Transportation Research Part B 132 (2020) 136–151

Contents lists available at ScienceDirect

Transportation Research Part B

journal homepage: www.elsevier.com/locate/trb

Stabilisation strategy for unstable transport systems under general evolutionary dynamics☆

Takamasa Iryo[a,*], Michael J. Smith[b], David Watling[c]

[a] Graduate School of Engineering, Kobe University, 1-1, Rokkodai, Nada, Kobe, 657-8501, Japan
[b] Department of Mathematics, University of York, York YO10 5DD, United Kingdom
[c] Institute for Transport Studies, University of Leeds, Leeds LS2 9JT, United Kingdom

ARTICLE INFO

Article history:
Received 1 December 2018
Revised 23 May 2019
Accepted 27 May 2019
Available online 13 June 2019

Keywords:
Evolutionary game theory
Day-to-day dynamics
Dynamic traffic assignment
Stability

ABSTRACT

Stability of equilibria in transport systems has been discussed for decades. Even in deterministic cases, where stochasticity is ignored, stability is not a general property; a counterexample has been found in (within-day) dynamic traffic assignment problems. Instability can be a source of uncertainty of travel time and although pricing may stabilise an unstable transport system, pricing is not always acceptable to the public. This study aims to develop a pricing strategy that stabilises a transport system *with minimum tolls*. We show that with our stabilising pricing system tolls are bounded above and converge to zero when the error in estimation of a no-toll equilibrium converges to zero. We then show that the proposed toll scheme stabilises a wide range of evolutionary dynamics. We also propose a heuristic procedure to minimise the toll level. The procedure can also be viewed as a method of finding a possibly unstable equilibrium solution of a transport system. This suggests that, while we have not provided a rigorous proof, we may be able to find an equilibrium solution of any transport problem including problems which arise in dynamic traffic assignment (DTA); in these DTA cases, how to construct a solution algorithm that always converges to an equilibrium solution is still an open question. The methods proposed here will be extended so that they apply in more realistic behavioural settings in future work.

1. Introduction

Stability of equilibria in transport systems has been discussed for decades. This discussion actually dates back to Beckmann et al. (1956), who stated that *'an equilibrium would be just an extreme state of rare occurrence if it were not stable - that is, if there were no forces which tended to restore equilibrium as soon as small deviations from it occurred'*. Later, Smith (1984a) investigated the stability of a conventional (but non-separable) static traffic assignment problem by using a Lyapunov function approach and showed that, in this case, an equilibrium is asymptotically stable when drivers' day-to-day

☆ This paper has been accepted for a podium presentation at the 23rd International Symposium on Transportation and Traffic Theory (ISTTT23) July 24–26, 2019 in Lausanne, CH.

* Corresponding author.
E-mail address: iryo@kobe-u.ac.jp (T. Iryo).

https://doi.org/10.1016/j.trb.2019.05.021

adjustment process is modelled by a certain formulation, called Smith dynamic (named in Hofbauer and Sandholm, 2009). This dynamic describes drivers' day-to-day adjustment process by using an ordinary differential equation (ODE), in which the date is described by a continuous number. We call such a dynamic an *evolutionary dynamic*. A number of different evolutionary dynamics have been used in transport studies to investigate stability. See, for example, Dupuis and Nagurney (1993), Friesz et al. (1994), Zhang and Nagurney (1996), Mounce (2006), Yang and Zhang (2009), He et al. (2010), Smith and Mounce (2011), Guo et al. (2013) and Iryo (2016).

Stability is, however, not a property of a general transport problem; in part because travel cost functions are not always monotone. Moreover a counterexample has been found in dynamic traffic assignment (DTA) problems: Iryo (2008) showed numerically that there exists a case in which the Smith dynamic fails to cause convergence to an equilibrium in Vickrey's departure-time-choice problem (Vickrey, 1969). Iryo (2019) also showed mathematically that the system is not guaranteed to converge to an equilibrium point if it follows the replicator dynamic (Schuster and Sigmund, 1983). Instablity of the departure-time-choice problem has also been investigated by Guo et al. (2018).

Instability of an evolutionary dynamic implies that we may face a situation in which the state of a transport system changes over days even if there is no variation in the demand pattern and the supply performance. Thus, as Iryo (2008) mentions, instability can reduce the travel time reliability of a transport system even if there is no variation in the demand pattern and the supply performance. While travel time unreliability is recognised as a common problem in many cities suffering from congestion, there appears to be no empirical evidence quantifying the impact instability has on travel time unreliability. Nevertheless, a methodology which reduces or even eliminates that component of uncertainty caused by instability would be welcome. The proportion of uncertainty caused by instability may currently be small. But, in the future, when other factors causing uncertainty are reduced or even eliminated; perhaps by an advanced travel-time estimation system; the proportion of uncertainty caused by instability might be much greater. Moreover, new technologies in transport services associated with the latest information and communication technologies, such as mobility-as-a-service, could incorporate complex interactions between users, which may cause complex travel cost functions; these functions are unlikely to be monotone and may increase instability and hence travel time uncertainty. It would thus be natural to design effective instability countermeasures before these new systems are widely implemented.

The issue of instability is also relevant to the issue of multiple equilibria. Bie and Lo (2010) mentioned the existence of an unstable equilibrium solution that separates two domains of attraction associated with two equilibria. The non-uniqueness of traffic equilibria is correlated to the asymmetric interactions between links (Watling, 1996), which has also been discussed by Heydecker (1983) and Smith (1984b). The existence of multiple equilibria is also known in the DTA context (Iryo, 2011; Iryo, 2015). Transport systems affected by social or economic factors, such as economics of scale or social interactions, may also be affected by the non-uniqueness issue (see for example the Mohring effect (Mohring, 1972) and Ying and Yang (2005) for economics of scale in public transport systems and Fukuda and Morichi (2007) for social interactions). If there are multiple equilibria then it will often be the case that transport authorities and operators prefer one of these equilibria (perhaps the equilibrium with the lowest total travel cost or the least environmental impact). However, realising the preferred equilibrium without any controlling scheme may be very difficult or impossible, especially if the desirable equilibrium is unstable.

In this study, we suppose given a transport system with a preferred equilibrium which is not globally asymptotically stable. (This preferred equilibrium may be the only equilibrium or it may be one of many equilibria.) We then employ a toll scheme as a control to eliminate instability in the transport system; so that the preferred equilibrium becomes globally asymptotically stable.

In a typical context of transport studies, tolls are considered as a measure to alleviate congestion or obtain revenue. Alleviating congestion would also alleviate travel time unreliability; a trial-and-error toll implementation such as those of Yang et al. (2004, 2010) would be suitable for maintaining stability by alleviating congestion.

It is well-known that introducing a toll scheme is not always acceptable to the public, as was found in the 2005 Edinburgh case (Gaunt et al., 2007). The toll scheme proposed in this paper is not aimed at reducing congestion directly or raising revenue; it is aimed at improving travel time reliability or for realising a preferable, but unstable, equilibrium. To achieve this goal without public opposition we seek a toll scheme that stabilises a preferred equilibrium without high charges. Such a scheme is likely to be acceptable. Charging a small toll is also important for reasons of equity — pricing methods are inherently inequitable, but this problem is minimised by making the toll as small as possible.

This paper proposes a pricing strategy that stabilises the evolutionary dynamics of a transport system whose preferred equilibrium would, without the pricing strategy, be unstable. We first introduce a toll scheme which causes evolutionary trajectories to converge to a possibly unstable, target, equilibrium. The toll imposed on travellers at this preferred equilibrium is zero (no traveller has to pay a toll). We analyse the monotonicity of the travel cost function including the toll and utilise the results to show that the proposed toll scheme stabilises the transport system for a wide range of evolutionary dynamics following the work of Hofbauer and Sandholm (2009) and Sandholm (2010b). We then prove that the toll level is always bounded at all times regardless of the settings of the toll and the target traffic pattern.

We consider also the case where the target flow pattern chosen for the toll-scheme design is not known to be an equilibrium; we show that in this case the total toll paid converges to zero when the target flow pattern converges to a no-toll equilibrium. Finally, we propose a heuristic procedure to find a target traffic pattern that is a no-toll equilibrium, and so minimises the total toll paid. Although certain tolls would be imposed on travellers during this iteration process, these will become much smaller than the non-toll part of the travel cost as the iteration proceeds.

The proposed heuristic procedure can also be recognised as a solution method to find an equilibrium solution regardless of its stability. This suggests that we may be able to find an equilibrium solution (with no toll) of any transport problem, including those of DTA; in these cases, how to construct a solution algorithm that always converges to an equilibrium solution in general problems is still an open question (Iryo, 2013).

The present paper consists of six sections including this introduction section. The model specification will be made in Section 2. In Section 3, the asymptotic stability of the proposed toll scheme is shown for a wide range of evolutionary dynamics. In Section 4, the upper bounds of the toll imposed on travellers are analytically calculated. A heuristic procedure to minimise the toll level is then proposed in Section 5. A numerical test is given in Section 6, followed by Section 7, which concludes the results of the paper and indicates future directions.

2. Model specification

2.1. Travellers and transport facilities

A transport system consisting of travellers and transport facilities is considered. Travellers recurrently make trips in the transport system everyday. Travellers are subdivided into groups; those in the same group have identical properties (e.g. origin-destination pair, value of time, and schedule preference). The set of all groups is denoted by G.

The term *transport facility* in this paper refers to anything providing a transport service. Travellers choose a transport facility or a combination of transport facilities to accomplish their trip. A transport facility typically corresponds to a physical means of transport but is not limited to this. For example, in a typical network traffic assignment problem, we can consider links to be the transport facilities and travellers choose a connected sequence of links (i.e. a route) to travel from an origin to a destination. On the other hand, in the same problem, we can also consider routes to be the transport facilities and travellers choose one from them. In the latter case, although routes may not be considered as a physical transport service, we regard them as 'logical' transport services that take travellers to their destinations. In the DTA context, we may also consider each time slot entering a link or a route as an individual transport facility (i.e. it is distinguished not only spatially but also temporally). The set of transport facilities is denoted by L. Any transport facility in L has the following properties:

1. The number of travellers using transport facility $l \in L$ is denoted by y_l, which is a non-negative real number. Its vector form is denoted by $\mathbf{y}$. It is referred to as *facility flow pattern* or *facility traffic volume*, inspired by the typical terms in the transport literature, such as link flow pattern and link traffic volume.
2. The facility flow pattern (i.e. $\mathbf{y}$) is observable by the administrator of the transport system.
3. Any traveller in group $g \in G$ chooses a combination of transport facilities, which is also referred to as an *alternative*. The set of alternatives available for travellers in group g is denoted by C^g.
4. The set of transport facilities associated with alternative $i \in C^g$ is denoted by L_i^g. We also define 0–1 constants δ_{il}^g, whose value is 1 if $l \in L_i^g$ and 0 otherwise.

Travellers' choice behaviour is denoted by a real non-negative number x_i^g, which indicates the number of travellers belonging to group $g \in G$ and choosing alternative $i \in C^g$. Its vector form (including all x_i^g for all groups and alternatives) is denoted by $\mathbf{x}$. The vector of x_i^g including group g only is denoted by $\mathbf{x}^g$. Note that $\mathbf{x}$ does not have to be observable by the administrator of the transport system. $\mathbf{x}$ is often associated with personal information such as income or schedule constraints, which are mostly difficult to observe by physical means such as traffic counters, probe-vehicle systems or online systems.

The relationship between x_i^g and y_l is denoted by

$$y_l = \sum_{g \in G} \sum_{i \in C^g} \delta_{il}^g x_i^g \quad \text{or} \quad \mathbf{y} = A\mathbf{x}, \tag{1}$$

where A is a 0–1 matrix corresponding to δ_{il}^g. The feasible regions of $\mathbf{x}$, $\mathbf{x}^g$, and $\mathbf{y}$ are denoted by X, X^g, and Y, respectively, which represent demand and non-negativity constraints. These sets are convex.

$\mathbf{x}$ and $\mathbf{y}$ change over days. In this study, unless otherwise mentioned, a day is described by a continuous number, denoted by t. The dependency of the state variable on t is denoted by e.g. $\mathbf{x}(t)$ and $\mathbf{y}(t)$, whereas the argument t is often omitted in the following descriptions for simplicity. The derivative with respect to t is denoted by a dot above the variable, e.g. $\dot{\mathbf{x}}$.

We suppose that travellers try to choose an alternative that minimises travel cost. Travel cost of travellers belonging to group $g \in G$ and choosing alternative i is denoted by $c_i^g(\mathbf{y})$. $c_i^g(\mathbf{y})$ does not include tolls, to be introduced in the next section. The term *travel cost without toll* is used to specify $c_i^g(\mathbf{y})$. We assume that it is Lipschitz continuous, finite, and depends on $\mathbf{y}$, not on $\mathbf{x}$. Its vector form is denoted by $\mathbf{c}(\mathbf{y})$ (for all users) and $\mathbf{c}^g(\mathbf{y})$ (for group g).

2.2. Toll scheme and target facility flow pattern

In this paper, we suppose that an authority of the transport system plans to introduce a toll scheme to stabilise the transport system if it is unstable. We employ a linear form for simplicity; other monotone functions may also be applicable.

The toll is applicable for all travellers using each transport facility $l \in L$ and is defined by

$$p_l(\alpha, y_l, \bar{y}_l) = \alpha(y_l - \bar{y}_l), \tag{2}$$

where α is a positive constant (if the toll is imposed; if not, α is set to zero) and $\bar{y}_l$ is a non-negative constant referred to as *target facility traffic volume*. These two parameters will be set externally by the authority. Note that a negative toll is allowed in this setting. Practically, negative tolls may be implemented by discounting existing tolls, fares, fuel taxes, etc. The vector form of $\bar{y}_l$, denoted by $\bar{\mathbf{y}}$, is referred to as *target facility flow pattern*. $\bar{\mathbf{y}}$ must be included in Y, the set of feasible facility flow patterns. The term 'target flow' implies that the authority intends to drive the facility flow pattern so that it is close to the target facility flow pattern, by using the proposed toll scheme above. The vector form of $p_l(\alpha, y_l, \bar{y}_l)$ is denoted by $\mathbf{p}(\alpha, \mathbf{y}, \bar{\mathbf{y}})$, which satisfies

$$\mathbf{p}(\alpha, \mathbf{y}, \bar{\mathbf{y}}) = \alpha(\mathbf{y} - \bar{\mathbf{y}}). \tag{3}$$

We also use the term 'toll$(\alpha, \bar{\mathbf{y}})$' to specify a toll scheme with specific parameters.

The toll is directly added to $\mathbf{c}(\mathbf{y})$, and, hence, the units of $\mathbf{p}$ and $\mathbf{c}(\mathbf{y})$ must be the same (typically, in a monetary unit such as dollars or a unit of time). The sum of them is denoted by $\pi_i^g(\alpha, \mathbf{y}, \bar{\mathbf{y}})$, or π_i^g for short, where

$$\pi_i^g(\alpha, \mathbf{y}, \bar{\mathbf{y}}) = c_i^g(\mathbf{y}) + \sum_{l \in L_i^g} \alpha(y_l - \bar{y}_l). \tag{4}$$

π_i^g is referred to as *travel cost with toll* or *travel cost*. If we intend to specify $c_i^g(\mathbf{y})$, the phrase *without toll* must always accompany it. We also use the vector form of π_i^g as follows:

$$\boldsymbol{\pi}(\alpha, \mathbf{y}, \bar{\mathbf{y}}) = \mathbf{c}(\mathbf{y}) + \alpha A^T(\mathbf{y} - \bar{\mathbf{y}}), \tag{5}$$

where A^T is the transpose of matrix A. The vector $\boldsymbol{\pi}^g(\alpha, \mathbf{y}, \bar{\mathbf{y}})$ is also used to combine $\pi_i^g(\alpha, \mathbf{y}, \bar{\mathbf{y}})$, which is associated with a certain group of travellers. In the following, the forms without arguments, i.e. $\boldsymbol{\pi}$ and $\boldsymbol{\pi}^g$, are also used for simplicity.

We now define a concept related to the strict monotonicity of travel cost as follows:

Definition. The travel cost function $\boldsymbol{\pi}(\alpha, \mathbf{y}, \bar{\mathbf{y}})$ is said to be *strictly monotone over Z* if and only if the following holds, where $Z \subseteq X \times X$:

$$\{\boldsymbol{\pi}(\alpha, A\mathbf{x}_2, \bar{\mathbf{y}}) - \boldsymbol{\pi}(\alpha, A\mathbf{x}_1, \bar{\mathbf{y}})\}^T(\mathbf{x}_2 - \mathbf{x}_1) > 0 \quad \forall(\mathbf{x}_1, \mathbf{x}_2) \in Z \ \text{ such that } \ \mathbf{x}_1 \neq \mathbf{x}_2. \tag{6}$$

Note that this definition is identical to the typical definition of the strict monotonicity when $Z = X \times X$.

2.3. Equilibrium

$\mathbf{x}^*$ will be called a Nash equilibrium if and only if the following variational inequality holds:

$$\boldsymbol{\pi}(\alpha, A\mathbf{x}^*, \bar{\mathbf{y}})^T(\mathbf{x} - \mathbf{x}^*) \geq 0 \quad \forall \mathbf{x} \in X. \tag{7}$$

Any $\mathbf{x}^*$ satisfying Eq. (7) is denoted by $\mathbf{x}_\alpha^*(\bar{\mathbf{y}})$ (or $\mathbf{x}_\alpha^*$ for short). If there is no toll, it is denoted by $\mathbf{x}_0^*$. We also use $\mathbf{y}_\alpha^* = A\mathbf{x}_\alpha^*$ and $\mathbf{y}_0^* = A\mathbf{x}_0^*$ to denote the equilibrated facility flow patterns with and without toll, respectively.

2.4. Evolutionary dynamics

In this study, we examine the stability of the chosen evolutionary dynamics; to judge the stability of an equilibrium solution. To do so, we need to specify the evolutionary dynamics in an explicit form. This is described by the following ODE:

$$\dot{x}_i^g(t) = \sum_{j \in C^g} \left\{ x_j^g(t)\rho_{ji}^g(\mathbf{x}^g(t), \boldsymbol{\pi}^g(t)) - x_i^g(t)\rho_{ij}^g(\mathbf{x}^g(t), \boldsymbol{\pi}^g(t)) \right\}, \tag{8}$$

where $\boldsymbol{\pi}^g(t) = \boldsymbol{\pi}^g(\alpha, A\mathbf{x}(t), \mathbf{y}(t))$. $\rho_{ij}^g(\mathbf{x}^g(t), \boldsymbol{\pi}^g(t))$ is called the *revision protocol* (Sandholm, 2010a), which describes how fast users change their choices from alternative i to j. The above dynamical system (8) is said to be globally asymptotically stable if and only if $\mathbf{x}$ converges to a Nash equilibrium point that is Lyapunov stable as $t \to \infty$, regardless of the initial value of $\mathbf{x}$.

The day is described by a continuous number in this model, although the day might be more naturally represented by a discrete number in the real world. To assess this issue, we present a discrete day-to-day dynamical model that can be correlated with the continuous dynamics defined by Eq. (8). First, we assume that travellers consider updating the current selection with positive probability ϕ every day; otherwise, they continue selecting the same alternative (this reflects the so-called inertia effect). Then, we formulate the day-to-day change of $\mathbf{x}$ as

$$x_i^g(\tau + 1) = (1 - \phi)x_i^g(\tau) + \phi\left\{ x_i^g(\tau) + \sum_{j \in C^g} x_j^g(\tau)\bar{\rho}_{ji}^g(\mathbf{x}^g(\tau), \boldsymbol{\pi}^g(\tau)) - x_i^g(\tau)\sum_{j \in C^g} \bar{\rho}_{ij}^g(\mathbf{x}^g(\tau), \boldsymbol{\pi}^g(\tau)) \right\}, \tag{9}$$

where τ is an integer indicating the day and $\bar{\rho}^g_{ij}(\mathbf{x}^g(\tau), \boldsymbol{\pi}^g(\tau))$ is the probability that a traveller currently choosing i changes his/her choice to j on the next day. The first term on the right-hand side of Eq. (9) indicates the number of travellers who do not consider updating their current selection. On the other hand, the second term (including all terms in the curly bracket) indicates the number of travellers who consider updating their current selection. The first term in the curly bracket is the number of travellers currently selecting alternative i. The second term is the number of travellers currently selecting other alternatives, but are going to select i on the next day. The third term is the number of travellers currently selecting i who are going to select other alternatives on the next day.

We then show how the ODE-based formulation in Eq. (8) corresponds to the discrete day-to-day dynamical model when $\phi \to 0$. Eq. (9) can be rewritten as

$$x^g_i(\tau+1) - x^g_i(\tau) = \phi \sum_{j\in C^g} \left\{ x^g_j(\tau)\bar{\rho}^g_{ji}(\mathbf{x}^g(\tau), \boldsymbol{\pi}^g(\tau)) - x^g_i(\tau)\bar{\rho}^g_{ij}(\mathbf{x}^g(\tau), \boldsymbol{\pi}^g(\tau)) \right\}. \tag{10}$$

We introduce yet another variable describing a day. This is $\bar{\tau}$, where $\bar{\tau} = \phi\tau$. We then have

$$\frac{x^g_i(\bar{\tau}+\phi) - x^g_i(\bar{\tau})}{\phi} = \sum_{j\in C^g} \left\{ x^g_j(\bar{\tau})\bar{\rho}^g_{ji}(\mathbf{x}^g(\bar{\tau}), \boldsymbol{\pi}^g(\bar{\tau})) - x^g_i(\tau)\bar{\rho}^g_{ij}(\mathbf{x}^g(\bar{\tau}), \boldsymbol{\pi}^g(\bar{\tau})) \right\}. \tag{11}$$

Taking the limit as $\phi \to 0$, we finally have

$$\frac{\mathrm{d}}{\mathrm{d}\bar{\tau}} x^g_i(\bar{\tau}) = \sum_{j\in C^g} \left\{ x^g_j(\bar{\tau})\bar{\rho}^g_{ji}(\mathbf{x}^g(\bar{\tau}), \boldsymbol{\pi}^g(\bar{\tau})) - x^g_i(\tau)\bar{\rho}^g_{ij}(\mathbf{x}^g(\bar{\tau}), \boldsymbol{\pi}^g(\bar{\tau})) \right\}, \tag{12}$$

which is mathematically identical to Eq. (8).

The analysis performed above implies that, if the proportion of travellers seeking to change their choice on a given day is sufficiently small, then the ODE-based evolutionary dynamics can be considered as a good approximation of the discrete day-to-day dynamical model defined by Eq. (9). In the present paper, we assume that such a condition holds in the mathematical analysis to maintain simplicity of the analysis and compatibility with the series of existing transport studies starting with Smith (1984a) and with evolutionary dynamics in game theory. We will also perform a numerical analysis to see whether the toll scheme to be proposed in this paper is applicable when the discrete formulation of the day-to-day model is employed instead of the ODE-based evolutionary dynamics.

3. Stabilisation by toll

This section explains how the proposed toll scheme stabilises an unstable transport system. First, a motivational example is shown in Section 3.1. Then, a formal proof is given in Section 3.2.

3.1. A motivational example

In this section, we give a simple network cost function determining the travel costs (without toll) on three routes in terms of the flows on all three routes. There is just one unstable equilibrium, and it is shown how this equilibrium may be stabilised. This example is chosen because it illustrates the stabilisation concepts introduced in this paper; this illustration is the simplest we have been able to discover. (More realistic but also more complicated examples leading to unstable equilibria, which might be stabilised, have been suggested elsewhere, e.g. Watling, 1996). Later in Section 6 we will consider a more realistic example, involving departure time choice dynamics but without knowledge of an unstable equilibrium.

Consider three routes (or modes) connecting the single origin-destination pair. The origin-destination demand is fixed at 3. Therefore, the route-flow vector $\mathbf{x} = (x_1, x_2, x_3)$ satisfies

$$x_1 + x_2 + x_3 = 3, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

We consider the situation in which the three routes have cyclic externality, i.e. the travel cost on route i is most affected by route $i+1$ flow and least affected by route $i-1$ flow, where we define that $i+1=1$ if $i=3$ and $i-1=3$ if $i=1$. Such a situation may occur at a roundabout. The (without toll) cost function is as follows:

$$c_1 = 2x_1 + 4x_2 + x_3, \quad c_2 = x_1 + 2x_2 + 4x_3, \quad c_3 = 4x_1 + x_2 + 2x_3. \tag{13}$$

(1, 1, 1) is an equilibrium since, putting $x_1 = x_2 = x_3 = 1$, all routes have the same cost, i.e. 7. The Jacobian of the cost function is a circulant matrix and consequently eigenvalues are 7, $2+4\omega+\omega^2$, and $2+\omega+4\omega^2$, where $\omega = 0.5(-1+\sqrt{3}i)$. As $\mathrm{Re}(2+4\omega+\omega^2) = \mathrm{Re}(2+\omega+4\omega^2) = -0.5$, the proposed transport system is unstable under a certain evolutionary dynamics such as the Smith dynamic (Smith, 1984a).

We wish to stabilise this unique equilibrium. Following (2), we consider using the following tolls or prices to ensure the stability of the equilibrium (1, 1, 1):

$$p_1 = \alpha(x_1 - 1), \quad p_2 = \alpha(x_2 - 1), \quad p_3 = \alpha(x_3 - 1). \tag{14}$$

Here, p_i is the toll charged to traverse route i. With tolls given by (14), the new with-toll cost function is

$$\begin{aligned} c_1 + p_1 &= 2x_1 + 4x_2 + x_3 + \alpha(x_1 - 1) = (2+\alpha)x_1 + 4x_2 + x_3 - \alpha \\ c_2 + p_2 &= x_1 + 2x_2 + 4x_3 + \alpha(x_2 - 1) = x_1 + (2+\alpha)x_2 + 4x_3 - \alpha \\ c_3 + p_3 &= 4x_1 + x_2 + 2x_3 + \alpha(x_3 - 1) = 4x_1 + x_2 + (2+\alpha)x_3 - \alpha. \end{aligned} \tag{15}$$

If we now choose α so that $\alpha > 0.5$, this modified cost function (15) is strictly monotone and (1, 1, 1) is still an equilibrium; however, this equilibrium is now (with the tolls (14)) globally asymptotically stable. The total of the tolls charged at the equilibrium is zero; therefore, travellers pay nothing at equilibrium for the added stability.

3.2. Formal proof for general cases

The toll defined by Eq. (2) is clearly strictly monotone with respect to $\mathbf{y}$ since α is positive, which implies that a sufficiently large α ensures that $\boldsymbol{\pi}$ is a strictly monotone function with respect to $\mathbf{x}$. To describe this property precisely, we first define set X_σ^2 as

$$X_\sigma^2 = \{(\mathbf{x}_1, \mathbf{x}_2) \mid \mathbf{x}_1 \in X, \mathbf{x}_2 \in X \quad \text{and} \quad |\mathbf{x}_1 - \mathbf{x}_2| \le \sigma |A\mathbf{x}_1 - A\mathbf{x}_2|\}, \tag{16}$$

where σ is a positive number, and introduce the following theorem:

Theorem 1. *For any σ such that X_σ^2 is not empty (note that such σ always exists unless Y is a singleton set), there exists $\alpha > 0$ guaranteeing that the travel cost with toll is strictly monotone over X_σ^2.*

Proof. Consider $\forall(\mathbf{x}_1, \mathbf{x}_2) \in X_\sigma^2$. Eq. (6), which defines the strict monotonicity over X_σ^2, is identical to

$$\alpha(\mathbf{x}_2 - \mathbf{x}_1)^T A^T A(\mathbf{x}_2 - \mathbf{x}_1) = \alpha|\mathbf{y}_2 - \mathbf{y}_1|^2 > -\{\mathbf{c}(\mathbf{y}_2) - \mathbf{c}(\mathbf{y}_1)\}^T(\mathbf{x}_2 - \mathbf{x}_1) \qquad \forall(\mathbf{x}_1, \mathbf{x}_2) \in X_\sigma^2 \tag{17}$$

and

$$\alpha > \frac{\{\mathbf{c}(\mathbf{y}_2) - \mathbf{c}(\mathbf{y}_1)\}^T(\mathbf{x}_1 - \mathbf{x}_2)}{|\mathbf{y}_2 - \mathbf{y}_1|^2} \qquad \forall(\mathbf{x}_1, \mathbf{x}_2) \in X_\sigma^2. \tag{18}$$

We also obtain

$$\frac{\{\mathbf{c}(\mathbf{y}_2) - \mathbf{c}(\mathbf{y}_1)\}^T(\mathbf{x}_1 - \mathbf{x}_2)}{|\mathbf{y}_2 - \mathbf{y}_1|^2} \le \frac{|\mathbf{c}(\mathbf{y}_2) - \mathbf{c}(\mathbf{y}_1)||\mathbf{x}_1 - \mathbf{x}_2|}{|\mathbf{y}_2 - \mathbf{y}_1|^2} \le \beta\sigma \qquad \forall(\mathbf{x}_1, \mathbf{x}_2) \in X_\sigma^2, \tag{19}$$

where β is a Lipschitz constant of function $\mathbf{c}(\mathbf{y})$ defined by

$$\beta = \sup_{\mathbf{y}_1 \ne \mathbf{y}_2} \frac{|\mathbf{c}(\mathbf{y}_2) - \mathbf{c}(\mathbf{y}_1)|}{|\mathbf{y}_2 - \mathbf{y}_1|}. \tag{20}$$

Therefore, any α greater than $\beta\sigma$ ensures Inequality (18) □

If A is defined so that $\mathbf{x}_1 \ne \mathbf{x}_2$ always implies $A\mathbf{x}_1 \ne A\mathbf{x}_2$, there always exists σ such that $X_\sigma^2 = X \times X$. Otherwise, $X_\sigma^2 \subset X \times X$. In the following, we first consider the former case, and then the latter case.

When $X_\sigma^2 = X \times X$, Theorem 1 implies that there exists α guaranteeing that the travel cost function with toll is strictly monotone in the normal sense (i.e. Eq. (6) is satisfied $\forall \mathbf{x}_1 \in X$ and $\forall \mathbf{x}_2 \in X$ unless $\mathbf{x}_1 = \mathbf{x}_2$). Therefore, we can easily claim that the equilibrium solution will be unique and stable under the proposed toll scheme. First, the strict monotonicity of travel cost guarantees the uniqueness of the equilibrium solution with toll (Smith, 1979). Second, the following corollary, which is directly invoked from the strict monotonicity of travel cost, guarantees stability of the system:

Corollary 1.1. *Consider any tolled transport system satisfying the assumptions in Section 2 and suppose that the evolutionary dynamics follows one of the following forms:*

- *Brown-von Neumann-Nash (BNN) dynamic,*
- *Best response dynamic,*
- *Impartial pairwise comparison dynamic (including Smith dynamic),*
- *Projection dynamic.*

Then, when there exists σ such that $X_\sigma^2 = X \times X$, there exists a positive number α such that the dynamics are globally asymptotically stable.

Proof. This is a straightforward consequence of applying Theorem 1 to the results for these forms of dynamics given in Hofbauer and Sandholm (2009) and Sandholm (2010b). □

Remark. The limitation to the four types of dynamics in Corollary 1.1 is almost certainly overly restrictive. Indeed, it is tempting to consider weakening the conditions so that the result applies to any evolutionary dynamics satisfying the following:

1. $\dot{\mathbf{x}} = \mathbf{0}$ if and only if the system is in Nash equilibrium.
2. $\Sigma_{i\in C^g} \dot{x}_i^g \pi_i^g < 0$ for any $g \in G$ at all times.

The first condition is called *Nash stationary* (Sandholm, 2010a). The second condition is called *rational behavioural adjustment process* (Yang and Zhang, 2009) or *positive correlation* (Sandholm, 2010a). However, the combination of these two properties and strict monotonicity (as in Theorem 1) is not in fact sufficient, as is established in the counterexample of Sandholm (2010b) (p. 236).

We then analyse uniqueness when $X_\sigma^2 \subset X \times X$. Assume that we have two equilibria denoted by $\mathbf{x}_\alpha^{*1}$ and $\mathbf{x}_\alpha^{*2}$. Let $\mathbf{y}_\alpha^{*1} = A\mathbf{x}_\alpha^{*1}$, $\mathbf{y}_\alpha^{*2} = A\mathbf{x}_\alpha^{*2}$, $\boldsymbol{\pi}_\alpha^{*1} = \boldsymbol{\pi}(\alpha, \mathbf{y}_\alpha^{*1}, \bar{\mathbf{y}})$, and $\boldsymbol{\pi}_\alpha^{*2} = \boldsymbol{\pi}(\alpha, \mathbf{y}_\alpha^{*2}, \bar{\mathbf{y}})$. Owing to Eq. (7), we have

$$\left(\boldsymbol{\pi}_\alpha^{*1} - \boldsymbol{\pi}_\alpha^{*2}\right)^T \left(\mathbf{x}_\alpha^{*2} - \mathbf{x}_\alpha^{*1}\right) \geq 0. \tag{21}$$

Inequality (21) contradicts the inequality of monotonicity defined by (6) unless $(\mathbf{x}_\alpha^{*1}, \mathbf{x}_\alpha^{*2}) \notin X_\sigma^2$, i.e.

$$|\mathbf{y}_\alpha^{*1} - \mathbf{y}_\alpha^{*2}| < \frac{|\mathbf{x}_\alpha^{*1} - \mathbf{x}_\alpha^{*2}|}{\sigma}. \tag{22}$$

Therefore, if we use σ that is sufficiently large, we can ensure that any two different facility flow patterns in different two equilibria are sufficiently close to each other. This property practically implies uniqueness of the facility flow pattern in equilibrium.

Regarding stability when $X_\sigma^2 \subset X \times X$, we first assume that there exists a positive constant κ such that:

$$|\dot{\mathbf{y}}| \geq \kappa |\dot{\mathbf{x}}|. \tag{23}$$

This assumption implies that, if travellers' choices are changing (i.e. $|\dot{\mathbf{x}}| > 0$), the facility flow pattern is also changing ($|\dot{\mathbf{y}}| > 0$). If it does not hold, i.e. $|\dot{\mathbf{y}}| < \kappa|\dot{\mathbf{x}}|\ \forall \kappa > 0$, we will observe a situation in which the facility flow pattern does not change while travellers' choices are changing. Such a situation is not likely because travellers do not coordinate their revisions of choices in the dynamics listed in Corollary 1.1 so as to keep $\mathbf{y}$ constant whereas $\mathbf{x}$ changes over time. In case we found such a situation, it is a singular situation because adding a small perturbation onto the cost function is sufficient to break the coordinated motions of $\mathbf{x}$. Therefore, assuming the existence of κ such that Inequality (23) holds is practically acceptable. In Sandholm (2010b), except for the projection dynamic, when the global asymptotic stability of the dynamics listed in Corollary 1.1 is proven, the strict monotonicity is used only to guarantee that $\dot{\mathbf{x}}^T J(\mathbf{x})\dot{\mathbf{x}} > 0$, where $J(\mathbf{x})$ is the Jacobian matrix of $\boldsymbol{\pi}(\alpha, A\mathbf{x}, \bar{\mathbf{y}})$. Once we let $\sigma \geq 1/\kappa$, owing to Theorem 1, there exists α that guarantees $\dot{\mathbf{x}}^T J(\mathbf{x})\dot{\mathbf{x}} > 0$ at all times whenever Inequality (23) holds. Therefore, we can conclude that there exists a positive number α such that the dynamics listed in Corollary 1.1 are globally asymptotically stable except for the projection dynamic.

For the projection dynamic, according to Sandholm (2010b),

$$\{\boldsymbol{\pi}(\alpha, A\mathbf{x}_\alpha^*, \bar{\mathbf{y}}) - \boldsymbol{\pi}(\alpha, A\mathbf{x}, \bar{\mathbf{y}})\}^T (\mathbf{x}_\alpha^* - \mathbf{x}) > 0 \quad \forall \mathbf{x} \in X \tag{24}$$

is sufficient to prove the existence of a Lyapunov function. Owing to Theorem 1, either Inequality (24) or

$$|\mathbf{y}_\alpha^* - \mathbf{y}| < \frac{|\mathbf{x}_\alpha^* - \mathbf{x}|}{\sigma} \qquad \forall \mathbf{x} \in X \tag{25}$$

is satisfied. Therefore, if we use σ that is sufficiently large, $\mathbf{y}$ always converges to $\mathbf{y}^*$ or sufficiently close to $\mathbf{y}^*$. This property practically implies asymptotic stability of the facility flow in equilibrium.

To summarise the above analyses, we state the following corollary, which is an extension of Corollary 1.1:

Corollary 1.2. *Consider any tolled transport system satisfying the assumptions in Section 2. Suppose that the evolutionary dynamic follows one of the four forms listed in Corollary 1.1. If the projection dynamic is not being considered suppose also that Eq. (23) holds. Then, there exists a positive number α such that the dynamic is globally asymptotically stable.*

Estimating a value of α so that $\alpha > \beta\sigma$ is satisfied may not be easy in real life; because detailed information on the cost function is required. So it would be natural for the transport authority to utilise a trial-and-error process, in which the value of α is updated until the system reaches a stable equilibrium point.

4. Evaluating the toll level actually imposed on travellers

Given an objective to toll as little as possible, then clearly the toll level actually imposed on travellers would be zero in an ideal situation, in which $\bar{\mathbf{y}} = \mathbf{y}_0^*$ holds. However, this is not possible in a realistic situation for two reasons: (i) the error in estimating $\mathbf{y}_0^*$, and (ii) the error in representing the facility flow pattern as an equilibrated flow pattern (which may occur, for example, when the system is in a transient state just after a change in the toll pattern). Since α, as defined in Eq. (2), may have to be very large, the toll level may also have to be very large due to these errors. Analysing how these errors affect the toll level is necessary to evaluate whether the proposed pricing strategy could successfully stabilise a realistic transport system with a small toll level.

In the subsequent sections, we develop a series of results. Firstly, in Section 4.1, we establish three lemmas, which are used as foundation results for subsequent sections. Then, in Section 4.2, we establish an upper bound on the toll assuming that the transport system is in equilibrium. (It is noted that while non-equilibrium states are not theoretically analysed in the present paper, their possible effects are qualitatively discussed in Section 7).

4.1. Lemmas for analysing the toll level

We first state three important lemmas to be used in the subsequent sections as follows:

Lemma 1. *The following inequality holds for any* $\mathbf{x}_\alpha^*$ *and* $\mathbf{x}_0^*$ *unless* $\mathbf{y}_\alpha^* = \mathbf{y}_0^*$*:*

$$|\mathbf{y}_\alpha^* - \mathbf{y}_0^*| \le -\frac{\Delta \mathbf{c}_\alpha^T(\mathbf{x}_\alpha^* - \mathbf{x}_0^*)}{\alpha|\mathbf{y}_\alpha^* - \mathbf{y}_0^*|} + \frac{(\bar{\mathbf{y}} - \mathbf{y}_0^*)^T(\mathbf{y}_\alpha^* - \mathbf{y}_0^*)}{|\mathbf{y}_\alpha^* - \mathbf{y}_0^*|} \le -\frac{\Delta \mathbf{c}_\alpha^T(\mathbf{x}_\alpha^* - \mathbf{x}_0^*)}{\alpha|\mathbf{y}_\alpha^* - \mathbf{y}_0^*|} + |\bar{\mathbf{y}} - \mathbf{y}_0^*| \quad \text{where} \quad \Delta \mathbf{c}_\alpha = \mathbf{c}(\mathbf{y}_\alpha^*) - \mathbf{c}(\mathbf{y}_0^*). \tag{26}$$

Proof. We use the following inequality, which is derived by substituting $\mathbf{x}_0^*$ into $\mathbf{x}$ in variational inequality (7):

$$\left\{\mathbf{c}(\mathbf{y}_\alpha^*) + \alpha A^T(\mathbf{y}_\alpha^* - \bar{\mathbf{y}})\right\}^T(\mathbf{x}_0^* - \mathbf{x}_\alpha^*) \ge 0. \tag{27}$$

This implies

$$\mathbf{c}(\mathbf{y}_0^*)^T(\mathbf{x}_0^* - \mathbf{x}_\alpha^*) + \left\{\Delta \mathbf{c}_\alpha + \alpha A^T(\mathbf{y}_\alpha^* - \bar{\mathbf{y}})\right\}^T(\mathbf{x}_0^* - \mathbf{x}_\alpha^*) \ge 0. \tag{28}$$

Applying inequality $\mathbf{c}(\mathbf{y}_0^*)^T(\mathbf{x}_0^* - \mathbf{x}_\alpha^*) \le 0$ (this can be obtained by substituting $\mathbf{x} = \mathbf{x}_\alpha^*$ and $\alpha = 0$ in Inequality (7)), we obtain

$$\left\{\Delta \mathbf{c}_\alpha + \alpha A^T(A\mathbf{x}_\alpha^* - \bar{\mathbf{y}})\right\}^T(\mathbf{x}_0^* - \mathbf{x}_\alpha^*) \ge 0. \tag{29}$$

Hence:

$$\left\{\Delta \mathbf{c}_\alpha + \alpha A^T(A\mathbf{x}_\alpha^* - A\mathbf{x}_0^*)\right\}^T(\mathbf{x}_0^* - \mathbf{x}_\alpha^*) \ge \alpha(\bar{\mathbf{y}} - \mathbf{y}_0^*)^T A(\mathbf{x}_0^* - \mathbf{x}_\alpha^*) \tag{30}$$

and therefore

$$\alpha|\mathbf{y}_\alpha^* - \mathbf{y}_0^*|^2 \le -\Delta \mathbf{c}_\alpha^T(\mathbf{x}_\alpha^* - \mathbf{x}_0^*) + \alpha(\bar{\mathbf{y}} - \mathbf{y}_0^*)^T(\mathbf{y}_\alpha^* - \mathbf{y}_0^*), \tag{31}$$

which establishes Eq. (26) as required. □

Lemma 2. *The following inequality holds for any* $\mathbf{x}_\alpha^*$ *and* $\bar{\mathbf{x}}$ *unless* $\mathbf{y}_\alpha^* = \bar{\mathbf{y}}$*:*

$$|\mathbf{p}(\alpha, \mathbf{y}_\alpha^*, \bar{\mathbf{y}})| = \alpha|\mathbf{y}_\alpha^* - \bar{\mathbf{y}}| \le \frac{\mathbf{c}(\mathbf{y}_\alpha^*)^T(\bar{\mathbf{x}} - \mathbf{x}_\alpha^*)}{|\bar{\mathbf{y}} - \mathbf{y}_\alpha^*|}, \tag{32}$$

where $\bar{\mathbf{x}}$ *is an arbitrary vector satisfying* $\bar{\mathbf{x}} \in X$ *and* $\bar{\mathbf{y}} = A\bar{\mathbf{x}}$*.*

Proof. Consider the following inequality:

$$\left\{\mathbf{c}(\mathbf{y}_\alpha^*) + \alpha A^T(\mathbf{y}_\alpha^* - \bar{\mathbf{y}})\right\}^T(\bar{\mathbf{x}} - \mathbf{x}_\alpha^*) \ge 0. \tag{33}$$

This is derived by substituting $\bar{\mathbf{x}}$ into $\mathbf{x}$ in variational inequality (7). Inequality (33) thus implies

$$\mathbf{c}(\mathbf{y}_\alpha^*)^T(\bar{\mathbf{x}} - \mathbf{x}_\alpha^*) \ge \alpha(\bar{\mathbf{y}} - \mathbf{y}_\alpha^*)^T A(\bar{\mathbf{x}} - \mathbf{x}_\alpha^*) = \alpha|\bar{\mathbf{y}} - \mathbf{y}_\alpha^*|^2, \tag{34}$$

which establishes Inequality (32) as required. □

Lemma 3. *For any* $\mathbf{y}_1$, $\mathbf{y}_2 \in Y$*, there exist* $\mathbf{x}_1$, $\mathbf{x}_2 \in X$ *such that* $A\mathbf{x}_1 = \mathbf{y}_1$, $A\mathbf{x}_2 = \mathbf{y}_2$*, and* $|\mathbf{x}_1 - \mathbf{x}_2| \le \eta|\mathbf{y}_1 - \mathbf{y}_2|$*, where*

$$\eta = \lim_{\varepsilon \to +0} \left\{ \max_{\mathbf{y}_1, \mathbf{y}_2 \in Y, |\mathbf{y}_1 - \mathbf{y}_2| \ge \varepsilon} \frac{\xi(\mathbf{y}_1 - \mathbf{y}_2)}{|\mathbf{y}_1 - \mathbf{y}_2|} \right\} < \infty, \tag{35}$$

$$\xi(\Delta \mathbf{y}) = \min|\mathbf{x}_1 - \mathbf{x}_2| \quad \text{subject to} \quad \Delta \mathbf{y} = A(\mathbf{x}_1 - \mathbf{x}_2) \quad \text{and} \quad \mathbf{x}_1, \mathbf{x}_2 \in X. \tag{36}$$

Proof. If $\mathbf{y}_1 = \mathbf{y}_2$, $|\mathbf{x}_1 - \mathbf{x}_2| \le \eta|\mathbf{y}_1 - \mathbf{y}_2|$ apparently holds by letting $\mathbf{x}_1 = \mathbf{x}_2$. Otherwise, for any $\mathbf{y}_1$, $\mathbf{y}_2 \in Y$, there exist $\mathbf{x}_1$, $\mathbf{x}_2 \in X$ such that $\xi(\mathbf{y}_1 - \mathbf{y}_2) = |\mathbf{x}_1 - \mathbf{x}_2|$ and hence $|\mathbf{x}_1 - \mathbf{x}_2| \le \eta|\mathbf{y}_1 - \mathbf{y}_2|$ is established owing to Eq. (35). On the other hand, Eq. (36) implies that $\xi(a\Delta\mathbf{y}) \le a\xi(\Delta\mathbf{y})$ for $0 \le a \le 1$. This can be proven by replacing $\mathbf{x}_1$ by $(0.5 + 0.5a)\mathbf{x}_1 + (0.5 - 0.5a)\mathbf{x}_2$ and $\mathbf{x}_2$ by $(0.5 - 0.5a)\mathbf{x}_1 + (0.5 + 0.5a)\mathbf{x}_2$ in the definition of $\xi(\Delta\mathbf{y})$, which also replaces $\Delta\mathbf{y}$ by $a\Delta\mathbf{y}$ and $|\mathbf{x}_1 - \mathbf{x}_2|$ by $a|\mathbf{x}_1 - \mathbf{x}_2|$. Note that the replaced $\mathbf{x}_1$ and $\mathbf{x}_2$ are in X as it is a convex set. Consequently, the denominator does not decrease faster than the numerator in Eq. (35) when $\varepsilon \to +0$, implying that η is finite at all times. □

4.2. Establishing the upper bound on the toll

We now introduce two theorems to show the upper bound on the toll imposed on travellers. In Theorem 2, it is proven that the proposed toll converges to zero when the error in estimation of the no-toll equilibrium converges to zero. On the other hand, in Theorem 3, it is proven that the toll is always upper bounded by a finite number.

The toll can be upper bounded by a value that is proportional to the distance between $\bar{\mathbf{y}}$ and $\mathbf{y}_0^*$, which represents the error in the estimation of $\mathbf{y}_0^*$. This implies that the toll converges to zero by improving the accuracy in estimating $\mathbf{y}_0^*$. The following theorem formally states this relationship:

Theorem 2. *There exists a positive finite number, denoted by* γ*, satisfying*

$$|\mathbf{p}(\alpha, \mathbf{y}^*_\alpha, \bar{\mathbf{y}})| \le \gamma |\bar{\mathbf{y}} - \mathbf{y}^*_0| \tag{37}$$

if α *is greater than a certain finite positive number.*

Proof. Lemmas 1 and 3 imply

$$\begin{aligned}\alpha|\mathbf{y}^*_\alpha - \mathbf{y}^*_0| &\le \frac{\Delta\mathbf{c}^T_\alpha(\mathbf{x}^*_0 - \mathbf{x}^*_\alpha)}{|\mathbf{y}^*_\alpha - \mathbf{y}^*_0|} + \alpha|\bar{\mathbf{y}} - \mathbf{y}^*_0| \le \frac{|\Delta\mathbf{c}_\alpha||\mathbf{x}^*_0 - \mathbf{x}^*_\alpha|}{|\mathbf{y}^*_\alpha - \mathbf{y}^*_0|} + \alpha|\bar{\mathbf{y}} - \mathbf{y}^*_0| \le \beta|\mathbf{x}^*_\alpha - \mathbf{x}^*_0| + \alpha|\bar{\mathbf{y}} - \mathbf{y}^*_0| \\ &\le \beta\eta|\mathbf{y}^*_\alpha - \mathbf{y}^*_0| + \alpha|\bar{\mathbf{y}} - \mathbf{y}^*_0|.\end{aligned} \tag{38}$$

Setting the value of α so that $\alpha > \beta\eta$ is satisfied, we obtain

$$|\mathbf{y}^*_\alpha - \mathbf{y}^*_0| \le \frac{\alpha}{\alpha - \beta\eta}|\bar{\mathbf{y}} - \mathbf{y}^*_0|, \tag{39}$$

which implies

$$|\mathbf{p}(\alpha, \mathbf{y}^*_\alpha, \bar{\mathbf{y}})| = \alpha|\mathbf{y}^*_\alpha - \bar{\mathbf{y}}| \le \alpha(|\mathbf{y}^*_\alpha - \mathbf{y}^*_0| + |\bar{\mathbf{y}} - \mathbf{y}^*_0|) \le \alpha\left\{\frac{\alpha}{\alpha - \beta\eta} + 1\right\}|\bar{\mathbf{y}} - \mathbf{y}^*_0|. \tag{40}$$

Because α and $\beta\eta$ are finite, the coefficient of $|\bar{\mathbf{y}} - \mathbf{y}^*_0|$ on the right-hand side is also finite. □

The toll can also be upper bounded by a positive constant that is always finite even if $\alpha \to \infty$, as stated by the following theorem:

Theorem 3.

$$|\mathbf{p}(\alpha, \mathbf{y}^*_\alpha, \bar{\mathbf{y}})| \le \eta|\mathbf{c}(\mathbf{y}^*_\alpha)| \quad \forall\alpha > 0. \tag{41}$$

Proof. When $\bar{\mathbf{y}} = \mathbf{y}^*_\alpha$, $\mathbf{p}(\alpha, \mathbf{y}^*_\alpha, \bar{\mathbf{y}}) = \mathbf{0}$ and consequently Inequality (41) is satisfied. Otherwise, Lemmas 2 and 3 imply

$$|\mathbf{p}(\alpha, \mathbf{y}^*_\alpha, \bar{\mathbf{y}})| = \alpha|\mathbf{y}^*_\alpha - \bar{\mathbf{y}}| \le \frac{\mathbf{c}(\mathbf{y}^*_\alpha)^T(\bar{\mathbf{x}} - \mathbf{x}^*_\alpha)}{|\bar{\mathbf{y}} - \mathbf{y}^*_\alpha|} \le |\mathbf{c}(\mathbf{y}^*_\alpha)|\frac{|\bar{\mathbf{x}} - \mathbf{x}^*_\alpha|}{|\bar{\mathbf{y}} - \mathbf{y}^*_\alpha|} \le \eta|\mathbf{c}(\mathbf{y}^*_\alpha)|. \tag{42}$$

□

5. Heuristic procedure to minimise the toll level

A heuristic procedure is proposed to minimise the toll level actually imposed on the travellers by finding a target facility flow pattern that is close to $\mathbf{y}^*_0$. Although this procedure is applicable to an actual transport system in the real world, it can also be recognised as an algorithm to find an equilibrium solution without toll in an arbitrary form of the traffic assignment problem.

The proposed procedure assumes that the authority can observe the facility flow pattern $\mathbf{y}$ at all times. On the other hand, $\mathbf{x}$ (i.e. travellers' choices) and $\mathbf{c}$ (i.e. the travel cost function) do not have to be observable. When the proposed procedure is utilised as a solution algorithm, the term 'authority' shall be replaced by another appropriate term such as 'transport planner' or 'transport scholar'. In such a context, these assumptions imply that the detailed structure of the travel cost function does not have to be known beforehand. This property should be useful especially when one is performing a traffic assignment problem in a complicated situation using a simulator.

The proposed procedure is as follows:

1. (Initialise) Let $k = 0$ and set the initial target facility flow pattern $\bar{\mathbf{y}}^0 \in Y$.
2. (Applying the toll) Set the toll$(\alpha, \bar{\mathbf{y}}^k)$.
3. (Observation) Observe $\mathbf{y}$ until it converges to an equilibrium point. Let $\mathbf{y}^k_{obs}$ be the facility flow pattern that is observed after the convergence.
4. (Convergence check) If the toll level (i.e. $\alpha|\mathbf{y}^k_{obs} - \bar{\mathbf{y}}^k|$) is sufficiently small, finish the calculation.
5. (Update) Let $\bar{\mathbf{y}}^{k+1} = \mathbf{y}^k_{obs}$, increment k, and go back to Step 2.

Note that $\mathbf{y}^k_{obs} = \mathbf{y}^*_\alpha(\bar{\mathbf{y}}^k)$ if $\mathbf{y}$ completely converges to a stationary point. It implies that $\bar{\mathbf{y}}^{k+1} = \mathbf{y}^*_\alpha(\bar{\mathbf{y}}^k)$ at Step 5. In addition, note that in the procedure above $\bar{\mathbf{y}}^0$ is arbitrary, but setting it close to $\mathbf{y}^*_0$ would generally be expected to be beneficial in finishing the iteration process earlier without a large toll. For example, $\bar{\mathbf{y}}^0$ may be determined by taking an average of $\mathbf{y}$ for a certain period before the toll is imposed.

Although proving that $\bar{\mathbf{y}}^k$ converges to $\mathbf{y}^*_0$ is difficult, we mathematically examine how the proposed procedure works by utilising Lemma 1. Recall Inequality (26). Substituting $\bar{\mathbf{y}} = \bar{\mathbf{y}}^k$ and $\bar{\mathbf{y}}^{k+1} = \mathbf{y}^*_\alpha(\bar{\mathbf{y}}^k)$, we have

$$|\bar{\mathbf{y}}^{k+1} - \mathbf{y}^*_0| \le -\frac{\Delta\mathbf{c}^T_\alpha(\bar{\mathbf{x}}^{k+1} - \mathbf{x}^*_0)}{\alpha|\bar{\mathbf{y}}^{k+1} - \mathbf{y}^*_0|} + |\bar{\mathbf{y}}^k - \mathbf{y}^*_0|\cos\theta, \tag{43}$$

where $\bar{\mathbf{x}}^k$ is any arbitrary vector satisfying $\bar{\mathbf{x}}^k \in X$ and $A\bar{\mathbf{x}}^k = \bar{\mathbf{y}}^k$ and θ is the angle between $(\bar{\mathbf{y}}^k - \mathbf{y}_0^*)$ and $(\bar{\mathbf{y}}^{k+1} - \mathbf{y}_0^*)$, which is mathematically defined as

$$\cos\theta = \frac{(\bar{\mathbf{y}}^k - \mathbf{y}_0^*)^T(\bar{\mathbf{y}}^{k+1} - \mathbf{y}_0^*)}{|\bar{\mathbf{y}}^k - \mathbf{y}_0^*||\bar{\mathbf{y}}^{k+1} - \mathbf{y}_0^*|}. \tag{44}$$

Let G be the first term of the right-hand side of Inequality (43) (including the leading minus sign) and consider the case where $G < 0$ (case 1) and $G \geq 0$ (case 2). In case 1,

$$|\bar{\mathbf{y}}^{k+1} - \mathbf{y}_0^*| < |\bar{\mathbf{y}}^k - \mathbf{y}_0^*| \tag{45}$$

is apparently satisfied at all times because $\cos\theta \leq 1$. Inequality (45) implies that $\bar{\mathbf{y}}^k$ always approaches $\mathbf{y}_0^*$ in the iteration process. In case 2, Inequality (45) may or may not hold depending on the value of G and $\cos\theta$. It is likely to hold when both G and $\cos\theta$ are small. At least, using a larger α should imply a smaller G, but it may increase $\cos\theta$ on the other hand.

The above analysis implies that, once we face the situation in which Inequality (45) does not hold during the iteration, using a different value for α may establish Inequality (45) again. As $\mathbf{y}_0^*$ is not known in a practical situation, we need to use another index to judge whether $\bar{\mathbf{y}}^k$ is approaching $\mathbf{y}_0^*$. Checking whether the toll level is decreasing is a good practical option.

6. Numerical tests

6.1. Model set-ups

A set of numerical tests were performed to observe how the proposed procedure finds a toll scheme that stabilises a transport system. For the numerical tests, we constructed a departure time choice problem with heterogeneous travellers by using the following settings:

- A single origin-destination network with a single bottleneck is considered.
- Travellers recurrently commute from the origin to the destination every morning.
- The total number of travellers is set to one, and the capacity of the bottleneck is set to one per unit time.
- Travel cost is shown in equivalent time units, given the value of time. It is the sum of delay at the bottleneck, schedule cost at the destination, and the toll charged by the proposed toll scheme divided by the value of time (VOT).
- Travellers are classified into five groups, denoted by group g, where $1 \leq g \leq 5$.
- A piecewise-linear schedule cost is assumed; the early penalty cost is $\gamma_e^g(t_w^g - t_d)$ for $t_d \leq t_w^g$ and the late penalty cost is $\gamma_l^g(t_d - t_w^g)$ for $t_d \geq t_w^g$, where t_d is the actual departure time from the bottleneck and t_w^g is the desired departure time from the bottleneck.
- Those in the same group have the same γ_e^g, γ_l^g, and t_w^g. The population share in each group is the same (i.e. 1/5=0.2).
- The VOT within each group is identical but varies between the different groups. It is assumed that one unit of time is valued as g monetary unit for travellers in group g. This implies that group 1 has the lowest VOT, whereas group 5 has the highest VOT.
- Each traveller chooses a departure time from the origin. Travel time from the origin to the bottleneck is set to zero. The study time duration is from -1.0 to 1.0. This duration is divided into 100 time slots. Each slot is referred to as a *departure time slot*.

In this setting, a transport facility corresponds to a departure time slot. t_w^g, γ_e^g, and γ_l^g are set from the following values for each g:

- $t_w^g \in \{-0.2, -0.1, 0, 0.1, 0.2\}$,
- $\gamma_e^g \in \{0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$,
- $\gamma_l^g \in \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$.

Note that different groups have different values of t_w^g. On the other hand, γ_e^g and γ_l^g are randomly chosen from the above sets, and the same value can be chosen for different groups.

For the evolutionary dynamics, the Smith dynamic and a mixture of the Smith and replicator dynamics (called *mixed dynamic* in this study) were considered. The explicit descriptions of these dynamics in the numerical test are:

$$\rho_{ij}^g = \left[\pi_i^g - \pi_j^g\right]_+ \tag{46}$$

for the Smith dynamic and

$$\rho_{ij}^g = \frac{1}{2}\left[\pi_i^g - \pi_j^g\right]_+ + 3x_j^g\left[\pi_i^g - \pi_j^g\right]_+ \tag{47}$$

for the mixed dynamic, where i, j indicate departure time slots and $[x]_+ = x$ for $x \geq 0$ and $[x]_+ = 0$ otherwise. The mixed dynamic was introduced to observe how the proposed method works with another formulation of evolutionary dynamics not listed in Corollary 1.1. Although the global asymptotic stability of this dynamic has not been proven, it is locally asymptotically stable around an equilibrium point if the travel cost function is strictly monotone (Iryo, 2016). The numerical calculations of these ODEs were performed by using the Runge-Kutta method, where the step size of t is 0.001.

The toll scheme was implemented as follows. We tested each of $\alpha = 10, 100, 1000$ for both dynamics, to observe how the value of α affects the behaviour of the dynamics. From $t = 0$ to $t = 20$, no toll was imposed on travellers. At $t = 20$, $\bar{\mathbf{y}}^k$ was set as an average of $\mathbf{y}$ between $t = 0$ and $t = 20$. Then, $\bar{\mathbf{y}}^k$ was updated at a constant interval of t, which was $\Delta t = 20$ (i.e. updates were made at $t = 20, 40, 60, \ldots$). At each update, the value of $\mathbf{y}$ was observed just before the update was used for the next $\bar{\mathbf{y}}^k$.

We checked the stability of the transport system by observing $|\dot{\mathbf{x}}|$. This will be referred to as the *change speed*. A larger value of the change speed implies a faster change of $\mathbf{x}$. On the other hand, when the change speed is close to zero, the system is close to a stationary point of the dynamics.

In order to compare the toll level across different tests, we evaluated the ratio of the total toll level paid by all travellers to the total travel cost of all travellers. This value is called the *toll-cost ratio*. It is defined by $\sum_{g=1}^{5}\sum_{i=1}^{100}|p_i/g| / \sum_{g=1}^{5}\sum_{i=1}^{100}\pi_i^g$, where p_i is the toll of departure time slot i (in monetary units). We use the absolute value of p_i instead of p_i itself to avoid a situation in which a positive toll cancels out a negative one.

We set the initial condition of the dynamics as $x_i^g = 1/100/5 = 0.002$ for all g and i at $t = 0$. The study duration was $0 \le t \le 300$. Note that the values of t (e.g. $t = 300$) do not imply days in the real world, owing to the continuous approximation involved in the evolutionary dynamics. To clarify this point, we do not call t day or date, but call $t/20$ (not t itself) as the *revision time scale*. Note that the revision time scale takes an integer number when the toll is updated.

In the numerical tests, the transport system is considered to be stabilised if the change speed always converges to zero even when a perturbation is applied to the system. While we did not add any artificial perturbation in the numerical calculation, we can recognise each update of the toll scheme (i.e. the change of $\bar{\mathbf{y}}$) as a perturbation.

6.2. Results of an example parameter set

An example parameter set consisting of the parameters $t_w^g = -0.2, -0.1, 0.0, 0.1, 0.2$, $\gamma_e^g = 0.7, 0.2, 0.3, 0.4, 0.7$, and $\gamma_l^g = 18, 19, 9, 14, 17$ (numbers are listed in the order of g) was used to perform a numerical test. All values of α (i.e. 10, 100, 1000) were used and their results were compared under both the Smith and the mixed dynamic.

Fig. 1 depicts the changes in the toll-cost ratio, whereas Fig. 2 depicts the changes in the change speed. They clearly show that, when $\alpha = 100$, the proposed toll scheme successfully stabilised the transport system (which is unstable without the toll) after the toll was applied. The toll-cost ratio tended to decrease as the target traffic flow pattern ($\bar{\mathbf{y}}^k$) was updated, implying that the iterative procedure worked well in this case. The toll-cost ratio was less than 5% when $\alpha = 100$ and the revision time scale was 15. For $\alpha = 1000$ the toll-cost ratio was relatively higher than that for $\alpha = 100$ and the iterative process did not reduce the toll-cost ratio. On the other hand, when $\alpha = 10$, the change speed did not converge to zero, implying that the transport system was not stabilised in this setting. Qualitatively similar results were found with both kinds of dynamics, i.e. the Smith dynamic and the mixed dynamic.

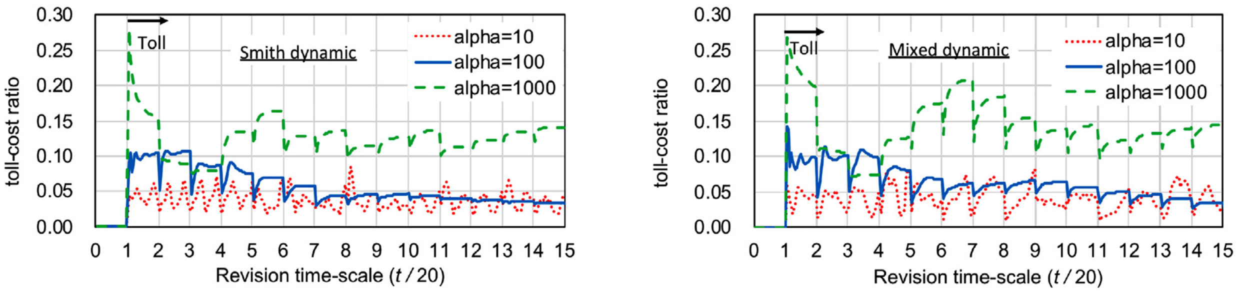

Fig. 1. Changes in the toll-cost ratio; left: Smith dynamic, right: mixed dynamic.

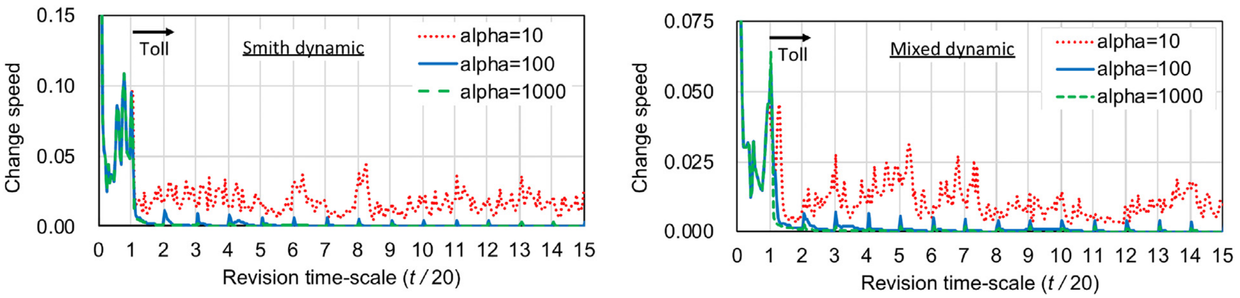

Fig. 2. Changes in the change speed; left: Smith dynamic, right: mixed dynamic.

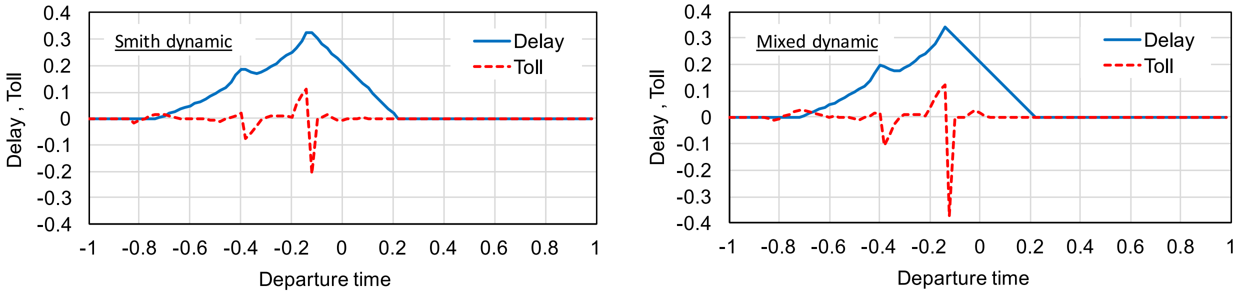

Fig. 3. Delay at the bottleneck and toll of group 1; left: Smith dynamic, right: mixed dynamic.

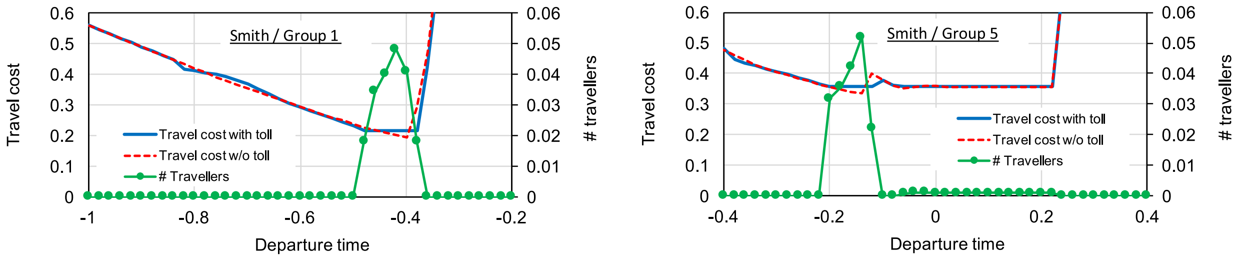

Fig. 4. Travel cost (with and without toll) and number of travellers of groups 1 (left) and 5 (right) by Smith dynamic.

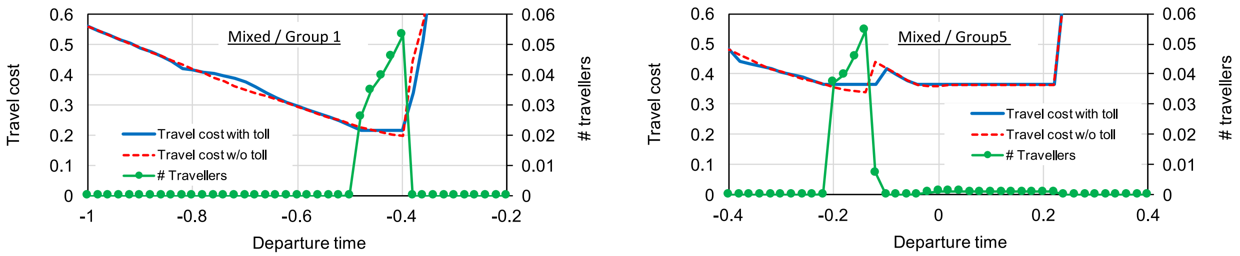

Fig. 5. Travel cost (with and without toll) and number of travellers of groups 1 (left) and 5 (right) by mixed dynamic.

Fig. 3 depicts the delays and tolls in each departure time slot when $\alpha = 100$ and the revision time scale was 15 (i.e. the toll pattern was revised 15 times, including the first introduction of the toll). The toll pattern in Fig. 3 is for group 1. Dividing the toll by g, we obtain the toll pattern for group g. We can see that the tolls are substantially smaller than the delays in both graphs, except at one time slot (-0.12 for both cases).

Figs. 4 and 5 show the travel costs (with and without toll) and the number of travellers choosing each departure time slot in groups 1 and 5, when $\alpha = 100$ and the revision time scale was 15. We can see that the travel costs with tolls and those without tolls were very similar, implying that the toll level imposed on travellers was substantially smaller than the non-toll part of the travel cost.

6.3. Results of 100 parameter sets

Fig. 6 depicts the changes in the toll-cost ratio and in the change speed of randomly generated 100 parameter sets (referred to as cases 1 – 100). α was set to 100. We can see that the toll scheme stabilised the transport system. On the other hand, although the toll-cost ratios of many cases tended to decrease, those in several cases did not successfully decrease.

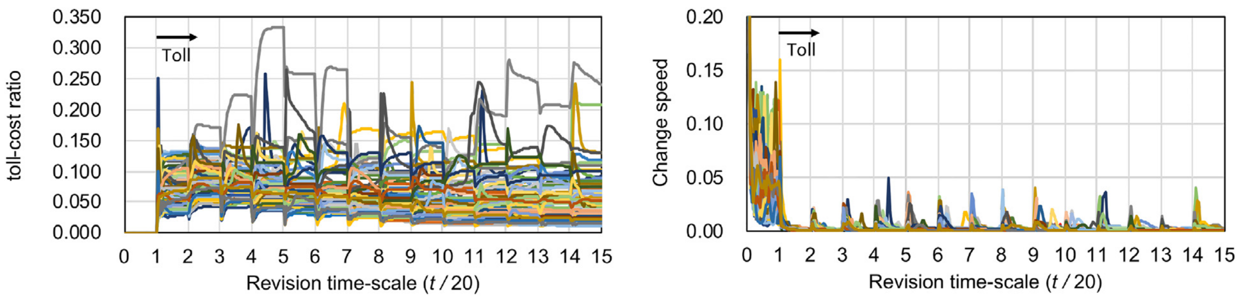

Fig. 6. Changes in the toll-cost ratio (left) and changes in the change speed for 100 cases by using the Smith dynamic ($\alpha = 100$).

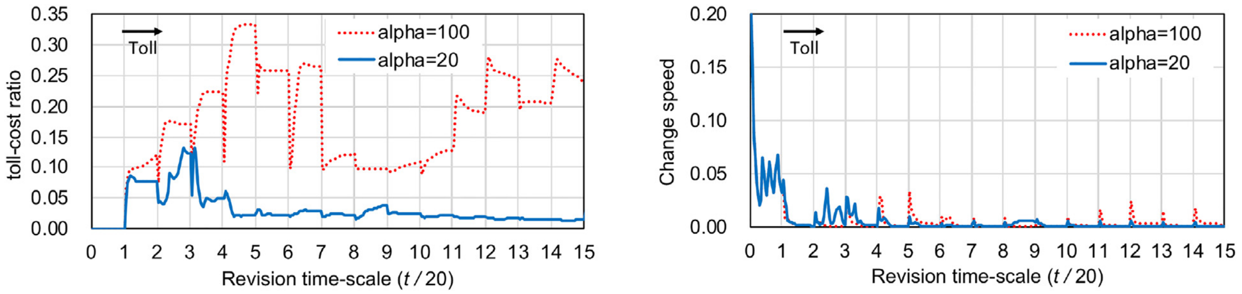

Fig. 7. Changes in the toll-cost ratio (left) and changes in the change speed for case 76 with two values of α by using the Smith dynamic.

As the analysis in Section 5 suggests that the value of α may affect the convergence behaviour of the iteration process, we employed $\alpha = 20$ for case 76, whose convergence behaviour of the toll-cost ratio was particularly bad, to explore whether a different choice of α might lead to convergence of the toll-cost ratio. Fig. 7 depicts the changes in the toll-cost ratio and in the change speed for case 76. The results for both $\alpha = 20$ and $\alpha = 100$ are shown in the graph. We can see that, at least for this case, changing the value of α was beneficial to ensure the convergence of the toll-cost ratio.

These results suggest that selecting an appropriate value of α is important for obtaining the best results from the stabilisation strategy, in which a smaller toll-cost ratio is realised within a reasonable number of updates of the target traffic flow pattern ($\bar{\mathbf{y}}^k$). Considering the results shown in Figs. 1 and 7, selecting α smaller is generally better as long as stability holds. If these findings are applicable in more general situations, they suggest a sensible procedure is to start with a small α and then increment it until the transport system is stabilised.

6.4. Discrete day-to-day dynamical model

In this section, the discrete version of the day-to-day dynamical model defined by Eq. (9) was employed for the numerical tests, instead of the evolutionary dynamics defined by Eq. (8). We define $\bar{\rho}_{ij}^{g}$ as

$$\bar{\rho}_{ij}^{g} = \min\left\{1, [\pi_i^g - \pi_j^g]_+\right\} \tag{48}$$

such that it is not greater than one at all times (otherwise, it cannot be considered as a probability). ϕ is set to 0.05 (i.e. five percent of the travellers reconsider their choices every day). Cases 1 and 10 were used. α was set to 50 for case 1 and 30 for case 10. The toll was not applied for the first 40 days (from day 0 to day 39) and was subsequently applied on day 40. The toll was then updated every 40 days until day 600. Instead of the change speed (i.e. $|\mathbf{x}|$), we use $|\mathbf{x}(\tau) - \mathbf{x}(\tau - 1)|$ (called 'change per day') as a measure of instability. The results are shown in Figs. 8 (tolls) and 9 (change per day). Fig. 9 also

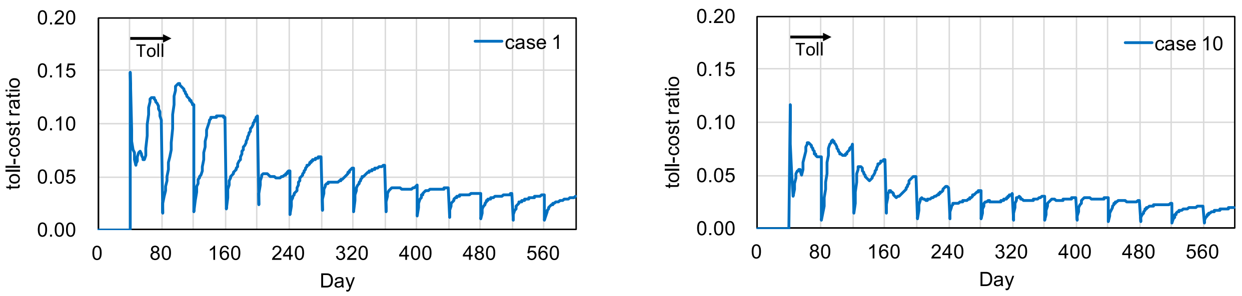

Fig. 8. Changes in the toll-cost ratio for cases 1 (left) and 10 (right) with the discrete day-to-day dynamical model.

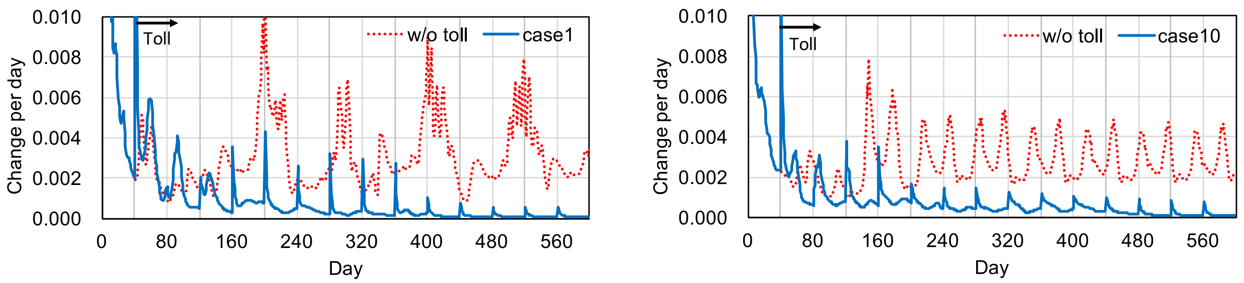

Fig. 9. Changes in the change per day for cases 1 (left) and 10 (right) with the discrete day-to-day dynamical model.

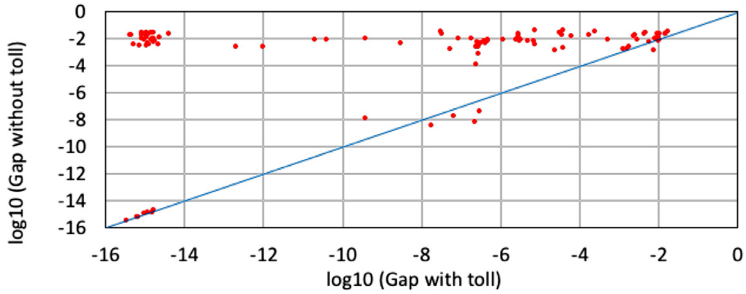

Fig. 10. Gaps of 100 cases at $t = 10{,}000$. The calculation was made for $\alpha = 2, 3, 5, 10, 20$, and the smallest gap was used to draw the graph.

includes the results without tolls for comparison. In both cases, after 500 days, the change per day converges to zero and the toll-cost ratio stabilises at around 0.02–0.03. These results illustrate that the proposed toll scheme can perform as expected, even when a model of discrete dynamics is adopted.

6.5. Long-run behaviour of the iterative process

In this section, we regard the proposed iterative process not as a toll scheme but as a solution algorithm, i.e. an algorithm to find an (unstable) equilibrium solution without tolls. For such an application, the maximum number of iterations (15) assumed in the numerical examples above is apparently insufficient for obtaining a precise solution. For the present example (i.e. the multi-class departure-time-choice problem), we actually already have an analytical solution method (Lindsey, 2004). Nevertheless, investigating the properties of the proposed iteration algorithm for this problem is still beneficial, in terms of understanding how it might apply to solve more general problems.

We observed the long-run convergence behaviour of the 100 cases, while we set the number of time slots as 20 (was 100) to reduce the calculation time and increase the number of iterations. The step size of t in the Runge-Kutta method was set to 0.01. The Smith dynamic was used. The values of α used in the calculation were 2, 3, 5, 10, and 20. No heterogeneity of the VOT was incorporated as it is not necessary for finding an equilibrium point. The convergence to an equilibrium solution without toll was evaluated by a gap function defined by

$$\xi = \sum_{g \in G} \sum_{i \in C^g} x_i^g \left\{ c_i^g(\mathbf{y}) - c^{*g}(\mathbf{y}) \right\} \quad \text{where} \quad c^{*g}(\mathbf{y}) = \min_{i \in C^g} c_i^g(\mathbf{y}). \tag{49}$$

To evaluate how the proposed iterative process reduces the value of the gap function, we calculated the gaps for all cases with the tolls imposed and without the tolls imposed at $t = 10{,}000$ and compared them with and without the tolls for each case. Fig. 10 shows the results; note that a log-log plot is used in this graph. For the gaps with tolls, we used the minimum value of the gap among different α values for each case. Sufficiently smaller gaps were observed when no toll was imposed in some of the 100 cases, perhaps owing to the smaller number of time slots (we did not observe the convergence behaviour when the number of time slots was 100, as shown in Fig. 6). On the other hand, larger gaps were still observed in many cases when no toll was imposed. We can see that the proposed iterative process reduced these gaps. The amount of the reduction varied among cases. In some cases, the reduction was significant, whereas in other cases, it was not very significant. This result implies that an accurate equilibrium solution without toll may be obtained by the iterative process proposed, by selecting an appropriate value of α.

7. Conclusions and future directions

In the present study, we have proposed a toll scheme that stabilises an unstable equilibrium solution in a transport system. We first analysed the monotonicity of the travel cost functions with the proposed toll scheme, and showed that it is strictly monotone over X_σ^2, as defined by Eq. (16), for any type of transport system, including those having unstable or multiple equilibria. Then, we showed that many evolutionary dynamical systems become globally asymptotically stable when the proposed toll scheme is imposed. We also investigated the properties of the equilibrium solution with toll and showed that the toll level actually imposed on travellers is always upper bounded. Furthermore, we showed that the toll converges to zero when the error in estimation of the no-toll equilibrium converges to zero. Finally, we proposed a heuristic procedure to minimise the toll level, which can also be recognised as a solution algorithm for finding an equilibrium solution of the transport system with no toll.

The potential benefits of the proposed method are both practical and theoretical. In a practical context, the proposed method constitutes a novel control scheme to realise a better status of a transport system with small charges, especially when there exists an equilibrium solution that is preferable but is not realised owing to its instability. When multiple equilibria exist, the proposed method can be utilised to select a preferable but unstable one among them. In a theoretical context, the proposed method may be used as a solution methodology for general equilibrium problems in transport science. As discussed in the DTA context (Iryo, 2013), such a useful method has yet to be developed.

The proposed method deals with an idealised situation that may not directly reflect all phenomena in the real world. Nevertheless, we believe our results are useful for understanding how to develop a toll strategy to stabilise a realistic transport system. It has similarities with a first-best toll scheme for fixed demand, which represents a toll scheme to minimise congestion in an idealised situation. In this sense, like the concept of the first-best toll scheme, the method developed in the present study can be viewed as a reference point for future studies dealing with more realistic situations.

The present study employed ODE-based dynamics, which only considers dynamics in a deterministic way. The inclusion of stochastic effects in the proposed method are a major issue to be addressed in future studies. Using evolutionary dynamics has the advantage of maintaining mathematical tractability, thanks to its continuous form (i.e. a day is described by a continuous number). We could assume a more realistic day-to-day dynamic model that incorporates both stochastic changes in travellers' behaviour and a discrete description of a day. A further potential generalisation is to allow the use of a stochastic user-equilibrium (SUE) model, as an alternative to the deterministic user-equilibrium form. Expanding the proposed method to the SUE formulation would not be very difficult; indeed, many parts of the present study did not rely on the ODE-based dynamics, but relied much more on the variational inequality system. This observation may well prove useful when we come to discuss stochastic effects.

The proposed toll has a linear formulation with respect to the traffic flow volume. Using other monotone-increasing functions aside from the linear form may be useful in avoiding overcharging, especially when the difference between the actual traffic flow volume and the target flow volume is large.

Many of the analytical parts of this study are only applicable when the transport system is in equilibrium (with tolls). Because the toll scheme stabilises an equilibrium point of the transport system, assuming that equilibrium is basically maintained is reasonable. On the other hand, when we impose a new toll pattern on travellers, the system enters a transient state, in which travellers have to re-adjust their choices with respect to the new situation. The transient state is not permanent if the system is stable, and, consequently, its impact is basically not large. Nevertheless, travellers may dislike the transient situation because they cannot predict future situations, including the toll level they will actually pay tomorrow — if they are unexpectedly and highly charged, the acceptability of the proposed toll scheme cannot be ensured. Announcing the toll level beforehand to travellers may well be a good strategy to avoid such an undesirable situation. By doing so, we can certainly help travellers avoid an expensive option. On the other hand, this strategy will affect travellers' day-to-day adjustment processes and, consequently, expansions of the dynamic model may be necessary for quantitative analyses.

Although the results of the numerical tests suggest that the proposed method may be useful to find an unstable equilibrium solution, we do not have any rigorous proof of the convergence thus far. In addition, the convergence behaviour of the proposed iterative process depends on the value of α, and we have no good way to select an α that ensures cheaper tolls that stabilise the system. Expanding the analysis in Section 5 would provide more insights on these issues.

Acknowledgement

This study was financially supported by Japan Society for the Promotion of Science KAKENHI Grant Number 16H02368.

References

Beckmann, M., McGuire, C.B., Winsten, C.B., 1956. Studies in the Economics of Transportation. Yale University Press, New Haven.
Bie, J., Lo, H.K., 2010. Stability and attraction domains of traffic equilibria in a day-to-day dynamical system formulation. Transport. Res. Part B 44 (1), 90–107.
Dupuis, P., Nagurney, A., 1993. Dynamical systems and variational inequalities. Ann. Oper. Res. 44, 7–42.
Friesz, T.L., Bernstein, D., Mehta, N.J., Tobin, R.L., 1994. Day-to-day dynamic network disequilibria and idealized traveler information systems. Oper. Res. 42 (6), 1120–1136.
Fukuda, D., Morichi, S., 2007. Incorporating aggregate behavior in an individual's discrete choice: an application to analyzing illegal bicycle parking behavior. Transport. Res. Part A 41 (4), 313–325.
Gaunt, M., Rye, T., Allen, S., 2007. Public acceptability of road user charging: the case of Edinburgh and the 2005 referendum. Transport Rev. 27 (1), 85–102.
Guo, R.Y., Yang, H., Huang, H.J., 2013. A discrete rational adjustment process of link flows in traffic networks. Transport. Res. Part C 34, 121–137.
Guo, R.-Y., Yang, H., Huang, H.-J., 2018. Are we really solving the dynamic traffic equilibrium problem with a departure time choice? Transport. Sci. 52 (3), 603–620.
He, X., Guo, X., Liu, H.X., 2010. A link-based day-to-day traffic assignment model. Transport. Res. Part B 44 (4), 597–608.
Heydecker, B.G., 1983. Some consequences of detailed junction modeling in road traffic assignment. Transport. Sci. 17 (3), 263–281.
Hofbauer, J., Sandholm, W.H., 2009. Stable games and their dynamics. J. Econ. Theory 144 (4), 1665–1693.
Iryo, T., 2008. An analysis of instability in a departure time choice problem. J. Adv. Transport. 42 (3), 333–356.
Iryo, T., 2011. Multiple equilibria in a dynamic traffic network. Transport. Res. Part B 45 (6), 867–879.
Iryo, T., 2013. Properties of dynamic user equilibrium solution: existence, uniqueness, stability, and robust solution methodology. Transportmetr. B 1 (1), 52–67.
Iryo, T., 2015. Investigating factors for existence of multiple equilibria in dynamic traffic network. Netw. Spat. Econ. 15 (3), 599–616.
Iryo, T., 2016. Day-to-day dynamical model incorporating an explicit description of individuals' information collection behaviour. Transport. Res. Part B 0, 1–16.
Iryo, T., 2019. Instability of departure time choice problem: a case with replicator dynamics. Transport. Res. Part B, doi:10.1016/j.trb.2018.08.005. in press
Lindsey, R., 2004. Existence, uniqueness, and trip cost function properties of user equilibrium in the bottleneck model with multiple user classes. Transport. Sci. 38 (3), 293–314.
Mohring, H., 1972. Optimization and scale economies in urban bus transportation. Am. Econ. Rev. 62 (4), 591–604.
Mounce, R., 2006. Convergence in a continuous dynamic queueing model for traffic networks. Transport. Res. Part B 40 (9), 779–791.
Sandholm, W.H., 2010. Pairwise comparison dynamics and evolutionary foundations for Nash equilibrium. Games 1 (1), 3–17.
Sandholm, W.H., 2010b. Population Games and Evolutionary Dynamics. MIT Press, Cambridge.
Schuster, P., Sigmund, K., 1983. Replicator dynamics. J. Theor. Biol. 100, 533–538.

Smith, M., Mounce, R., 2011. A splitting rate model of traffic re-routeing and traffic control. Transport. Res. Part B 45 (9), 1389–1409.
Smith, M.J., 1979. The existence, uniqueness and stability of traffic equilibria. Transport. Res. Part B 13 (4), 295–304.
Smith, M.J., 1984a. The stability of a dynamic model of traffic assignment - an application of a method of Lyapunov. Transport. Sci. 18 (3), 245–252.
Smith, M.J., 1984b. Two alternative definitions of traffic equilibrium. Transport. Res. Part B 18 (1), 63–65.
Vickrey, W.S., 1969. Congestion theory and transport investment. Am. Econ. Rev. 59 (2), 251–260.
Watling, D.P., 1996. Asymmetric problems and stochastic process models of traffic assignment. Transport. Res. Part B 30 (5), 339–357.
Yang, F., Zhang, D., 2009. Day-to-day stationary link flow pattern. Transport. Res. Part B 43 (1), 119–126.
Yang, H., Meng, Q., Lee, D.H., 2004. Trial-and-error implementation of marginal-cost pricing on networks in the absence of demand functions. Transport. Res. Part B 38 (6), 477–493.
Yang, H., Xu, W., sheng He, B., Meng, Q., 2010. Road pricing for congestion control with unknown demand and cost functions. Transport. Res. Part C 18 (2), 157–175.
Ying, J.Q., Yang, H., 2005. Sensitivity analysis of stochastic user equilibrium flows in a bi-modal network with application to optimal pricing. Transport. Res. Part B 39 (9), 769–795.
Zhang, D., Nagurney, A., 1996. On the local and global stability of a travel route choice adjustment process. Transport. Res. Part B 30 (4), 245–262.