



This is a repository copy of *Condensation in preferential attachment models with location-based choice*.

White Rose Research Online URL for this paper:
<https://eprints.whiterose.ac.uk/146779/>

Version: Accepted Version

Article:

Haslegrave, J., Jordan, J. and Yarrow, M. (2020) Condensation in preferential attachment models with location-based choice. *Random Structures and Algorithms*, 56 (3). pp. 775-795. ISSN 1042-9832

<https://doi.org/10.1002/rsa.20889>

This is the peer reviewed version of the following article: Haslegrave, J, Jordan, J, Yarrow, M. Condensation in preferential attachment models with location-based choice. *Random Struct Alg.* 2019, which has been published in final form at <https://doi.org/10.1002/rsa.20889>. This article may be used for non-commercial purposes in accordance with Wiley Terms and Conditions for Use of Self-Archived Versions.

Reuse

Items deposited in White Rose Research Online are protected by copyright, with all rights reserved unless indicated otherwise. They may be downloaded and/or printed for private study, or other acts as permitted by national copyright laws. The publisher or other rights holders may allow further reproduction and re-use of the full text version. This is indicated by the licence information on the White Rose Research Online record for the item.

Takedown

If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing eprints@whiterose.ac.uk including the URL of the record and the reason for the withdrawal request.



eprints@whiterose.ac.uk
<https://eprints.whiterose.ac.uk/>

Condensation in preferential attachment models with location-based choice

Haslegrave, John
University of Warwick
J.Haslegrave@cantab.net

Jordan, Jonathan
University of Sheffield
Jonathan.Jordan@sheffield.ac.uk

Yarrow, Mark
University of Sheffield
MAYarrow1@sheffield.ac.uk

30th May 2019

Abstract

We introduce a model of a preferential attachment based random graph which extends the family of models in which condensation phenomena can occur. Each vertex has an associated uniform random variable which we call its location. Our model evolves in discrete time by selecting r vertices from the graph with replacement, with probabilities proportional to their degrees plus a constant α . A new vertex joins the network and attaches to one of these vertices according to a given probability associated to the ranking of their locations. We give conditions for the occurrence of condensation, showing the existence of phase transitions in α below which condensation occurs. The condensation in our model differs from that in preferential attachment models with fitness in that the condensation can occur at a random location, that it can be due to a persistent hub, and that there can be more than one point of condensation.

Keywords: Preferential Attachment, Fitness, Location, Random Graphs, Phase Transition

1 Introduction

Preferential attachment graphs were developed as an extension of Erdős and Rényi's random graph model in order to model evolving networks that exhibited a power law in their degree distributions. The standard preferential attachment method discussed by Barabási and Albert [2] evolves from an initial graph G_0 with n_0 vertices v_{1-n_0}, \dots, v_0 . For each $n \geq 0$ the graph G_{n+1} is formed by a new vertex v_{n+1} joining G_n and attaching to an existing vertex $V \in \{v_{1-n_0}, \dots, v_n\}$ according to

$$P(V = v_i) = \frac{\deg_{G_n}(v_i) + \alpha}{\sum_{j=1-n_0}^n (\deg_{G_n}(v_j) + \alpha)}, \quad (1)$$

for some $\alpha > -1$. Equation (1) gives the form of preferential attachment developed by Dorogovtsev, Mendes and Samukhin in [7] as a generalisation of the Barabási and Albert model found in [2], and we shall use this more general form. However, several of the papers referred to in this section, including [2], only consider the case $\alpha = 0$.

It is clear to see from (1) that vertices with a higher degree have a higher probability of attracting new edges. Some commonly mentioned applications of preferential attachment graphs include the number of links to a website and the growth of the number of connections on social networks.

It is observable in real world networks that the growth in influence of an individual vertex is affected by more factors than just its degree. How attractive the vertex is by itself or in comparison to the others also plays a large part. A model incorporating this notion was introduced by Bianconi and Barabási in [3], where they gave each vertex a multiplicative fitness value in their version of (1). They did this in order to add an extra dimension to the competition between vertices that joined the vertex using a generalised preferential attachment mechanism. As a consequence, new, fitter, vertices can still compete against vertices which are well-established in the existing network. A particularly interesting feature of preferential attachment with fitness is the so-called condensation phenomena, where at time n a single vertex or a set of vertices of size $o(n)$ (which vertices can depend on time) can have a total degree of order n . Condensation for preferential attachment with fitness is studied in detail by Borgs et al. [4], Dereich and Ortgiese [6] and Dereich, Mailler and Mörters [5].

Another variant of preferential attachment is the choice model introduced by Malyshev and Paquette in [11, 12] and Krapivsky and Redner in [10]. Here when a new vertex joins the network it first selects several candidate existing vertices at random using (1), then attaches to one of the candidates according to a deterministic rule, such as always attaching to the candidate of highest degree. In these papers, depending on the parameters, as the number of vertices increases linear or approximately linear growth can be observed in the degree of the largest vertex. Preferential attachment with degree-based choice was further studied by Haslegrave and Jordan [9], who showed that condensation can also occur when choosing a lower-ranked vertex.

The choice model is combined with fitness in the model studied by Freeman and Jordan [8] in which each vertex has its own fitness value; the new vertex joins the graph G_n by using preferential attachment to select r vertices from G_n . The new vertex forms an edge between itself and the vertex with the highest fitness of the r selections. It is shown in [8] that again condensation can occur in this model.

In this paper we generalise the model of [8] to allow for choices other than the largest or smallest, for example selecting the middle vertex of a selection of three. Informally, an example of when the middle of three model might be appropriate is when voting for a political candidate; one might prefer to avoid voting for a candidate is too far left or right, and so decide to cast their vote in the middle. We will also allow for randomised choice rules based on the ranking. Because we are no longer selecting the largest value,

we will use the term location rather than fitness. In this paper the locations will be uniform random variables on $[0, 1]$; note that as we are only using the order of locations, this is equivalent to any measure without atoms.

Using stochastic approximation techniques, we will show that the normalised empirical measure on the location space given by weighting the location of each vertex of the graph by its degree plus α converges almost surely to a limit. In some cases, this limit is random and has an atom, the emergence of the atom corresponding to condensation occurring in the system. We will show that the atom appears at a random location, as opposed to the results of [4, 8] where condensation can only occur at the supremum of the fitness distribution. In addition we will show that condensation in our model can be due to a single vertex which acts as a persistent hub, which is not possible in fitness models. In fact, for some choices of our parameters, condensation has probability 1, but persistent hub behaviour has probability strictly between 0 and 1, so condensation can occur in at least two different ways, each with positive probability.

We will also show that in some cases condensation can occur simultaneously at more than one point; again this is quite different behaviour from that seen in [4, 8]. Indeed, there are situations where multiple condensation occurs almost surely, and others in which it occurs with probability strictly between 0 and 1. When using a deterministic choice rule, however, multiple condensation cannot occur and there is always a single phase transition between condensation occurring with probability 0 and 1; we give an exact expression for this phase transition for every deterministic rule.

The remainder of this article will start with a discussion of our model and a summary of the main results in Section 2. Our proofs are in Section 3, and finally Section 4 includes some specific examples that highlight some of the important aspects of our main results.

2 Our model and results

2.1 The model

Fix a parameter $r \in \mathbb{N}$ with $r \geq 2$, a vector $\Xi \in \mathbb{R}^r$ such that $\Xi_i \in [0, 1]$ and $\sum_{i=1}^r \Xi_i = 1$, and a real number $\alpha > -1$.

We start with a tree G_0 containing $n_0 \geq 2$ vertices which we will denote by $V(G_0) = \{v_0, v_{-1}, \dots, v_{-(n_0-1)}\}$. Every vertex v_i in G_0 has its own location x_i in $(0, 1)$; we will assume that these locations are distinct. Given G_n , at time $n + 1$ we form G_{n+1} by adding a new vertex v_{n+1} to the network with a single edge. This vertex has its own uniform random location $x_{n+1} \sim \text{Uni}[0, 1]$, which is independent of the other x_i , and chooses where to attach to at time $n + 1$ by selecting a sample of r pre-existing vertices in G_n with replacement with probabilities proportional to their degrees plus α as given by equation (1).

Let the r selected vertices at time $n + 1$ be denoted by $V_1^{(n+1)}, \dots, V_r^{(n+1)}$ with locations

$x_1^{(n+1)}, \dots, x_r^{(n+1)}$ respectively, and renumber if necessary so that the locations satisfy $x_1^{(n+1)} \leq x_2^{(n+1)} \leq \dots \leq x_r^{(n+1)}$. For definiteness we specify that if two or more vertices in the selection have the same location, we rank them in the order they were selected; note, however, that with probability 1 the only way for this to occur is if the same vertex is selected more than once. The probability that v_{n+1} attaches to vertex $V_i^{(n+1)}$ is then given by Ξ_i ; this choice is independent of the vertices selected and of all choices made at previous time steps. However, this choice need not be independent of x_i ; while independence between these two variables may seem a natural assumption it is not necessary, and relaxing this assumption may better fit some motivations.

For example if $r = 3$ and $\Xi = (0, 1, 0)$ then the new vertex selects a sample of size 3, and connects to the selected vertex of median rank. The model can be thought of as generalising the case of the model of [8] with fixed sample size; that model is obtained by taking our model with $\Xi_r = 1$ (or equivalently with $\Xi_1 = 1$).

Throughout the paper, we shall assume that G_0 is a tree. While this requirement is not necessary, and does not change the results, trees are the most natural starting graphs since the attachment process preserves the tree structure. With this assumption, we also have $\sum_{j=1}^n (\deg_{G_n}(v_j) + \alpha) = 2(n + n_0 - 1) + \alpha(n + n_0) = (n + n_0 - 1)(2 + \alpha) + \alpha$.

2.2 Results

We define $\Psi_n(x)$ to be the probability that a vertex selected randomly from G_n according to the law (1) has location less than or equal to x , that is

$$\Psi_n(x) = \frac{1}{(n + n_0 - 1)(2 + \alpha) + \alpha} \sum_{v_i \in V(G_n): x_i \leq x} (\deg_{G_n}(v_i) + \alpha).$$

Clearly $\Psi_n(0) = 0$, almost surely, and $\Psi_n(1) = 1$. We can think of Ψ_n as being the distribution function of the normalised empirical measure on the location space given by weighting the location of each vertex of the graph by its degree plus α ; we will label this measure ν_n .

Our first result is on the convergence of the measures ν_n . For a fixed x and choice of Ξ , define

$$F_1(y; x, \Xi) = x(\alpha + 1) - (2 + \alpha)y + \sum_{l=1}^r \Xi_l \sum_{i=l}^r \binom{r}{i} y^i (1 - y)^{r-i},$$

to be considered as a function of y for $y \in [0, 1]$.

We will say that $p \in (0, 1)$ is a *stable zero* of $F_1(y; x, \Xi)$ if $F_1(p; x, \Xi) = 0$ where there exists an ϵ such that for $y \in (p - \epsilon, p)$ we have $F_1(y; x, \Xi) > 0$ and for $y \in (p, p + \epsilon)$ we have $F_1(y; x, \Xi) < 0$. Similarly $p \in (0, 1)$ is an *unstable zero* if $F_1(p; x, \Xi) = 0$ and there exists ϵ such that for $y \in (p - \epsilon, p)$ we have $F_1(y; x, \Xi) < 0$ and for $y \in (p, p + \epsilon)$ we have $F_1(y; x, \Xi) > 0$, and $p \in (0, 1)$ is a *touchpoint* if $F_1(p; x, \Xi) = 0$ and there exists ϵ such that we have either $F_1(y; x, \Xi) < 0$ for all $y \in (p - \epsilon, p + \epsilon) \setminus \{p\}$ or $F_1(y; x, \Xi) > 0$ for all $y \in (p - \epsilon, p + \epsilon) \setminus \{p\}$.

Remark 2.1. Since $F_1(y; x, \Xi)$ is a polynomial, every root in $(0, 1)$ is either a stable zero, an unstable zero or a touchpoint. Also, for $x \in (0, 1)$ we have $F_1(0; x, \Xi) > 0 > F_1(1; x, \Xi)$, so 0 and 1 are not roots.

Theorem 2.2. As $n \rightarrow \infty$, the sequence of measures converges weakly, almost surely, to a (possibly random) probability measure on $[0, 1]$, whose distribution function we will call Ψ . Furthermore, the following properties hold for any given $x \in (0, 1)$.

1. Almost surely, $\Psi(x)$ is a zero of the function $F_1(y; x, \Xi)$.
2. For any stable zero or touchpoint y of $F_1(y; x, \Xi)$, there is positive probability that $\Psi(x) = y$.
3. Any unstable zero y of $F_1(y; x, \Xi)$ has probability zero that $\Psi(x) = y$.

Depending on the parameters of the model, the limit Ψ may be continuous or discontinuous; for example in Section 4.1 we will show that the model mentioned above where $r = 3$ and $\Xi = (0, 1, 0)$ exhibits a phase transition where Ψ is almost surely continuous for $\alpha \geq -\frac{1}{2}$ and almost surely has a discontinuity for $\alpha < -\frac{1}{2}$.

Discontinuity of Ψ implies that Ψ_n increases by $\Theta(1)$ on an interval of length $o(1)$ as $n \rightarrow \infty$; this corresponds to a condensation phenomenon whereby a small number of vertices with locations in a range of size $o(1)$ have a $\Theta(1)$ probability of being selected. A consequence of Theorem 2.2 is that where there is an interval of x values for which $F_1(y; x, \Xi)$ has more than one stable root, the discontinuity occurs at a random location, as any stable root has positive probability of being the limit for each x in the interval.

It does not immediately follow from discontinuity of Ψ that a single vertex has linear degree; however, the next result shows that this occurs with positive probability. Without loss of generality, we will focus on the vertex v_0 , present in the graph from the start, and label its location as z . We define

$$D_n = \frac{\alpha + \deg_{G_n}(v_0)}{(n + n_0 - 1)(2 + \alpha) + \alpha}, \quad (2)$$

which would be the probability of selecting v_0 for attachment under the preferential attachment rule.

Theorem 2.3. Let z be the location of vertex v_0 . If $y_i \geq y_j$ are two stable fixed points of $F_1(y; z, \Xi)$, then there is positive probability that $(\Psi_n(z), D_n) \rightarrow (y_i, y_i - y_j)$ as $n \rightarrow \infty$.

Theorem 2.3 shows that if there are two distinct stable fixed points of $F_1(y; z, \Xi)$ then condensation can occur at a persistent hub in the sense that a vertex of the initial graph with location z has its degree growing linearly with n with positive probability. The condensation phenomenon that occurs in this case is thus different from that found for preferential attachment with multiplicative fitness, where Dereich, Mailler and Mörters [5] show that the maximum degree divided by n converges to zero in probability; it is also distinct from that found by Freeman and Jordan [8] where, although the maximum

degree is usually of linear order in n , any individual vertex only dominates for a finite time before being displaced by fitter vertices.

However, the next result shows that for some choices of our parameters there is also positive probability that the condensation phenomenon is not due to a persistent hub, as it implies there is positive probability of the condensation occurring at a specific location, where the probability of there being a vertex is zero. This suggests that in this case the condensation phenomenon is more like one of those found in [5] or [8], in that vertices whose location is close to the condensation location are replaced over time in the condensate by those which are even closer.

Theorem 2.4. *Let $x \in (0, 1)$ and Ξ be such that there exists $p \in (0, 1)$ which is a touchpoint of $F_1(y; x, \Xi)$. Then there is positive probability that condensation occurs at p in the sense that Ψ has a discontinuity at p .*

One natural question is whether it is possible to have more than one discontinuity in Ψ , implying more than one point of condensation. The following result shows that this is not possible in the case where the same rank is always chosen.

Theorem 2.5. *Whenever $\Xi_k = 1$ for some $k \in \{1, 2, \dots, r\}$, it is impossible to have more than one point of condensation.*

An example of a choice of Ξ for which more than one point of condensation is possible appears in section 4.2.

In the case where the same rank is always chosen, we can give a precise description of the phase transition.

Theorem 2.6. *If $\Xi_k = 1$ then condensation almost surely occurs if $\alpha < \alpha_c$ and almost surely does not occur if $\alpha \geq \alpha_c$, where $\alpha_c = r - 2$ if $k = 1$ or $k = r$, and*

$$\alpha_c = r \binom{r-1}{k-1} \frac{(k-1)^{k-1} (r-k)^{r-k}}{(r-1)^{r-1}} - 2 \quad (3)$$

otherwise.

In particular, $\alpha_c > -1$ for all values of r and k , meaning that there is always a genuine phase transition. In fact $\alpha_c > 0$, giving condensation even for linear preferential attachment, except in a few small cases: first (or second) of two choices; middle of three; second (or third) of four; and middle of five.

3 Proofs

For the majority of this section we will restrict the model to the case where the choice between the r selected vertices is deterministic, i.e. v_{n+1} always attaches to the selected vertex with the k th lowest location for some fixed k . In the formal notation given above this model can be written as $\Xi = (0, 0, \dots, 1, \dots, 0, 0)$ where the 1 is in the k th position; write $\mathbf{e}_k^{(r)}$ for this vector. We will deal with general Ξ in Section 3.2.

A key technique we use in our proofs is that of stochastic approximation algorithms, originally developed by Robbins and Monro [15]. Stochastic approximation methods appear naturally in preferential attachment models, and have been used, for example, by Malyshkin and Paquette [11] and Dereich and Ortgiese [6]. Stochastic approximation processes operate in discrete time with standard notation, based on Pemantle [14],

$$X_{n+1} - X_n = \gamma_n(F(X_n) + \xi_{n+1} + R_n), \quad (4)$$

where $\{X_n, n \geq 1\}$ is a sequence of random variables on \mathbb{R}^d , γ_n are step sizes satisfying $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\sum_{n=1}^{\infty} \gamma_n^2 < \infty$, F is a function from \mathbb{R}^d to itself, R_n are remainder terms which must tend to zero and satisfy $\sum_{n=1}^{\infty} n^{-1}|R_n| < \infty$, and ξ_{n+1} are noise terms satisfying $\mathbb{E}(\xi_{n+1} | \mathcal{F}_n) = 0$.

We will mainly use results found in Section 2 of [14], which show that under certain conditions the process will converge almost surely to an equilibrium of F , that stable equilibria have positive probability of being the limit and that unstable equilibria usually do not.

3.1 Proofs of Theorems 2.2 to 2.4 for deterministic choice rules

Let \mathcal{F}_n be the σ -algebra generated by the graphs G_n and the locations of their vertices up until time n , i.e. $\mathcal{F}_n = \sigma(G_i, x_i; i \leq n)$. For $x \in [0, 1]$, the probability of attaching to a vertex with location in $[0, x]$ at time $n + 1$, conditional on \mathcal{F}_n , is given by

$$g(\Psi_n(x); \mathbf{e}_k^{(r)}) = \sum_{i=k}^r \binom{r}{i} \Psi_n(x)^i (1 - \Psi_n(x))^{r-i}. \quad (5)$$

We can now formulate the first stochastic approximation equation associated to our model, which will allow us to show that as the network grows the total weight of vertices with location in $[0, x]$ grows linearly. If G_0 has n_0 vertices and e_0 edges then G_n has $n_0 + n$ vertices and $e_0 + n$ edges, giving the normalising constant $\gamma_n = (2(e_0 + n) + \alpha(n_0 + n))^{-1}$. Assuming G_0 is a tree, we have the simpler form $\gamma_n = ((n + n_0 - 1)(2 + \alpha) + \alpha)^{-1}$.

Lemma 3.1. *For a fixed $x \in [0, 1]$, we have the stochastic approximation equation*

$$\Psi_{n+1}(x) - \Psi_n(x) = \gamma_{n+1} \left(F_1(\Psi_n(x); x, \mathbf{e}_k^{(r)}) + \xi_{n+1} \right), \quad (6)$$

where $F_1(y; x, \mathbf{e}_k^{(r)}) = g(y; \mathbf{e}_k^{(r)}) - (2 + \alpha)y + x(1 + \alpha)$ and $\mathbb{E}(\xi_{n+1} | \mathcal{F}_n) = 0$.

Proof. Equation (6) is in the form of (4) with F mapping y to $F_1(y; x, \mathbf{e}_k^{(r)})$ and $R_n \equiv 0$; clearly γ_n has the required properties and so it suffices to show that a suitable ξ_{n+1} may be defined.

Note that $\gamma_n^{-1}\Psi_n(x)$ is the total weight at time n of all vertices located in $[0, x]$, where each vertex v has weight $\deg_{G_n}(v) + \alpha$. Thus we consider the change to this total weight from the new vertex and edge; say that v_{n+1} attaches to w_{n+1} .

The vertex w_{n+1} has location at most x with probability $g\left(\Psi_n(x); \mathbf{e}_k^{(r)}\right)$ from (5) as the probability of attaching to the vertex with rank k of r selections. The expected addition to the total weight arising from the location of v_{n+1} is $x(1 + \alpha)$. Therefore

$$\begin{aligned}\mathbb{E}(\gamma_{n+1}^{-1}\Psi_{n+1}(x) \mid \mathcal{F}_n) &= \gamma_n^{-1}\Psi_n(x) + x(1 + \alpha) + g\left(\Psi_n(x); \mathbf{e}_k^{(r)}\right) \\ &= \gamma_{n+1}^{-1}\Psi_n(x) - (2 + \alpha)\Psi_n(x) + x(1 + \alpha) + g\left(\Psi_n(x); \mathbf{e}_k^{(r)}\right) \\ &= \gamma_{n+1}^{-1}\Psi_n(x) + F_1\left(\Psi_n(x); x, \mathbf{e}_k^{(r)}\right),\end{aligned}$$

and so

$$\mathbb{E}(\Psi_{n+1}(x) \mid \mathcal{F}_n) = \Psi_n(x) + \gamma_{n+1}F_1\left(\Psi_n(x); x, \mathbf{e}_k^{(r)}\right).$$

Defining ξ_{n+1} as

$$\xi_{n+1} = \frac{\Psi_{n+1}(x) - \mathbb{E}(\Psi_{n+1}(x) \mid \mathcal{F}_n)}{\gamma_{n+1}},$$

we have (6) and clearly $\mathbb{E}(\xi_{n+1} \mid \mathcal{F}_n) = 0$. \square

Before continuing, we summarise the necessary results on one-dimensional stochastic approximations; recall the definitions of stable zeros, unstable zeros and touchpoints from Section 2.2. [14, Corollary 2.7] and [14, Theorem 2.8] apply on the assumption that F is bounded and continuous and $\mathbb{E}(\xi_{n+1}^2 \mid \mathcal{F}_n) \leq K$ for some finite K , and say, respectively, that X_n converges almost surely to a zero of F and that any stable zero has positive probability of being the limit. [14, Theorem 2.9] requires additionally that $\mathbb{E}(\xi_{n+1}^+ \mid \mathcal{F}_n)$ and $\mathbb{E}(\xi_{n+1}^- \mid \mathcal{F}_n)$ are bounded above and below by positive numbers in a neighbourhood of p , and says that if p is an unstable zero then it is almost surely not the limit. Finally, Theorem 2.5 of Antunović, Mossel and Rácz [1], based on work by Pemantle in [13], requires that F is continuously differentiable and $|\xi_{n+1}| \leq K$ almost surely, and says that each touchpoint has positive probability of being the limit.

Theorem 3.2. *Let $x \in (0, 1)$. The sequence of random variables $\Psi_n(x)$ converges almost surely to a zero of $F_1\left(y; x, \mathbf{e}_k^{(r)}\right)$. Any stable zero in $[0, 1]$ or touchpoint in $(0, 1)$ has positive probability of being the limit, while any unstable zero has probability zero of being the limit.*

Proof. First we note that $F_1\left(0; x, \mathbf{e}_k^{(r)}\right) > 0$ and $F_1\left(1; x, \mathbf{e}_k^{(r)}\right) < 0$. Therefore there must be at least one zero of $F_1\left(y; x, \mathbf{e}_k^{(r)}\right)$ in the interval $[0, 1]$.

Since $y \mapsto F_1\left(y; x, \mathbf{e}_k^{(r)}\right)$ is a polynomial, it is continuously differentiable. Consequently, in order to apply the results detailed above, it suffices to check that ξ_{n+1} has the required properties.

Recall that $\gamma_{n+1}^{-1}\Psi_{n+1}(x)$ is the total weight of vertices in $[0, x]$ at time $n + 1$, so ξ_{n+1} is the difference between the actual value of this quantity and its expectation at time

n . Recall that v_{n+1} has location x_{n+1} , and write z_{n+1} for the location of its neighbour in G_{n+1} . Given \mathcal{F}_n , the total weight only depends on x_{n+1} and z_{n+1} , and takes values in $[\gamma_n^{-1}\Psi_n(x), \gamma_n^{-1}\Psi_n(x) + 2 + \alpha]$. Thus $\gamma_{n+1}^{-1}\mathbb{E}(\Psi_{n+1}(x) \mid \mathcal{F}_n)$ is also in this interval, meaning that $|\xi_{n+1}| \leq 2 + \alpha$, and so $\mathbb{E}(\xi_{n+1} \mid \mathcal{F}_n) \leq (2 + \alpha)^2$.

We now verify the conditions on ξ_{n+1}^+ and ξ_{n+1}^- . Since $|\xi_n| = \xi_n^+ + \xi_n^-$, both of these are also bounded above. Fix an unstable zero $p \in (0, 1)$ and let ϵ be such that $0 < p - \epsilon < p + \epsilon < 1$. Provided $\Psi_n(x) \in (p - \epsilon, p + \epsilon)$, we have $\mathbb{P}(x_{n+1}, z_{n+1} \leq x \mid \mathcal{F}_n) = xg(\Psi_n(x); \mathbf{e}_k^{(r)})$ is bounded away from 0, and similarly for $\mathbb{P}(x_{n+1}, z_{n+1} > x \mid \mathcal{F}_n)$. It follows that $\mathbb{E}(\gamma_{n+1}^{-1}\Psi_{n+1}(x) \mid \mathcal{F}_n)$ is bounded away from $\gamma_n^{-1}\Psi_n(x)$ but attains this value with probability bounded away from 0, giving a lower bound on $\mathbb{E}(\xi_{n+1}^- \mid \mathcal{F}_n)$; similar reasoning applies to ξ_{n+1}^+ .

Thus all the results described above apply in this setting, giving almost sure convergence to the zero set, positive probability of convergence to each stable zero or touchpoint, and zero probability of convergence to each unstable zero. \square

The following result completes the proof of Theorem 2.2 in the case $\Xi = \mathbf{e}_k^{(r)}$.

Corollary 3.3. *The sequence of measures defined by Ψ_n converges weakly, almost surely, to a limit defined by a (possibly random) distribution function $\Psi : \mathbb{R} \rightarrow [0, 1]$.*

Proof. By definition, we have that for each n Ψ_n is a non-decreasing cadlag function with $\Psi_n(1) = 1$ and, almost surely, $\Psi_n(0) = 0$. We apply Theorem 3.2 to a countable dense set of $x \in (0, 1)$ and for x in this set we define $\hat{\Psi}(x) = \lim_{n \rightarrow \infty} \Psi_n(x)$. We then define a cadlag function Ψ by defining $\Psi(x) = \inf_{y > x} \hat{\Psi}(y)$ for $x \in [0, 1]$, $\Psi(x) = 0$ for $x < 0$ and $\Psi(x) = 1$ for $x \geq 1$. By this construction, the probability measure with distribution function Ψ is a weak limit of the sequence of probability measures with distribution functions Ψ_n . \square

We now move towards proving Theorem 2.3 in the case $\Xi = \mathbf{e}_k^{(r)}$. To do this, we consider a two dimensional stochastic approximation for $(\Psi_n(z), D_n)$, where D_n is given by (2). Let χ_n be the location of a selected vertex under preferential attachment from G_n . Assuming that v_0 is the only vertex at location z , which occurs almost surely, we have

$$\begin{aligned} P(\chi_n = z \mid \mathcal{F}_n) &= D_n, \\ P(\chi_n < z \mid \mathcal{F}_n) &= \Psi_n(z) - D_n, \\ P(\chi_n > z \mid \mathcal{F}_n) &= 1 - \Psi_n(z). \end{aligned}$$

The probability of the k th ranked location being z , and hence of selecting vertex v_0 for v_{n+1} to attach to is given by

$$h(\Psi_n(z), D_n; \mathbf{e}_k^{(r)}) = \sum_{j=0}^{k-1} \sum_{i=k}^r \binom{r}{i} \binom{i}{j} (\Psi_n(z) - D_n)^j D_n^{i-j} (1 - \Psi_n(z))^{r-i}. \quad (7)$$

We can now form our two dimensional stochastic approximation.

Lemma 3.4. *We have the stochastic approximation equation*

$$D_{n+1} - D_n = \gamma_{n+1} \left(F_2 \left(\Psi_n(z), D_n; \mathbf{e}_k^{(r)} \right) + \zeta_{n+1} \right), \quad (8)$$

where $F_2(y, d; \mathbf{e}_k^{(r)}) = h(y, d; \mathbf{e}_k^{(r)}) - (2 + \alpha)d$ and $\mathbb{E}(\zeta_{n+1} | \mathcal{F}_n) = 0$.

Proof. As in the proof of Lemma 3.1, (8) is a stochastic approximation with $R_n \equiv 0$ provided that a suitable ζ_{n+1} may be defined. Writing w_{n+1} for the vertex v_{n+1} attaches to, we have

$$\begin{aligned} \mathbb{E}(D_{n+1} | \mathcal{F}_n) &= \gamma_{n+1} (\mathbb{E}(\deg_{G_{n+1}}(v_0)) + \alpha) \\ &= \gamma_{n+1} (\deg_{G_n}(v_0) + \alpha + \mathbb{P}(w_{n+1} = v_0 | \mathcal{F}_n)) \\ &= \gamma_{n+1} \left(\gamma_n^{-1} D_n + h \left(\Psi_n(z), D_n; \mathbf{e}_k^{(r)} \right) \right), \end{aligned}$$

using (7). Since $\gamma_n^{-1} \gamma_{n+1} = 1 - (2 + \alpha)\gamma_{n+1}$, it follows that

$$\mathbb{E}(D_{n+1} | \mathcal{F}_n) = D_n + \gamma_{n+1} F_2 \left(\Psi_n(z), D_n; \mathbf{e}_k^{(r)} \right).$$

Defining

$$\zeta_{n+1} = \frac{D_{n+1} - \mathbb{E}(D_{n+1} | \mathcal{F}_n)}{\gamma_{n+1}},$$

we have (8) with $\mathbb{E}(\zeta_{n+1} | \mathcal{F}_n) = 0$. □

We have now formed a two dimensional system of stochastic approximation equations represented by

$$\begin{pmatrix} \Psi_{n+1}(z) \\ D_{n+1} \end{pmatrix} - \begin{pmatrix} \Psi_n(z) \\ D_n \end{pmatrix} = \gamma_{n+1} \begin{pmatrix} F_1(\Psi_n(z); x, \Xi) \\ F_2(\Psi_n(z), D_n; \Xi) \end{pmatrix} + \gamma_{n+1} \begin{pmatrix} \xi_{n+1} \\ \zeta_{n+1} \end{pmatrix}. \quad (9)$$

The following relationship between F_1 and F_2 will be useful for identifying stationary points of the vector field associated to our two dimensional stochastic approximation.

Theorem 3.5. *We have that*

$$F_1(y - d; x, \mathbf{e}_k^{(r)}) = F_1(y; x, \mathbf{e}_k^{(r)}) - F_2(y, d; \mathbf{e}_k^{(r)}). \quad (10)$$

Proof. We use induction on k . For $k = 1$,

$$\begin{aligned} F_1(y - d; x, \mathbf{e}_1^{(r)}) &= 1 - (1 - y + d)^r - (2 + \alpha)(y - d) + x(1 + \alpha) \\ &= - (2 + \alpha)(y) + x(1 + \alpha) + \sum_{i=1}^r \binom{r}{i} y^i (1 - y)^{r-i} \end{aligned}$$

$$\begin{aligned}
& - \left(-(2 + \alpha)d + \sum_{i=1}^r \binom{r}{i} d^i (1 - y)^{r-i} \right) \\
& = F_1(y; x, \mathbf{e}_1^{(r)}) - F_2(y, d; \mathbf{e}_1^{(r)})
\end{aligned}$$

Assuming (10) holds for k ,

$$\begin{aligned}
F_1(y - d; x, \mathbf{e}_{k+1}^{(r)}) &= \sum_{i=k+1}^r \binom{r}{i} (y - d)^i (1 - y + d)^{r-i} - (2 + \alpha)(y - d) + x(\alpha + 1) \\
&= F_1(y - d; x, \mathbf{e}_k^{(r)}) - \binom{r}{k} (y - d)^k (1 - y + d)^{r-k} \\
&= F_1(y; x, \mathbf{e}_k^{(r)}) - F_2(y, d; \mathbf{e}_k^{(r)}) - \binom{r}{k} (y - d)^k (1 - y + d)^{r-k} \\
&= F_1(y; x, \mathbf{e}_{k+1}^{(r)}) - F_2(y, d; \mathbf{e}_k^{(r)}) - \binom{r}{k} (y - d)^k (1 - y + d)^{r-k} \\
&\quad + \binom{r}{k} y^k (1 - y)^{r-k}
\end{aligned}$$

where in the last line we have separated out the first term in the sum in the definition of $F_1(y; x, \mathbf{e}_k^{(r)})$. By the definition of $F_2(y, d; \mathbf{e}_k^{(r)})$ and binomial expansion of $(1 - y + d)^{r-k}$ and $y^k = (y - d + d)^k$, we get that $F_1(y - d; x, \mathbf{e}_{k+1}^{(r)})$ is equal to

$$\begin{aligned}
& F_1(y; x, \mathbf{e}_{k+1}^{(r)}) - (2 + \alpha)d - \sum_{j=0}^{k-1} \sum_{i=k}^r \binom{r}{i} \binom{i}{j} (y - d)^j d^{i-j} (1 - y)^{r-i} \\
& - \sum_{i=k}^r \binom{r}{i} \binom{i}{k} (y - d)^k d^{i-k} (1 - y)^{r-i} + \sum_{j=0}^k \binom{r}{k} \binom{k}{j} (y - d)^j d^{k-j} (1 - y)^{r-k}.
\end{aligned}$$

Re-arranging the sums gives that

$$\begin{aligned}
F_1(y - d; x, \mathbf{e}_{k+1}^{(r)}) &= F_1(y; x, \mathbf{e}_{k+1}^{(r)}) - \sum_{j=0}^{k-1} \sum_{i=k}^r \binom{r}{i} \binom{i}{j} (y - d)^j d^{i-j} (1 - y)^{r-i} \\
&\quad - \sum_{j=k}^k \sum_{i=k}^r \binom{r}{i} \binom{i}{j} (y - d)^j d^{i-j} (1 - y)^{r-i} \\
&\quad + \sum_{j=0}^k \sum_{i=k}^k \binom{r}{i} \binom{i}{j} (y - d)^j d^{i-j} (1 - y)^{r-i} - (2 + \alpha)d \\
&= F_1(y; x, \mathbf{e}_{k+1}^{(r)}) - \left(\sum_{j=0}^k \sum_{i=k+1}^r \binom{r}{i} \binom{i}{j} (y - d)^j d^{i-j} (1 - y)^{r-i} + (2 + \alpha)d \right) \\
&= F_1(y; x, \mathbf{e}_{k+1}^{(r)}) - F_2(y, d; \mathbf{e}_{k+1}^{(r)}),
\end{aligned}$$

completing the proof. \square

Let $\mathcal{Y}(z)$ be the set $\{y : F_1(y; z, \mathbf{e}_k^{(r)}) = 0\}$, and write its elements $y_1, y_2, \dots, y_{|\mathcal{Y}(z)|}$. It then follows from Theorem 3.5 that $F_2(y_i, y_i - y_j; z, \mathbf{e}_k^{(r)}) = 0$ and that the solutions to $F_1(y; z, \mathbf{e}_k^{(r)}) = F_2(y, d; \mathbf{e}_k^{(r)}) = 0$ all take the form $(y, d) = (y_i, y_i - y_j)$ where $i, j \in \{1, 2, \dots, |\mathcal{Y}(z)|\}$. Note also that if $y < \frac{(1+\alpha)x}{2+\alpha}$ then $F_1(y; z, \mathbf{e}_k^{(r)}) > 0$, and similarly if $y > 1 - \frac{(1+\alpha)(1-x)}{2+\alpha}$ then $F_1(y; z, \mathbf{e}_k^{(r)}) < 0$, so any $y \in \mathcal{Y}(z)$ satisfies

$$\frac{(1+\alpha)x}{2+\alpha} \leq y \leq 1 - \frac{(1+\alpha)(1-x)}{2+\alpha}. \quad (11)$$

To investigate the stability of the stationary points, we will now calculate the Jacobian M of the two dimensional system. We can observe that M is an upper triangular matrix because $F_1(y; z, \mathbf{e}_k^{(r)})$ does not depend on d so $\frac{\partial F_1}{\partial d} = 0$. Therefore the eigenvalues of our system are

$$\begin{aligned} \lambda_1(y; \mathbf{e}_k^{(r)}) &= \sum_{i=k}^r \binom{r}{i} y^{i-1} (1-y)^{r-i-1} (i-ry) - (2+\alpha), \\ \lambda_2(y, d; \mathbf{e}_k^{(r)}) &= \sum_{j=0}^{k-1} \sum_{i=k}^r \binom{r}{i} \binom{i}{j} Q_{i,j}^{(r)}(y, d) (y-d)^{j-1} d^{i-j-1} (1-y)^{r-i-1} - (2+\alpha), \end{aligned}$$

where $Q_{i,j}^{(r)}(y, d) = (y-d)(i-iy+rd-id) + j(1-y)(2d-y)$.

Theorem 3.6. *For any $y_i, y_j \in \mathcal{Y}(z)$ such that $y_i - y_j \geq 0$ and*

$$\frac{\partial}{\partial y} F_1(y; z, \mathbf{e}_k^{(r)}) = \lambda_1(y; \mathbf{e}_k^{(r)}) < 0$$

is satisfied for both $y = y_i$ and $y = y_j$, we have that $(y_i, y_i - y_j)$ is a stable equilibrium of the vector field $(F_1(y; z, \mathbf{e}_k^{(r)}), F_2(y, d; \mathbf{e}_k^{(r)}))$.

Proof. By rearranging (10) we can see that

$$F_2(y, d; \mathbf{e}_k^{(r)}) = F_1(y; z, \mathbf{e}_k^{(r)}) - F_1(y-d; z, \mathbf{e}_k^{(r)})$$

and can deduce that

$$\lambda_2(y, d; \mathbf{e}_k^{(r)}) = \frac{\partial}{\partial d} (F_1(y; z, \mathbf{e}_k^{(r)}) - F_1(y-d; z, \mathbf{e}_k^{(r)})).$$

Here $F_1(y; z, \mathbf{e}_k^{(r)})$ does not depend on d , giving $\frac{\partial}{\partial d} F_1(y; z, \mathbf{e}_k^{(r)}) = 0$. Consequently, $\lambda_2(y, d; \mathbf{e}_k^{(r)}) = -\frac{\partial}{\partial d} F_1(y - d; z, \mathbf{e}_k^{(r)}) = \lambda_1(y - d; \mathbf{e}_k^{(r)})$. All roots of $F_1(y; z, \mathbf{e}_k^{(r)}) = F_2(y, d; \mathbf{e}_k^{(r)}) = 0$ are of the form $(y_i, y_i - y_j)$. Evaluating eigenvalues at this point gives $\lambda_1(y_i; \mathbf{e}_k^{(r)})$ and $\lambda_2(y_i, d_i; \mathbf{e}_k^{(r)}) = \lambda_1(y_j; \mathbf{e}_k^{(r)})$, which, referring to our initial conditions, are both negative. Therefore the pair y_i and y_j form the possible limit $(y_i, y_i - y_j)$. \square

Corollary 3.7. *If $y_i \geq y_j$ are two stable fixed points of $F_1(y; z, \mathbf{e}_k^{(r)})$, then there is positive probability of $(\Psi_n(z), D_n) \rightarrow (y_i, y_i - y_j)$ as $n \rightarrow \infty$.*

Proof. Theorem 3.6 shows that $(y_i, y_i - y_j)$ is a stable stationary point of the vector field $(F_1(y; z, \mathbf{e}_k^{(r)}), F_2(y, d; \mathbf{e}_k^{(r)}))$. The conclusion then follows from Theorem 2.16 of Pemantle [14], as long as we can show that there is no t for which $(\Psi_{t+n}(z), D_{t+n})_{n \geq 0}$ almost surely avoids some neighbourhood of $(y_i, y_i - y_j)$. To see this, note that D_{t+n} depends only on the number of vertices added up to time $t + n$ which connect to the vertex v_0 , and that $\Psi_{t+n}(z)$ depends only on the number of vertices added up to time $t + n$ which connect to vertices with location less than or equal to z and the number of those vertices which have locations less than or equal to z themselves.

Any integer between $\deg_{G_0}(v_0)$ and $\deg_{G_0}(v_0) + t + n$ inclusive has positive probability as a value with for $\deg_{G_{t+n}}(v_0)$; by the definition of D_{t+n} this ensures that for sufficiently large n there is positive probability of D_{t+n} being in any given subinterval of $[0, \frac{1}{2+\alpha}]$. By similar arguments for $\Psi_{t+n}(z)$, $(\Psi_{t+n}(z), D_{t+n})_{n \geq 0}$ can, with positive probability, approach $(y_i, y_i - y_j)$ arbitrarily closely, as long as $0 \leq y_i \leq 1$ and $0 \leq y_i - y_j \leq \frac{1}{2+\alpha}$. That these conditions are satisfied follows from (11) and the assumption that $y_i \geq y_j$. \square

This completes the proof of Theorem 2.3 in the case $\Xi = \mathbf{e}_k^{(r)}$.

To prove Theorem 2.4, we first note that where p is a touchpoint of $F_1(y; x, \mathbf{e}_k^{(r)})$ with $F_1(y; x, \mathbf{e}_k^{(r)})$ non-positive in a neighbourhood of p there will be a neighbourhood of p which contains no zeros of $F_1(y; x - u, \Xi)$ for positive u . Hence the probability of $\Psi(x - u)$ being in this neighbourhood of p is zero, but we know from Theorem 3.2 that there is positive probability that $\lim_{n \rightarrow \infty} \Psi_n(x) = p$. Hence there is positive probability of a discontinuity at x . The same applies, with $x - u$ replaced by $x + u$, if $F_1(y; x, \mathbf{e}_k^{(r)})$ is non-negative in a neighbourhood of p , completing the proof.

3.2 Proofs of Theorems 2.2 to 2.4 in the general case

We now extend the proofs of Theorems 2.2 to 2.4 in the case where Ξ is not necessarily equal to $\mathbf{e}_k^{(r)}$. We can derive

$$\begin{aligned} F_1(y; x, \Xi) &= \sum_{l=1}^r \Xi_l F_1(y; x, \mathbf{e}_l^{(r)}) \\ &= x(\alpha + 1) - (2 + \alpha)y + \sum_{l=1}^r \Xi_l \sum_{i=l}^r \binom{r}{i} y^i (1 - y)^{r-i} \end{aligned}$$

and extend the definition of F_2 from Lemma 3.4 as

$$\begin{aligned} F_2(y, d; \Xi) &= \sum_{l=1}^r \Xi_l F_2(y, d; \mathbf{e}_l^{(r)}) \\ &= \left(\sum_{l=1}^r \Xi_l \sum_{j=0}^{l-1} \sum_{i=l}^r \binom{r}{i} \binom{i}{j} (y - d)^j d^{i-j} (1 - y)^{r-i} \right) - (2 + \alpha)d. \end{aligned}$$

We can see that Lemmas 3.1 and 3.4 still hold, and the arguments for Theorem 3.2 and Corollary 3.3 work in the same way as for the case $\Xi = \mathbf{e}_k^{(r)}$, completing the proof of Theorem 2.2.

By considering the above expressions for $F_1(y; x, \Xi)$ and $F_2(y; x, \Xi)$ as weighted sums of $F_1(y; x, \mathbf{e}_l^{(r)})$ and $F_2(y; x, \mathbf{e}_l^{(r)})$ respectively, we can see that Theorem 3.5 still holds, so if we let $\mathcal{Y}(z) = \{y_1, y_2, \dots, y_{|\mathcal{Y}(z)|}\}$ be the set of zeros of $F_1(y; z, \Xi)$, the stationary points are still of the form $(y_i, y_i - y_j)$. It is easy to see that the eigenvalues of the Jacobian are now

$$\begin{aligned} \lambda_1(y; \Xi) &= -(2 + \alpha) + \sum_{l=1}^r \Xi_l \frac{\partial}{\partial y} F_1(y; z, \mathbf{e}_l^{(r)}) \\ &= -(2 + \alpha) + \sum_{l=1}^r \Xi_l \lambda_1(y; \mathbf{e}_l^{(r)}); \\ \lambda_2(y, d; \Xi) &= -(2 + \alpha) + \sum_{l=1}^r \Xi_l \frac{\partial}{\partial d} F_2(y, d; \mathbf{e}_l^{(r)}) \\ &= -(2 + \alpha) + \sum_{l=1}^r \Xi_l \lambda_1(y - d; \mathbf{e}_l^{(r)}). \end{aligned}$$

Consequently Theorem 3.6 can also be extended to this case: if we have $y_i, y_j \in \mathcal{Y}(z)$ with $y_i \geq y_j$ that satisfy $\lambda_1(y_i; \Xi) < 0$ and $\lambda_1(y_j; \Xi) < 0$ then $(y_i, y_i - y_j)$ is a stable equilibrium of the vector field given by $F_1(y; z, \Xi)$ and $F_2(y, d; \Xi)$. This completes the proof of Theorem 2.3.

Finally, the proof of Theorem 2.4 is the same as for the case $\Xi = \mathbf{e}_k^{(r)}$.

3.3 Proofs of Theorems 2.5 and 2.6

Proof of Theorem 2.5. By differentiating $F_1(y; x, \mathbf{e}_k^{(r)})$ we show that for every x there are at most two values of $\Psi(x)$ which occur with positive probability, and where there are two such values that they occur in two disjoint intervals which do not depend on x . Thus a point of condensation must almost surely involve a jump between these regions. Since $\Psi(x)$ is increasing by definition, this can happen at most once.

We have

$$\begin{aligned} \frac{\partial}{\partial y} g(y; \mathbf{e}_k^{(r)}) &= \frac{\partial}{\partial y} \sum_{i=k}^r \binom{r}{i} y^i (1-y)^{r-i} \\ &= \sum_{i=k}^r i \binom{r}{i} y^{i-1} (1-y)^{r-i} - \sum_{i=k}^{r-1} (r-i) \binom{r}{i} y^i (1-y)^{r-i-1} \\ &= \sum_{i=k}^r r \binom{r-1}{i-1} y^{i-1} (1-y)^{r-i} - \sum_{i=k}^{r-1} r \binom{r-1}{i} y^i (1-y)^{r-i-1} \\ &= r \binom{r-1}{k-1} y^{k-1} (1-y)^{r-k}, \end{aligned}$$

because all other terms cancel. So

$$\frac{\partial}{\partial y} F_1(y; x, \mathbf{e}_k^{(r)}) = r \binom{r-1}{k-1} y^{k-1} (1-y)^{r-k} - (2 + \alpha);$$

note that this does not depend on x .

If $k = r$ then $\frac{\partial^2}{\partial y^2} F_1(y; x, \mathbf{e}_k^{(r)})$ is positive on $(0, 1)$, and if $k = 1$ then it is negative on $(0, 1)$. Otherwise,

$$\begin{aligned} \frac{\partial^2}{\partial y^2} F_1(y; x, \mathbf{e}_k^{(r)}) &= r \binom{r-1}{k-1} ((k-1)(1-y) - (r-k)y) y^{k-2} (1-y)^{r-k-1} \\ &= r \binom{r-1}{k-1} y^{k-2} (1-y)^{r-k-1} ((k-1) - (r-1)y), \end{aligned}$$

which is positive for $y \in (0, \frac{k-1}{r-1})$ and negative for $y \in (\frac{k-1}{r-1}, 1)$. It follows that, for any choice of k , the equation

$$r \binom{r-1}{k-1} y^{k-1} (1-y)^{r-k} - (2 + \alpha) = 0 \tag{12}$$

has at most two roots in $(0, 1)$, and that if it has exactly two such roots $z_1 < z_2$ then the left-hand side is positive for $y \in (z_1, z_2)$.

Suppose (12) has two roots $z_1 < z_2$. Then for any x we have that $F_1(y; x, \mathbf{e}_k^{(r)})$ is strictly decreasing on the intervals $[0, z_1)$ and $(z_2, 1]$, but strictly increasing on (z_1, z_2) .

Consequently, $F_1(y; x, \mathbf{e}_k^{(r)}) = 0$ has at most one root $y_1(x) \in [0, z_1]$, at most one root $y_2(x) \in (z_1, z_2)$ (which, if it exists, is an unstable zero), and at most one root $y_3(x) \in [z_2, 1]$. Further, for each i , $y_i(x)$ is continuous on the range of x for which it exists. Note that $F_1(0; 0, \mathbf{e}_k^{(r)}) = F_1(1; 1, \mathbf{e}_k^{(r)}) = 0$ and so $y_1(0) = 0$ and $y_3(1) = 1 = \Psi(1)$. By Theorem 2.2, almost surely for almost all x we have $\Psi(x) \in \{y_1(x), y_3(x)\}$. Let $x^* = \inf\{x \in [0, 1] : \Psi(x) = y_3(x)\}$; since Ψ is increasing and $y_1(x)$ and $y_3(x)$ are continuous we have $\Psi(x) = y_1(x)$ for all $x \in [0, x^*)$ and $\Psi(x) = y_3(x)$ for all $x \in (x^*, 1]$. It follows that Ψ is continuous everywhere except at x^* , as required.

If (12) has exactly one root, z , in $(0, 1)$, then the equation $F_1(y; x, \mathbf{e}_k^{(r)}) = 0$ has at most one root $y_1(x) \in [0, z]$ and at most one root $y_2(x) \in (z, 1]$ for every $x \in [0, 1]$; again we must have $y_1(0) = 0$ and $y_2(1) = 1 = \Psi(1)$. Defining x^* as above, almost surely $\Psi(x) = y_1(x)$ for all $x \in [0, x^*)$ and $\Psi(x) = y_2(x)$ for all $x \in (x^*, 1]$, so again Ψ is continuous except possibly at x^* . Finally, if (12) has no roots in $[0, 1]$, then the equation $F_1(y; x, \mathbf{e}_k^{(r)}) = 0$ has exactly one root $y_1(x) \in [0, 1]$ for every $x \in [0, 1]$ and we must have $y_1(0) = 0$ and $y_1(1) = 1 = \Psi(1)$. Thus we almost surely have $\Psi(x) \equiv y_1(x)$, and there are no points of condensation. \square

Proof of Theorem 2.6. First, suppose $k = r$ (the case $k = 1$ is similar). Then

$$\frac{\partial}{\partial y} F_1(y; x, \mathbf{e}_k^{(r)}) = ry^{r-1} - (2 + \alpha),$$

and $ry^{r-1} - (2 + \alpha) = 0$ has one root in $(0, 1)$ if $\alpha > r - 2$ and none otherwise. The proof of Theorem 2.5 shows that almost surely condensation does not occur in the latter case. In the former case, let the root be z . Then $F_1(y; x, \mathbf{e}_k^{(r)})$ is strictly increasing on $(z, 1)$ meaning that any root of $F_1(y; x, \mathbf{e}_k^{(r)}) = 0$ in this region is an unstable zero. Consequently Theorem 2.2 implies that almost surely $\Psi(x) \leq z$ for all $x < 1$, i.e. condensation occurs at 1.

Secondly, suppose $1 < k < r$. Then the left-hand side of (12) is $-(2 + \alpha) < 0$ for $y = 0$ and $y = 1$, and is strictly increasing on $y \in (0, \frac{k-1}{r-1})$ and decreasing for $y \in (\frac{k-1}{r-1}, 1)$. Thus if it is negative for $y = \frac{k-1}{r-1}$ then (12) has no roots in $(0, 1)$, and, as before, condensation almost surely does not occur. If the left-hand side is positive for $y = \frac{k-1}{r-1}$ then (12) has two roots $z_1 < z_2$ in $(0, 1)$, and any root of $F_1(y; x, \mathbf{e}_k^{(r)}) = 0$ in (z_1, z_2) is an unstable zero, meaning that almost surely $\Psi(x) \notin (z_1, z_2)$ for any x , so condensation occurs. If the left-hand side vanishes for $y = \frac{k-1}{r-1}$ then $F_1(y; x, \mathbf{e}_k^{(r)})$ is strictly decreasing on $[0, 1]$ and so $F_1(y; x, \mathbf{e}_k^{(r)}) = 0$ has a single continuously-varying root $y_1(x)$ satisfying $y_1(0) = 0$ and $y_1(1) = 1$; almost surely $\Psi(x) \equiv y_1(x)$ and so there is no condensation.

Since the sign of the left-hand side of (12) changes precisely at α_c , this completes the proof. \square

4 Examples

In this section we consider some examples of choices of Ξ which illustrate how the results of Theorems 2.2 and 2.3 can apply to different cases.

4.1 Middle of three

As $r = 1$ gives standard preferential attachment, and the cases with $r = 2$ and either $\Xi = (0, 1)$ or $\Xi = (1, 0)$ are equivalent to cases of the choice-fitness model of [8], which has rather different behaviour, the simplest case which illustrates our results is the “middle of three” model given by $r = 3$ and $\Xi = (0, 1, 0)$. Here Theorem 2.6 predicts a phase transition at $\alpha_c = -1/2$.

We can express our functions $F_1(y; x, \mathbf{e}_k^{(r)})$ and $F_2(y, d; \mathbf{e}_k^{(r)})$ as

$$F_1(y; x, \mathbf{e}_2^{(3)}) = -2y^3 + 3y^2 - (2 + \alpha)y + x(\alpha + 1)$$

and

$$F_2(y, d; \mathbf{e}_2^{(3)}) = -2d^3 + 6d^2y - 3d^2 - 6dy^2 + 6dy - d(2 + \alpha).$$

For $x \in (0, 1)$, define $\{\psi_1(x), \psi_2(x), \psi_3(x)\}$ such that $\psi_1(x) \leq \psi_2(x) \leq \psi_3(x)$ as the three real roots of $F_1(y; x, \mathbf{e}_2^{(3)}) = 0$ when three exist and $\psi(x)$ as the single root when only one exists. We have $F_1'(y; x, \mathbf{e}_2^{(3)}) = -6y^2 + 6y - (2 + \alpha)$, and so $F_1(y; x, \mathbf{e}_2^{(3)})$ is decreasing in y whenever $\alpha \geq -\frac{1}{2}$, but has turning points at $y = \frac{1}{2} \pm \sqrt{-(1 + 2\alpha)/12}$ for $\alpha \in (-1, -\frac{1}{2})$. At $x = \frac{1}{2}$ we have $F_1(y; x, \mathbf{e}_2^{(3)}) = (\frac{1}{2} - y)(F_1'(y; x, \mathbf{e}_2^{(3)}) + 1 + 2\alpha)/3$. Consequently the values of $F_1(y; \frac{1}{2}, \mathbf{e}_2^{(3)})$ at the turning points are $\mp \sqrt{-(1 + 2\alpha)^3/108}$, and the corresponding values of $F_1(y; \frac{1}{2}, \mathbf{e}_2^{(3)})$ are given by $(1 + \alpha)(x - \frac{1}{2}) \mp \sqrt{-(1 + 2\alpha)^3/108}$. It follows that there are multiple roots if and only if $|x - \frac{1}{2}| \leq \sqrt{\frac{-(1 + 2\alpha)^3}{108(1 + \alpha)}}$; write $s = s(\alpha)$ for this quantity. Note that $s < \frac{1}{2}$ if and only if $\alpha > -\frac{7}{8}$.

Figure 1 plots $F_1(y; x, \mathbf{e}_2^{(3)})$ against $y \in [0, 1]$ for the value $\alpha = -\frac{3}{4}$ and $x = \frac{1}{2} \pm s$, and Figure 2 plots the roots of $F_1(y; x, \mathbf{e}_2^{(3)})$ against $x \in [0, 1]$. There is exactly one real root when $x \in \left\{ \left[0, \frac{9 + \sqrt{6}}{18}\right) \cup \left(\frac{9 - \sqrt{6}}{18}, 1\right] \right\}$ and three real roots when $x \in \left[\frac{9 - \sqrt{6}}{18}, \frac{9 + \sqrt{6}}{18} \right]$.

In this setting, Theorem 2.2 becomes the following.

Theorem 4.1. *For a fixed location $x \in (0, 1)$, the random variable $\Psi_n(x)$ converges*

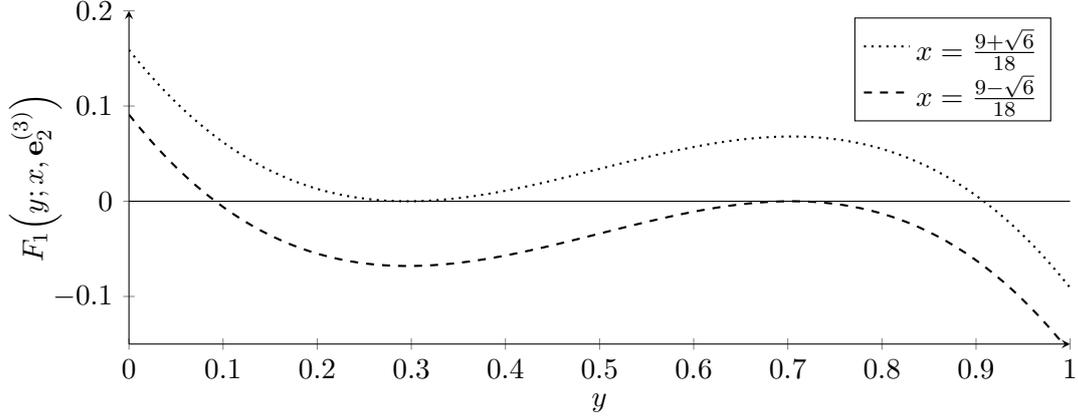


Figure 1: $F_1(y; x, \mathbf{e}_2^{(3)})$ for $\alpha = -0.75$ and $x = \frac{1}{2} \pm s$.

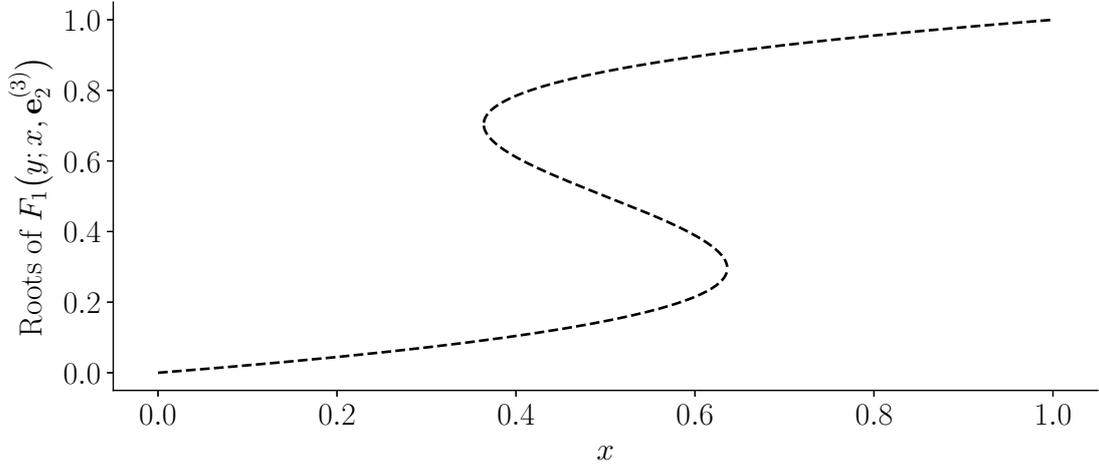


Figure 2: The roots of $F_1(y; x, \mathbf{e}_2^{(3)})$ for $x \in [0, 1]$ and $\alpha = -0.75$.

pointwise as $n \rightarrow \infty$ almost surely to the following limits.

$$\lim_{n \rightarrow \infty} \Psi_n(x) = \begin{cases} \psi(x), & \text{if } \alpha \geq -\frac{1}{2} \\ \psi(x), & \text{if } \alpha \in (-\frac{7}{8}, -\frac{1}{2}) \text{ and } x \notin [\frac{1}{2} - s, \frac{1}{2} + s] \\ \psi_1(x) \text{ or } \psi_3(x), & \text{if } \alpha \in (-\frac{7}{8}, -\frac{1}{2}) \text{ and } x \in [\frac{1}{2} - s, \frac{1}{2} + s] \\ \psi_1(x) \text{ or } \psi_3(x), & \text{if } \alpha \leq -\frac{7}{8}. \end{cases}$$

We can see there is a phase transition at $\alpha = -\frac{1}{2}$: when $\alpha \geq -\frac{1}{2}$, Ψ is almost surely continuous, whereas when $\alpha < -\frac{1}{2}$, Ψ follows the lower root $\psi_1(x)$ until a random point in $[\frac{1}{2} - s, \frac{1}{2} + s]$ at which it jumps to the upper root $\psi_3(x)$, giving a point of condensation.

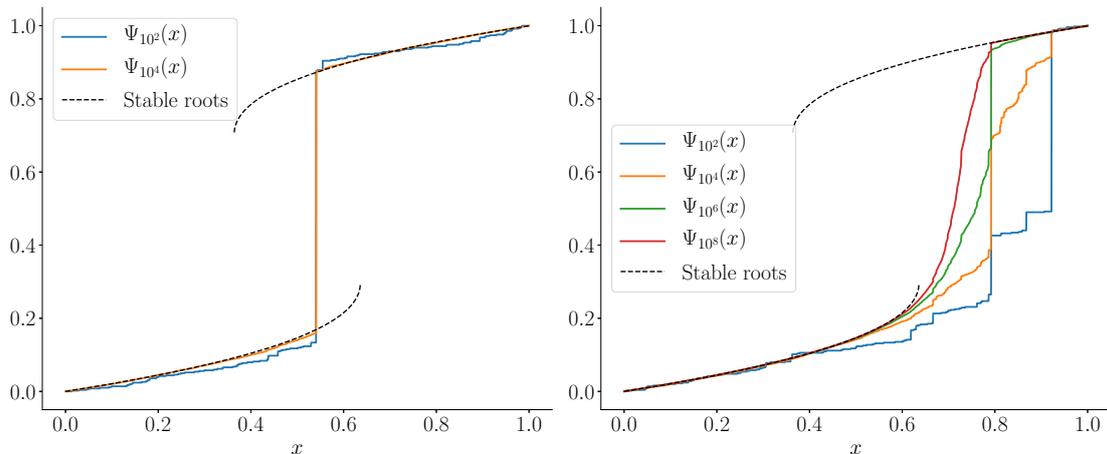


Figure 3: Results from simulations for $\alpha = -0.75$

If $-\frac{7}{8} < \alpha < -\frac{1}{2}$, Theorem 2.3 implies that this point of condensation is with positive probability caused by a persistent hub occurring at a random location which has full support on $(\frac{1}{2} - s, \frac{1}{2} + s)$. However, Theorem 2.4 implies that the point of condensation also has positive probability of occurring at each of the endpoints $\frac{1}{2} - s$ and $\frac{1}{2} + s$; since almost surely these values are not locations of any vertex, it follows that there is also a positive probability that there is no persistent hub. Figure 3 shows the results of two simulations for $\alpha = -0.75$ with different behaviour: in the first simulation there is rapid convergence of Ψ_n to a limit with condensation occurring via a persistent hub, whereas in the second Ψ_n shows much slower convergence, apparently towards condensation at $\frac{1}{2} + s$. If $\alpha \leq -\frac{7}{8}$, Theorem 2.3 implies that the location of the jump has full support on $(0, 1)$.

As we can now implement conditions on $F_1(y, x; \mathbf{e}_k^{(r)})$ using x and α to control whether we have one or three real roots we can solve

$$F_1(y, x; \mathbf{e}_k^{(r)}) = F_2(y, d, x; \mathbf{e}_k^{(r)}) = 0$$

by assuming $F_1(y, x; \mathbf{e}_k^{(r)}) = 0$ has three real roots $\{\psi_1(x), \psi_2(x), \psi_3(x)\}$. We therefore can solve $F_2(y, d, x; \mathbf{e}_k^{(r)}) = 0$ to get $\delta_1 = 0$ and δ_2 and δ_3 given by $\frac{3}{4}(2\psi_i(x) - 1) \pm \frac{1}{4}\sqrt{-12\psi_i(x)^2 + 12\psi_i(x) - 7 - 8\alpha}$.

Figure 4 illustrates Theorem 3.6 in this case, showing the eigenvalues of the Jacobian at the stationary points. The solid curves show $F_1(y, x; \mathbf{e}_2^{(3)})$ for $y \in [0, 1]$ at $x = \frac{1}{2} \pm s$, the upper and lower limits of the region of x where there are three real roots. In this same region are plotted $\lambda_1(y; \mathbf{e}_2^{(3)})$ (the parabola), $\lambda_2(y, \delta_2; \mathbf{e}_2^{(3)})$ and $\lambda_2(y, \delta_3; \mathbf{e}_2^{(3)})$ (dashed and dotted lines respectively). The two regions where the eigenvalues are both negative

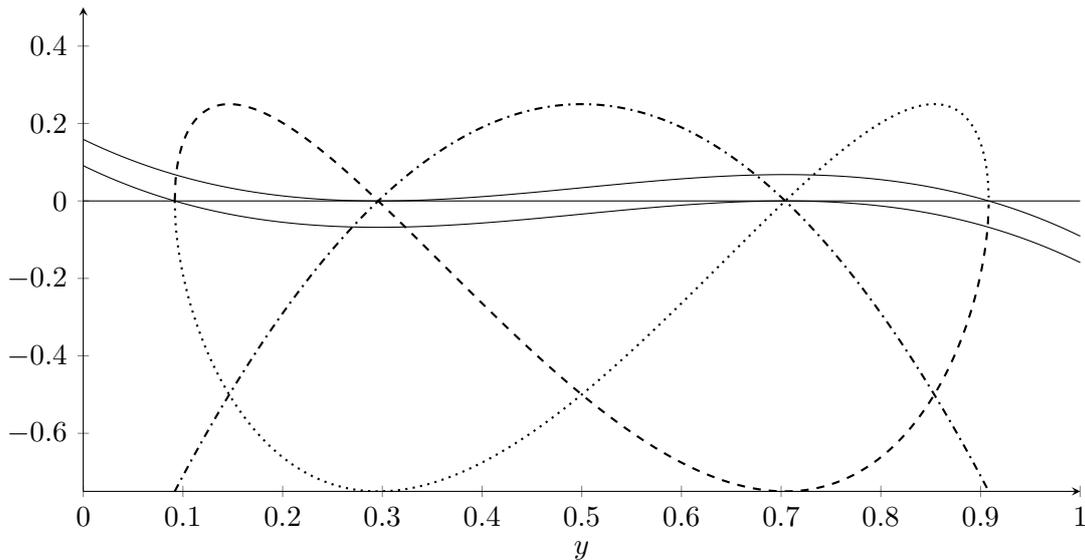


Figure 4: Plot of eigenvalues of the Jacobian when $\alpha = -0.75$

overlap with where the roots of $F_1(y; x, \mathbf{e}_k^{(r)})$ would be as x increases from the lower limit to the upper limit.

4.2 Second or sixth of seven

The second example we will discuss makes use of the vector notation introduced in Section 2. The “middle of three” model of Section 4.1 is an example of selecting the k th highest location from r selections, and demonstrates a phase transition below which condensation must occur at a single point. By Theorem 2.5, no such model can have condensation occurring simultaneously at more than one point. We now consider whether models in which more than one rank has positive probability of being selected can demonstrate multiple points of condensation. If there are three (or more) stable roots of $F_1(y; x, \Xi)$ for some range of x then by Theorem 2.3 there is a positive probability of a jump from the first to the second in that range, and in this case since there are still higher stable roots, another jump must occur. If there are two disjoint ranges with two or more stable roots, separated by a range in which there is only one, then at least one jump must occur in each of these ranges. In this section we give an example which (for different values of α) demonstrates that both of these can occur, even for models where only two ranks have positive probability. A real-life example of when two points of condensation might be expected is that of a bipartisan election, where two candidates from different regions of the location parameter (which might represent political position) may both attract a given proportion of the votes.

The distribution we shall use is $\Xi = (0, \frac{1}{2}, 0, 0, 0, \frac{1}{2}, 0)$, that is, each new vertex is equally likely to connect to the second or sixth rank of seven candidates; this was the simplest

example we could find which allowed for two points of condensation. With the motivating example above, it would be reasonable to suppose that vertices whose own location is higher are more likely to choose the sixth rank of their candidate pool; recall that this type of dependence is permitted in the model.

In this setting, (5) gives the following expression for $F_1(y; x, \Xi)$:

$$\begin{aligned} F_1(y; x, \Xi) &= \frac{1}{2}F_1(y; x, \mathbf{e}_2^{(7)}) + \frac{1}{2}F_1(y; x, \mathbf{e}_6^{(7)}) \\ &= \frac{1}{2}\left(\sum_{i=2}^7 \binom{7}{i} y^i (1-y)^{7-i}\right) + \frac{1}{2}\left(\sum_{i=6}^7 \binom{7}{i} y^i (1-y)^{7-i}\right) \\ &\quad - (2 + \alpha)y + x(\alpha + 1) \\ &= -6y^7 + 21y^6 - 42y^5 + \frac{105}{2}y^4 - 35y^3 + \frac{21}{2}y^2 - (2 + \alpha)y + x(\alpha + 1). \end{aligned}$$

The middle of three model features two phase transitions, at $\alpha = -\frac{1}{2}$ and $\alpha = -\frac{7}{8}$. To discuss phase transitions in this model, we note that the derivative (with respect to y) $F_1'(y; x, \Xi)$ does not depend on x and is decreasing in α for fixed y ; we also note that the symmetry in the system means that $F_1'(y; x, \Xi) = F_1'(1-y; x, \Xi)$. We can thus define

$$\alpha_1 = \inf\{\alpha : F_1'(y; x, \Xi) \leq 0 \forall y \in (0, 1)\};$$

for $\alpha \geq \alpha_1$, $F_1(y; x, \Xi)$ is a decreasing function of y and so for all $x \in (0, 1)$ there is a unique root of $F_1(y; x, \Xi) = 0$ in $(0, 1)$, whereas for $\alpha < \alpha_1$ there is at least one interval of values of x which have at least three roots of $F_1(y; x, \Xi) = 0$ in $(0, 1)$. Hence our results show that condensation occurs almost surely if and only if $\alpha < \alpha_1$. We can calculate α_1 explicitly, since

$$F_1''(y; x, \Xi) = -\frac{7}{2}(2y-1)\left(6y^2 - 6y + 4 - \sqrt{10}\right)\left(6y^2 - 6y + 4 + \sqrt{10}\right)$$

does not depend on α . It is easy to verify that $F_1'(y; x, \Xi)$ is maximised at $y = \frac{1}{2} \pm \frac{1}{6}\sqrt{6\sqrt{10} - 15}$, and the maximum value is positive if and only if $\alpha < \frac{35\sqrt{10}-116}{9}$.

For $\alpha \in (-1, \alpha_1)$ $F_1(y; x, \Xi)$ has, in $(0, 1)$, two local minima at $\eta_1(\alpha)$ and $\eta_3(\alpha)$ and two local maxima at $\eta_2(\alpha)$ and $\eta_4(\alpha)$, where $\eta_1(\alpha) < \eta_2(\alpha) < \eta_3(\alpha) < \eta_4(\alpha)$; these values depend on α but not on x . Set $\alpha_2 = \sup\{\alpha : F_1(\eta_2(\alpha); x, \Xi) \geq F_1(0; x, \Xi)\}$; then the set of values of x which have at least three roots includes 0 and 1 if and only if $\alpha \leq \alpha_2$. Next, set $\alpha_3 = \sup\{\alpha : F_1(\eta_4(\alpha); x, \Xi) \geq F_1(\eta_1(\alpha); x, \Xi)\}$; then for $\alpha < \alpha_3$ there is a range of values of x such that there are five roots of $F_1(y; x, \Xi) = 0$ in $(0, 1)$, whereas for $\alpha > \alpha_3$ there are always at most three, and there are two disjoint intervals of x where there are three. Hence for $\alpha \in (\alpha_3, \alpha_1)$ there will almost surely be two points of condensation, whereas for $\alpha < \alpha_3$ there will be positive probability of there being a single point of condensation.

Finally, set $\alpha_4 = \sup\{\alpha : F_1(\eta_4(\alpha); x, \Xi) \geq F_1(0; x, \Xi)\}$; then there are five roots of $F_1(y; x, \Xi) = 0$ in $(0, 1)$ for all $x \in (0, 1)$ if and only if $\alpha \leq \alpha_4$, and hence for this range

of α a single point of condensation can occur at a location which is fully supported on $(0, 1)$.

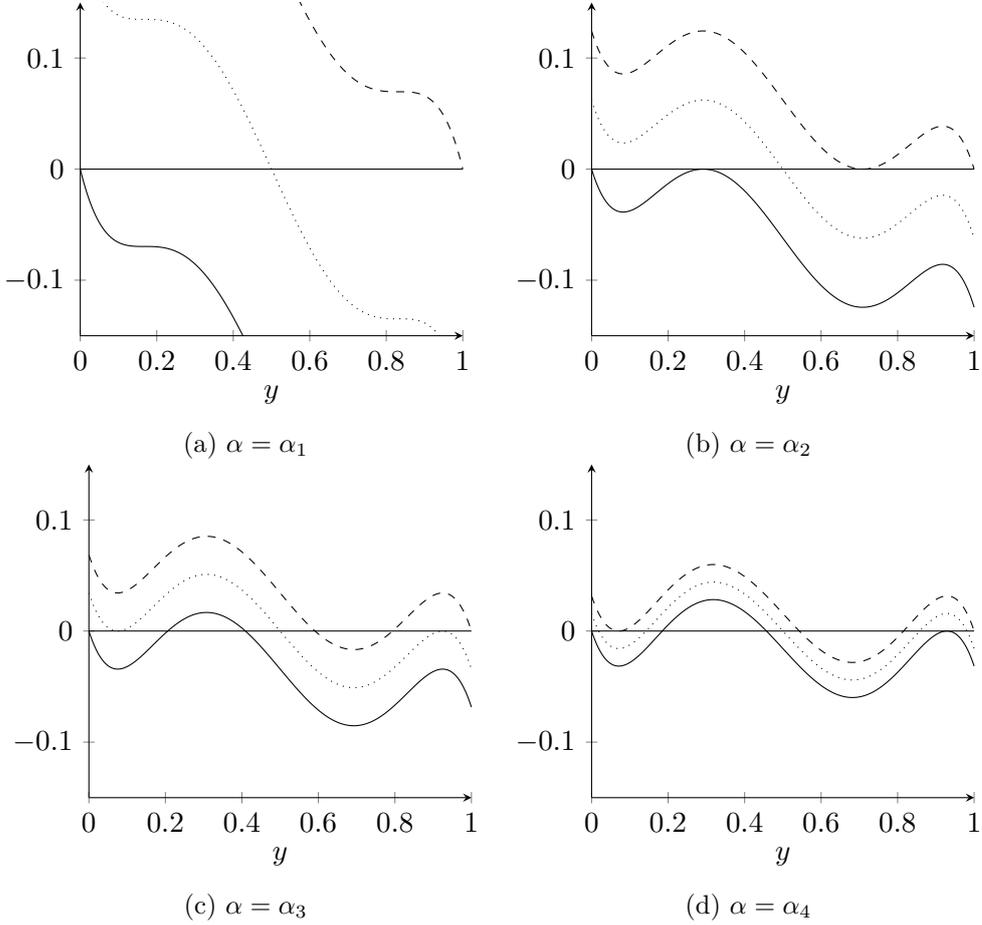


Figure 5: $F_1(y; x, \Xi)$ evaluated at four different values of α corresponding to the phase transitions that appear for this choice of Ξ , and at three different values of x : from top to bottom, $x = 1$, $x = \frac{1}{2}$ and $x = 0$.

The four transition points satisfy $\alpha_1 = \frac{35\sqrt{10}-116}{9} \approx -0.59114$, $\alpha_2 \approx -0.87562$, $\alpha_3 \approx -0.93144$ and $\alpha_4 \approx -0.96842$. Plots of $F_1(y; x, \Xi)$ for each of the transition points α_1 , α_2 , α_3 and α_4 are shown in Figure 5, and plots showing the roots of $F_1(y; x, \Xi) = 0$ for two specific values of α ($\alpha = -0.85 \in (\alpha_2, \alpha_1)$ and $\alpha = -0.95 \in (\alpha_4, \alpha_3)$) appear in Figure 6.

For $\alpha = -0.85$ the stable roots and touchpoints are given by three continuous partial functions of x : ψ_1 defined on $(0, \beta_2]$, ψ_3 defined on $[\beta_1, 1 - \beta_1]$, and ψ_5 defined on $[1 - \beta_2, 1)$, where $0 < \beta_1 < \beta_2 < 1/2$. (In fact we have $\beta_1 \approx 0.0492$ and $\beta_2 \approx 0.2721$.) Each function gives a stable root on the interior of its domain and a touchpoint on the boundary. Consequently, by Theorem 2.2, almost surely $\Psi(x)$ takes the value $\psi_1(x)$ on

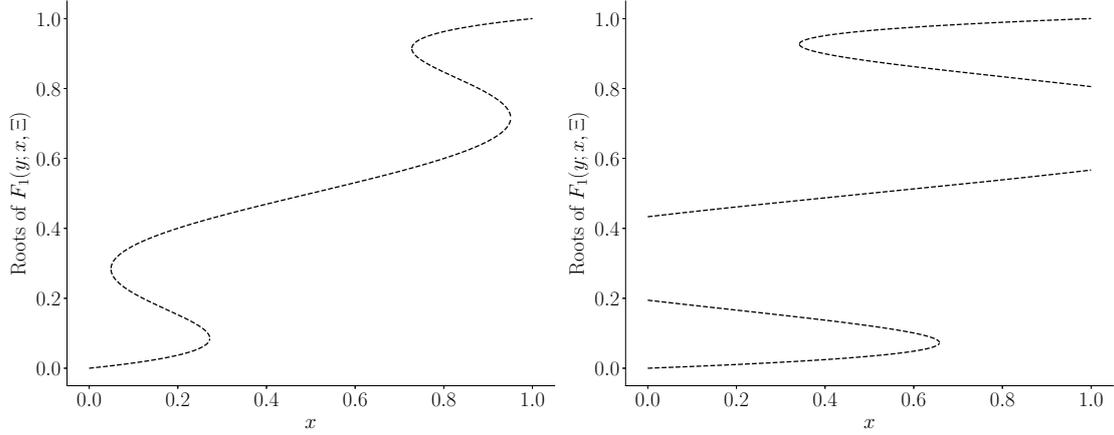


Figure 6: The roots of $F_1(y; x, \Xi)$ for $\alpha = -0.85$ (left) and $\alpha = -0.95$ (right).

some interval containing $(0, \beta_1)$, the value $\psi_3(x)$ on some interval containing $(\beta_2, 1 - \beta_2)$, and the value $\psi_5(x)$ on some interval containing $(1 - \beta_1, 1)$. Thus there are almost surely two points of condensation. By Theorem 2.3, each of these points of condensation is caused by a persistent hub with positive probability. However, Theorem 2.4 implies there is also a positive probability of condensation occurring without a persistent hub at β_1 , β_2 , $1 - \beta_2$ or $1 - \beta_1$. Figure 7 shows the results of two simulations: in the first simulation, both points of condensation arise from persistent local hubs, but in the second the upper part of Ψ_n shows much slower convergence, apparently towards condensation at $1 - \beta_2$. Our results do not give any bounds on the relative probabilities of these types of behaviour; however, simulations suggest that it is relatively uncommon to have early hubs forming in both the feasible regions (β_1, β_2) and $(1 - \beta_2, 1 - \beta_1)$.

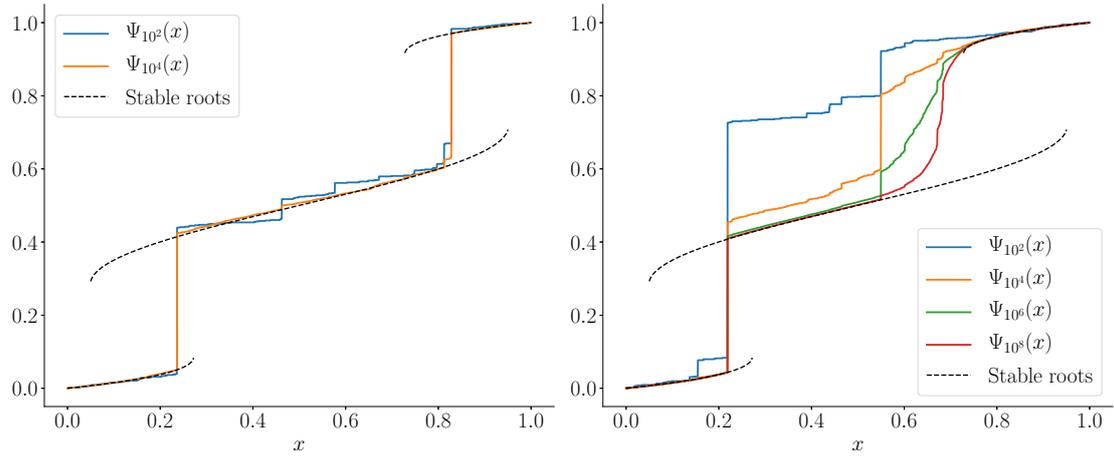


Figure 7: Results from simulations for $\alpha = -0.85$

For $\alpha = -0.95$ the corresponding partial functions of x are ψ_1 defined on $(0, 1 - \beta]$, ψ_3 defined on $(0, 1)$, and ψ_5 defined on $[1 - \beta, 1)$, where $\beta \approx 0.3420$. In this case Theorem 2.2 implies that there are nontrivial regions on which $\Psi(x)$ takes the values $\psi_1(x)$ and $\psi_5(x)$, but there need not be any x for which $\Psi(x) = \psi_3(x)$, since that is never the only stable root. Consequently there may be two points of condensation if such an x exists, and one point of condensation corresponding to a jump from ψ_1 to ψ_5 otherwise. Theorem 2.3 implies that each of these types of behaviour has positive probability, and Figure 8 shows the results of two simulations exhibiting the two types of behaviour. As before, Theorem 2.4 implies that there is a positive probability of non-persistent condensation occurring at β or $1 - \beta$.

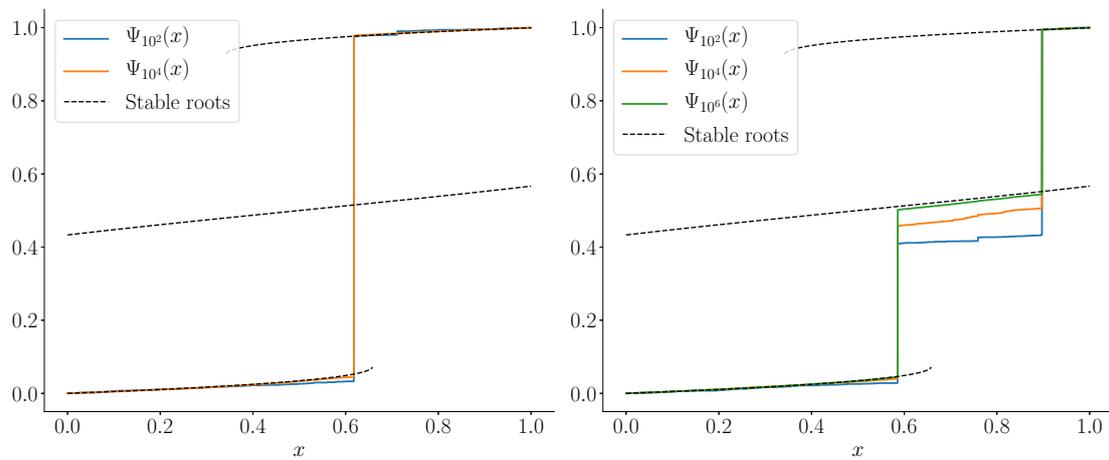


Figure 8: Results from simulations for $\alpha = -0.95$

5 Acknowledgments

The third author acknowledges support from the University of Sheffield for funding their research in the form of a graduate teaching assistant studentship, and would also like to thank Dr Nic Freeman for all of his feedback during the writing of this paper.

The first author acknowledges support from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 639046).

References

- [1] T. Antunović, E. Mossel, and M. Rácz. Coexistence in preferential attachment networks. *Combinatorics, Probability and Computing*, 25:797–822, 2016.
- [2] A.-L. Barabási and R. Albert. Emergence of scaling in random networks. *Science*, 286(5439):509–512, 1999.

- [3] G. Bianconi and A.-L. Barabási. Competition and multiscaling in evolving networks. *EPL (Europhysics Letters)*, 54(4):436, 2001.
- [4] C. Borgs, J. Chayes, C. Daskalakis, and S. Roch. First to market is not everything: an analysis of preferential attachment with fitness. In *Proceedings of the thirty-ninth annual ACM symposium on Theory of Computing*, pages 135–144. ACM, 2007.
- [5] S. Dereich, C. Mailler, and P. Mörters. Nonextensive condensation in reinforced branching processes. *The Annals of Applied Probability*, 27(4):2539–2568, 2017.
- [6] S. Dereich and M. Ortgiese. Robust analysis of preferential attachment models with fitness. *Combinatorics, Probability and Computing*, 23(3):386–411, 2014.
- [7] S. N. Dorogovtsev, J. F. F. Mendes, and A. N. Samukhin. Structure of growing networks with preferential linking. *Phys. Rev. Lett.*, 85:4633–4636, 2000.
- [8] N. Freeman and J. Jordan. Extensive condensation in a model of preferential attachment with fitnesses. *arXiv preprint arXiv:1812.06946*, 2018.
- [9] J. Haslegrave and J. Jordan. Preferential attachment with choice. *Random Structures & Algorithms*, 48(4):751–766, 2016.
- [10] P. Krapivsky and S. Redner. Choice-driven phase transition in complex networks. *Journal of Statistical Mechanics: Theory and Experiment*, 2014(4):P04021, 2014.
- [11] Y. Malyshekin and E. Paquette. The power of choice combined with preferential attachment. *Electronic Communications in Probability*, 19, 2014.
- [12] Y. Malyshekin and E. Paquette. The power of choice over preferential attachment. *ALEA*, 12(2):903–915, 2015.
- [13] R. Pemantle. When are touchpoints limits for generalized Pólya urns? *Proc. Amer. Math. Soc.*, 113:235–243, 1991.
- [14] R. Pemantle. A survey of random processes with reinforcement. *Probab. Surv.*, 4:1–79, 2007.
- [15] H. Robbins and S. Monro. A stochastic approximation method. *The Annals of Mathematical Statistics*, pages 400–407, 1951.