the expansion of this product equals

$$\big\langle\, T^{(r_1)}_{i_1 j_1} \ldots T^{(r_m)}_{i_m j_m}\,,\, T^{(-s_1)}_{k_1 l_1} \ldots T^{(-s_d)}_{k_d l_d} \big(\delta_{k_{d+1} l_{d+1}} + T^{(-1)}_{k_{d+1} l_{d+1}}\big) \ldots \big(\delta_{k_n l_n} + T^{(-1)}_{k_n l_n}\big)\big\rangle\,.$$

The value (11.4) is then the coefficient of (11.6) in the expansion of the product

$$\prod_{1\leqslant b\leqslant d}^{\rightarrow} \; \prod_{1\leqslant a\leqslant m}^{\rightarrow} \big(\, 1 - \sum_{r=1}^{\infty} u_a^{-r} v_b^{\,r-1} P_{a,b+m} \,\big) \;\times \tag{11.7}$$

$$\prod_{d<b\leqslant n}^{\rightarrow} \Big(\prod_{1\leqslant a\leqslant m}^{\rightarrow} \big(\, 1 - \sum_{r=1}^{\infty} u_a^{-r} v_b^{\,r-1} P_{a,b+m} \,\big) - 1 \Big)\,.$$

If that coefficient here is non-zero, then $r_1 + \cdots + r_m \geqslant s_1 + \cdots + s_d + n - d\,.$ $\square$

12. Non-degeneracy of the pairing

In Section 10 we equipped the algebra $\mathrm{Y}^*(\mathfrak{gl}_N)$ with a descending filtration. Now consider the corresponding graded algebra $\mathrm{gr}\,\mathrm{Y}^*(\mathfrak{gl}_N)$. Its component of degree s will be denoted by $\mathrm{gr}_s \mathrm{Y}^*(\mathfrak{gl}_N)$. For any $s \geqslant 1$ denote by $\widetilde{T}^{(-s)}_{ij}$ the image of $T^{(-s)}_{ij}$ in $\mathrm{gr}_s \mathrm{Y}^*(\mathfrak{gl}_N)$. By (10.6) we immediately get

Lemma 12.1. *In the graded algebra* $\mathrm{gr}\,\mathrm{Y}^*(\mathfrak{gl}_N)$, *for any* $r, s \geqslant 1$ *we have*

$$[\,\widetilde{T}^{(-r)}_{ij}, \widetilde{T}^{(-s)}_{kl}] = \delta_{kj}\, \widetilde{T}^{(-r-s)}_{il} - \delta_{il}\, \widetilde{T}^{(-r-s)}_{kj}\,.$$

In Section 5 we equipped the algebra $\mathrm{Y}(\mathfrak{gl}_N)$ with an ascending filtration, such that the corresponding graded algebra $\mathrm{gr}\,\mathrm{Y}(\mathfrak{gl}_N)$ is commutative. Its subspace of all elements of degree s will be denoted by $\mathrm{gr}_s \mathrm{Y}(\mathfrak{gl}_N)$. Keeping to the notation of Section 5, for any $s \geqslant 1$ let $\widehat{T}^{(s)}_{ij}$ be the image of the generator $T^{(s)}_{ij}$ in $\mathrm{gr}_s \mathrm{Y}(\mathfrak{gl}_N)$.

We can define a bilinear pairing

$$\langle\ ,\ \rangle :\ \mathrm{gr}\,\mathrm{Y}(\mathfrak{gl}_N) \times \mathrm{gr}\,\mathrm{Y}^*(\mathfrak{gl}_N) \to \mathbb{C} \tag{12.1}$$

by making its value

$$\big\langle\, \widehat{T}^{(r_1)}_{i_1 j_1} \ldots \widehat{T}^{(r_m)}_{i_m j_m}\,,\, \widetilde{T}^{(-s_1)}_{k_1 l_1} \ldots \widetilde{T}^{(-s_n)}_{k_n l_n} \,\big\rangle \tag{12.2}$$

equal to (11.4) if $r_1 + \ldots + r_m = s_1 + \ldots + s_n$ and by making it equal to zero otherwise. Here $r_1, \ldots, r_m, s_1, \ldots, s_n \geqslant 1$ and $m, n \geqslant 0$. The indices $i_1, j_1, \ldots, i_m, j_m$ and $k_1, l_1, \ldots, k_n, l_n$ may be arbitrary. This definition is self-consistent. Namely, if

$$r_1 + \cdots + r_m = s_1 + \ldots + s_n = s \tag{12.3}$$

for some $s \geqslant 1$, then by Lemma 11.2 we have

$$\big\langle T^{(r_1)}_{i_1 j_1} \ldots T^{(r_m)}_{i_m j_m} + X, T^{(-s_1)}_{k_1 l_1} \ldots T^{(-s_n)}_{k_n l_n} + Y \big\rangle = \big\langle T^{(r_1)}_{i_1 j_1} \ldots T^{(r_m)}_{i_m j_m}, T^{(-s_1)}_{k_1 l_1} \ldots T^{(-s_n)}_{k_n l_n} \big\rangle$$

for any $X \in \mathrm{Y}(\mathfrak{gl}_N)$ and $Y \in \mathrm{Y}^*(\mathfrak{gl}_N)$ of degrees respectively less and more than s.

Proposition 12.2. *For any index* $s \geqslant 0$, *the restriction of the pairing* (12.1) *to* $\mathrm{gr}_s \mathrm{Y}(\mathfrak{gl}_N) \times \mathrm{gr}_s \mathrm{Y}^*(\mathfrak{gl}_N)$ *is non-degenerate.*

Proof. Fix an integer $s \geqslant 0$. In each of two vector spaces $\mathrm{gr}_s \mathrm{Y}(\mathfrak{gl}_N)$ and $\mathrm{gr}_s \mathrm{Y}^*(\mathfrak{gl}_N)$ we will choose a basis so that the matrix of the bilinear pairing (12.1) relative to these bases is lower triangular, with non-zero diagonal entries. In particular, we will prove that these two vector spaces are of the same dimension.

Let $r_1, \ldots, r_m$ and $s_1, \ldots, s_n$ be non-increasing sequences of positive integers satisfying (12.3). In other words, these two sequences are partitions of s. We will equip the set of all partitions of s with the *inverse* lexicographical ordering. In this ordering, the sequence $r_1,\ldots,r_m$ precedes the sequence $s_1,\ldots,s_n$ if for some $c\geqslant 0$

$$r_m=s_n\,,\,r_{m-1}=s_{n-1}\,,\,\ldots\,,r_{m-c+1}=s_{n-c+1}\quad\text{while}\quad r_{m-c}<s_{n-c}\,.$$

Suppose that $s_1,\ldots,s_d\geqslant 2$ while $s_{d+1},\ldots,s_n=1$ for $d\geqslant 0$. Unlike in the proof of Lemma 11.2, now we do not exclude the case $d=n$. Take the coefficient at

$$u_1^{-r_1}\ldots u_m^{-r_m}\,v_1^{s_1-1}\ldots v_n^{s_n-1} \tag{12.4}$$

in the expansion of the product (11.7) as a series in $u_1^{-1},\ldots,u_m^{-1},v_1\,\ldots,v_n$. This coefficient is an element of the algebra $(\operatorname{End}\mathbb{C}^N)^{\otimes(m+n)}$. If this coefficient is non-zero, then equality

$$r_1+\dots+r_m=s_1+\ldots+s_n$$

implies that each of the indices $r_1,\ldots,r_m$ in (12.4) is a sum of some of the indices $s_1,\ldots,s_n$. Moreover, then each of the indices $s_1,\ldots,s_n$ appears in these sums only once. If a sequence $r_1,\ldots,r_m$ obtained by this summation precedes the sequence $s_1,\ldots,s_n$ in the inverse lexicographical ordering, then the two sequences must coincide. That is, $m=n$ and $r_a=s_a$ for every index $a=1,\ldots,m$.

For $r=1,2,\ldots$ denote by $\mathcal{S}_r$ the segment of the sequence $1,\ldots,m$ consisting of all indices a such that $s_a=r$. If the sequences $r_1,\ldots,r_m$ and $s_1,\ldots,s_n$ coincide, then the coefficient at (12.4) in the expansion of the product (11.7) equals

$$(-1)^m\prod_{r\geqslant 1}\Big(\sum_p\prod_{a\in\mathcal{S}_r}P_{a,p(a)+m}\Big)$$

where p runs through the set of all permutations of the sequence $\mathcal{S}_r$. Note that in the products over r and a above, all the factors pairwise commute.

The graded algebra $\operatorname{gr}\mathrm{Y}(\mathfrak{gl}_N)$ is free commutative with the generators $\widehat{T}_{ij}^{(r)}$ where $r\geqslant 1$, see Corollary 8.4. Choose the basis in the vector space $\operatorname{gr}_s\mathrm{Y}(\mathfrak{gl}_N)$ consisting of the monomials

$$\widehat{T}_{i_1j_1}^{(r_1)}\ldots\widehat{T}_{i_mj_m}^{(r_m)}. \tag{12.5}$$

The ordering of factors in (12.5) is irrelevant, let us order them in any way such that $r_1\geqslant\dots\geqslant r_m$. Choose any linear ordering of these basis vectors, subordinate to the inverse lexicographical ordering of the corresponding sequences $r_1,\ldots,r_m$. The above arguments imply, that for any two basis elements,

$$\widehat{T}_{i_1j_1}^{(r_1)}\ldots\widehat{T}_{i_mj_m}^{(r_m)}\quad\text{and}\quad\widehat{T}_{k_1l_1}^{(s_1)}\ldots\widehat{T}_{k_ml_m}^{(s_m)}$$

such that the the sequence $r_1,\ldots,r_m$ precedes the sequence $s_1,\ldots,s_n$, the pairing value (12.2) is non-zero only if $m=n$ and for every index $a=1,\ldots,m$ we have

$$r_a=s_a\quad\text{and}\quad i_a=l_a\,,\,j_a=k_a\,.$$

Then the value (12.2) equals $(-1)^m\,g!\,h!\,\ldots$ where $g,h,\ldots$ are the multiplicities in the sequence of the triples $(r_1,i_1,j_1),\ldots,(r_m,i_m,j_m)$. Therefore the monomials

$$\widetilde{T}_{j_1i_1}^{(-r_1)}\ldots\widetilde{T}_{j_mi_m}^{(-r_m)} \tag{12.6}$$

in $\operatorname{gr}_s\mathrm{Y}^*(\mathfrak{gl}_N)$ corresponding to the basis elements (12.5) of vector space $\operatorname{gr}_s\mathrm{Y}(\mathfrak{gl}_N)$, are linearly independent. These monomials also span the vector space $\operatorname{gr}_s\mathrm{Y}^*(\mathfrak{gl}_N)$. The latter result follows from Lemma 12.1 by using induction on m. Hence these monomials form a basis in $\operatorname{gr}_s\mathrm{Y}^*(\mathfrak{gl}_N)$. The matrix of the pairing (12.1) relative to the two bases is then lower triangular, with non-zero diagonal entries. $\square$

The graded algebra $\operatorname{gr} \mathrm{Y}^*(\mathfrak{gl}_N)$ inherits from $\mathrm{Y}^*(\mathfrak{gl}_N)$ the bialgebra structure. Namely, using (10.8), for any $r \geqslant 1$ we get

$$\Delta\big(\widetilde{T}_{ij}^{(-r)}\big) = \widetilde{T}_{ij}^{(-r)} \otimes 1 + 1 \otimes \widetilde{T}_{ij}^{(-r)} \quad \text{and} \quad \varepsilon\big(\widetilde{T}_{ij}^{(-r)}\big) = 0. \tag{12.7}$$

Although the antipode S is defined only on the completion $\mathrm{Y}^\circ(\mathfrak{gl}_N)$ of $\mathrm{Y}^*(\mathfrak{gl}_N)$, it still induces a well-defined anipodal map on the graded algebra $\operatorname{gr} \mathrm{Y}^*(\mathfrak{gl}_N)$,

$$\mathrm{S} : \widetilde{T}_{ij}^{(-r)} \mapsto -\widetilde{T}_{ij}^{(-r)}. \tag{12.8}$$

Hence $\operatorname{gr} \mathrm{Y}^*(\mathfrak{gl}_N)$ becomes a Hopf algebra.

Now consider the subalgebra $z\,\mathfrak{gl}_N[z] \cong \mathfrak{gl}_N \otimes (\,z\,\mathbb{C}[z]\,)$ in the polynomial current Lie algebra $\mathfrak{gl}_N[z]$. The next proposition indicates the difference between the graded algebras $\operatorname{gr} \mathrm{Y}(\mathfrak{gl}_N)$ and $\operatorname{gr} \mathrm{Y}^*(\mathfrak{gl}_N)$, cf. Proposition 5.1.

Proposition 12.3. *The Hopf algebra* $\operatorname{gr} \mathrm{Y}^*(\mathfrak{gl}_N)$ *is isomorphic to the universal enveloping algebra* $\mathrm{U}(\,z\,\mathfrak{gl}_N[z]\,)$.

Proof. Lemma 12.1 implies that the assignment $E_{ij}\,z^r \mapsto \widetilde{T}_{ij}^{(-r)}$ for $r \geqslant 1$ defines a surjective homomorphism

$$\mathrm{U}(\,z\,\mathfrak{gl}_N[z]\,) \to \operatorname{gr} \mathrm{Y}^*(\mathfrak{gl}_N)\,. \tag{12.9}$$

The kernel of this homomorphism is trivial, because the monomials (12.6) in $\widetilde{T}_{ij}^{(-r)}$ corresponding to basis elements (12.5) of the free commutative algebra $\operatorname{gr} \mathrm{Y}(\mathfrak{gl}_N)$ form a basis in $\operatorname{gr} \mathrm{Y}^*(\mathfrak{gl}_N)$. This was shown in the proof of Proposition 12.2. By comparing the definitions (12.7),(12.8) with (5.4),(5.5) we complete the proof. $\square$

We state the main property of the pairing $\langle\ ,\ \rangle$ as the following theorem.

Theorem 12.4. *The map* (11.1) *is a non-degenerate bialgebra pairing.*

Proof. By Lemma 11.2 and Proposition 12.2 the pairing $\langle\ ,\ \rangle$ is non-degenerate. Let us show that under the pairing (11.1), the multiplication and comultiplication on $\mathrm{Y}(\mathfrak{gl}_N)$ become dual respectively to the comultiplication and multiplication on $\mathrm{Y}^*(\mathfrak{gl}_N)$. We have to prove that

$$\langle\, X, Z\,W\,\rangle = \langle\, \Delta(X), Z \otimes W\,\rangle \quad \text{and} \quad \langle\, X\,Y, Z\,\rangle = \langle\, X \otimes Y, \Delta(Z)\rangle \tag{12.10}$$

for any elements $X, Y \in \mathrm{Y}(\mathfrak{gl}_N)$ and $Z, W \in \mathrm{Y}^*(\mathfrak{gl}_N)$. Here we use the convention

$$\langle\, X \otimes Y, Z \otimes W\,\rangle = \langle\, X, Z\rangle\,\langle\, Y, W\,\rangle\,.$$

For instance, let us prove the first equality in (12.10). To this end it suffices to substitute the series $T_{i_1j_1}(u_1)\dots T_{i_mj_m}(u_m)$ and

$$T^*_{k_1l_1}(v_1)\dots T^*_{k_dl_d}(v_d)\ ,\ T^*_{k_{d+1}l_{d+1}}(v_{d+1})\dots T^*_{k_nl_n}(v_n)$$

for X and Z, W respectively. Here $0 \leqslant d \leqslant n$. If $d = 0$ or $d = n$, then we substitute 1 respectively for Z or for W. After these substitutions, we will have to prove that

$$\big\langle\, T_{i_1j_1}(u_1)\dots T_{i_mj_m}(u_m)\,,\, T^*_{k_1l_1}(v_1)\dots T^*_{k_nl_n}(v_n)\,\big\rangle \tag{12.11}$$

equals the sum

$$\sum_{h_1,\dots,h_m=1}^{N} \big\langle\, T_{i_1h_1}(u_1)\dots T_{i_mh_m}(u_m)\,,\, T^*_{k_1l_1}(v_1)\dots T^*_{k_dl_d}(v_n)\,\big\rangle \times \tag{12.12}$$
$$\big\langle\, T_{h_1j_1}(u_1)\dots T_{h_mj_m}(u_m)\,,\, T^*_{k_{d+1}l_{d+1}}(v_1)\dots T^*_{k_nl_n}(v_n)\,\big\rangle\,.$$

To prove the latter equality, let us multiply (12.11) and (12.12) by the element

$$e_{i_1 j_1} \otimes \ldots \otimes e_{i_m j_m} \otimes e_{k_1 l_1} \otimes \ldots \otimes e_{k_n l_n} \in (\operatorname{End} \mathbb{C}^N)^{\otimes (m+n)},$$

taking the sum over the indices $i_1, j_1, \ldots, i_m, j_m$ and $k_1, l_1, \ldots, k_n, l_n$. In this way, from (12.11) we obtain the product

$$\prod_{1 \leqslant a \leqslant m}^{\rightarrow} \prod_{1 \leqslant b \leqslant n}^{\rightarrow} R_{a, b+m}(u_a - v_b)$$

due to the definition (11.2). From (12.12) we obtain the product

$$\prod_{1 \leqslant b \leqslant d}^{\rightarrow} \prod_{1 \leqslant a \leqslant m}^{\rightarrow} R_{a, b+m}(u_a - v_b) \cdot \prod_{d < b \leqslant n}^{\rightarrow} \prod_{1 \leqslant a \leqslant m}^{\rightarrow} R_{a, b+m}(u_a - v_b)$$

which is evidently equal to the previous product.

We have already noted the equality $\langle\, 1 \,, 1 \,\rangle = 1$. Moreover, by setting $n = 0$ the definition (11.2), for any $r_1, \ldots, r_m \geqslant 1$ we get the equality

$$\langle\, T_{i_1 j_1}^{(r_1)} \ldots T_{i_m j_m}^{(r_m)} \,, 1 \,\rangle = 0 \quad \text{if} \quad m \geqslant 1\,.$$

Thus $\langle\, X \,, 1 \,\rangle = \varepsilon(X)$ for any element $X \in \mathrm{Y}(\mathfrak{gl}_N)$. By setting $m = 0$ in (11.2) and using the induction on n or, alternatively, by using Lemma 11.2, we obtain for any $s_1, \ldots, s_n \geqslant 1$ the equality

$$\langle\, 1 \,, T_{k_1 l_1}^{(-s_1)} \ldots T_{k_n l_n}^{(-s_n)} \,\rangle = 0 \quad \text{if} \quad n \geqslant 1.$$

Thus $\langle\, 1 \,, Z \,\rangle = \varepsilon(Z)$ for any element $Z \in \mathrm{Y}^*(\mathfrak{gl}_N)$. Therefore the counit and the unit maps for the bialgebra $\mathrm{Y}(\mathfrak{gl}_N)$ are dual respectively to the unit and the counit maps for the bialgebra $\mathrm{Y}^*(\mathfrak{gl}_N)$. □

Due to Theorem 8.3, the vector space $\mathrm{Y}(\mathfrak{gl}_N)$ has a basis parameterized by all multisets of triples $(r_1, i_1, j_1), \ldots, (r_m, i_m, j_m)$ where

$$r_1, \ldots, r_m \in \{1, 2, \ldots\} \quad \text{and} \quad i_1, j_1, \ldots, i_m, j_m \in \{1, \ldots, N\}$$

while $m = 0, 1, 2, \ldots$. The corresponding basis vector in $\mathrm{Y}(\mathfrak{gl}_N)$ is the monomial

$$T_{i_1 j_1}^{(r_1)} \ldots T_{i_m j_m}^{(r_m)}. \tag{12.13}$$

The ordering of the factors in this monomial can be chosen arbitrarily. Suppose that here $r_1 \geqslant \cdots \geqslant r_m$. Then the sequence $r_1, \ldots, r_m$ can be regarded as a partition of $r_1 + \cdots + r_m$. Equip the set of all partitions of $0, 1, 2, \ldots$ with the following ordering. If $r < s$, the partitions of r precede those of s. For any given r, the set of partitions of r is equipped with the inverse lexicographical ordering; see the proof of Proposition 12.2. Choose any linear ordering of the basis elements (12.13), subordinate to the above described ordering of their sequences $r_1, \ldots, r_m$. The proof of Proposition 12.2 implies that the monomials

$$T_{j_1 i_1}^{(-r_1)} \ldots T_{j_m i_m}^{(-r_m)}$$

corresponding to the basis elements (12.13) form a basis of the vector space $\mathrm{Y}^*(\mathfrak{gl}_N)$. The matrix of the pairing (11.1) relative to these two bases is lower triangular with non-zero diagonal entries; see also Lemma 11.2. Here the basis elements of $\mathrm{Y}^*(\mathfrak{gl}_N)$ are linearly ordered as the corresponding basis elements (12.13) of $\mathrm{Y}(\mathfrak{gl}_N)$.

13. Universal R-matrix

Consider the formal completion $\mathrm{Y}^\circ(\mathfrak{gl}_N)$ of the filtered algebra $\mathrm{Y}^*(\mathfrak{gl}_N)$ defined in Section 10. By Proposition 11.2 the canonical pairing (11.1) extends to a pairing

$$\langle\ ,\ \rangle : \mathrm{Y}(\mathfrak{gl}_N) \times \mathrm{Y}^\circ(\mathfrak{gl}_N) \to \mathbb{C}\,.$$

Choose any basis $X_1, X_2, \ldots$ in the vector space $\mathrm{Y}(\mathfrak{gl}_N)$.

Proposition 13.1. *The completion $\mathrm{Y}^\circ(\mathfrak{gl}_N)$ does contain the system of elements $X_1', X_2', \ldots$ dual to $X_1, X_2, \ldots$ so that $\langle X_r\,, X_s' \rangle = \delta_{rs}$ for any r and s.*

Proof. As we explained at the end of Section 10, one can choose a basis $Y_1, Y_2, \ldots$ in $\mathrm{Y}(\mathfrak{gl}_N)$ and a basis $Y_1^*, Y_2^*, \ldots$ in $\mathrm{Y}^*(\mathfrak{gl}_N)$ so that the filtration degree

$$\deg Y_s^* \to \infty \quad \text{when} \quad s \to \infty\,, \tag{13.1}$$

and so that the matrix of the pairing (11.1) relative to these bases is lower triangular with non-zero diagonal entries. Let $[\,g_{rs}]$ be its inverse matrix. The formal sums

$$Y_s' = \sum_{r=1}^{\infty} g_{rs}\, Y_r^* \tag{13.2}$$

satisfy the equations $\langle Y_r\,, Y_s' \rangle = \delta_{rs}$ for all indices r and s. Each of these sums is contained in $\mathrm{Y}^\circ(\mathfrak{gl}_N)$ due to (13.1). Moreover, because the the matrix $[g_{rs}]$ is also lower triangular, the property (13.1) implies that

$$\deg Y_s' \to \infty \quad \text{when} \quad s \to \infty\,. \tag{13.3}$$

Now let $X_1, X_2, \ldots$ be any basis in $\mathrm{Y}(\mathfrak{gl}_N)$. Let $[h_{rs}]$ be the coordinate change matrix from the basis $Y_1, Y_2, \ldots$ so that for any index r we have

$$Y_s = \sum_{r=1}^{\infty} h_{rs}\, X_r\,.$$

This sum must be finite, so that for any fixed index s there are only finitely many non-zero coefficients h_{rs}. The sums

$$X_r' = \sum_{s=1}^{\infty} h_{rs}\, Y_s' \tag{13.4}$$

satisfy the equations $\langle X_r\,, X_s' \rangle = \delta_{rs}$ as required. Each of these sums is contained in the completion $\mathrm{Y}^\circ(\mathfrak{gl}_N)$ due to the property (13.3). ☐

Consider an infinite sum of elements of the tensor product $\mathrm{Y}^\circ(\mathfrak{gl}_N) \otimes \mathrm{Y}(\mathfrak{gl}_N)$

$$\mathcal{R} = \sum_{r=1}^{\infty} X_r' \otimes X_r\,. \tag{13.5}$$

This sum does not depend on the choice of the basis $X_1, X_2, \ldots$ in the vector space $\mathrm{Y}(\mathfrak{gl}_N)$ in the following sense. Let $Y_1, Y_2, \ldots$ be the basis in $\mathrm{Y}(\mathfrak{gl}_N)$ used in the proof of Proposition 13.1. Using the formula (13.4) for every $r = 1, 2, \ldots$ expand the vectors $X_1', X_2', \ldots$ in (13.5). Then fix an index s and consider the sum of terms

$$\sum_{r=1}^{\infty} (\,h_{rs}\, Y_s') \otimes X_r$$

corresponding to the vector Y'_s in (13.4). Only finite number of these terms are non-zero, and their sum is equal to $Y'_s \otimes Y_s$. In this sense, the sum in (13.5) equals

$$\sum_{s=1}^{\infty} Y'_s \otimes Y_s\,.$$

The infinite sum $\mathcal{R}$ is called the *universal R-matrix* for the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$.

Any element of the vector space $\mathrm{Y}^\circ(\mathfrak{gl}_N) \otimes \mathrm{Y}(\mathfrak{gl}_N)$ determines a linear operator on the vector space $\mathrm{Y}(\mathfrak{gl}_N)$. If A is the operator corresponding to an element $Z \otimes Y \in \mathrm{Y}^\circ(\mathfrak{gl}_N) \otimes \mathrm{Y}(\mathfrak{gl}_N)$, then

$$A\,(X) = \langle\, X\,, Z\,\rangle\; Y \quad \text{ for any } \quad X \in \mathrm{Y}(\mathfrak{gl}_N). \tag{13.6}$$

By the above argument, the series of operators corresponding to (13.5) pointwise converges to the identity operator $\mathrm{id} : X \mapsto X$ on the vector space $\mathrm{Y}(\mathfrak{gl}_N)$.

Proposition 13.2. *For the comultiplication on* $\mathrm{Y}(\mathfrak{gl}_N)$ *and* $\mathrm{Y}^\circ(\mathfrak{gl}_N)$ *we have*

$$(\,\mathrm{id} \otimes \Delta)\,(\mathcal{R}) = \mathcal{R}_{12}\,\mathcal{R}_{13} \quad \textit{and} \quad (\,\Delta \otimes \mathrm{id})\,(\mathcal{R}) = \mathcal{R}_{13}\,\mathcal{R}_{23} \tag{13.7}$$

where

$$\mathcal{R}_{12} = \sum_{r=1}^{\infty} X'_r \otimes X_r \otimes 1\,, \quad \mathcal{R}_{13} = \sum_{r=1}^{\infty} X'_r \otimes 1 \otimes X_r\,, \quad \mathcal{R}_{23} = \sum_{r=1}^{\infty} 1 \otimes X'_r \otimes X_r\,.$$

Proof. Let us prove the first of the two equalities (13.7). This is an equality of infinite sums of elements from the tensor product $\mathrm{Y}^\circ(\mathfrak{gl}_N) \otimes \mathrm{Y}(\mathfrak{gl}_N) \otimes \mathrm{Y}(\mathfrak{gl}_N)$. It means the equality of the corresponding operators $\mathrm{Y}(\mathfrak{gl}_N) \to \mathrm{Y}(\mathfrak{gl}_N) \otimes \mathrm{Y}(\mathfrak{gl}_N)$. By applying the linear operator corresponding to the infinite sum $(\,\mathrm{id} \otimes \Delta)\,(\mathcal{R})$ to any fixed element $X \in \mathrm{Y}(\mathfrak{gl}_N)$ we get the element $\Delta(X)$. By applying to X the operator corresponding to $\mathcal{R}_{12}\,\mathcal{R}_{13}$ we obtain the sum

$$\sum_{r,s=1}^{\infty} \langle\, X\,, Y'_r\, Y'_s\,\rangle\, Y_r \otimes Y_s = \sum_{r,s=1}^{\infty} \langle\, \Delta(X)\,, Y'_r \otimes Y'_s\,\rangle\, Y_r \otimes Y_s = \Delta(X).$$

Here we used the first equality in (12.10), and non-degeneracy of the pairing (11.1). The property (13.3) guarantees that in both sums over r and s displayed above, only finite number of summands are non-zero when X is fixed; see Lemma 11.2. We have thus proved the first equality in (13.7). The second equality is deduced from the second equality in (12.10) in a similar way. $\square$

Proposition 13.3. *For the counit maps on* $\mathrm{Y}(\mathfrak{gl}_N)$ *and* $\mathrm{Y}^\circ(\mathfrak{gl}_N)$*,*

$$(\,\mathrm{id} \otimes \varepsilon)\,(\mathcal{R}) = 1 \quad \textit{and} \quad (\,\varepsilon \otimes \mathrm{id})\,(\mathcal{R}) = 1.$$

Proof. Because $\varepsilon(X) = \langle\, X\,, 1\,\rangle$ for any element $X \in \mathrm{Y}(\mathfrak{gl}_N)$ by Theorem 12.4,

$$(\,\mathrm{id} \otimes \varepsilon)\,(\mathcal{R}) = \sum_{s=1}^{\infty} \langle\, Y_s\,, 1\,\rangle\, Y'_s = 1\,.$$

Similarly, because $\varepsilon(Z) = \langle\, 1\,, Z\,\rangle$ for any element $Z \in \mathrm{Y}^\circ(\mathfrak{gl}_N)$, we also have

$$(\,\varepsilon \otimes \mathrm{id})\,(\mathcal{R}) = \sum_{s=1}^{\infty} \langle\, 1\,, Y'_s\,\rangle\, Y_s = 1$$

where only finitely many summands are non-zero due to (13.3), see Lemma 11.2. $\square$

The infinite sum in (13.5) can be also regarded as an element of a completion of the tensor product $\mathrm{Y}^*(\mathfrak{gl}_N)\otimes\mathrm{Y}(\mathfrak{gl}_N)$. Namely, let us extend the descending filtration from the algebra $\mathrm{Y}^*(\mathfrak{gl}_N)$ to the tensor product $\mathrm{Y}^*(\mathfrak{gl}_N)\otimes\mathrm{Y}(\mathfrak{gl}_N)$ by giving the degree r to each element of the form $T_{ij}^{(-r)}\otimes X$ where $X\in\mathrm{Y}(\mathfrak{gl}_N)$ and $r\geqslant 1$. The element $1\otimes X$ of $\mathrm{Y}^*(\mathfrak{gl}_N)\otimes\mathrm{Y}(\mathfrak{gl}_N)$ is given the zero degree. Take the formal completion of the algebra $\mathrm{Y}^*(\mathfrak{gl}_N)\otimes\mathrm{Y}(\mathfrak{gl}_N)$ relative to this filtration. This completion contains the tensor product $\mathrm{Y}^\circ(\mathfrak{gl}_N)\otimes\mathrm{Y}(\mathfrak{gl}_N)$, but does not coincide with it because the algebra $\mathrm{Y}(\mathfrak{gl}_N)$ is infinite-dimensional.

The next corollary shows in particular, that the sum in (13.5) is invertible as an element of the completion of the algebra $\mathrm{Y}^*(\mathfrak{gl}_N)\otimes\mathrm{Y}(\mathfrak{gl}_N)$.

Corollary 13.4. *For the antipodal maps on* $\mathrm{Y}(\mathfrak{gl}_N)$ *and* $\mathrm{Y}^\circ(\mathfrak{gl}_N)$ *we have*

$$(\mathrm{id}\otimes\mathrm{S})\,(\mathcal{R})=\mathcal{R}^{-1}\quad\textit{and}\quad(\mathrm{S}\otimes\mathrm{id})\,(\mathcal{R})=\mathcal{R}^{-1}.$$

Proof. Regard the first equality in (13.7) as that of the elements of the completion of the algebra $\mathrm{Y}^*(\mathfrak{gl}_N)\otimes\mathrm{Y}(\mathfrak{gl}_N)\otimes\mathrm{Y}(\mathfrak{gl}_N)$. On this algebra, the descending filtration is defined by giving the degree r to each element of the form $T_{ij}^{(-r)}\otimes X\otimes Y$ where $X,Y\in\mathrm{Y}(\mathfrak{gl}_N)$ and $r\geqslant 1$. The element $1\otimes X\otimes Y$ is then given the degree zero.

Let $\mu:\mathrm{Y}(\mathfrak{gl}_N)\otimes\mathrm{Y}(\mathfrak{gl}_N)\to\mathrm{Y}(\mathfrak{gl}_N)$ be the map of algebra multiplication, and $\delta:\mathbb{C}\to\mathrm{Y}(\mathfrak{gl}_N)$ be the unit map: $\delta\,(1)=1$. Let us apply the map $\mathrm{id}\otimes\mathrm{S}\otimes\mathrm{id}$, and then the map $\mathrm{id}\otimes\mu$ to to both sides of the first equality in (13.7). At the right hand side we get the element $((\mathrm{id}\otimes\mathrm{S})(\mathcal{R}))\cdot\mathcal{R}$. At the left hand side we get the element of the tensor product $\mathrm{Y}^\circ(\mathfrak{gl}_N)\otimes\mathrm{Y}(\mathfrak{gl}_N)$,

$$(\,(\mathrm{id}\otimes\mu)\,(\mathrm{id}\otimes\mathrm{S}\otimes\mathrm{id})\,(\mathrm{id}\otimes\Delta)\,)\,(\mathcal{R})=(\,(\mathrm{id}\otimes\delta)\,(\mathrm{id}\otimes\varepsilon)\,)\,(\mathcal{R})=1\otimes 1\,.$$

Here we used the first axiom of antipode from Section 4 in the case $\mathrm{A}=\mathrm{Y}(\mathfrak{gl}_N)$, and the first equality of Proposition 13.3. Hence the first equality of Corollary 13.4 follows from the first equality in (13.7).

Similarly, using the first axiom of antipode in the case $\mathrm{A}=\mathrm{Y}^\circ(\mathfrak{gl}_N)$ and the second equality of Proposition 13.3, the second equality of Corollary 13.4 follows from the second equality in (13.7). The last equality should be regarded here as that of the elements of the completion of the algebra $\mathrm{Y}^*(\mathfrak{gl}_N)\otimes\mathrm{Y}^*(\mathfrak{gl}_N)\otimes\mathrm{Y}(\mathfrak{gl}_N)$. On this algebra a descending filtration is defined by giving the degree $r+s$ to any element of the form

$$T_{ij}^{(-r)}\otimes T_{kl}^{(-s)}\otimes X\,.$$

Then the elements $T_{ij}^{(-r)}\otimes 1\otimes X$ and $1\otimes T_{ij}^{(-r)}\otimes X$ are given the degree r, while the element $1\otimes 1\otimes X$ is given the degree zero. Here $X\in\mathrm{Y}(\mathfrak{gl}_N)$ and $r,s\geqslant 1$. The argument is completed in the same way as for the first equality. $\square$

Let us now replace the complex parameter c in the definition (6.2) of a covector representation ρ_c of $\mathrm{Y}(\mathfrak{gl}_N)$ by the formal variable v. Then we get a homomorphism

$$\rho_v:\mathrm{Y}(\mathfrak{gl}_N)\to\mathrm{End}\,\mathbb{C}^N\,[v]\,; \tag{13.8}$$

it is defined by the assignment $T(u)\mapsto R(u-v)$ of formal power series in u^{-1}.

Similarly, the assignment $T^*(v)\mapsto R(u-v)$ of formal power series in v defines a homomorphism

$$\rho^*_u:\mathrm{Y}^*(\mathfrak{gl}_N)\to\mathrm{End}\,\mathbb{C}^N\,[u^{-1}]\,. \tag{13.9}$$

To prove the homomorphism property using the matrix form (10.5) of the defining relations of the algebra $\mathrm{Y}^*(\mathfrak{gl}_N)$, we have to check the equality of rational functions in the variables u, v and w with the values in the algebra $(\mathrm{End}\,\mathbb{C}^N)^{\otimes 3}$,

$$R_{01}(u-v)\,R_{02}(u-w)\,R_{12}(v-w) = R_{12}(v-w)\,R_{02}(u-w)\,R_{01}(u-v)\,.$$

This equality follows from (2.6). Here we use the indices $0,1,2$ instead of $1,2,3$ to label the tensor factors of $(\mathrm{End}\,\mathbb{C}^N)^{\otimes 3}$. By comparing the expansions (10.2) and (11.5), we see that

$$\rho^*_u : T^{(-r)}_{ij} \mapsto -u^{-r} e_{ji} \quad \text{for any} \quad r \geqslant 1\,. \tag{13.10}$$

Obviously, the homomorphism ρ^*_u extends to a homomorphism

$$\mathrm{Y}^\circ(\mathfrak{gl}_N) \to \mathrm{End}\,\mathbb{C}^N\,[[u^{-1}]]\,.$$

We shall keep the notation ρ^*_u for the extended homomorphism.

Proposition 13.5. *We have equalities of formal power series in u^{-1} and v,*

$$(\,\rho^*_u \otimes \mathrm{id}\,)\,(\mathcal{R}) = T(u) \quad \textit{and} \quad (\,\mathrm{id} \otimes \rho_v\,)\,(\mathcal{R}) = T^*(v)\,.$$

Proof. By the definition (11.2) of the pairing $\mathrm{Y}(\mathfrak{gl}_N) \otimes \mathrm{Y}^*(\mathfrak{gl}_N) \to \mathbb{C}$ for any $n \geqslant 0$ the element $T(u) \in \mathrm{End}\,(\mathbb{C}^N) \otimes \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]]$ has the property that

$$T(u) \otimes T^*_1(v_1) \ldots T^*_n(v_n) \mapsto R_{12}(u-v_1) \ldots R_{1,n+1}(u-v_n)$$

under the linear map

$$\mathrm{id} \otimes \beta \otimes \mathrm{id} : \mathrm{End}\,\mathbb{C}^N \otimes \mathrm{Y}(\mathfrak{gl}_N) \otimes \mathrm{Y}^*(\mathfrak{gl}_N) \otimes (\mathrm{End}\,\mathbb{C}^N)^{\otimes n} \to (\mathrm{End}\,\mathbb{C}^N)^{\otimes\,(n+1)}.$$

Because our pairing is non-degenerate, the same property for the element

$$(\,\rho^*_u \otimes \mathrm{id}\,)\,(\mathcal{R}) = \sum_{s=1}^{\infty} \rho^*_u\,(Y'_s) \otimes Y_s$$

will imply the first equality of Proposition 13.5. Note that when $s \to \infty$, then the degree in u^{-1} of the image $\rho^*_u\,(Y'_s)$ tends to infinity due to (13.3) and (13.10). Hence the above displayed sum over $s = 1, 2, \ldots$ is contained in $\mathrm{End}\,(\mathbb{C}^N) \otimes \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]]\,$.

Thus to prove the first equality of Proposition 13.5, we have to show that under the linear map $\mathrm{id} \otimes \beta \otimes \mathrm{id}\,$,

$$\sum_{s=1}^{\infty} \rho^*_u\,(Y'_s) \otimes Y_s \otimes T^*_1(v_1) \ldots T^*_n(v_n) \mapsto R_{12}(u-v_1) \ldots R_{1,n+1}(u-v_n)\,.$$

Since the system of vectors $Y'_1, Y'_2, \ldots$ is dual to the basis $Y_1, Y_2, \ldots$ of $\mathrm{Y}(\mathfrak{gl}_N)$, this is equivalent to showing that

$$T^*_1(v_1) \ldots T^*_n(v_n) \mapsto R_{12}(u-v_1) \ldots R_{1,n+1}(u-v_n)$$

under the linear map

$$\rho^*_u \otimes \mathrm{id} : \mathrm{Y}^*(\mathfrak{gl}_N) \otimes (\mathrm{End}\,\mathbb{C}^N)^{\otimes n} \to (\mathrm{End}\,\mathbb{C}^N)^{\otimes\,(n+1)}[u^{-1}]\,.$$

The latter property follows directly from the definition of the homomorphism ρ^*_u. The proof of the second equality of Proposition 13.5 is similar and is omitted. ☐

Corollary 13.6. *We have the equality of formal power series in u^{-1} and v,*

$$(\,\rho^*_u \otimes \rho_v\,)\,(\mathcal{R}) = R(u-v)\,.$$

14. Double Yangian

Let Δ' be the comultiplication on $\mathrm{Y}^*(\mathfrak{gl}_N)$ *opposite* to the comultiplication Δ defined by (10.7). By definition, the map

$$\Delta' : \mathrm{Y}^*(\mathfrak{gl}_N) \to \mathrm{Y}^*(\mathfrak{gl}_N) \otimes \mathrm{Y}^*(\mathfrak{gl}_N)$$

is the composition of the comultiplication Δ with the linear operator on the tensor product $\mathrm{Y}^*(\mathfrak{gl}_N) \otimes \mathrm{Y}^*(\mathfrak{gl}_N)$ exchanging the tensor factors.

The *double Yangian* of $\mathfrak{gl}_N$ is defined as an associative unital algebra $\mathrm{DY}(\mathfrak{gl}_N)$ over $\mathbb{C}$ generated by the elements of $\mathrm{Y}(\mathfrak{gl}_N)$ and $\mathrm{Y}^*(\mathfrak{gl}_N)$ subject to the relations

$$\mathcal{R}\,\Delta(W) = \Delta'(W)\,\mathcal{R} \quad \text{for every} \quad W \in \mathrm{Y}^*(\mathfrak{gl}_N)\,. \tag{14.1}$$

In the rest of this section we will provide a more explicit description of the algebra $\mathrm{DY}(\mathfrak{gl}_N)$, see Theorem 14.4 below. In Section 15 we will show that the defining homomorphisms of $\mathrm{Y}(\mathfrak{gl}_N)$ and $\mathrm{Y}^*(\mathfrak{gl}_N)$ to $\mathrm{DY}(\mathfrak{gl}_N)$ are in fact embeddings. At the end of that section we will also provide an equivalent definition of the $\mathrm{DY}(\mathfrak{gl}_N)$.

In (14.1) we have an equality of infinite sums of elements of the tensor product $\mathrm{Y}^\circ(\mathfrak{gl}_N) \otimes \mathrm{DY}(\mathfrak{gl}_N)$. It means the equality of the corresponding linear operators $\mathrm{Y}(\mathfrak{gl}_N) \to \mathrm{DY}(\mathfrak{gl}_N)$, cf. (13.6). For instance, let us consider the infinite sum

$$\mathcal{R}\,\Delta(W) = \sum_{s=1}^{\infty} (Y'_s \otimes Y_s)\,\Delta(W)$$

at the right hand side of the equality postulated in (14.1). Note that for any fixed $X \in \mathrm{Y}(\mathfrak{gl}_N)$ and $Z \in \mathrm{Y}^*(\mathfrak{gl}_N)$, only finitely many summands in the infinite sum

$$\sum_{s=1}^{\infty} \langle\, X\,, Y'_s Z \,\rangle\, Y_s$$

are non-zero; see Lemma 11.2 and the property (13.3). This observation shows that the linear operator $\mathrm{Y}(\mathfrak{gl}_N) \to \mathrm{DY}(\mathfrak{gl}_N)$ corresponding to the infinite sum $\mathcal{R}\,\Delta(W)$ is well-defined for any element $W \in \mathrm{Y}^*(\mathfrak{gl}_N)$.

Now take the pair of homomorphisms ρ_u and ρ^*_u where we use the same formal variable u, see (13.8) and (13.9).

Proposition 14.1. *The associative algebra homomorphisms $\rho_u\,, \rho^*_u$ extend to a homomorphism $\mathrm{DY}(\mathfrak{gl}_N) \to \mathrm{End}\,\mathbb{C}^N[\,u, u^{-1}]$.*

Proof. Using (14.1), for any $W \in \mathrm{Y}^*(\mathfrak{gl}_N)$ we have to check the equality

$$(\,\mathrm{id} \otimes \rho_u\,)\,(\mathcal{R})\,(\,\mathrm{id} \otimes \rho^*_u\,)\,(\Delta\,(W)) = (\,\mathrm{id} \otimes \rho^*_u\,)\,(\Delta'(W))\,(\,\mathrm{id} \otimes \rho_u\,)\,(\mathcal{R})$$

of formal series in u with coefficients in the algebra $\mathrm{Y}^\circ(\mathfrak{gl}_N) \otimes \mathrm{End}\,\mathbb{C}^N$. It suffices to substitute here the series $T^*_{ij}(v)$ for the element W. Due to the definition (10.7) and to Proposition 13.5, the result of the substitution is the relation

$$\sum_{k=1}^{N} \,T^*(u)\,(T^*_{ik}(v) \otimes \rho^*_u(T^*_{kj}(v))) = \sum_{k=1}^{N} \,(T^*_{kj}(v) \otimes \rho^*_u(T^*_{ik}(v)))\,T^*(u)\,.$$

Let us take the tensor products of both sides of the latter relation with the element $e_{ij} \in \mathrm{End}\,\mathbb{C}^N$, and then sum over $i, j = 1\ldots, N$. Using the identity $e_{ij} = e_{ik}\,e_{kj}$

we then get the relation

$$\sum_{i,j,k=1}^{N} (T^*(u)\otimes 1)\,(T^*_{ik}(v)\otimes \rho^*_u(T^*_{kj}(v))\otimes e_{ik}\,e_{kj}\,) \;= \tag{14.2}$$
$$\sum_{i,j,k=1}^{N} (T^*_{kj}(v)\otimes \rho^*_u(T^*_{ik}(v))\otimes e_{ik}\,e_{kj}\,)\,(T^*(u)\otimes 1)$$

of formal power series in u, v with the coefficients in $\mathrm{Y}^*(\mathfrak{gl}_N)\otimes \mathrm{End}\,\mathbb{C}^N\otimes \mathrm{End}\,\mathbb{C}^N$. Note that by the definition of the homomorphism (13.9),

$$\sum_{i,j=1}^{N} \rho^*_u(T^*_{ij}(v))\otimes e_{ij} = R(u-v)\,.$$

Therefore the relation (14.2) can be rewritten as

$$T^*_1(u)\,T^*_2(v)\,(\,1\otimes R(u-v)) = (\,1\otimes R(u-v))\,T^*_2(v)\,T^*_1(u)\,.$$

But this is just the defining relation for the algebra $\mathrm{Y}^*(\mathfrak{gl}_N)$, see (10.5). □

Let c be any non-zero complex number. In Proposition 14.1, we can specialize the formal variable u to c. Then we obtain a representation $\mathrm{DY}(\mathfrak{gl}_N)\to \mathrm{End}\,\mathbb{C}^N$. We call it a *covector representation* of the algebra $\mathrm{DY}(\mathfrak{gl}_N)$, it extends the covector representation (6.2) of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$.

The vector representation (6.3) of $\mathrm{Y}(\mathfrak{gl}_N)$ can be extended to a representation of $\mathrm{DY}(\mathfrak{gl}_N)$, by mapping $T^*(v)\mapsto R^t(v-u)$. We call it a *vector representation* of the algebra $\mathrm{DY}(\mathfrak{gl}_N)$ and denote it by σ_c. Note that then

$$\sigma_c : T^{(-r)}_{ij}\mapsto c^{-r}\,e_{ij} \quad \text{for any} \quad r\geqslant 1\,. \tag{14.3}$$

The proof that these assignments together with (6.3) define a representation of the algebra $\mathrm{DY}(\mathfrak{gl}_N)$ is similar to that of Proposition 14.1, and is omitted here.

To write down commutation relations in the algebra $\mathrm{DY}(\mathfrak{gl}_N)$, we will use the tensor product $\mathrm{End}\,\mathbb{C}^N\otimes \mathrm{DY}(\mathfrak{gl}_N)\otimes \mathrm{End}\,\mathbb{C}^N$. There is a natural embedding of the algebra $\mathrm{End}\,\mathbb{C}^N\otimes \mathrm{End}\,\mathbb{C}^N$ into this tensor product, such that $x\otimes y\mapsto x\otimes 1\otimes y$ for any elements $x, y\in \mathrm{End}\,\mathbb{C}^N$. In the next proposition, the Yang R-matrix (2.4) is identified with its image relative to this embedding.

Proposition 14.2. *In the algebra* $\mathrm{End}\,\mathbb{C}^N\otimes \mathrm{DY}(\mathfrak{gl}_N)\otimes \mathrm{End}\,\mathbb{C}^N\,[[u^{-1}, v]]$ *we have*

$$(T(u)\otimes 1)\,R(u-v)\,(1\otimes T^*(v)) = (1\otimes T^*(v))\,R(u-v)\,(T(u)\otimes 1)\,. \tag{14.4}$$

Proof. Let us substitute $T^*_{ij}(v)$ for W in the equality in (14.1), and then apply the homomorphism $\rho^*_u\otimes \mathrm{id}$ to the resulting equality. Due to the definition (10.7) and to Proposition 13.5, we get an equality of formal power series in u^{-1} and v with the coefficients from $\mathrm{End}\,\mathbb{C}^N\otimes \mathrm{DY}(\mathfrak{gl}_N)$,

$$\sum_{k=1}^{N} T(u)\,(\rho^*_u(T^*_{ik}(v))\otimes T^*_{kj}(v)) = \sum_{k=1}^{N} (\rho^*_u(T^*_{kj}(v))\otimes T^*_{ik}(v))\,T(u)\,.$$

Let us now take the tensor products of both sides of this equality with the element $e_{ij}\in \mathrm{End}\,\mathbb{C}^N$, and then sum over $i, j = 1\ldots, N$. Using the identity $e_{ij} = e_{ik}\,e_{kj}$

we obtain an equality of series with coefficients from $\mathrm{End}\,\mathbb{C}^N \otimes \mathrm{DY}(\mathfrak{gl}_N) \otimes \mathrm{End}\,\mathbb{C}^N$

$$\sum_{i,j,k=1}^{N} (\,T(u) \otimes 1\,)\,(\,\rho^*_u(T^*_{ik}(v)) \otimes T^*_{kj}(v) \otimes e_{ik}\,e_{kj}\,) \;= \tag{14.5}$$

$$\sum_{i,j,k=1}^{N} (\,\rho^*_u(T^*_{kj}(v)) \otimes T^*_{ik}(v) \otimes e_{ik}\,e_{kj}\,)\,(\,T(u) \otimes 1\,)\,.$$

By using the definition of ρ^*_u the equality (14.5) can be rewritten as (14.4). □

Proposition 14.3. *Relation* (14.4) *is equivalent to the collection of relations* (14.1).

Proof. By Proposition 14.2 the relation (14.4) follows from (14.1). Let $u_1, \ldots, u_m$ be independent variables. Define the homomorphism

$$\rho^*_{u_1 \ldots u_m} : \mathrm{Y}^*(\mathfrak{gl}_N) \to (\mathrm{End}\,\mathbb{C}^N)^{\otimes m}\,[u_1^{-1}, \ldots, u_m^{-1}] \tag{14.6}$$

as the composition of the m-fold comultiplication $\mathrm{Y}^*(\mathfrak{gl}_N) \to \mathrm{Y}^*(\mathfrak{gl}_N)^{\otimes m}$ and of the tensor product of the homomorphisms (13.9) where $u = u_1, \ldots, u_m$. By using the descending filtration on $\mathrm{Y}^*(\mathfrak{gl}_N)$ and the surjective homomorphism (12.9) we can prove that when the number m vary, the kernels of all homomorphisms $\rho^*_{u_1 \ldots u_m}$ have only zero intersection. The proof is similar that of Proposition 8.1 and is omitted here. It now suffices to derive from (14.4) that for any $W \in \mathrm{Y}^*(\mathfrak{gl}_N)$

$$(\,\rho^*_{u_1 \ldots u_m} \otimes \mathrm{id}\,)\,(\,\mathcal{R}\,\Delta(W)) = (\,\rho^*_{u_1 \ldots u_m} \otimes \mathrm{id}\,)\,(\,\Delta'(W)\,\mathcal{R}\,)\,. \tag{14.7}$$

Here the homomorphism (14.6) is extended to a homomorphism

$$\mathrm{Y}^\circ(\mathfrak{gl}_N) \to (\mathrm{End}\,\mathbb{C}^N)^{\otimes m}\,[[u_1^{-1}, \ldots, u_m^{-1}]]$$

and the extension is still denoted by $\rho^*_{u_1 \ldots u_m}$. Using Propositions 13.2 and 13.5, the relation (14.7) can be rewritten as

$$T_1(u_1) \ldots T_m(u_m)\,(\,\rho^*_{u_1 \ldots u_m} \otimes \mathrm{id}\,)\,(\,\Delta(W))$$
$$= (\,\rho^*_{u_1 \ldots u_m} \otimes \mathrm{id}\,)\,(\,\Delta'(W))\,T_1(u_1) \ldots T_m(u_m)\,.$$

It suffices to verify the latter relation for each of the series $T^*_{ij}(v)$ being substituted for the element W. By the definition (10.7), the substitution yields the relation of the formal power series in $u_1^{-1}, \ldots, u_m^{-1}$ and v with the coefficients in the algebra $(\mathrm{End}\,\mathbb{C}^N)^{\otimes m} \otimes \mathrm{DY}(\mathfrak{gl}_N)$,

$$T_1(u_1) \ldots T_m(u_m) \;\times$$

$$\sum_{k_1, \ldots, k_m = 1}^{N} \rho^*_{u_1}(T^*_{ik_1}(v)) \otimes \rho^*_{u_2}(T^*_{k_1 k_2}(v)) \otimes \ldots \otimes \rho^*_{u_m}(T^*_{k_{m-1} k_m}(v)) \otimes T^*_{k_m j}(v) \;=$$

$$\sum_{k_1, \ldots, k_m = 1}^{N} \rho^*_{u_1}(T^*_{k_1 k_2}(v)) \otimes \ldots \otimes \rho^*_{u_{m-1}}(T^*_{k_{m-1} k_m}(v)) \otimes \rho^*_{u_m}(T^*_{k_m j}(v)) \otimes T^*_{ik_1}(v)$$

$$\times \;\; T_1(u_1) \ldots T_m(u_m)\,.$$

Let us now take the tensor products of both sides of this relation with the element $e_{ij} \in \mathrm{End}\,\mathbb{C}^N$, and then sum over the indices $i, j = 1 \ldots, N$. By using the identity

$$e_{ij} = e_{ik_1} e_{k_1 k_2} \ldots e_{k_{m-1} k_m} e_{k_m j}$$

in $\operatorname{End}\mathbb{C}^N$, we arrive at the following relation of series with the coefficients from the tensor product $(\operatorname{End}\mathbb{C}^N)^{\otimes m}\otimes \mathrm{DY}(\mathfrak{gl}_N)\otimes \operatorname{End}\mathbb{C}^N$:

$$
\begin{aligned}
&(T_1(u_1)\dots T_m(u_m)\otimes 1)\,R_{1,m+1}(u_1-v)\dots R_{m,m+1}(u_m-v)\,(1\otimes T^*(v))\\
=\;&(1\otimes T^*(v))\,R_{1,m+1}(u_1-v)\dots R_{m,m+1}(u_m-v)\,(T_1(u_1)\dots T_m(u_m)\otimes 1)\,.
\end{aligned}
$$

Here the subscript $m+1$ labels the last tensor factor $\operatorname{End}\mathbb{C}^N$, which comes after $\mathrm{DY}(\mathfrak{gl}_N)$. This relation can be proved by using (14.1) repeatedly, i.e. m times. □

We have now established the following theorem explicitly decribing $\mathrm{DY}(\mathfrak{gl}_N)$.

Theorem 14.4. *The algebra* $\mathrm{DY}(\mathfrak{gl}_N)$ *is generated by elements* $T_{ij}^{(r)}, T_{ij}^{(-r)}$ *with* $1\leqslant i,j\leqslant N$ *and* $r\geqslant 1$ *subject only to the relations* (2.8),(10.5) *and* (14.4).

Note that the relation (14.4) is equivalent to the collection of relations

$$
(u-v)\,[T_{ij}(u),T^*_{kl}(v)]=\sum_{m=1}^{N}\Big(\delta_{jk}\,T_{im}(u)\,T^*_{ml}(v)-\delta_{il}\,T^*_{km}(v)\,T^*_{mj}(u)\Big)
$$

for all $i,j,k,l=1,\dots,N$. We omit the proof of the equivalence, as it is very similar to the proof of Proposition 2.2. The last displayed relation can be rewritten as

$$
[T_{ij}(u),T^*_{kl}(v)]=\sum_{p=0}^{\infty}\sum_{m=1}^{N}u^{-p-1}v^{p}\,\Big(\delta_{jk}\,T_{im}(u)\,T^*_{ml}(v)-\delta_{il}\,T^*_{km}(v)\,T_{mj}(u)\Big).
$$

Expanding here the series in u,v and equating the coefficients at $u^{-r}v^{s-1}$ we get

$$
\begin{aligned}
[\,T_{ij}^{(r)},T_{kl}^{(-s)}]=\sum_{a=\max(1,r-s+1)}^{r}&\Big(\delta_{jk}\Big(\delta_{a,r-s+1}\,T_{il}^{(r-s)}+\sum_{m=1}^{N}T_{im}^{(a-1)}\,T_{ml}^{(r-s-a)}\Big)\\
&-\delta_{il}\Big(\delta_{a,r-s+1}\,T_{kj}^{(r-s)}+\sum_{m=1}^{N}T_{km}^{(r-s-a)}\,T_{mj}^{(a-1)}\Big)\Big)
\end{aligned}
$$

for any indices $r,s\geqslant 1$. Here we keep to the notation $T_{ij}^{(0)}=\delta_{ij}$.

We will complete this section with describing a bialgebra structure on $\mathrm{DY}(\mathfrak{gl}_N)$. The algebra $\mathrm{DY}(\mathfrak{gl}_N)$ is generated by its two subalgebras, $\mathrm{Y}(\mathfrak{gl}_N)$ and $\mathrm{Y}^*(\mathfrak{gl}_N)$. We have already shown that the assignments (4.1) and (10.7) define comultiplications on these two subalgebras, while the assignments $\varepsilon: T(u)\mapsto 1$ and $\varepsilon: T^*(v)\mapsto 1$ define counit maps on them; see Propositions 4.1 and 10.1. Let us now replace the comultiplication Δ on $\mathrm{Y}^*(\mathfrak{gl}_N)$ by its opposite comultiplication Δ'.

Proposition 14.5. *The double Yangian* $\mathrm{DY}(\mathfrak{gl}_N)$ *is a bialgebra over* $\mathbb{C}$ *with the comultiplication defined by extending* Δ *on* $\mathrm{Y}(\mathfrak{gl}_N)$ *and* Δ' *on* $\mathrm{Y}^*(\mathfrak{gl}_N)$, *and with the counit defined by mapping* $T(u),T^*(v)\mapsto 1$.

Proof. Using the equivalent form (14.4) of the defining relations (14.1), the proof is similar to that of the proof of Proposition 4.1. Here we omit the details. □

15. Filtration on the double Yangian

In Section 5 we explained that the associative algebra $\mathrm{Y}(\mathfrak{gl}_N)$ can be regarded as a flat deformation of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N[z])$. Our explanation was based on Proposition 5.1. In the present section we establish an analogue of that result for the double Yangian $\mathrm{DY}(\mathfrak{gl}_N)$.

In order to do so, let us replace the descending filtration on the algebra $\mathrm{Y}^*(\mathfrak{gl}_N)$ by an ascending filtration, such that any generator $T_{ij}^{(-r)}$ with $r \geqslant 1$ has the degree $-r$. Relative to this ascending filration on $\mathrm{Y}^*(\mathfrak{gl}_N)$, the subspace of elements of degree not more than $-r$ coincides with the subspace of the elements of degree not less than r relative to the descending filtration. Let us now combine the ascending filtration on $\mathrm{Y}^*(\mathfrak{gl}_N)$ with the ascending filtration on $\mathrm{Y}(\mathfrak{gl}_N)$ used in Section 8. That is, now introduce an ascending $\mathbb{Z}$-filtration on the algebra $\mathrm{DY}(\mathfrak{gl}_N)$ by setting

$$\deg' T_{ij}^{(r)} = r-1 \quad \text{and} \quad \deg' T_{ij}^{(-r)} = -r$$

for each index $r \geqslant 1$. Denote by $\mathrm{gr}'\,\mathrm{DY}(\mathfrak{gl}_N)$ the corresponding $\mathbb{Z}$-graded algebra. Keeping to the notation of Section 8, for any $r \geqslant 1$ let $\widetilde{T}_{ij}^{(r)}$ be the image of $T_{ij}^{(r)}$ in the degree $r-1$ component of $\mathrm{gr}'\,\mathrm{DY}(\mathfrak{gl}_N)$. Since we are now using an ascending filtration on $\mathrm{Y}^*(\mathfrak{gl}_N)$ instead of the descending one, for any $r \geqslant 1$ we will denote by $\widetilde{T}_{ij}^{(-r)}$ the image of $T_{ij}^{(-r)}$ in the degree $-r$ component of $\mathrm{gr}'\,\mathrm{DY}(\mathfrak{gl}_N)$. So $\widetilde{T}_{ij}^{(-r)}$ now formally gets a new meaning, which should not cause any confusion however.

Lemma 15.1. *In the graded algebra* $\mathrm{gr}'\,\mathrm{DY}(\mathfrak{gl}_N)$ *for any* $r,s \geqslant 1$ *we have*

$$[\,\widetilde{T}_{ij}^{(r)},\widetilde{T}_{kl}^{(-s)}] \;=\; \begin{cases} \delta_{kj}\,\widetilde{T}_{il}^{(r-s)} - \delta_{il}\,\widetilde{T}_{kj}^{(r-s)} & \textit{if} \quad r-s>0, \\[6pt] \delta_{kj}\,\widetilde{T}_{il}^{(r-s-1)} - \delta_{il}\,\widetilde{T}_{kj}^{(r-s-1)} & \textit{if} \quad r-s\leqslant 0. \end{cases}$$

Proof. This follows from the relation displayed in Section 14 last. Indeed, relative to the ascending filtration on $\mathrm{DY}(\mathfrak{gl}_N)$ the commutator at the left hand side of that relation has the degree $r-s-1$ for any $r,s \geqslant 1$. For $r-s>0$ the sum at the right hand side equals

$$\delta_{jk}\,T_{il}^{(r-s)} - \delta_{il}\,T_{kj}^{(r-s)}$$

plus terms of degree not more that $r-s-2$. For $r-s=0$ that sum equals

$$\delta_{jk}\,\bigl(\,\delta_{il}+T_{il}^{(-1)}\,\bigr) - \delta_{il}\,\bigl(\,\delta_{kj}+T_{kj}^{(-1)}\,\bigr) = \delta_{jk}\,T_{il}^{(-1)} - \delta_{il}\,T_{kj}^{(-1)}$$

plus terms of degree not more that -2. Finally, for $r-s<0$ that sum equals

$$\delta_{jk}\,T_{il}^{(r-s-1)} - \delta_{il}\,T_{kj}^{(r-s-1)}$$

plus terms of degree not more that $r-s-2$. □

The graded algebra $\mathrm{gr}'\,\mathrm{DY}(\mathfrak{gl}_N)$ inherits from $\mathrm{DY}(\mathfrak{gl}_N)$ a bialgebra structure, see Proposition 14.5. Moreover $\mathrm{gr}'\,\mathrm{DY}(\mathfrak{gl}_N)$ is a Hopf algebra, see the remarks we made just before Proposition 12.3.

Proposition 15.2. *The graded Hopf algebra* $\mathrm{gr}'\,\mathrm{DY}(\mathfrak{gl}_N)$ *is isomorphic to universal enveloping algebra* $\mathrm{U}(\,\mathfrak{gl}_N[z,z^{-1}]\,)$.

Proof. Consider the subalgebras $\mathrm{gr}'\,\mathrm{Y}(\mathfrak{gl}_N)$ and $\mathrm{gr}'\,\mathrm{Y}^*(\mathfrak{gl}_N)$ of the graded algebra $\mathrm{gr}'\,\mathrm{DY}(\mathfrak{gl}_N)$. We have an isomorphism (5.7) of graded algebras defined by the assignments (5.6). Further, due to Lemma 12.1 a surjective homomorphism

$$\mathrm{U}(\,z^{-1}\,\mathfrak{gl}_N[z^{-1}]\,) \to \mathrm{gr}'\,\mathrm{Y}^*(\mathfrak{gl}_N)$$

can be defined by

$$E_{ij}\,z^{-r} \mapsto \widetilde{T}_{ij}^{(-r)} \quad \text{for} \quad r \geqslant 1\,.$$

Lemma 15.1 ensures that these two homomorphisms extend to a homomorphism

$$\mathrm{U}(\,\mathfrak{gl}_N[z,z^{-1}]\,) \to \mathrm{gr}'\,\mathrm{DY}(\mathfrak{gl}_N)\,. \tag{15.1}$$

This homomorphism is surjective and we will prove that it is injective as well. Our proof will be similar to the proof of injectivity of the homomorphism (5.7) given at the end of Section 8. But now we will use Propositions 14.1 and 14.5.

Take any finite linear combination C of the products

$$(E_{i_1j_1}\,z^{\,s_1})\dots(E_{i_mj_m}\,z^{\,s_m})\in \mathrm{U}(\mathfrak{gl}_N[z,z^{-1}])$$

with certain coefficients

$$C^{\,s_1\dots s_m}_{\,i_1j_1\dots i_mj_m}\in\mathbb{C}$$

where the indices $s_1,\dots,s_m\in\mathbb{Z}$ and the number $m\geqslant 0$ may vary; the indices $i_1,j_1,\dots,i_m,j_m$ may vary as well. Suppose $C\neq 0$ as an element of $\mathrm{U}(\mathfrak{gl}_N[z,z^{-1}])$. The algebra $\mathrm{U}(\mathfrak{gl}_N[z,z^{-1}])$ comes with a natural $\mathbb{Z}$-grading such that for any integer s the generator $E_{ij}\,z^s$ has the degree s. The homomorphism (15.1) preserves this grading. Without loss of generality, suppose that the element C is homogeneous of degree d with respect to this grading. That is,

$$C^{\,s_1\dots s_m}_{\,i_1j_1\dots i_mj_m}=0\quad\text{ if }\quad s_1+\dots+s_m\neq d\,.$$

Now define the element $A\in\mathrm{DY}(\mathfrak{gl}_N)$ as the sum

$$\sum_{s_1+\dots+s_m=d} C^{\,s_1\dots s_m}_{\,i_1j_1\dots i_mj_m}\,T^{(r_1)}_{i_1j_1}\dots T^{(r_m)}_{i_mj_m}$$

where for every $k=1,\dots,m$ we set $r_k=s_k$ if $s_k<0$, and $r_k=s_k+1$ if $s_k\geqslant 0$. Let B be the image of A in the d-th component of the graded algebra $\mathrm{gr}'\,\mathrm{DY}(\mathfrak{gl}_N)$. The element B coincides with the image of C under the homomorhism (15.1).

For any non-zero complex number c the evaluation representation (6.4) of the algebra $\mathrm{U}(\mathfrak{gl}_N[z])$ can be extended to a representation $\widetilde{\sigma}_c$ of $\mathrm{U}(\mathfrak{gl}_N[z,z^{-1}])$ so that

$$\widetilde{\sigma}_c: E_{ij}\,z^s\mapsto c^{\,s}\,e_{ij}\quad\text{ for any }\quad s\in\mathbb{Z}\,.$$

Then by (6.3) and (14.3) we have

$$\widetilde{\sigma}_c(\,E_{ij}\,z^s\,)=\begin{cases}\sigma_c(T^{(s)}_{ij}\,) & \text{if}\quad s<0\,,\\ \sigma_c(T^{(s+1)}_{ij}\,) & \text{if}\quad s\geqslant 0\,.\end{cases}$$

Now let $c_1,\dots,c_n$ be any non-zero complex numbers. Let $D\in(\mathrm{End}\,\mathbb{C}^N)^{\otimes n}$ be the image of C under the tensor product of the representations $\widetilde{\sigma}_{c_1},\dots,\widetilde{\sigma}_{c_n}$ of the algebra $\mathrm{U}(\mathfrak{gl}_N[z,z^{-1}])$. Denote by $\sigma_{c_1\dots c_n}$ the tensor product of the representations $\sigma_{c_1},\dots,\sigma_{c_n}$ of the algebra $\mathrm{DY}(\mathfrak{gl}_n)$; here we use Proposition 14.5. The image of $A\in\mathrm{DY}(\mathfrak{gl}_N)$ under the representation $\sigma_{c_1\dots c_n}$ is a Laurent polynomial in $c_1,\dots,c_n$. The degree of this polynomial does not exceed d, see (4.4) and (10.8). The sum of the terms of degree d of this polynomial equals D, see the proof of Proposition 8.1.

For any finite-dimensional Lie algebra $\mathfrak{a}$ there is an analogue of Lemma 7.1 for $\mathfrak{a}\,[z,z^{-1}]$ instead of $\mathfrak{a}\,[z]$. The proof of that analogue is similar to that of Lemma 7.1 itself and is omitted here. Using that analogue, we can choose n and $c_1,\dots,c_n\neq 0$ so that $D\neq 0$. Then $\deg' A=d$. Indeed, if we had $\deg' A<d$ then the degree of the Laurent polynomial $\sigma_{c_1\dots c_n}(A)$ would be also less then d. This would contradict to the non-vanishing of D. By the definition of the element $B\in\mathrm{gr}'\,\mathrm{Y}(\mathfrak{gl}_N)$, the equality $\deg' A=d$ means that $B\neq 0$. So the homomorphism (15.1) is injective.

Comparing the definitions (5.2),(5.3) and (12.7),(12.8) with general definitions (5.4),(5.5) now completes the proof of the proposition. □

By applying the Poincaré–Birkhoff–Witt theorem [2, Section 2.1] to the current Lie algebra $\mathfrak{gl}_N[z,z^{-1}]$ we now obtain its analogue for the double Yangian $\mathrm{DY}(\mathfrak{gl}_N)$.

Theorem 15.3. *Given any linear ordering of the set of generators $T_{ij}^{(r)}$ and $T_{ij}^{(-r)}$ with $r \geqslant 1$, any element of the algebra $\mathrm{DY}(\mathfrak{gl}_N)$ can be uniquely written as a linear combination of ordered monomials in these generators.*

Corollary 15.4. *The defining homomorphisms of the algebras $\mathrm{Y}(\mathfrak{gl}_N)$ and $\mathrm{Y}^*(\mathfrak{gl}_N)$ to $\mathrm{DY}(\mathfrak{gl}_N)$ are embeddings.*

We will now use our ascending filtration on $\mathrm{DY}(\mathfrak{gl}_N)$ to show that in the initial definition of this algebra, the relations (14.1) can be replaced by the relations

$$\Delta(X)\,\mathcal{R} = \mathcal{R}\,\Delta'(X) \quad \text{for every} \quad X \in \mathrm{Y}(\mathfrak{gl}_N)\,. \tag{15.2}$$

Here Δ' is the comultiplication on $\mathrm{Y}(\mathfrak{gl}_N)$ opposite to (4.1). The infinite sums at both sides of the relations (15.2) can be regarded as elements of the tensor product of $\mathrm{Y}(\mathfrak{gl}_N)$ and of the completion of $\mathrm{DY}(\mathfrak{gl}_N)$ relative to our ascending filtration. The completion of $\mathrm{Y}^*(\mathfrak{gl}_N)$ as a subalgebra of $\mathrm{DY}(\mathfrak{gl}_N)$ then coincides with $\mathrm{Y}^\circ(\mathfrak{gl}_N)$.

Proposition 15.5. *Relations* (15.2) *in the algebra* $\mathrm{DY}(\mathfrak{gl}_N)$ *are equivalent to* (14.1).

Proof. Let $Y_1, Y_2, \ldots$ be the basis of $\mathrm{Y}(\mathfrak{gl}_N)$ from the proof of Proposition 13.1. Let

$$Y_p\,Y_q = \sum_{r=1}^{\infty} a_{pq}^r\,Y_r \quad \text{and} \quad \Delta\,(Y_r) \;=\; \sum_{p,q=1}^{\infty} b_{pq}^r\,Y_p \otimes Y_q$$

so that $a_{pq}^r\,, b_{pq}^r \in \mathbb{C}$ are the structure constants of the bialgebra $\mathrm{Y}(\mathfrak{gl}_N)$ relative to this basis. Since the system of vectors $Y_1', Y_2', \ldots$ of $\mathrm{Y}^\circ(\mathfrak{gl}_N)$ is dual to the system $Y_1, Y_2, \ldots$ relative to the bialgebra pairing (11.1), we also have the equalities

$$Y_p'\,Y_q' = \sum_{r=1}^{\infty} b_{pq}^r\,Y_r' \quad \text{and} \quad \Delta\,(Y_r') = \sum_{p,q=1}^{\infty} a_{pq}^r\,Y_p' \otimes Y_q'\,.$$

Here we extend the comultiplication Δ on $\mathrm{Y}^*(\mathfrak{gl}_N)$ to $\mathrm{Y}^\circ(\mathfrak{gl}_N)$ as we did just after stating Proposition 10.1.

It suffices to take $X = Y_r$ with $r = 1, 2, \ldots$ in the relations (15.2). Hence we get

$$\sum_{p,q,s=1}^{\infty} b_{pq}^r\,(Y_p\,Y_s') \otimes (Y_q\,Y_s) \;=\; \sum_{p,q,s=1}^{\infty} b_{pq}^r\,(Y_s'\,Y_q) \otimes (Y_s\,Y_p)$$

or

$$\sum_{p,q,s,t=1}^{\infty} a_{qs}^t\,b_{pq}^r\,(Y_p\,Y_s') \otimes Y_t \;=\; \sum_{p,q,s,t=1}^{\infty} a_{sp}^t\,b_{pq}^r\,(Y_s'\,Y_q) \otimes Y_t\,.$$

So the relations (15.2) are equivalent to the relations in our completion of $\mathrm{DY}(\mathfrak{gl}_N)$

$$\sum_{p,q,s=1}^{\infty} a_{qs}^t\,b_{pq}^r\,Y_p\,Y_s' \;=\; \sum_{p,q,s=1}^{\infty} a_{sp}^t\,b_{pq}^r\,Y_s'\,Y_q \quad \text{where} \quad r\,,t = 1, 2, \ldots\,. \tag{15.3}$$

The vectors $Y_1', Y_2', \ldots$ have been determined by (13.2) using a basis $Y_1^*, Y_2^*, \ldots$ of $\mathrm{Y}^*(\mathfrak{gl}_N)$. We also have the equalities

$$Y_s^* = \sum_{r=1}^{\infty} f_{rs}\,Y_r' \tag{15.4}$$

where $f_{rs} = \langle Y_r \,, Y_s^* \rangle$. The matrix $[\, g_{rs}]$ used in (13.2) is inverse to $[\, f_{rs}]$. Due to (13.2) and (15.4) we can replace $W \in \mathrm{Y}^*(\mathfrak{gl}_N)$ by $Y_t' \in \mathrm{Y}^\circ(\mathfrak{gl}_N)$ with $t = 1, 2, \ldots$ in the relations (14.1). In this way we get

$$\sum_{p,q,s=1}^{\infty} a_{pq}^t \, (Y_s' \, Y_p') \otimes (Y_s \, Y_q') \; = \sum_{p,q,s=1}^{\infty} a_{pq}^t \, (Y_q' \, Y_s') \otimes (Y_p' \, Y_s)$$

or

$$\sum_{p,q,r,s=1}^{\infty} a_{pq}^t \, b_{sp}^r \, Y_r' \otimes (Y_s \, Y_q') \; = \sum_{p,q,r,s=1}^{\infty} a_{pq}^t \; b_{qs}^r \, Y_r' \otimes (Y_p' \, Y_s) \, .$$

So the relations (14.1) are equivalent to the relations in our completion of $\mathrm{DY}(\mathfrak{gl}_N)$

$$\sum_{p,q,s=1}^{\infty} a_{pq}^t \, b_{sp}^r \, Y_s \, Y_q' \; = \sum_{p,q,s=1}^{\infty} a_{pq}^t \; b_{qs}^r \, Y_p' \, Y_s \quad \text{where} \quad r \,, t = 1, 2, \ldots \, .$$

By cyclically permuting the summation indices in these relations we get (15.3). □

Corollary 15.6. *The coefficients of the series $Z(u)$ lie in the centre of* $\mathrm{DY}(\mathfrak{gl}_N)$.

Proof. The coefficients of $Z(u)$ lie in the centre of $\mathrm{Y}(\mathfrak{gl}_N)$ by Lemma 3.3. To prove that they commute with the elements of $\mathrm{Y}^*(\mathfrak{gl}_N)$ as a subalgebra of $\mathrm{DY}(\mathfrak{gl}_N)$ let us substitute the series $Z(u)$ for $X \in \mathrm{Y}(\mathfrak{gl}_N)$ in (15.2). Due to Proposition 4.2 we get

$$\sum_{s=1}^{\infty} (\, Z(u) \, Y_s' \,) \otimes (\, Z(u) \, Y_s) \; = \sum_{s=1}^{\infty} (Y_s' \, Z(u)) \otimes (Y_s \, Z(u)) \, .$$

As the coefficients of $Z(u)$ are central in $\mathrm{Y}(\mathfrak{gl}_N)$, dividing this by $1 \otimes Z(u)$ yields

$$\sum_{s=1}^{\infty} (\, Z(u) \, Y_s' \,) \otimes Y_s \; = \sum_{s=1}^{\infty} (Y_s' \, Z(u)) \otimes Y_s \, .$$

It follows that the coefficients of $Z(u)$ commute with every Y_s' in our completion of the algebra $\mathrm{DY}(\mathfrak{gl}_N)$. By using the relations (15.4) we now get the corollary. □

Now consider the series $Z^\circ(v)$ appearing in Lemma 10.2. Arguing as in the proof of the Corollary 15.6, but using the relations (14.1) and Proposition 10.4 instead of the relations (15.2) and Proposition 4.2, we can show that the coefficients of $Z^\circ(v)$ belong to the centre of our completion of the algebra $\mathrm{DY}(\mathfrak{gl}_N)$. However, in general these coefficients do not belong to the algebra $\mathrm{DY}(\mathfrak{gl}_N)$ itself, see Section 10 again.

Our completion of the algebra $\mathrm{DY}(\mathfrak{gl}_N)$ can also be used to rewrite the relations (10.5) and (14.4) similarly to (2.8). Take the element $T^\natural(v)$ inverse to $T^*(v)$. In the notation analogous to (10.4) the equality (10.5) of series in u and v with coefficients in $\mathrm{Y}^*(\mathfrak{gl}_N) \otimes (\mathrm{End}\, \mathbb{C}^N)^{\otimes 2}$ can be then rewritten as the equality

$$R(u-v) \, T_1^\natural(u) \, T_2^\natural(v) = T_2^\natural(v) \, T_1^\natural(u) \, R(u-v)$$

of series with coefficients in $\mathrm{Y}^\circ(\mathfrak{gl}_N) \otimes (\mathrm{End}\, \mathbb{C}^N)^{\otimes 2}$. The (14.4) can be rewritten as

$$R(u-v) \, (\, T(u) \otimes 1 \,) \, (\, 1 \otimes T^\natural(v)) = (\, 1 \otimes T^\natural(v)) \, (\, T(u) \otimes 1 \,) \, R(u-v) \, .$$

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