Journal of King Saud University – Science 32 (2020) 928–933

Contents lists available at ScienceDirect

Journal of King Saud University – Science

journal homepage: www.sciencedirect.com

Reconstruction of the timewise conductivity using a linear combination of heat flux measurements

M.J. Huntul [a,*], D. Lesnic [b]

[a] Department of Mathematics, College of Science, Jazan University, Jazan, Saudi Arabia
[b] Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, UK

ARTICLE INFO

Article history:
Received 14 March 2019
Revised 3 May 2019
Accepted 19 May 2019
Available online 22 May 2019

Keywords:
Coefficient identification
Two-dimensional heat equation
Nonlocal observation
Nonlinear optimization

ABSTRACT

The reconstruction of the timewise conductivity in the heat equation from an observation consisting of a linear combination of heat flux measurement data is considered. This inverse formulation results in a local uniquely solvable problem. The two-dimensional inverse problem is discretized using an alternating direction explicit method. The resulting constrained optimization problem is minimized iteratively by employing a MATLAB toolbox subroutine.

1. Introduction

The determination of coefficients from nonlocal boundary information has received significant attention from many researchers in recent years, see Cannon et al. (1990), Cannon and Matheson (1993) and Cannon and Yin (1989) to mention only a few. Nonlocal problems arise naturally in modeling various phenomena such as groundwater transport in porous media (Dagan, 1994), nuclear radioactive decay (Shelukhin, 1993), viscoelasticity (Renardy et al., 1987) and semiconductor devices (Allegretto et al., 1999).

One-dimensional inverse problems concerning the reconstruction of timewise conductivity coefficient from various nonlocal conditions were investigated in Hussein et al. (2016) and Huntul and Lesnic (2017). The inverse problems investigated in this paper have already been proved to be locally uniquely solvable in Ivanchov and Sahaidak (2004) and Kinash (2018), but no numerical reconstruction has been attempted so far, and it is the objective of the current research to undertake the numerical realization of these problems.

* Corresponding author.
E-mail addresses: mhantool@jazanu.edu.sa (M.J. Huntul), amt5ld@maths.leeds.ac.uk (D. Lesnic).

Peer review under responsibility of King Saud University.

ELSEVIER Production and hosting by Elsevier

2. Inverse formulations

Let $Q_T = \Omega \times (0, T)$, where Ω is the rectangle $(0, h) \times (0, l)$, and consider the determination of the coefficient $a(t) > 0$ in

$$u_t = a(t)\nabla^2 u + \underline{b}(x, y, t) \cdot \nabla u + c(x, y, t)u + f(x, y, t), \quad (x, y, t) \in Q_T, \tag{1}$$

where $\underline{b} = (b_1, b_2)$ is a known velocity representing convective flow, c is a known absorption coefficient and f is a known source, with unknown $u(x, y, t)$, subject to

$$u|_{t=0} = \varphi, \quad \text{in} \quad [0, h] \times [0, l] = \overline{\Omega}, \tag{2}$$

$$u|_{x=0} = \mu_{11}, \quad u|_{x=h} = \mu_{12}, \quad \text{in} \quad [0, l] \times [0, T], \tag{3}$$

$$u|_{y=0} = \mu_{21}, \quad u|_{y=l} = \mu_{22}, \quad \text{in} \quad [0, h] \times [0, T], \tag{4}$$

together with the non-local over-determination

$$a(t)[\nu_1(t)u_x(0, Y_0, t) + \nu_2(t)u_x(h, Y_0, t)] = \kappa(t), \quad t \in [0, T], \tag{5}$$

where Y_0 is a fixed point within the interval $(0, l)$, and $\boldsymbol{\varphi}, \boldsymbol{\mu}_{1i}, \boldsymbol{\mu}_{2i}, \nu_i$ for $i = 1, 2$ and κ are given functions. Physical situations in which the diffusivity $a(t)$ depends on time (and is unknown) occur in damage and radioactive decay applications, (Cannon, 1984). In case $\nu_1 = 1, \nu_2 = 0$, Eq. (5) becomes

$$a(t)u_x(0, Y_0, t) = \kappa(t), \quad t \in [0, T] \tag{6}$$

and the local unique solvability of the problem given by (1)–(4) and (6) reads as stated below (Ivanchov and Sahaidak, 2004).

https://doi.org/10.1016/j.jksus.2019.05.006
1018-3647/

Theorem 1. *Assume that:*

(A1) $\varphi \in C^2(\overline{\Omega})$, $\mu_{1i} \in C^{2,1}([0,l] \times [0,T])$, $\mu_{2i} \in C^{2,1}([0,h] \times [0,T])$, $i = 1,2$, $\kappa \in C[0,T]$, $f \in C^{1,0}(\overline{Q_T})$, $b_i \in C(\overline{Q_T})$, $i = 1,2$, $c \in C(\overline{Q_T})$, *and* b_1, b_2 *and* c are *Hölder continuous in space;*
(A2) $\kappa > 0$, $\varphi_x > 0$;
(A3) The Dirichlet and initial data (2)–(4) are consistent at $t = 0$ *on the boundary* $\partial\Omega$;
(A4) Eq. (1) holds at $t = 0$ *for* $(x,y) \in \partial\Omega$.

Then, there exists $T_0 \in (0,T]$, *for which the solution* $(0 < a(t), u(x,y,t)) \in C[0,T_0] \times C^{2,1}(\overline{Q_{T_0}})$ *of the problem (1)–(4) and (6) exists.*

Theorem 2. *Assume:*

(A5) b_1, b_2 *and* $c \in C(\overline{Q_T})$ *are Hölder continuous in space*
(A6) $\kappa > 0$.

Then, the problem (1)–(4) and (6) has at most one solution in the class $(0 < a(t), u(x,y,t)) \in C[0,T] \times C^{2,1}(\overline{Q_T})$.

We finally remark that in Ivanchov and Sahaidak (2004) it is stated that the local existence and uniqueness also hold in case the local heat flux measurement (6) is replaced by the non-local condition

$$a(t)\left[u_x(0,Y_0,t) + u_y(X_0,0,t)\right] = \nu(t), \quad t \in [0,T],$$

where X_0 is a fixed point in the interval $(0,h)$.

2.1. Statement of a related inverse problem

Assume, for simplicity, that $b_1 = b_2 = c = 0$, such that (1) becomes

$$u_t = a(t)\nabla^2 u + f(x,y,t), \quad (x,y,t) \in Q_T. \tag{7}$$

If instead of the nonlocal heat flux condition (5) we prescribe the nonlocal-derivative combination

$$\nu_1(t)u_x(0,Y_0,t) + \nu_2(t)u_x(h,Y_0,t) = \chi(t), \quad t \in [0,T], \tag{8}$$

Table 1
The exact solution ((14) and (18)) and numerical results for $\kappa(t)$, for the direct probelm (Examples 1 and 2).

Example 1	t	0.1	0.2	0.3	...	0.8	0.9	$rmse(\kappa)$
	$\kappa(t)$	4.4033	4.8044	5.2052	...	7.2075	7.6080	6.2E−3
	Exact	4.4000	4.8000	5.2000	...	7.2000	7.6000	0
Example 2	t	0.1	0.2	0.3	...	0.8	0.9	$rmse(\kappa)$
	$\kappa(t)$	8.4000	7.8000	7.2000	...	7.8000	8.7000	2.5E−14
	Exact	4.4000	7.8000	7.2000	...	7.8000	8.7000	0

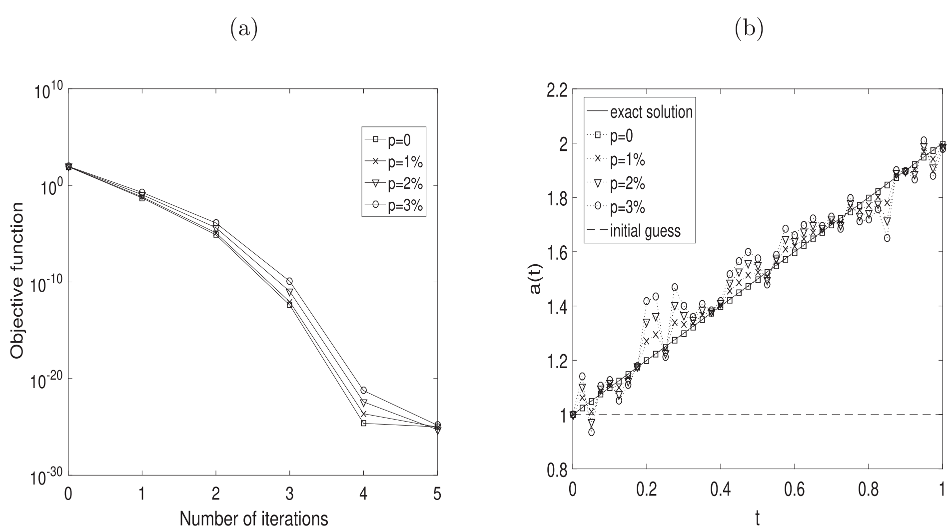

Fig. 1. (a) The objective function (10), and (b) the solutions for $a(t)$, for $p \in \{0, 1\%, 2\%, 3\%\}$ noise (Example 1).

Table 2
Computational details.

Example 1	$p = 0$	$p = 1\%$	$p = 2\%$	$p = 3\%$
Minimum value of (10)	1.0E−25	9.4E−26	5.1E−26	1.4E−25
$rmse(a)$	2.4E−3	0.0301	0.0605	0.0911
Example 2	$p = 0$	$p = 1\%$	$p = 2\%$	$p = 3\%$
Minimum value of (10)	6.3E−26	6.3E−26	7.4E−26	8.4E−26
$rmse(a)$	5.3E−15	0.0209	0.0419	0.0628

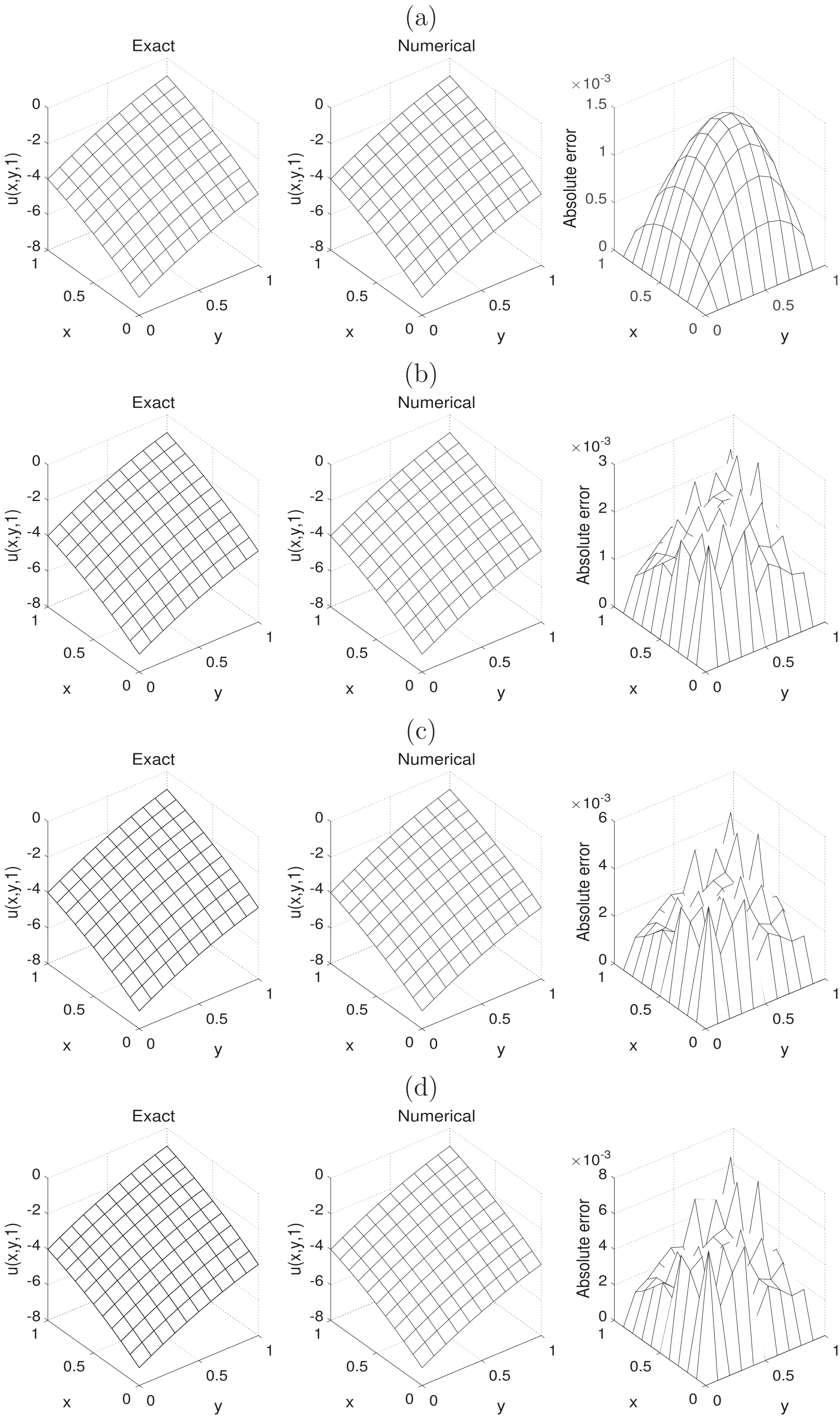

Fig. 2. The solutions for $u(x, y, 1)$, for: (a) $p = 0$, (b) $p = 1\%$, (c) $p = 2\%$, and (d) $p = 3\%$ (Example 1).

then the local unique solvability of the problem given (2)–(4), (7) and (8) read as stated in the following two theorems (Kinash, 2018).

Theorem 3. *Assume that (A3) is satisfied and that:*

(A7) $f \in C\left(\overline{Q_T}\right)$, $\varphi \in C^2\left(\overline{\Omega}\right)$, $\Psi \in C^{2,1}\left(\overline{Q_T}\right)$, $\tilde{\chi}, \nu_1, \nu_2 \in C^1([0,T])$, and $f, \nabla^2\varphi$,
$\nabla^2\Psi$ and Ψ_t *are Hölder continuous with exponent* $\alpha \in (0,1)$ *in space, where*

$$\begin{aligned}\tilde{\chi}(t) := {} & \chi(t) - \nu_1(t)(\varphi_x(0,Y_0) + \Psi_x(0,Y_0,t)) - \nu_2(t)(\varphi_x(h,Y_0)\\ & +\Psi_x(h,Y_0,t)), \Psi(x,y,t) := \mu_{11}(y,t) - \mu_{11}(y,0)\\ & +\tfrac{x}{h}(\mu_{12}(y,t) - \mu_{12}(y,0) - \mu_{11}(y,t) + \mu_{11}(y,0))\\ & +\mu_{21}(x,t) - \mu_{21}(x,0) - [\mu_{11}(0,t) - \mu_{11}(0,0)\\ & +\tfrac{x}{h}(\mu_{12}(0,t) - \mu_{12}(0,0) - \mu_{11}(0,t) + \mu_{11}(0,0))]\\ & +\tfrac{y}{l}[\mu_{22}(x,t) - \mu_{22}(x,0) - \mu_{11}(l,t) + \mu_{11}(l,0)\\ & -\tfrac{x}{h}(\mu_{12}(l,t) - \mu_{12}(l,0) - \mu_{11}(l,t) + \mu_{11}(l,0)) - \mu_{21}(x,t)\\ & +\mu_{21}(x,0) + \mu_{11}(0,t) - \mu_{11}(0,0) + \tfrac{x}{h}(\mu_{12}(0,t) - \mu_{12}(0,0)\\ & -\mu_{11}(0,t) + \mu_{11}(0,0))];\end{aligned}$$

(A8) $-\nu_1(t)\left(\nabla^2\varphi(0,Y_0) + \nabla^2\Psi(0,Y_0,t)\right) + \nu_2(t)\left(\nabla^2\varphi(h,Y_0) + \nabla^2\Psi(h,Y_0,t)\right) > 0$, $\quad -\nu_2(t)(f(h,Y_0,t) - \Psi_t(h,Y_0,t)) + \nu_1(t)(f(0,Y_0,t) - \Psi_t(0,Y_0,t)) > 0$, $\quad t \in [0,T]$;
(A9) $\nu_1(0)\varphi\prime(0,Y_0) + \nu_2(0)\varphi\prime(h,Y_0) = \tilde{\chi}(0)$.

Then, there exists $T_0 \in (0,T]$, *for which the solution* $(0 < a(t), u(x,y,t)) \in C[0,T_0] \times C^{2,1}\left(\overline{Q_{T_0}}\right)$ *of the problem given (2)–(4), (7) and (8) exists.*

Theorem 4. *Let the assumption (A7) and*

$$U(t) := \nu_2(t)\left(\nabla^2\varphi(h,Y_0) + \nabla^2\Psi(h,Y_0,t)\right) - \nu_1(t)\left(\nabla^2\varphi(0,Y_0) + \nabla^2\Psi(0,Y_0,t)\right) \neq 0, t \in [0,T], \qquad (9)$$

be satisfied. Then, there exists $T_0 \in (0,T]$, *for which the solution of the problem (2)–(4), (7) and (8) is unique in the class* $(0 < a(t), u(x,y,t)) \in C[0,T] \times C^{2,1}\left(\overline{Q_T}\right)$, *and* $\nabla^2 u$ *is Hölder continuous in space.*

The numerical realisation of the inverse problem (2)–(4), (7) has been undertaken elsewhere, (Huntul, 2018), and therefore it is not considered further herein.

3. Inverse problem

The numerical solution of (1)–(5) is obtained by minimizing

$$F(\mathbf{a}) = \sum_{n=1}^{N}[a_n(\nu_1(t_n)u_x(0,Y_0,t_n) + \nu_2(t_n)u_x(h,Y_0,t_n)) - \kappa(t_n)]^2, \qquad (10)$$

where $t_n = n\Delta t, \Delta t = T/N$, N is the number of time steps and $a_n := a(t_n)$, and u solves numerically using the alternating direction explicit (ADE) method (Barakat and Clark, 1966; Campbell and Yin, 2007; Ozisik, 1994) the direct problem (1)–(4) for given $a(t)$. The minimization of (10) is accomplished using the *lsqnonlin* subroutine in MATLAB. Simple bounds on the variable $a > 0$ are specified as $10^{-5} \leqslant a \leqslant 10^2$.

The data (5) is subject to random noise as

$$\kappa^{\epsilon}(t_n) = \kappa(t_n) + \epsilon_n, \quad n = \overline{1,N}, \qquad (11)$$

where $(\epsilon_n)_{n=\overline{1,N}} = normrnd(0,\sigma,N), \sigma = p \times \max_{t\in[0,T]}|\kappa(t)|$, and p is the percentage of noise.

4. Results

Define

$$rmse(a) = \left[\frac{T}{N}\sum_{n=1}^{N}\left(a^{Numerical}(t_n) - a^{Exact}(t_n)\right)^2\right]^{1/2} \qquad (12)$$

and take $h = l = T = 1$ and $Y_0 = 1/2$.

Example 1. Input data $\nu_1(t) = 1, \nu_2(t) = 0$,

$$\begin{aligned}&\varphi(x,y) = -(x-2)^2 - (y-2)^2,\\ &b_1(x,y,t) = x+y+t,\\ &b_2(x,y,t) = -x-y-t,\\ &c(x,y,t) = x+y+t,\\ &\mu_{11}(y,t) = -4+t-(y-2)^2,\\ &\mu_{12}(y,t) = -1+t-(y-2)^2,\\ &\mu_{21}(x,t) = -4+t-(x-2)^2,\\ &\mu_{22}(x,t) = -1+t-(x-2)^2,\\ &f(x,y,t) = 1+4(1+t)\\ &-\left[t-(x-2)^2-(y-2)^2+2y-2x\right](t+x+y), \qquad (13)\end{aligned}$$

$$\kappa(t) = a(t)u_x(0,Y_0,t) = 4(1+t). \qquad (14)$$

(a) (b)

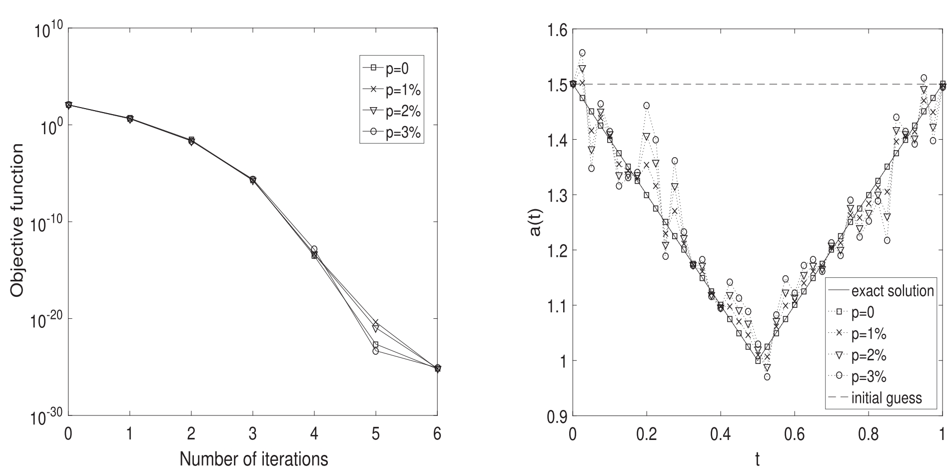

Fig. 3. (a) The objective function (10), and (b) the solutions for $a(t)$, for $p \in \{0, 1\%, 2\%, 3\%\}$ noise (Example 2).

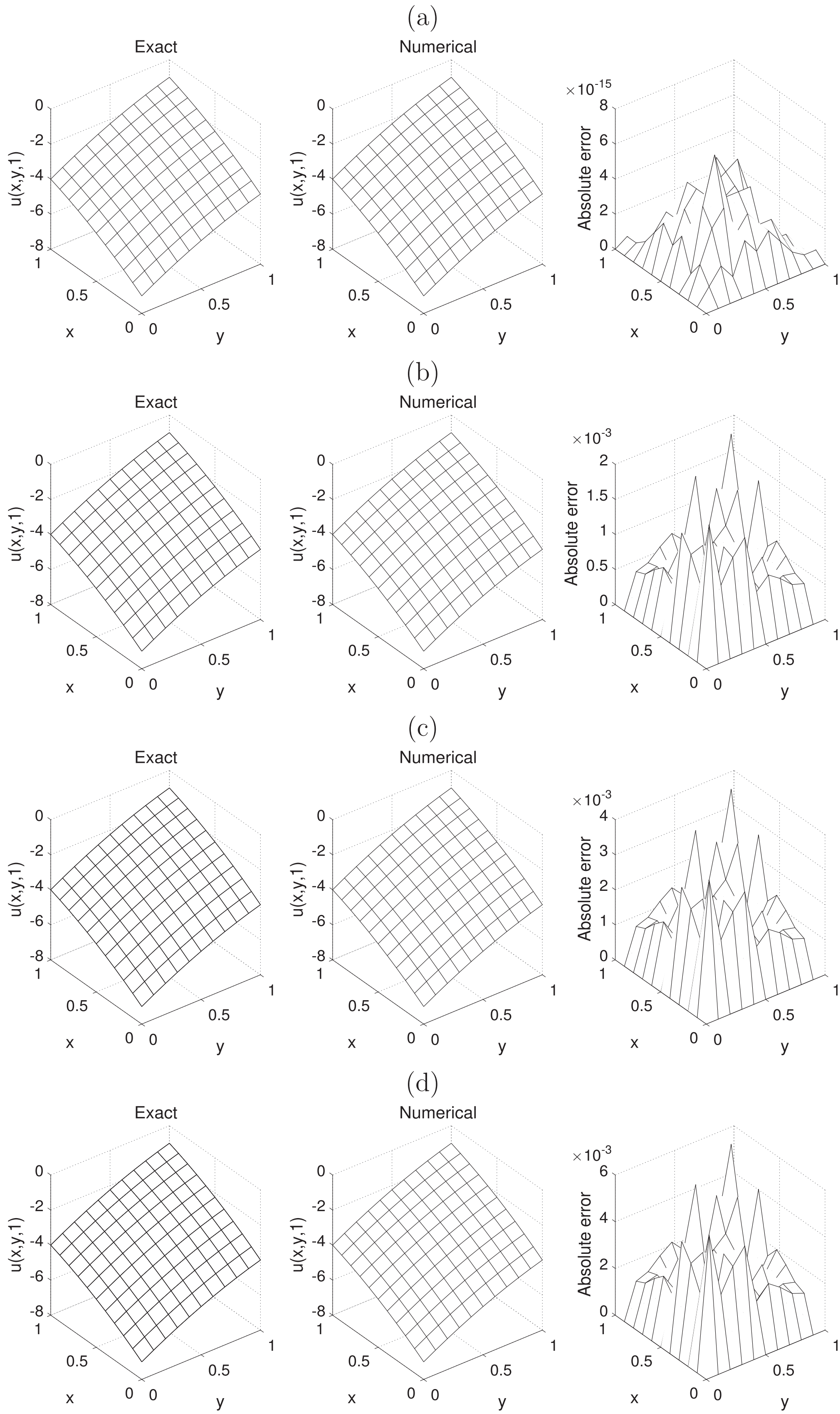

Fig. 4. The solutions for $u(x, y, 1)$, for: (a) $p = 0$, (b) $p = 1\%$, (c) $p = 2\%$, and (d) $p = 3\%$ noise (Example 2).

First, it can easily be checked that with this data, the conditions $(A1)$–$(A6)$ are satisfied, hence the local unique solvability of the inverse problem (1)–(4) and (6) is guaranteed. The analytical solution is

$$u(x,y,t) = t - (x-2)^2 - (y-2)^2, \tag{15}$$

$$a(t) = 1 + t. \tag{16}$$

First, we solve numerically the direct problem (1)–(4), when $a(t)$ is known and given by (16), using the ADE with the mesh sizes $\Delta x = \Delta y = 1/10$ and $\Delta t = 0.025$. The exact solution (13) for the heat flux over-specification $\kappa(t)$ compared with the numerical solutions in Table 1, shows on an excellent agreement.

Next, we solve the inverse problem (1)–(4) and (6) with the input (13) and (14) using the *lsqnonlin* minimization of the functional (10) with the initial guess for the vector $\mathbf{a} = (a(t_n))_{n=\overline{1,N}}$ given by

$$a^0(t_n) = a(0) = 1, \quad n = \overline{1,N}. \tag{17}$$

We take the same mesh size as in the direct problem above, but we add noise in the input data (14), as in Eq. (11). The objective function (10), is plotted in Fig. 1(a), where a monotonic decreasing convergence is obtained in 5 iterations in about 8 min CPU time. The related results for $a(t)$ are shown in Fig. 1(a), and in Table 2, and it can be observed that as the percentage of noise p decreases, the numerical solutions become more stable and accurate. Fig. 2 compares the exact and numerical solutions for $u(x,y,1)$ showing good agreement and stability. No regularization was found necessary to penalise the nonlinear least-squares objective functional (10) for amounts of noise up to $p = 3\%$, hinting towards the conclusion that the inverse problem under investigation is not severely ill-posed. Nevertheless, for higher amounts of noise the instability in retrieving the coefficient $a(t)$ will become apparent and regularization would need to be employed.

Example 2. We test the inverse problem given by (7) (i.e. we take $\underline{b} = \underline{0}$ and $c = 0$ in (1)) and (2)–(4) with the same initial and Dirichlet data as in (13) and the more general non-local over-specification (5) given by

$$\kappa(t) = a(t)[v_1(t)u_x(0,Y_0,t) + v_2(t)u_x(h,Y_0,t)] = 6(|t-0.5|+1),$$
$$v_1(t) = v_2(t) = 1. \tag{18}$$

We also take the source $f(x,y,t) = 5 + 4|t-0.5|$. The previous example has reconstructed the smooth timewise thermal conductivity $a(t)$ given by (15). In this example, we assess the numerical reconstruction of a non-differentiable conductivity given by

$$a(t) = |t-0.5| + 1. \tag{19}$$

We also have the same analytical solution for the temperature $u(x,y,t)$ given by (15). The initial guesses for the vector $\mathbf{a}$ for this example has been taken as

$$a^0(t_n) = a(0) = 1.5, \quad n = \overline{1,N}. \tag{20}$$

Note that the value of $a(0)$ is available from (2) and (5) as

$$a(0) = \frac{\kappa(0)}{v_1(0)\varphi_x(0,Y_0) + v_2(0)\varphi_x(h,Y_0)}. \tag{21}$$

As in Example 1, for the numerical discretisation we employ the ADE with the mesh sizes $\Delta x = \Delta y = 0.1$ and $\Delta t = 0.025$. Fig. 3(a) shows the rapidly decreasing functional (10) with the number of iterations. The results shown in Figs. 3(b) and 4, and in Table 2, yield the same conclusions as those obtained for Example 1.

5. Conclusions

The reconstruction of conductivity and temperature from the non-local linear combination of heat flux over-determination (5) has been achieved numerically by employing the ADE, as a direct solver, implemented in a constrained minimization procedure. Further work will consist in retrieving multiple coefficients.

Declaration of interest

None.

References

Allegretto, W., Lin, Y., Zhou, A., 1999. A box scheme for coupled systems resulting from microsensor thermistor problems. Dyn. Continuous Discrete Impulsive Syst. 5, 209–223.
Barakat, H.Z., Clark, A.J., 1966. On the solution of the diffusion equations by numerical methods. J. Heat Transfer 88, 421–427.
Campbell, L.J., Yin, B., 2007. On the stability of alternating-direction explicit methods for advection-diffusion equations. Numer. Methods Partial Differ. Eqs. 23, 1429–1444.
Cannon, J.R., 1984. The One-dimensional Heat Equation. Addison-Wesley, Menlo Park, California.
Cannon, J.R., Matheson, A.L., 1993. A numerical procedure for diffusion subject to the specification of mass. Int. J. Eng. Sci. 31, 347–355.
Cannon, J.R., Yin, H.M., 1989. A class of nonlinear nonclassical parabolic equations. J. Differ. Eqs. 79, 266–288.
Cannon, J.R., Lin, Y., Wang, S., 1990. An implicit finite difference scheme for the diffusion equation subject to mass specification. Int. J. Eng. Sci. 28, 573–578.
Dagan, D., 1994. The significance of heterogeneity of evolving scales to transport in porous formations. Water Resour. Res. 13, 33273–33336.
Huntul, M.J.M., 2018. Determination of Unknown Coefficients in the Heat Equation. University of Leeds (Ph.D. thesis).
Huntul, M.J., Lesnic, D., 2017. An inverse problem of finding the time-dependent thermal conductivity from boundary data. Int. Commun. Heat Mass Transfer 301, 147–154.
Hussein, M.S., Lesnic, D., Ismailov, M.I., 2016. An inverse problem of finding the time-dependent diffusion coefficient from an integral condition. Math. Methods Appl. Sci. 396, 546–554.
Ivanchov, M.I., Sahaidak, R.V., 2004. Inverse problem of determination of the leading coefficient of a two-dimensional parabolic equation. Matematychni Metody ta Fiziko-Mekhanichni Polya 47, 7–16.
Kinash, N. Ye., 2018. A nonlocal inverse problem for the two-dimensional heat-conduction equation. J. Math. Sci. 231, 558–571.
Ozisik, M.N., 1994. Finite Difference Methods in Heat Transfer. CRC Press, Boca Raton, FL.
Renardy, M., Hrusa, W.J., Nohel, J.A., 1987. Mathematical problems in viscoelasticity. Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 35. Longman Scientific & Technical, Harlow.
Shelukhin, V.V., 1993. A nonlocal in time model for radionuclide propagation in Stokes fluid. Dinamika Sploshn. Sredy 107, 180–193.