The (theta, wheel)-free graphs
Part II: structure theorem

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Abstract

A hole in a graph is a chordless cycle of length at least 4. A theta is a graph formed by three paths between the same pair of distinct vertices so that the union of any two of the paths induces a hole. A wheel is a graph formed by a hole and a node that has at least 3 neighbors in the hole. In this paper we obtain a decomposition theorem for the class of graphs that do not contain an induced subgraph isomorphic to a theta or a wheel, i.e. the class of (theta, wheel)-free graphs. The decomposition theorem uses clique cutsets and 2-joins. Clique cutsets are vertex cutsets that work really well in decomposition based algorithms, but are unfortunately not general enough to decompose more complex hereditary graph classes. A 2-join is an edge cutset that appeared in decomposition theorems of several complex classes, such as perfect graphs, even-hole-free graphs and others. In these decomposition theorems 2-joins are used together with vertex cutsets that are more general than clique cutsets, such as star cutsets and their generalizations (which are much harder to use in algorithms). This is a first example of a decomposition theorem that uses just the combination of

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clique cutsets and 2-joins. This has several consequences. First, we can easily transform our decomposition theorem into a complete structure theorem for (theta, wheel)-free graphs, i.e. we show how every (theta, wheel)-free graph can be built starting from basic graphs that can be explicitly constructed, and gluing them together by prescribed composition operations; and all graphs built this way are (theta, wheel)-free. Such structure theorems are very rare for hereditary graph classes, only a few examples are known. Secondly, we obtain an $\mathcal{O}(n^4m)$-time decomposition based recognition algorithm for (theta, wheel)-free graphs. Finally, in Parts III and IV of this series, we give further applications of our decomposition theorem.

1 Introduction

In this article, all graphs are finite and simple.

A *prism* is a graph made of three node-disjoint chordless paths $P_1 = a_1 \dots b_1$, $P_2 = a_2 \dots b_2$, $P_3 = a_3 \dots b_3$ of length at least 1, such that $a_1a_2a_3$ and $b_1b_2b_3$ are triangles and no edges exist between the paths except those of the two triangles. Such a prism is also referred to as a $3PC(a_1a_2a_3, b_1b_2b_3)$ or a $3PC(\Delta, \Delta)$ (3PC stands for *3-path-configuration*).

A *pyramid* is a graph made of three chordless paths $P_1 = a \dots b_1$, $P_2 = a \dots b_2$, $P_3 = a \dots b_3$ of length at least 1, two of which have length at least 2, node-disjoint except at a, and such that $b_1b_2b_3$ is a triangle and no edges exist between the paths except those of the triangle and the three edges incident to a. Such a pyramid is also referred to as a $3PC(b_1b_2b_3, a)$ or a $3PC(\Delta, \cdot)$.

A *theta* is a graph made of three internally node-disjoint chordless paths $P_1 = a \dots b$, $P_2 = a \dots b$, $P_3 = a \dots b$ of length at least 2 and such that no edges exist between the paths except the three edges incident to a and the three edges incident to b. Such a theta is also referred to as a $3PC(a, b)$ or a $3PC(\cdot, \cdot)$.

A *hole* in a graph is a chordless cycle of length at least 4. A *wheel* $W = (H, c)$ is a graph formed by a hole H (called the *rim*) together with a node c (called the *center*) that has at least three neighbors in the hole.

A *3-path-configuration* is a graph isomorphic to a prism, a pyramid or a theta. Observe that the lengths of the paths in the definitions of 3-path-configurations are designed so that the union of any two of the paths induce a hole. A *Truemper configuration* is a graph isomorphic to a prism, a pyramid, a theta or a wheel (see Figure 1). Observe that every Truemper configuration contains a hole.

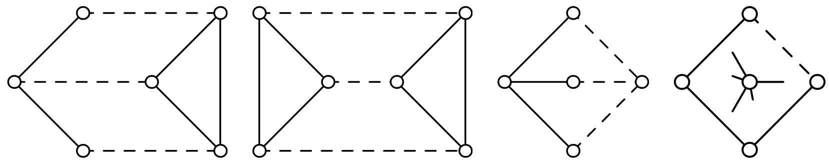

Figure 1: Pyramid, prism, theta and wheel (dashed lines represent paths)

If G and H are graphs, we say that G *contains* H when H is isomorphic to an induced subgraph of G. We say that G is *H-free* if it does not contain H. We extend this to classes of graphs with the obvious meaning (for instance, a graph is (theta, wheel)-free if it does not contain a theta and does not contain a wheel).

In this paper we prove a decomposition theorem for (theta, wheel)-free graphs, from which we obtain a full structure theorem and a polynomial time recognition algorithm. This is part of a series of papers that systematically study the structure of graphs where some Truemper configurations are excluded. This project is motivated and explained in more details in the first paper of the series [7]. In Parts III and IV of the series (see [11, 12]) we give several applications of the structure theorem.

The main result and the outline of the paper

A graph is *chordless* if all its cycles are chordless. By the following decomposition theorem proved in [7], to prove a decomposition theorem for (theta, wheel)-free graphs, it suffices to focus on graphs that contain a pyramid.

Theorem 1.1 ([7]) *If G is (theta, wheel, pyramid)-free, then G is a line graph of a triangle-free chordless graph or it has a clique cutset.*

In Section 2, we define a generalization of pyramids that we call P-graphs. The full definition is complex, but essentially, a P-graph is a graph that can be vertexwise partitioned into the line graph of a triangle-free chordless graph and a clique. Clearly, if a (theta, wheel)-free graph contains a pyramid, then it contains a P-graph. We consider such a maximal P-graph and prove that the rest of the graph attaches to it in a special way that entails a decomposition.

The decompositions that we use are the clique cutset and the 2-join (to be defined soon). Our main theorem is the following.

Theorem 1.2 *If G is (theta, wheel)-free, then G is a line graph of a triangle-free chordless graph or a P-graph, or G has a clique cutset or a 2-join.*

Clique cutsets are vertex cutsets that work really well in decomposition based algorithms, but are unfortunately not general enough to decompose more complex hereditary graph classes. A 2-join is an edge cutset that appeared in decomposition theorems of several complex classes, such as perfect graphs [3], even-hole-free graphs [6, 13] and others. In these decomposition theorems 2-joins are used together with vertex cutsets that are more general than clique cutsets, such as star cutsets and their generalizations (which are much harder to use in algorithms). This is the first example of a decomposition theorem that uses just the combination of clique cutsets and 2-joins. This has several consequences. First, we can easily transform our decomposition theorem into a complete structure theorem for (theta, wheel)-free graphs, i.e. we show how every (theta, wheel)-free graph can be built starting from basic graphs that can be explicitly constructed, and gluing them together by prescribed composition operations; and all graphs built this way are (theta, wheel)-free. Such structure theorems are very rare for hereditary graph classes, only a few examples are known, such as chordal graphs [8], universally-signable graphs [5], graphs that do not contain a cycle with a unique chord [14], claw-free graphs [4] and bull-free graphs [2] (for a survey see [15]).

The second consequence is the following theorem, and the remaining consequences are given in [11].

Theorem 1.3 *There exists an $O(n^4m)$-time algorithm that decides whether an input graph G is (theta, wheel)-free.*

In Section 2, we give all the definitions needed in the statement of Theorem 1.2. In particular, we define P-graphs and 2-joins. In Section 3, we study skeletons (the skeleton is the root-graph of the line graph part of a P-graph). In Section 4, we study the properties of P-graphs. In Section 5, we study attachments to P-graphs in (theta, wheel)-free graphs. In Section 6, we prove Theorem 1.2. In Section 7, we prove Theorem 1.3 and describe how a structure theorem is derived from our decomposition theorem.

Terminology and notations

A *clique* in a graph is a (possibly empty) set of pairwise adjacent vertices. We say that a clique is *big* if it is of size at least 3. A clique of size 3 is also referred to as a *triangle*, and is denoted by Δ. A *diamond* is a graph obtained from a clique of size 4 by deleting an edge. A *claw* is a graph induced by nodes u, v_1, v_2, v_3 and edges uv_1, uv_2, uv_3.

A *path* P is a sequence of distinct vertices $p_1p_2 \dots p_k$, $k \geq 1$, such that p_ip_{i+1} is an edge for all $1 \leq i < k$. Edges p_ip_{i+1}, for $1 \leq i < k$, are called the *edges of* P. Vertices p_1 and p_k are the *ends* of P. A cycle C is a sequence of vertices $p_1p_2 \dots p_kp_1$, $k \geq 3$, such that $p_1 \dots p_k$ is a path and p_1p_k is an edge. Edges p_ip_{i+1}, for $1 \leq i < k$, and edge p_1p_k are called the *edges of* C. Let Q be a path or a cycle. The vertex set of Q is denoted by $V(Q)$. The *length* of Q is the number of its edges. An edge $e = uv$ is a *chord* of Q if $u, v \in V(Q)$, but uv is not an edge of Q. A path or a cycle Q in a graph G is *chordless* if no edge of G is a chord of Q.

Let A and B be two disjoint node sets such that no node of A is adjacent to a node of B. A path $P = p_1 \dots p_k$ *connects* A *and* B if either $k = 1$ and p_1 has neighbors in both A and B, or $k > 1$ and one of the two endnodes of P is adjacent to at least one node in A and the other endnode is adjacent to at least one node in B. The path P is a *direct connection between* A *and* B if in $G[V(P) \cup A \cup B]$ no path connecting A and B is shorter than P. The direct connection P is said to be *from* A *to* B if p_1 is adjacent to a node of A and p_k is adjacent to a node of B.

Let G be a graph. For $x \in V(G)$, $N(x)$ is the set of all neighbors of x in G, and $N[x] = N(x) \cup \{x\}$. Let H and C be vertex-disjoint induced subgraphs of G. The *attachment of* C *over* H, denoted by $N_H(C)$, is the set of all vertices of H that have at least one neighbor in C. When C consists of a single vertex x, we denote the attachment of C over H by $N_H(x)$, and we say that it is an *attachment of* x *over* H. Note that $N_H(x) = N(x) \cap V(H)$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S.

When clear from the context, we will sometimes write G instead of $V(G)$.

2 Statement of the decomposition theorem

We start by defining the cutsets used in the decomposition theorem. In a graph G, a subset S of nodes and edges is a *cutset* if its removal yields a disconnected graph. A node cutset S is a *clique cutset* if S is a clique. Note that every disconnected graph has a clique cutset: the empty set.

For a graph G and disjoint sets $A, B \subseteq V(G)$, we say that a node cutset S of G *separates* A *and* B if $S \subseteq V(G) \setminus (A \cup B)$ and no vertex of A is in the same connected component of $G \setminus S$ as some vertex of B.

An *almost 2-join* in a graph G is a pair (X_1, X_2) that is a partition of

$V(G)$, and such that:

- For $i = 1, 2$, X_i contains disjoint nonempty sets A_i and B_i, such that every node of A_1 is adjacent to every node of A_2, every node of B_1 is adjacent to every node of B_2, and there are no other adjacencies between X_1 and X_2.
- For $i = 1, 2$, $|X_i| \geq 3$.

An almost 2-join (X_1, X_2) is a *2-join* when for $i \in \{1, 2\}$, X_i contains at least one path from A_i to B_i, and if $|A_i| = |B_i| = 1$ then $G[X_i]$ is not a chordless path.

We say that $(X_1, X_2, A_1, A_2, B_1, B_2)$ is a *split* of this 2-join, and the sets A_1, A_2, B_1, B_2 are the *special sets* of this 2-join.

A *star cutset* in a graph is a node cutset S that contains a node (called a *center*) adjacent to all other nodes of S. Note that a nonempty clique cutset is a star cutset.

Lemma 2.1 ([7]) *If G is a (theta, wheel)-free graph that has a star cutset, then G has a clique cutset.*

We now define the basic graphs. A graph G is *chordless* if no cycle of G has a chord, and it is *sparse* if for every edge $e = uv$, at least one of u or v has degree at most 2. Clearly all sparse graphs are chordless.

An edge of a graph is *pendant* if at least one of its endnodes has degree 1. A *branch vertex* in a graph is a vertex of degree at least 3. A *branch* in a graph G is a path of length at least 1 whose internal vertices are of degree 2 in G and whose endnodes are both branch vertices. A *limb* in a graph G is a path of length at least 1 whose internal vertices are of degree 2 in G and whose one endnode has degree at least 3 and the other one has degree 1. Two distinct branches are *parallel* if they have the same endnodes. Two distinct limbs are *parallel* if they share the same vertex of degree at least 3.

Cut vertices of a graph R that are also branch vertices are called the *attaching vertices* of R. Let x be an attaching vertex of a graph R, and let $C_1, \dots, C_t$ be the connected components of $R \setminus x$ that together with x are not limbs of R (possibly, $t = 0$, when all connected components of $R \setminus x$ together with x are limbs). If x is the end of at least two parallel limbs of R, let C_{t+1} be the subgraph of R formed by all the limbs of R with endnode x. The graphs $R[V(C_i) \cup \{x\}]$ (for $i = 1, \dots, t$, if $t \neq 0$) and the graph C_{t+1} (if it exists) are the *x-petals* of R.

For any integer $k \geq 1$, a *k-skeleton* is a graph R such that (see Figures 2, 3 and 4 for examples of k-skeletons for $k = 1, 2, 5$):

(i) R is connected, triangle-free, chordless and contains at least three pendant edges (in particular, R is not a path).

(ii) R has no parallel branches (but it may contains parallel limbs).

(iii) For every cut vertex u of R, every component of $R \setminus u$ has a vertex of degree 1 in R.

(iv) For every vertex cutset $S = \{a, b\}$ of R and for every component C of $R \setminus S$, either $R[C \cup S]$ is a chordless path from a to b, or C contains at least one vertex of degree 1 in R.

(v) For every edge e of a cycle of R, at least one of the endnodes of e is of degree 2 in R.

(vi) Each pendant edge of R is given one label, that is an integer from $\{1, \ldots, k\}$.

(vii) Each label from $\{1, \ldots, k\}$ is given at least once (as a label), and some label is used at least twice.

(viii) If some pendant edge whose one endnode is of degree at least 3 receives label i, then no other pendant edge receives label i.

(ix) If R has no branches then $k = 1$, and otherwise if two limbs of R are parallel, then their pendant edges receive different labels and at least one of these labels is used more than once.

(x) If $k > 1$ then for every attaching vertex x and for every x-petal H of R, there are at least two distinct labels that are used in H. Moreover, if H' is a union of at least one but not all x-petals, then there is a label i such that both H' and $(R \setminus H') \cup \{x\}$ have pendant edges with label i.

(xi) If $k = 2$, then both labels are used at least twice.

Note that if R is a k-skeleton, then it edgewise partitions into its branches and its limbs. To prove this, let e be an edge of R and $P = u \ldots v$, where $\deg(u) \geq \deg(v)$, the maximal path of R that contains e and whose internal vertices are of degree 2. If $\deg(u) = \deg(v) = 1$, then R is the chordless path induced by $V(P)$, which contradicts (i). If $\deg(v) = 2$, then, by the maximality of P, uv is an edge of R. Now, if $\deg(u) = 2$, then R is the chordless cycle induced by $V(P)$, which contradicts (i); if $\deg(u) \geq 3$, then

u is a cut vertex of R that contradicts (iii). So, $\deg(u), \deg(v) \geq 3$ and P is a branch of R, or $\deg(u) \geq 2$ and $\deg(v) = 1$ in which case $\deg(u) \geq 3$ (by the maximality of P) and P is a limb of R.

Also, there is a trivial one-to-one correspondence between the pendant edges of R and the limbs of R: any pendant edge belongs to a unique limb, and conversely any limb contains a unique pendant edge.

If R is a graph, then the *line graph* of R, denoted by $L(R)$, is the graph whose nodes are the edges of R and such that two nodes of $L(R)$ are adjacent in $L(R)$ if and only if the corresponding edges are adjacent in R.

A *P-graph* is any graph B that can be constructed as follows (see Figures 2, 3 and 4 for examples of P-graphs):

- Pick an integer $k \geq 1$ and a k-skeleton R.
- Build $L(R)$, the line graph of R. The vertices of $L(R)$ that correspond to pendant edges of R are called *pendant vertices* of $L(R)$, and they receive the same label as their corresponding pendant edges in R.
- Build a clique K with vertex set $\{v_1, \dots, v_k\}$, disjoint from $L(R)$.
- B is now constructed from $L(R)$ and K by adding edges between v_i and all pendant vertices of $L(R)$ that have label i, for $i = 1, \dots, k$.

We say that K is the *special clique* of B and R is the *skeleton* of B.

The next lemma, that is proved in Part I, allows us to focus on (theta, wheel, diamond)-free graphs in the remainder of the paper.

Lemma 2.2 ([7]) *If G is a wheel-free graph that contains a diamond, then G has a clique cutset.*

Observe that P-graphs are generalizations of pyramids (this is why we call them P-graphs). Let us explain this. A pyramid is *long* if all of its paths are of length greater than 1. Note that in a wheel-free graph all pyramids are long. Every long pyramid $\Pi = 3PC(x_1x_2x_3, y)$ is a P-graph, where $K = \{y\}$ and R is a tree that is obtained from a claw by subdividing each edge at least once and giving all pendant edges label 1 (see Figure 2). It can be checked that a pyramid whose one path is of length 1 (and that is therefore a wheel) is not a P-graph. This is a consequence of Lemma 4.2 to be proved soon, but let us sketch a direct proof: the apex of the pyramid is the center of a claw, so it must be in the special clique, which therefore has size 1 or 2. It follows that the skeleton must contain two pendant edges with the same label, and one of them contains a vertex of degree 3, a contradiction to condition (viii).

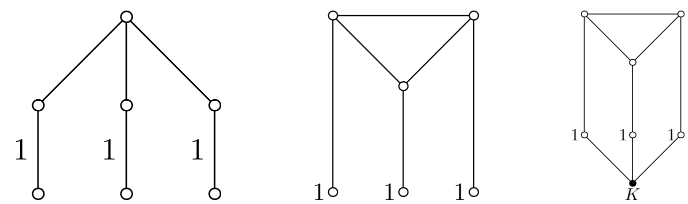

Figure 2: A 1-skeleton, its line graph and the corresponding P-graph.

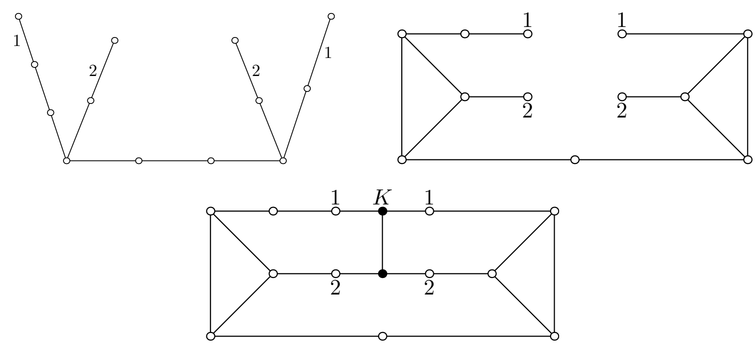

Figure 3: A 2-skeleton, its line graph and the corresponding P-graph.

Lemma 2.3 *A long pyramid is a P-graph.*

In fact, every P-graph contains a long pyramid. Formally we do not need this simple fact, we therefore just sketch the proof: consider three pendant edges of the skeleton for which at most two labels are used (this exists by (i) and (vii)). Consider a minimal connected subgraph T of R that contains these three edges. It is easy to check that T is a tree with three pendant edges and a unique vertex v of degree 3, and that adding to its line graph the vertices of K corresponding to at most two labels yields a long pyramid. To check that the pyramid is long condition (viii) is used, to check that two paths of T linking v to pendant edges with the same label have length at least 2.

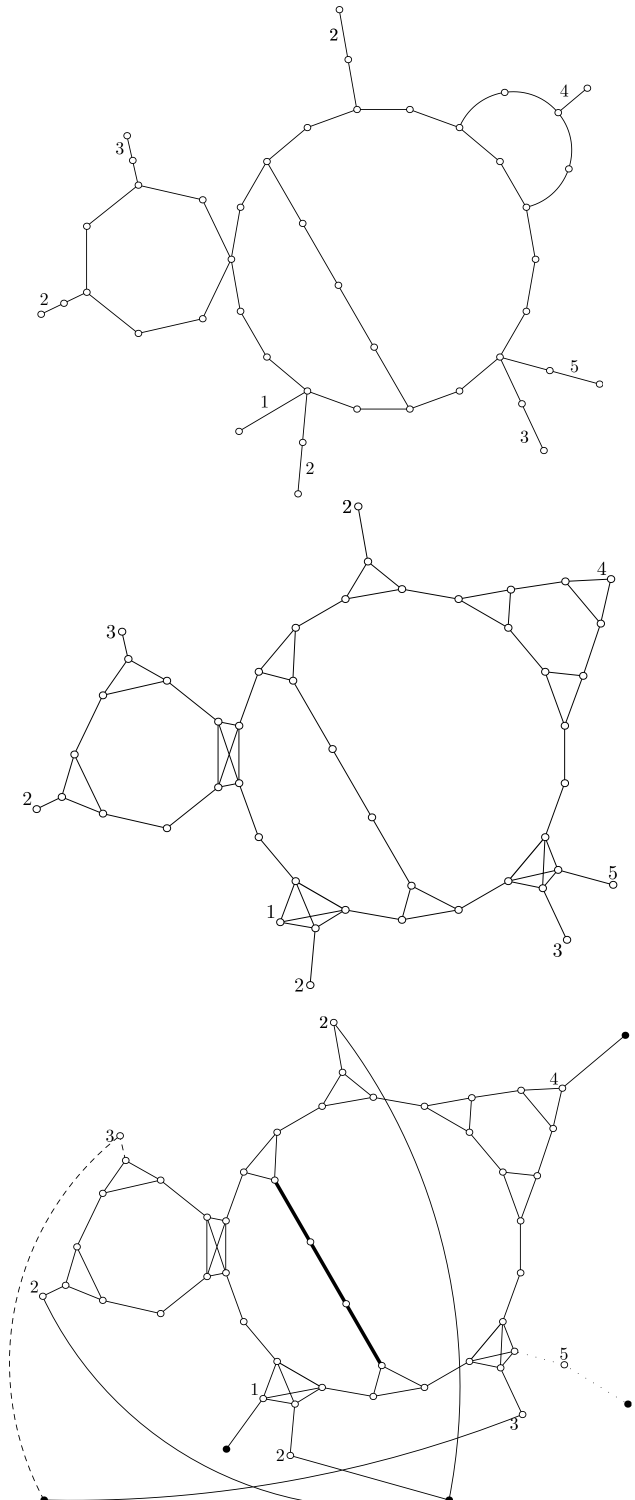

Figure 4: A 5-skeleton, its line graph and the corresponding P-graph (black vertices are the vertices of the special clique of this P-graph; edges between them are not drawn). An internal (resp. claw, clique) segment of this P-graph is represented by a bold (resp. dashed, dotted) line.

3 Connectivity of skeletons

In the following theorem we state versions of Menger's theorem that we use in this paper.

Theorem 3.1 *Let G be a graph.*

(i) Let u and v be non-adjacent vertices of G. Then the maximum number of internally vertex-disjoint paths from u to v is equal to the minimum size of a cutset S of G that separates $\{u\}$ and $\{v\}$.

(ii) Let A and B be disjoint subsets of $V(G)$. Then the maximum number of vertex-disjoint paths with one endnode in A and the other in B is equal to the minimum size of a cutset S of G that separates A and B.

(iii) Let $u \in V(G)$ and $B \subseteq V(G) \setminus \{u\}$. Then the maximum number of paths from u to B that are vertex-disjoint except at u is equal to the minimum size of a cutset S of G that separates $\{u\}$ and B.

Additionally, we will often use the following variant of Menger's theorem, which is due to Perfect [10].

Let G be a graph, $x \in V(G)$ and $Y \subseteq V(G) \setminus \{x\}$. A set of k paths $P_1, P_2, \ldots, P_k$ of G is a *k-fan from x to Y* if $V(P_i) \cap V(P_j) = \{x\}$, for $1 \leq i < j \leq k$, and $|V(P_i) \cap Y| = 1$, for $1 \leq i \leq k$. A *fan from x to Y* is a $|Y|$-fan from x to Y.

Lemma 3.2 ([1, 10]) *Let G be a graph, $x \in V(G)$ and $Y, Z \subseteq V(G) \setminus \{x\}$ such that $|Y| < |Z|$. If there are fans from x to Y and from x to Z, then there is a fan from x to $Y \cup \{z\}$, for some $z \in Z \setminus Y$.*

For distinct vertices $v_1, v_2, \ldots, v_k$ of G, and pairwise disjoint and non-empty subsets $W_1, W_2, \ldots, W_k$ of $V(G) \setminus \{v_1, v_2, \ldots, v_k\}$, we say that k vertex-disjoint paths $P_1, P_2, \ldots, P_k$ are *from* $\{v_1, v_2, \ldots, v_k\}$ *to* $\{W_1, W_2, \ldots, W_k\}$ if for some permutation $\sigma \in \mathbb{S}_k$, $P_i \cap \{v_1, v_2, \ldots, v_k\} = \{v_i\}$ and $P_i \cap (W_1 \cup W_2 \cup \ldots \cup W_k)$ is a vertex of $W_{\sigma(i)}$, for $1 \leq i \leq k$.

Lemma 3.3 *Let G be a connected graph, $v_1, v_2, \ldots, v_k$ distinct vertices of G and $W_1, W_2, \ldots, W_k$ pairwise disjoint and non-empty subsets of $V(G) \setminus \{v_1, v_2, \ldots, v_k\}$, such that all vertices of W_1 are of degree 1. The following holds:*

(1) if $k = 2$, and all vertices of W_2 are of degree 1 or $W_2 = \{w_2\}$, then there exist 2 vertex-disjoint paths from $\{v_1, v_2\}$ to $\{W_1, W_2\}$, or a vertex u that separates $\{v_1, v_2\}$ from $W_1 \cup W_2$;

(2) if $k = 3$, $W_2 = \{w_2\}$, $W_3 = \{w_3\}$ and there exist 2 vertex-disjoint paths from $\{v_2, v_3\}$ to $\{w_2, w_3\}$, then there exist 3 vertex-disjoint paths from $\{v_1, v_2, v_3\}$ to $\{W_1, \{w_2\}, \{w_3\}\}$, or there exist vertices u_1 and u_2 such that $\{u_1, u_2\}$ separates $\{v_1, v_2, v_3\}$ from $W_1 \cup \{w_2, w_3\}$.

PROOF — Let G' be the graph obtained from G by adding a vertex v ($v \notin V(G)$) and edges vv_i, for $1 \leq i \leq k$.

(1) By Menger's theorem, there is a vertex u that separates $\{v_1, v_2\}$ from $W_1 \cup W_2$, or two vertex-disjoint paths from $\{v_1, v_2\}$ to $W_1 \cup W_2$. If the first outcome holds, then we are done, so we may assume that there are vertex-disjoint paths P_1 and P_2 from $\{v_1, v_2\}$ to $W_1 \cup W_2$. If both W_1 and W_2 contain an endnode of P_1 and P_2, then we are again done. So we assume that both P_1 and P_2 have an endnode in w.l.o.g. W_1, and let these endnodes be v_1' and v_2'. This means that in G' there is a fan from v to $\{v_1', v_2'\}$. Since G' is connected, there is a fan from v to some $v'' \in W_2$, and therefore, by Lemma 3.2, there is a fan from v to $\{v', v''\}$, for some $v' \in \{v_1', v_2'\}$. This completes the proof of (1).

(2) By Menger's theorem, there are vertices u_1 and u_2 that separate $\{v_1, v_2, v_3\}$ from $W_1 \cup \{w_2, w_3\}$, or three vertex-disjoint paths such that each of them has one endnode in $\{v_1, v_2, v_3\}$ and the other in $W_1 \cup \{w_2, w_3\}$. If the first outcome holds, then we are done, so we may assume that there are vertex-disjoint paths P_1, P_2, P_3 such that each of them has one endnode in $\{v_1, v_2, v_3\}$ and the other in $W_1 \cup \{w_2, w_3\}$. Let the endnodes of paths P_1, P_2, P_3 that are in $W_1 \cup \{w_2, w_3\}$ be v_1', v_2', v_3'. This means that in G' there is a fan from v to $\{v_1', v_2', v_3'\}$. By the conditions of the lemma, there is also a fan from v to $\{w_2, w_3\}$, and therefore, by Lemma 3.2, there is a fan from v to $\{w, w_2, w_3\}$, for some $w \in \{v_1', v_2', v_3'\} \setminus \{w_2, w_3\}$. Since $\{v_1', v_2', v_3'\} \setminus \{w_2, w_3\}$ is a subset of W_1, this completes our proof. $\square$

Recall a standard notion: a *block* of a graph is an induced subgraph that is connected, has no cut vertices and is maximal with respect to these properties. Recall that every block of a graph is either 2-connected, or is a single edge. Recall that cut vertices of a graph R that are of degree at least 3 are called the *attaching vertices* of R.

Lemma 3.4 *Let R be a k-skeleton. If C is a 2-connected block of R, then no two vertices of C that are of degree at least 3 in R are adjacent. In particular, every 2-connected block of R is sparse, no two adjacent vertices of every cycle of R have degree at least 3, and if an edge of R is between two vertices of degree at least 3, then it is a cutedge of R.*

PROOF — This is equivalent to condition (v) in the definition of a P-graph, since an edge of R belongs to a cycle if and only if it belongs to a 2-connected block of R. □

Lemma 3.5 *Let R be a k-skeleton. If e_1 and e_2 are edges of R, then there exists a cycle of R that goes through e_1 and e_2, or there exists a path in R whose endnodes are of degree 1 (in R) and that goes through e_1 and e_2.*

PROOF — We set $e_1 = u_1v_1$ and $e_2 = u_2v_2$. We apply Menger's theorem to $\{u_1, v_1\}$ and $\{u_2, v_2\}$ (or their one-element subsets if these sets are not disjoint). If the outcome is a pair of vertex-disjoint paths, then we obtain the cycle whose existence is claimed. We may therefore assume that the outcome is a cut vertex x that separates e_1 from e_2. Hence, R is vertex-wise partitioned into X_1, $\{x\}$ and X_2, in such a way that $\{u_1, v_1\} \subseteq X_1 \cup \{x\}$ and $\{u_2, v_2\} \subseteq X_2 \cup \{x\}$ and there are no edges between X_1 and X_2. We now show that $R[X_1 \cup \{x\}]$ contains a path from a vertex of degree 1 in R to x that contains e_1. Since R is connected this is clearly true if an endnode of e_1 has degree 1 in R. So we may assume that both endnodes of e_1 are of degree greater than 1 in R. Let Y_1 be the set of all vertices in X_1 that have degree 1 in R. Note that $Y_1 \neq \emptyset$ by (iii) of the definition of the skeleton. Suppose $u_1 = x$. By (iii) of the definition of the skeleton, there exists a path in $R[X_1]$ from a vertex of degree 1 to v_1, and this path can be extended to a desired path by adding the edge v_1u_1. Therefore, by symmetry, we may assume that $x \notin \{u_1, v_1\}$. In $R[X_1 \cup \{x\}]$, we apply Lemma 3.3 to $\{u_1, v_1\}$ and $\{Y_1, x\}$. If we obtain a cut vertex y that separates $\{u_1, v_1\}$ from $\{x\} \cup Y_1$, then y is cut vertex of R (separating e_1 from $Y_1 \cup X_2$) and the component of $R \setminus y$ that contains e_1 contradicts (iii). Hence, we obtain two vertex-disjoint paths, whose union yields a path P_1 that contains e_1 from a vertex of degree 1 (in R) to x. A similar path P_2 exists in $R[X_2 \cup \{x\}]$. The union of P_1 and P_2 yields the path whose existence is claimed. □

Lemma 3.6 *Let R be a k-skeleton. Every 2-connected induced subgraph D of R has at least 3 distinct vertices that have neighbors outside D. In particular, every 2-connected block of R has at least 3 attaching vertices.*

PROOF — Let D be a 2-connected induced subgraph of R. Let u_1 be a degree 1 vertex of R (it exists by (i)). Since R is connected, there is a path $P_1 = u_1 \dots v_1$, where v_1 is the unique vertex of P_1 in D. In particular, v_1 is a vertex of D with a neighbor outside of D.

If v_1 is not a cut vertex of R that separates $P_1 \setminus v_1$ from $D \setminus v_1$, then there is a path $P_2 = u_1 \dots v_2$, where v_2 is the unique vertex of P_2 in D. Otherwise, by (iii), the component C of $R \setminus v_1$ that contains $D \setminus v_1$ has a vertex u_2 of degree 1 in R, and a path $P_2 = u_2 \dots v_2$, where v_2 is the unique vertex of P_2 from D. So in both cases we get a vertex v_2 distinct from v_1 such that both v_1 and v_2 have neighbors outside D. Since D is 2-connected, v_1 and v_2 are contained in a cycle of D, so by (v), v_1v_2 is not an edge of R.

Suppose that $\{v_1, v_2\}$ is not a cutset of R that separates $(P_1 \cup P_2) \setminus \{v_1, v_2\}$ from a vertex of D. Then there is a path $P_3 = u_3 \dots v_3$ in $R \setminus \{v_1, v_2\}$, where u_3 is a vertex of $(P_1 \cup P_2) \setminus \{v_1, v_2\}$ and v_3 is the unique vertex of D in P_3, and hence v_1, v_2, v_3 are the desired three vertices.

So we may assume that $\{v_1, v_2\}$ is a cutset of R that separates $(P_1 \cup P_2) \setminus \{v_1, v_2\}$ from a vertex of D. By (ii) there is a component C' of $R \setminus \{v_1, v_2\}$ such that $C' \cap D \neq \emptyset$ and $R[C' \cup \{v_1, v_2\}]$ is not a chordless path. By (iv), C' contains a vertex u_3 of degree 1 in R, and a path $P_3 = u_3 \dots v_3$, where v_3 is the unique vertex of P_3 in D. Hence v_1, v_2, v_3 are the desired three vertices.

Finally, observe that if D is a block then each of v_1, v_2, v_3 is a cut vertex of R, and hence D has at least three attaching vertices. □

Lemma 3.7 *Let R be a k-skeleton. Let x_1 and x_2 be branch vertices of R (not necessarily distinct). Then, there are two paths $P_1 = x_1 \dots y_1$ and $P_2 = x_2 \dots y_2$, vertex-disjoint (except at x_1 if $x_1 = x_2$) such that y_1 and y_2 both have degree 1 and are incident with edges with the same label.*

PROOF — First suppose that there exists a label i that is used at least twice in R, and such that there does not exist a vertex x and two sets $X, Y \subset V(R)$ such that $X, Y, \{x\}$ form a partition of $V(R)$, $x_1, x_2 \in X \cup \{x\}$, all degree 1 vertices from edges with label i are in Y, and there are no edges between X and Y. Then, by Menger's theorem there exist two vertex-disjoint paths (except at x_1 if $x_1 = x_2$) between $\{x_1, x_2\}$ and the set of all degree 1 vertices from edges with label i.

So, suppose that in R, for every label i that is used at least twice in R, there exists a vertex x and two sets $X, Y \subseteq V(R)$ such that $X, Y, \{x\}$ form a partition of $V(R)$, $x_1, x_2 \in X \cup \{x\}$, all degree 1 vertices from edges with label i are in Y, and there are no edges between X and Y. We then choose i, x, X and Y subject to the minimality of X. We claim that x is an attaching vertex of R. If $x \in \{x_1, x_2\}$, it is true by assumption. Otherwise, if x has a unique neighbor x' in X, then x' is a cut vertex that contradicts the minimality of X (it separates $X \setminus \{x'\}$ from $Y \cup \{x\}$). Hence, x has at least two neighbors in X, and at least one in Y, so it is indeed an attaching vertex.

Suppose that $X \cup \{x\}$ contains a limb of R ending at x. This limb cannot have x_1 or x_2 as its internal vertex, so we can move it to Y which contradicts the minimality of X. It follows that $X \cup \{x\}$ is an x-petal, or is the union of two x-petals X_1 (that contains x_1) and X_2 (that contains x_2). In this last case, by (x), there exists a label j that is used in both X_1 and $R \setminus X_1$. So, there exists a path from x_1 to an edge with label j in $X_1 \setminus \{x\}$ and a path in $R \setminus X_1$ from x_2 to an edge with label j, and the conclusion follows. When $X \cup \{x\}$ is an x-petal, we note that there exists another x-petal included in $Y \cup \{x\}$, because $Y \cup \{x\}$ cannot be a single limb since a label is used twice in Y. Hence, by (x), there exists a label j that is used in both X and Y. Let Z be the set of degree 1 vertices from X which are the degree 1 ends of edges with label j.

First suppose that $x = x_1$. Since $R[Y \cup \{x\}]$ is connected, it contains a path from x to a vertex incident to an edge labeled j. If $x_2 = x$ then similarly $R[X \cup \{x\}]$ contains a path from x to a vertex in Z, and the result holds. So we may assume that $x_2 \in X$. By connectivity of X there exists a path in $R[X]$ from x_2 to a vertex of Z, and the result holds. Therefore, by symmetry, we may assume that $x \notin \{x_1, x_2\}$. Now suppose that $x_1 = x_2$. If there are two paths from x_1 to $Z \cup \{x\}$, then the result holds (by possibly extending one of the paths from x, through Y, to a vertex incident to an edge labeled j). Otherwise, by Menger's theorem there is a cut vertex that contradicts the minimality of X. Therefore we may assume that $x_1 \neq x_2$.

We now apply Lemma 3.3 to $\{x_1, x_2\}$ and $\{Z, x\}$. If the conclusion is two disjoint paths, we are done (by extending the path ending in x to an edge with label j in Y). And if the outcome is a cut vertex x' that separates $\{x_1, x_2\}$ from $\{x\} \cup Z$, then we define X' as the union of the components of $R \setminus \{x'\}$ that contain x_1 and x_2. This contradicts the minimality of X. $\square$

Lemma 3.8 *Let R be a k-skeleton. Let $P = x_1 \dots x_2$ be a branch of R and x'_1 a neighbor of x_1 not in P. Then there are three paths $P_1 = x_1 \dots y_1$, $P'_1 = x_1 x'_1 \dots y'_1$ and $P_2 = x_2 \dots y_2$, vertex-disjoint except P_1 and P'_1 sharing x_1, and such that y_1, y'_1 and y_2 are degree 1 vertices incident with edges with at most two different labels.*

PROOF — By Lemma 3.7, there are vertices y_1 and y_2 of degree 1 incident with edges with the same label, such that there exist vertex-disjoint paths from $\{x_1, x_2\}$ to $\{y_1, y_2\}$. We define X as the set of all vertices of degree 1 in R, except y_1 and y_2. Note that $X \neq \emptyset$ by (i). We apply Lemma 3.3

to $\{x'_1, x_1, x_2\}$ and $\{X, y_1, y_2\}$. If the output is three vertex-disjoint paths, then the conclusion of the lemma holds (x_1 needs to be added to the path that starts at x'_1). Otherwise, there exists a cutset $\{a, b\}$ that separates $\{x'_1, x_1, x_2\}$ from $\{y_1, y_2\} \cup X$. This contradicts (iv). $\square$

4 Properties of P-graphs

For a P-graph B with special clique K and skeleton R, we use the following additional terminology. The cliques of $L(R)$ of size at least 3 are called the *big cliques* of $L(R)$. Note that they correspond to sets of edges in R that are incident to a vertex of degree at least 3. We denote by $\mathcal{K}$ the set that consists of K and all big cliques of $L(R)$. Remove from B the edges of cliques in $\mathcal{K}$. What remains are vertex-disjoint paths, except possibly those that meet at a vertex of K. These paths are *segments* of B; moreover, a segment is an *internal segment* if its endnodes belong to big cliques of $L(R)$, and otherwise it is a *leaf segment*. If S is a leaf segment and $u \in K$ is an endnode of S, we say that S is a *claw segment* if S is not the only segment with endnode u; otherwise we say that S is a *clique segment*. Observe that it is possible that a segment is of length 0, but then it must be an internal segment. Two segments $S_1 = s_1 \dots t_1$ and $S_2 = s_2 \dots t_2$ are *parallel* if s_1, t_1, s_2, t_2 are all distinct nodes and for some $K_1 \in \mathcal{K} \setminus \{K\}$, $s_1, s_2 \in K_1$ and $t_1, t_2 \in K$. Note also that every two cliques of B meet in at most one vertex (since R is triangle-free).

Lemma 4.1 *Let B be a graph that satisfies all the conditions of being a P-graph except that its skeleton fails to satisfies (v) or (viii). Then B contains a wheel.*

PROOF — Let R be the skeleton of B, and K its special clique.
Case 1: when R fails to satisfy (v). Suppose that in R there exists an edge $e = xy$ contained in a cycle C such that x and y are both of degree at least 3. If in $R \setminus e$, there are two internally vertex-disjoint paths from x to y, then R contains a cycle with a chord (namely e). So in $L(R)$, e is a vertex that is the center of a wheel. Hence, by Menger's theorem, we may assume that in $R \setminus e$, there is a cut vertex u that separates x from y (note that u is on C). Let X (resp. Y) be the connected component of $R \setminus \{e, u\}$ that contains x (resp. y). We claim that in $R[X \cup \{u\}]$ there exists a path $P_x = x'' \dots x \dots u$ such that x'' has degree 1 in R.

If x is a cut vertex of R, P_x can be constructed as the union of a path from x to u going through the component of $R \setminus x$ that contains u, and a path from a vertex x'' of degree 1 (that exists by (iii)) to x going through another component. So, we assume that x is not a cut vertex of R. Hence, from here on, we assume that $R \setminus x$ is connected.

We observe that $\{x, u\}$ is a cutset of R, which separates y from each neighbor of x distinct from y. We define u' as the vertex of $C \setminus e$ closest to x along $C \setminus e$ such that $\{x, u'\}$ is a cutset of R that separates y from each neighbor of x distinct from y. Let x' be a neighbor of x not in C (this exists since x has degree at least 3 by assumption). Since x is not a cut vertex of R, $u' \neq x$. Let X' be the connected component of $R \setminus \{x, u'\}$ that contains x'. Suppose $xu' \in E(R)$. Since $R \setminus x$ is connected there is a path P from x' to u' in $X' \cup \{u'\}$. Together with C this provides a cycle with a chord (namely xu'), which yields a wheel in B. So, $xu' \notin E(R)$. Let X_c be the connected component of $R \setminus \{x, u'\}$ that contains the vertices from $C \setminus e$ that are between x and u' (possibly, $X' = X_c$). Note that the vertices of $C \setminus e$ that are between u' and u are in the same connected component of $R \setminus \{x, u'\}$ as y, so none of them is in $X' \cup X_c$. In $R[X' \cup X_c \cup \{x, u'\}]$ there are two internally vertex-disjoint paths Q_1 and Q_2 from x to u', for otherwise, by Menger's theorem, a vertex u'' from C separates them, and $\{x, u''\}$ is a cutset that contradicts u' being closest to x ($\{x, u''\}$ also separates y from each neighbor of x distinct from y). Note that $X' \cup \{u', x\}$ or $X_c \cup \{u', x\}$ is not a chordless path, since otherwise they induce parallel branches contradicting (ii). Therefore, by (iv) one of them contains a vertex x'' of degree 1. So, there exists a cycle C' (made of Q_1 and Q_2) in $R[X' \cup X_c \cup \{x, u'\}]$, and a minimal path in $R[X' \cup X_c]$ from x'' to a vertex in C'. This proves that a path visiting in order x'', x and u' exists. We build P_x by extending this path to u along $C \setminus e$.

We can build a similar path P_y. In B, the paths P_x and P_y can be completed to a wheel via K (e is the center of this wheel).

Case 2: when R fails to satisfy (viii). Suppose for a contradiction that some edge xx' of R has label 1, where x has degree at least 3 and x' degree 1. Suppose moreover that another edge of R, say yy' where y' has degree 1, also receives label 1. Let Z be set of all degree 1 vertices of R, except x' and y'. We claim that in R, there exist two vertex-disjoint paths $P_y = x'' \dots y'$ and $P_z = x''' \dots z$, where $z \in Z$ and x'', x''' are some neighbors of x different from x'. For otherwise, by Lemma 3.3, there exists a cut vertex u in R that separates $\{x'', x'''\}$ from $Z \cup \{y'\}$. Then $\{u, x'\}$ is a cutset of R such that the connected component C of $R \setminus \{u, x'\}$ that contains x fails to satisfy (iv). Additionally, we may assume that P_y (resp. P_z) does not contain x, since otherwise instead of P_y (resp. P_z) we can take the subpath of P_y (resp. P_z) from a neighbor of x (on this path) to y' (resp. z). Now, in B, the two paths P_y and P_z together with x and vertices from K yield a hole, that is the rim of a wheel centered at the vertex xx' of $L(R)$. $\square$

Lemma 4.2 *Every P-graph is (theta, wheel, diamond)-free.*

PROOF — Let B be a P-graph with skeleton R and special clique K. By construction of B, none of the vertices of $L(R)$ can be centres of claws in B. So all centres of claws of B are contained in K and are therefore pairwise adjacent. It follows that B is theta-free. Since R is triangle-free and pendant vertices of $L(R)$ have unique neighbors in K, and by (viii), B is diamond-free.

Suppose that B contains a wheel (H, x). If $x \in K$ then some neighbor x_1 of x in H does not belong to K, and hence is a pendant vertex of $L(R)$. It follows that the neighborhood of x_1 in $L(R)$ is a clique and that x_1 has a unique neighbor in K. But this contradicts the assumption that x_1 belongs to the hole H of $B \setminus x$. Therefore, $x \notin K$.

Since x is a vertex of $L(R)$, it cannot be a center of a claw in B. Since B is diamond-free, x has neighbors x_1, x_2, x_3 in H, where x_2x_3 is an edge and x_1x_2 and x_1x_3 are not. Let x_1' and x_1'' be the neighbors of x_1 in H. Note that x has no neighbor in $H \setminus \{x_1, x_1', x_1'', x_2, x_3\}$ and it is adjacent to at most one vertex of $\{x_1', x_1''\}$.

Suppose $x_1 \in K$. Then w.l.o.g. $x_1' \notin K$. But then x_1' and x are pendant vertices of $L(R)$ that have the same labels. Since $\{x, x_2, x_3\}$ induce a triangle in $L(R)$, x corresponds to a pendant edge of R whose one endnode is of degree at least 3, contradicting (viii). Therefore $x_1 \notin K$, and hence it cannot be a center of a claw. Without loss of generality it follows that the neighbors of x in H are x_1', x_1, x_2, x_3 and none of them is in K. In particular, x is not a pendant vertex of $L(R)$.

Let e_x be the edge of R that corresponds to vertex x of $L(R)$. Note that the endnodes of e_x are of degree at least 3 in R. So by (v), e_x cannot be contained in a 2-connected block of R. It follows that x is a cut vertex of $L(R)$. Let C_1 and C_2 be connected components of $L(R) \setminus x$. Then w.l.o.g. $x_1', x_1 \in C_1$ and $x_2, x_3 \in C_2$, and every path in $B \setminus x$ from $\{x_1', x_1\}$ to $\{x_2, x_3\}$ must go through K. It follows that H must have a chord, a contradiction. $\square$

Lemma 4.3 *If B is a P-graph with special clique $K = \{v_1, \dots, v_k\}$ and v a vertex of an internal segment of B, then there exists a hole H in B that* *contains v, some vertex $v_i \in K$ and two neighbors of v_i in $B \setminus K$.*

PROOF — We view v as an edge of the skeleton R of B. The edge v belongs to a branch of R with ends x_1 and x_2. Let $P_1 = x_1 \dots y_1$ and $P_2 = x_2 \dots y_2$ be the two paths whose existence is proved in Lemma 3.7 applied to x_1 and x_2. Let i be the label of edges incident to y_1 and y_2. The hole whose existence is claimed is induced by v_i and the line graph of the union of P_1, P_2, and the branch of R from x_1 to x_2. □

Lemma 4.4 *Let B be a P-graph with special clique $K = \{v_1, \dots, v_k\}$. Let $K_1, K_2, K_3 \in \mathcal{K} \setminus \{K\}$ be three distinct big cliques. Then there exist three paths $P_1 = v \dots u_1$, $P_2 = v \dots u_2$ and $P_3 = v \dots u_3$, vertex-disjoint except at v, with no edges between them (except at v), such that $v \in K$ and for $i \in \{1, 2, 3\}$, $K_i \cap P_i = \{u_i\}$.*

PROOF — Each of the cliques K_1, K_2 and K_3 is a set of edges from R that share a common vertex. This defines three branch vertices x_1, x_2 and x_3 in R. By Lemma 3.7 there are vertex-disjoint paths from $\{x_1, x_2\}$ to $\{y_1, y_2\}$, where y_1 and y_2 are two vertices of R incident with edges that have the same label say 1. We denote by X the set of all the vertices of degree 1 from R different from y_1 and y_2 (X is not empty by (i)). We now apply Lemma 3.3 to $\{x_1, x_2, x_3\}$ and $\{X, y_1, y_2\}$. If three vertex-disjoint paths exist (up to a permutation, say $Q_1 = x_1 \dots y_1$, $Q_2 = x_2 \dots y_2$ and $Q_3 = x_3 \dots y_3$, where $y_3 \in X$ and w.l.o.g. y_3 has label 1 or 2), then we are done. Indeed, in $L(R)$, this yields three chordless paths with no edges between them, ending at three vertices with labels 1, 1, 1 or 1, 1, 2. By adding v_1 or v_1, v_2, we obtain the three paths whose existence is claimed.

We may therefore assume that the outcome of Lemma 3.3 is a set C of at most two vertices that separates $\{x_1, x_2, x_3\}$ and $X \cup \{y_1, y_2\}$. This contradicts (iii) or (iv). □

Lemma 4.5 *Let B be a P-graph with special clique $K = \{v_1, \dots, v_k\}$. Let S be a leaf segment of B, whose ends are in K and in $K_2 \in \mathcal{K} \setminus \{K\}$. Let $K_1 \neq K_2$ be a clique in $\mathcal{K} \setminus \{K\}$. Then there exist three paths $P_1 = v \dots u_1$, $P_2 = v \dots u_2$ and $P_S = v \dots u_S$, vertex-disjoint except at v, with no edges between them (except at v and for one edge in K_2), such that $v \in K$, u_S is the endnode of S in K_2, and for $i \in \{1, 2\}$, $K_i \cap P_i = \{u_i\}$. Moreover, $P_S = S$ or $P_S \setminus v = S$.*

PROOF — In skeleton R of B, the segment S corresponds to limb with a pendant edge e_S. Each of the cliques K_1 and K_2 is a set of edges from R that share a common vertex. This defines two vertices x_1 and x_2 in R.

We suppose first that e_S has a label that is used only once in the skeleton R. We apply Lemma 3.7 to x_1 and x_2. This yields paths P_1 and P_2 that have pendant edges with the same label, say 1. Then S, line graphs of P_1 and P_2 and vertex v_1, give the desired three paths.

We now suppose that the label of e_s, say 1, is used for another pendant edge with a vertex y of degree 1. We denote by X the set of all degree 1 vertices of R, except y and the end of e_s. We apply Lemma 3.3 to $\{x_1, x_2\}$ and $\{X, y\}$. If two paths are obtained, note that they do not intersect S (because S is a limb), so by adding S to corresponding paths in B, we obtain the paths that we need. Otherwise, we obtain a cut vertex, that together with any vertex of S yields a cutset of size 2 that contradicts (iv). □

Lemma 4.6 *Let B be a P-graph with special clique $K = \{v_1, \dots, v_k\}$ such that $k \geq 2$. Let S_1 and S_2 be leaf segments of B that have a common endnode v_i in K, and let their other endnodes be in K_1 and K_2, respectively ($K_1 \neq K_2$). Then there exist paths $P_1 = u' \dots u_1$ and $P_2 = u'' \dots u_2$, vertex-disjoint except maybe at a vertex of K (when $u' = u''$) and with no edges between them (except for one edge of K if $u' \neq u''$, or for edges incident to u' when $u' = u''$), such that for $i \in \{1, 2\}$, $K_i \cap P_i = \{u_i\}$, $u', u'' \in K \setminus \{v_i\}$ and $v_i \notin P_1 \cup P_2$.*

PROOF — In skeleton R of B, the segments S_1 and S_2 correspond to limbs with pendant edges e_1 and e_2, respectively. Each of the cliques K_1 and K_2 is a set of edges from R that share a common vertex. This defines two vertices x_1 and x_2 in R.

The label of e_1 and e_2 is i. We denote by X the set of all degree 1 vertices of R that are incident with an edge not labeled with i. We apply Menger's theorem to $\{x_1, x_2\}$ and X (by (vii) and (xi) we have $|X| \geq 2$). If two paths are obtained, then we are done. Otherwise, we obtain a cut vertex x, that separates $\{x_1, x_2\}$ from X. Since x_1 and x_2 are of degree 3 we may assume that x is an attaching vertex, which contradicts (x). □

Lemma 4.7 *Let B be a P-graph with special clique $K = \{v_1, \dots, v_k\}$ such that $k \geq 2$. Let S_1 be a leaf segment with endnode $v_i \in K$, and an endnode in $K_1 \in \mathcal{K} \setminus \{K\}$, and let $K_2 \in \mathcal{K} \setminus \{K, K_1\}$. Then there exist paths $P_1 = u_1 \dots u'$ and $P_2 = u_2 \dots u''$ vertex-disjoint except maybe at a vertex of K*

(when $u' = u''$) and with no edges between them (except for one edge of K if $u' \neq u''$, and for edges incident to u' when $u' = u''$), such that $u', u'' \in K \setminus \{v_i\}$, $v_i \notin P_1 \cup P_2$, $P_1 \cap K_1 = \{u_1\}$ and $P_2 \cap K_2 = \{u_2\}$.

PROOF — In skeleton R of B, the segment S_1 corresponds to a limb with pendant edge e_1. Each of the cliques K_1 and K_2 is a set of edges from R that share a common vertex. This defines two vertices x_1 and x_2 in R.

The label of e_1 is i. We denote by X the set of all degree 1 vertices of R that are incident with an edge not labeled with i. We apply Menger's theorem to $\{x_1, x_2\}$ and X (by (vii) and (xi) we have $|X| \geq 2$). If two paths are obtained, then we are done. Otherwise, we obtain a cut vertex x, that separates $\{x_1, x_2\}$ from X. Since x_1 and x_2 are of degree 3 we may assume that x is an attaching vertex, which contradicts (x). □

Lemma 4.8 *Let B be a P-graph with special clique $K = \{v_1\}$. If S is a leaf segment of B and S' an internal segment of B, with an endnode in $K' \in \mathcal{K}$ such that $S \cap K' = \emptyset$, then there exists a pyramid Π contained in B, such that S and S' are contained in different paths of Π and $|\Pi \cap K'| = 2$.*

PROOF — Let R be the skeleton of B. Let P_S (resp. $P_{S'}$) be the limb (resp. branch) of R that corresponds to S (resp. S'). Let x be the degree 1 vertex of P_S, let x_1 be the other endnode of P_S, and let y_1 and y_2 be the endnodes of $P_{S'}$, such that edges incident to y_1 correspond to nodes of K'. Then $x_1 \neq y_1$. Furthermore, let X be the set of all degree 1 vertices of R different from x.

If in R there exists a vertex z that separates $\{y_1, y_2\}$ from X, then for any internal vertex z' of P_S (it exists by (vii) and (viii)), the set $\{z, z'\}$ is a cutset of R that contradicts (iv). So, by Menger's theorem there are vertex-disjoint paths $P' = y_1 \ldots x'$ and $P'' = y_2 \ldots x''$, where $x', x'' \in X$. Suppose that in $R \setminus y_1$ there exists a path from x to $(P' \cup P'') \setminus \{y_1\}$, and let P''' be chosen such that it has the minimum length. Then $L(P' \cup P'' \cup P''' \cup P_{S'}) \cup \{v_1\}$ induces the desired pyramid.

So, we may assume that y_1 is a cut vertex of R, such that x and $(P' \cup P'') \setminus \{y_1\}$ are contained in different connected components of $R \setminus y_1$. Let C_x be the connected component of $R \setminus \{y_1\}$ that contains x, let e_x be the edge incident to x and let e_y be an edge of $P_{S'}$. By Lemma 3.5 there exists a path P in R that contains edges e_x and e_y whose endnodes are of degree 1 in R. Note that P contains $P_{S'}$. Let x'_1 be a node adjacent to x_1 that does not belong to P. Since $x_1 \neq y_1$, we have $\{x_1, x'_1\} \subseteq C_x$. Let us apply Lemma 3.3 in graph C_x to $\{x_1, x'_1\}$ and $\{X_1, x\}$, where X_1 is the set of all degree

1 (in R) nodes of C_x different from x (X_1 is non-empty, since otherwise for any internal vertex z' of P_S the set $\{z', y_1\}$ is a cutset of R that contradicts (iv)). If vertex-disjoint paths $P_1 = P_S$ and P'_1 are obtained, then $L(P \cup P'_1)$ and v_1 induce a desired pyramid Π. Otherwise, let z be a vertex of C_x that separates $\{x_1, x'_1\}$ from $X_1 \cup \{x\}$. But then $\{z, y_1\}$ is a cutset of R that contradicts (iv). □

Lemma 4.9 *Let B be a P-graph with special clique $K = \{v_1\}$. If S_1 and S_2 are leaf segments of B, then there exists a pyramid Π contained in B, such that S_1 and S_2 are contained in different paths of Π.*

PROOF — Let x_1 (resp. x_2) be degree 1 vertex of skeleton R of B incident to pendant edge that corresponds to a vertex of S_1 (resp. S_2). Furthermore, let X be the set of all degree 1 vertices of R different from x_1 and x_2. Note that by (i), $X \neq \emptyset$. Let P' be a direct connection from $\{x_1, x_2\}$ to X in R, and w.l.o.g. let x_1 be the neighbor of one endnode of P'. Let P'' be a direct connection from x_2 to P'. Then $L(P' \cup P'') \cup \{v_1\}$ induces the desired pyramid. □

Lemma 4.10 *Let B be a P-graph with special clique $K = \{v_1, \dots, v_k\}$. Let v be the vertex of an internal segment of length 0, let $K_1 \in \mathcal{K} \setminus \{K\}$ be such that $v \in K_1$ and let $u \in K_1 \setminus \{v\}$. Then B contains a pyramid $\Pi = 3PC(uvx, y)$ such that $x \in K_1$ and $y \in K$.*

PROOF — Let R be the skeleton of B, and let $e = x_1x_2$ be an edge of R that corresponds to vertex v. Let x'_1 be the neighbor of x_1 in R such that $x_1x'_1$ corresponds to vertex u. Let $P_1 = x_1 \dots y_1$, $P'_1 = x_1x'_1 \dots y'_1$ and $P_2 = x_2 \dots y_2$ be the three paths obtained by applying Lemma 3.8 to x_1, x'_1 and x_2. Then y_1, y'_1 and y_2 are vertices of degree 1 in R incident with edges with at most two different labels, say i and j. It follows that $L(\{x_1, x_2\} \cup P_1 \cup P'_1 \cup P_2)$ and $\{v_i, v_j\}$ induce the desired pyramid in B. □

5 Attachments to a P-graph

Lemma 5.1 ([7]) *Let G be a (theta, wheel)-free graph. If H is a hole of G and v a node of $G \setminus H$, then the attachment of v over H is a clique of size at most 2.*

Lemma 5.2 *In a P-graph B every pair of segments is contained in a hole. Also, every pair of vertices of B is contained in a hole.*

PROOF — Follows directly from Lemma 3.5 (note that every vertex of B is contained in a segment of B, and every segment contains a vertex that corresponds to an edge of skeleton R of B). $\square$

Lemma 5.3 *Let G be a (theta, wheel, diamond)-free graph and B a P-graph contained in G. If $v \in G \setminus B$, then either $|N_B(v)| \leq 1$ or $N_B(v)$ is a maximal clique of B.*

PROOF — Since G is diamond-free, it suffices to show that $N_B(v)$ is a clique. Assume not and let v_1 and v_2 be non adjacent neighbors of v in B. By Lemma 5.2, v_1 and v_2 are contained in a hole H of B. But then H and v contradict Lemma 5.1. $\square$

Let G be a (theta, wheel, diamond)-free graph and $\Pi = 3PC(x_1x_2x_3, y)$ be a pyramid contained in G. Then Π is a long pyramid and by Lemma 2.3 it is a P-graph with special clique $\{y\}$. For $i = 1, 2, 3$, we denote by S_i the branch of Π from y to x_i and we denote by y_i the neighbor of y on this path. By Lemma 5.3 it follows that the attachment of a node $v \in G \setminus \Pi$ over Π is a clique of size at most 3. For $i = 1, 2, 3$, we shall say that v is of *Type i w.r.t.* Π if $|N_\Pi(v)| = i$. We now define several kinds of paths that interact with Π.

- A *crossing* of Π is a chordless path $P = p_1 \dots p_k$ in $G \setminus \Pi$ of length at least 1, such that p_1 and p_k are of Type 1 or 2 w.r.t. Π, for some $i, j \in \{1, 2, 3\}$, $i \neq j$, $N_\Pi(p_1) \subseteq S_i$, $N_\Pi(p_k) \subseteq S_j$, p_1 has a neighbor in $S_i \setminus \{y\}$, p_k has a neighbor in $S_j \setminus \{y\}$, at least one of p_1, p_k has a neighbor in $(S_i \cup S_j) \setminus \{x_i, x_j\}$ and no node of $P \setminus \{p_1, p_k\}$ has a neighbor in Π.

- Let $P = p_1 \dots p_k$ be a crossing of Π such that for some $i, j \in \{1, 2, 3\}$, $i \neq j$, $N_\Pi(p_1) = \{y_i\}$ or $\{y_i, y\}$, p_k is of Type 2 w.r.t. Π and $N_\Pi(p_k) \subseteq S_j \setminus \{y, y_j\}$. Moreover, if $N_\Pi(p_1) = \{y_i\}$ then S_i has length at least 3. Then we say that P is a *crosspath* of Π (from y_i to S_j). We also say that P is a *y_i-crosspath* of Π.

- If $P = p_1 \dots p_k$ is a crossing of Π such that p_1 and p_k are of Type 2 w.r.t. Π and neither is adjacent to $\{y, y_1, y_2, y_3, x_1, x_2, x_3\}$, then P is a *loose crossing* of Π.

A long pyramid with a loose crossing is a P-graph. To see this, consider a 1-skeleton made of a chordless cycle C together with three chordless paths P_1, P_2, P_3, all of length at least 2, such that for $i \in \{1,2,3\}$, $P_i \cap C = \{v_i\}$, and v_1, v_2, v_3 are pairwise distinct and nonadjacent. The three pendant edges of the paths receive label 1, and the special clique has size 1.

A long pyramid with a crosspath is also a P-graph. The special clique K is $\{y_i, y\}$ (when $N_\Pi(p_1) = \{y_i\}$) or $\{y_i, y, p_1\}$ (when $N_\Pi(p_1) = \{y_i, y\}$), so it has size 2 or 3. It is easy to check that removing K yields the line graph of a tree that has two vertices of degree 3 and four pendant edges that receive labels 1, 1, 2, 2 when $|K| = 2$ and 1, 1, 2, 3, when $|K| = 3$.

Lemma 5.4 *Let G be a (theta, wheel, diamond)-free graph. If $P = p_1 \dots p_k$ is a crossing of a $\Pi = 3PC(x_1x_2x_3, y)$ contained in G, then P is a crosspath or a loose crossing of Π.*

PROOF — Assume w.l.o.g. that p_1 has a neighbor in $S_1 \setminus \{y\}$, and p_k in $S_2 \setminus \{y\}$. Not both p_1 and p_k can be adjacent to y, since otherwise $N_\Pi(p_1) = \{y_1, y\}$ and $N_\Pi(p_k) = \{y_2, y\}$, and hence $S_1 \cup S_2 \cup P$ induces a wheel with center y. Suppose that both p_1 and p_k are of Type 2 w.r.t. Π. If p_1 is adjacent to y, then P is a crosspath, since otherwise p_k is adjacent to y_2 and not to y, and hence $S_2 \cup S_3 \cup P$ induces a wheel with center y_2. So we may assume that neither p_1 nor p_k is adjacent to y. If p_1 is adjacent to y_1, then $G[(\Pi \setminus \{x_2\}) \cup P]$ contains a wheel with center y_1. So p_1 is not adjacent to y_1, and by symmetry p_k is not adjacent to y_2. If p_1 is adjacent to x_1, then $G[(\Pi \setminus \{y_2\}) \cup P]$ contains a wheel with center x_1. So p_1 is not adjacent to x_1, and by symmetry p_k is not adjacent to x_2. It follows that P is a loose crossing.

Without loss of generality we may now assume that p_1 is of Type 1 w.r.t. Π. If p_k is also of Type 1, then $S_1 \cup S_2 \cup P$ induces a theta. So p_k is of Type 2. If p_1 is not adjacent to y_1, then $G[(\Pi \setminus \{x_2\}) \cup P]$ contains a $3PC(y, \widetilde{y})$, where $\widetilde{y}$ is the only neighbor of p_1 on Π. So p_1 is adjacent to y_1. Since $S_1 \cup S_2 \cup P$ cannot induce a wheel with center y, p_k is not adjacent to y. Since $S_2 \cup S_3 \cup P \cup \{y_1\}$ cannot induce a wheel with center y_2, $N_\Pi(p_k) \subseteq S_2 \setminus \{y, y_2\}$. If S_1 is of length 2, then $G[(\Pi \setminus \{y_2\}) \cup P]$ contains a wheel with center x_1. Therefore S_1 is of length at least 3, and hence P is a crosspath. □

Lemma 5.5 *Let G be a (theta, wheel, diamond)-free graph. If G contains a pyramid Π with a crossing P, then $G[\Pi \cup P]$ is a P-graph.*

PROOF — Follows from Lemma 5.4 and the fact already mentioned that a pyramid together with a loose crossing or a crosspath is a P-graph. □

Let S be a segment of a P-graph B such that its endnodes are in K_1 and K_2. Then we say that $S \cup K_1 \cup K_2$ is an *extended segment* of B.

Lemma 5.6 *Let B be a P-graph with special clique K which is contained in a (theta,wheel,diamond)-free graph G. Let $P = u \dots v$ be a path in $G \setminus B$ whose interior nodes have no neighbors in B and one of the following holds:*

(1) $N_B(u)$ and $N_B(v)$ are cliques of size at least 2 in $B \setminus K$ which are not contained in the same extended segment of B.

(2) $N_B(u) = K$, where $|K| \geq 2$, and $N_B(v)$ is a clique of size at least 2 which is in $B \setminus K$, but not in an extended clique segment of B.

(3) $N_B(u) = \{w\} \subseteq K$, and $N_B(v)$ is a clique of size at least 2 in $B \setminus K$ which is not in a extended segment of B incident with w.

Then $G[B \cup P]$ is a P-graph contained in G.

PROOF — Let $K = \{v_1, \dots, v_k\}$ and let R be the skeleton of B. In all three cases neighbors of v in B are in fact in $L(R)$, and they correspond to some edges of R all incident to a single vertex $k_2 \in R$. By Lemma 5.3, v is adjacent to all vertices that correspond to edges incident to k_2. We now consider each of the cases.

(1) Let k_1 be the vertex of R whose incident edges correspond to vertices of the clique $N_B(u)$ in $L(R)$. Note that by Lemma 5.3, u is adjacent to all vertices that correspond to edges incident to k_1. Construct graph R' from R by adding a branch P_R between k_1 and k_2, of length one more than the length of P. We prove that R' is a k-skeleton.

By Lemma 4.1, it suffices to check that all conditions other than (v) and (viii) are met. Since P is of length at least 1, P_R is of length at least 2, and thus (i) holds. Since $N_B(u)$ and $N_B(v)$ are not contained in the same extended segment of B, no branch of R contains both k_1 and k_2, and hence (ii) holds.

Note that R and R' have the same degree 1 vertices and the same limbs. It follows that (vi), (vii), (ix) and (xi) hold for R'.

Let x be a cut vertex of R'. Since R is connected, x is not an internal vertex of P_R. Hence, x is also a cut vertex of R and every component of $R' \setminus x$ contains a union of components of $R \setminus x$. It follows that (iii) holds.

Also, every x-petal of R' is a union of some x-petals of R and some vertices of P_R, and therefore (x) holds.

To prove (iv) let $\{a, b\}$ be a cutset of R'. If a and b are in the interior of P_R, one component of $R' \setminus \{a, b\}$ is a chordless path from a to b, and the other contains all the vertices of R' of degree 1, so (iv) holds. If one of a or b, say a, is in the interior of P_R, and the other (so, b) is not, then b is a cut vertex of R. Also, every component of $R' \setminus \{a, b\}$ contains a component of $R \setminus b$. Hence (iv) holds because (iii) holds for R. Finally, if none of a and b is in the interior of P_R, then $\{a, b\}$ is also a cutset of R, and every components of $R' \setminus \{a, b\}$ contains a component of $R \setminus \{a, b\}$. Therefore, (iv) holds for R' because it holds for R. Thus (iv) holds, and our claim is proven.

(2) Construct graph R' from R by adding a chordless path P_R of the same length as P, whose one endnode is k_2 and the remaining nodes are new. Note that pendant edges of R are also pendant edges of R', and R' has one new pendant edge (the one incident to the vertex of degree 1 in R' that is in P_R). Let us assign label $k + 1$ to the new pendant edge. We claim that R' is a skeleton. By Lemma 4.1, there is no need to check (v) and (viii). Since P_R is a limb, (i), (ii), (vi), (vii) and (xi) hold for R' because they hold for R and since in this case $k \geq 2$.

Let us show that (ix) holds. It could be that the limb that we add to R to build R' is in fact parallel to a limb Q of R, that corresponds to a clique segment S' of B. If the label of pendant edge of Q is used only once, then $N_B(v)$ is contained in an extended clique segment of B (namely extended segment of S'), a contradiction. So (ix) holds.

The conditions (iii), (iv) and (x) hold for R' because they hold for R. Indeed, in R', we added a limb, this only possibly adds a vertex of degree 1 to a component, making the condition easier to satisfy.

(3) Let $w = v_i$. We build a path P_R of the same length as P and we consider the graph R' obtained from R by attaching P_R at k_2. Hence, in P_R there is a pendant edge, and we give it label i. We claim that R' is a skeleton. By Lemma 4.1, there is no need to check (v) and (viii). Since P_R is a limb, (i), (ii), (vi), (vii) and (xi) hold for R' because they hold for R.

Condition (ix) also holds, since the limb that we add to build R' has pendant edge with label i that is now used at least twice, and it is not parallel to some other limb with pendant edge i by the condition of the lemma.

The conditions (iii), (iv) and (x) hold for R' because they hold for R. Indeed, in R' we added a limb, which only possibly adds a vertex of degree 1 to a component, making the condition easier to satisfy. □

Lemma 5.7 *Let G be a (theta, wheel, diamond)-free graph, and let B be the P-graph contained in G with special clique $K = \{v_1, \dots, v_k\}$ and skeleton R, such that k is maximum, and among all P-graphs contained in G and with special clique of size k, B has the maximum number of segments. Let $P = u \dots v$ be a chordless path in $G \setminus B$ such that u and v both have neighbors in B and no interior node of P has a neighbor in B. Then one of the following holds:*

(1) $N_B(P) \subseteq K'$, where $K' \in \mathcal{K}$.

(2) There exists a segment S of B, of length at least 1, whose endnodes are in $K_1 \cup K_2$ where $K_1, K_2 \in \mathcal{K}$, such that $N_B(P) \subseteq K_1 \cup K_2 \cup S$. Moreover, if u (resp. v) has a neighbor in $K_i \setminus S$, for some $i \in \{1, 2\}$, then u (resp. v) is complete to K_i.

PROOF — Before proving the theorem, note that in the proof, conclusion (2) can be replaced by a weaker conclusion :

(2') There exists a segment S of B, of length at least 1, whose endnodes are in $K_1 \cup K_2$ where $K_1, K_2 \in \mathcal{K}$, such that $N_B(P) \subseteq K_1 \cup K_2 \cup S$.

Indeed, if (2') is satisfied, then (1) or (2) is satisfied. Let us prove this. Suppose that (2') holds, but neither (1) nor (2) does. Up to symmetry, and by Lemma 5.3, this means that $N_B(u)$ is a single vertex u' of $K_1 \setminus S$. If $N_B(v)$ is also a single vertex v', then by Lemma 5.2, P together with a hole that goes through u' and v' forms a theta (note that since (1) does not hold, $v' \in (S \cup K_2) \setminus K_1$ and hence since R has no parallel branches by (ii), $u'v'$ is not an edge). By Lemma 5.3, we may therefore assume that $N_B(v) = K_2$ or $N_B(v)$ is a clique of size 2 in S.

We first suppose that $K \notin \{K_1, K_2\}$. In R, $N_B(u)$ is an edge $y_1y'_1$, where y'_1 is a branch vertex and S corresponds to a branch $P' = y'_1 \dots y'_2$. We apply Lemma 3.8 to y_1, y'_1 and y'_2. Let P_1, P_2 and P_3 be the three paths obtained and suppose that label i is used on pendant edges of two of these paths. Then the graph induced by $L(P_1 \cup P_2 \cup P_3)$ together with $S \setminus K_1$, P and K contains a $3PC(u', v_i)$ (note that by (viii), $u'v_i$ is not an edge).

Next suppose that $K_1 = K$ and let $u' = v_i$. First observe that if $N_B(v) = K_2$ and there exists a segment S' of B with endnode v_i and an endnode in K_2, then P satisfies (2) w.r.t. S'. So this cannot happen. It follows that if $N_B(v) \cap K = \emptyset$ then by part (3) of Lemma 5.6, the maximality of B is contradicted. So let $v_j \in N_B(v) \cap K$, where $v_j \neq v_i$, and let S' be a segment of B with endnode v_i. Let Q be a direct connection from S' to S in $B \setminus K$. Then $G[S \cup S' \cup P \cup Q]$ is a wheel with center v_j, a contradiction.

Therefore $K_2 = K$. First suppose that u' is a vertex of an internal segment of B. Then by Lemma 4.3, there exists a hole H that contains u' and a vertex $v_j \in K$ such that neighbors of v_j in H are in $B \setminus K$. If S is not contained in H, then $G[H \cup P \cup (S \setminus K_1)]$ contains a $3PC(u', v_j)$ (note that since u' belongs to an internal segment of B, $u'v_j$ is not an edge). So S is contained in H, and hence v_j is an endnode of S. If $N_B(v) = K$ then $G[H \cup P]$ is a theta. So $N_B(v) \neq K$. In R, u' is an edge $y_1 y_1'$, where y_1' is a branch vertex, and S corresponds to a limb $P' = y_1' \dots y_2'$. Let X be the set of all degree 1 vertices of R incident with pendant edges labeled with j not including y_2' (note that X is nonempty) and Y the set of all other degree 1 vertices of R not including y_2'. If in $(R \setminus P') \cup \{y_1'\}$ there are vertex-disjoint paths P_1 and P_2 from $\{y_1', y_1\}$ to $\{X, Y\}$, then $G[L(P_1 \cup P_2) \cup P \cup (S \setminus K_1) \cup \{u'\}]$ contains a $3PC(u', v_j)$. So, by Lemma 3.3, there is a vertex x in R that separates $\{y_1', y_1\}$ from $X \cup Y$ in $(R \setminus P') \cup \{y_1'\}$, and therefore $\{y_2', x\}$ is a cutset of R that contradicts (iv).

It follows that u' is an endnode of a leaf segment S' of B. Since (2) does not hold for P and S', $N_B(v) \neq K$ and hence $N_B(v)$ is a clique of size 2 in S. Let v_i (resp. v_j) be the endnode of S (resp. S') in K. Suppose $v_i = v_j$. Then by (ix), R has no branches, so by (i), $G[(B \setminus (S \cap K_1)) \cup P]$ contains a $3PC(u', v_i)$ (note that $u'v_i$ is not an edge by (viii)). So $v_i \neq v_j$. By (ix) there is a segment $S'' \notin \{S, S'\}$ with an endnode in $\{v_i, v_j\}$. Note that S'' does not have an endnode in K_1. Let Q be a direct connection from S'' to K_1 in $B \setminus K$. Then $G[(S \setminus K_1) \cup S' \cup S'' \cup P \cup Q]$ either contains a $3PC(u', v_i)$ (if S'' has endnode v_i) or $3PC(u', v_j)$ (if S'' has endnode v_j, note that in this case by (viii), $u'v_j$ is not an edge). Therefore, if (2') holds then (1) or (2) holds.

We are now back to the main proof. Suppose the conclusion of the theorem fails to be true. By Lemma 5.3, it suffices to consider the following cases.

Case 1: For some $K_1, K_2 \in \mathcal{K} \setminus \{K\}$, $N_B(u) = K_1$ and $N_B(v) = K_2$.

Since (1) does not hold, $K_1 \neq K_2$. Let us first prove that no segment of B has endnodes in $K_1 \cup K_2$.

Suppose to the contrary that some segment S of B has endnodes in $K_1 \cup K_2$. Since (2') does not hold, S is of length 0, say $S = x$. So S is an internal segment of B. Let e_x be the edge of R that corresponds to x. By

Lemma 3.4, e_x is a cut edge of R, and hence x is a cut vertex of $L(R)$. For $i = 1, 2$, let C_i be the connected component of $L(R) \setminus x$ that contains $K_i \setminus x$. Note that the endnodes of e_x in R are cut vertices of R, and hence by (iii), C_i has a pendant vertex, for $i = 1, 2$. It follows that B contains a chordless wz-path Q, where $w \in K_1 \setminus x$, $z \in K_2 \setminus x$ and no interior node of Q has a neighbor in $K_1 \cup K_2$. But then $P \cup Q \cup \{x\}$ induces a wheel with center x. Therefore, no segment of B has an endnode in $K_1 \cup K_2$.

Now, by part (1) of Lemma 5.6, this contradicts the maximality of B.

Case 2: For some $K_1 \in \mathcal{K} \setminus \{K\}$, $N_B(u) = K_1$ and $N_B(v) = K$.

Since (2') does not hold, there is no (leaf) segment with endnodes in K_1 and K, and so by parts (2) and (3) of Lemma 5.6 and maximality of B, this case is impossible.

Case 3: For some segment S of B, $N_B(u) = K$ and $N_B(v) \subseteq S$.

Since (2') does not hold, S is an internal segment of B. Let v' be a neighbor of v in S. Apply Lemma 4.3 to B and v'. This provides a hole H in B that contains v' and a single node of K. Note that H contains S because S is a segment. If v' is the only neighbor of v in S, then H and P form a theta, a contradiction. So, by Lemma 5.1, for some vertex v'' of S adjacent to v', $N_B(v) = \{v', v''\}$. By parts (2) and (3) of Lemma 5.6 this contradicts the maximality of B.

Case 4: For some $K_1 \in \mathcal{K} \setminus \{K\}$ and some internal segment S of B, $N_B(u) = K_1$ and $N_B(v) \subseteq S$.

Let K_2 and K_3 be the end cliques of S. Since (1) and (2') do not hold, $K_1 \notin \{K_2, K_3\}$. We apply Lemma 4.4 to K_1, K_2 and K_3. This provides three paths P_1, P_2 and P_3. If $N_B(v) = \{v'\}$ then P_1, P_2, P_3, S and P induce a theta. So by Lemma 5.3 $N_B(v) = \{v', v''\}$ where v' and v'' are two adjacent vertices of S. By part (1) of Lemma 5.6 this contradicts the maximality of B.

Case 5: For some $K_1 \in \mathcal{K} \setminus \{K\}$ and some leaf segment S of B, $N_B(u) = K_1$ and $N_B(v) \subseteq S$.

Let the endnodes of S be in cliques K and $K_2 \in \mathcal{K} \setminus \{K\}$. Since S is a leaf segment of B, it is of length at least 1. Since (2') does not hold, $K_1 \neq K_2$. Let v' be a neighbor of v in S, and let $P_1 = w_1 \dots w$,

$P_2 = w_2 \dots w$ and $P_S = w_s \dots w$ be paths obtained when Lemma 4.5 is applied to segment S and clique K_1.

First, let us assume that $N_B(v) = \{v'\}$. If $v' \neq w$ and v' is not adjacent to w, then $G[P_1 \cup P_2 \cup P_S \cup P]$ induces a $3PC(v', w)$, a contradiction. So, $v' = w$ or $v'w$ is an edge. If $v' \in K$, then by part (3) of Lemma 5.6 and maximality of B, there is a segment S' with one endnode in K_1 and the other v'. But then P and S' satisfy condition (2'). So, $v' \notin K$, and hence $v'w$ is an edge. Suppose $k = 1$. Let z be a node of K_1 that belongs to an internal segment of B (note that since $K_1 \neq K_2$, and since R is connected by (i), it follows that R has a branch and z exists by (ix)). By Lemma 4.8 there exists a pyramid Π contained in B such that S and z belong to different paths of Π and $|\Pi \cap K_1| = 2$. So, $N_\Pi(u)$ is an edge of a path of Π that contains z. Note that since G is wheel-free, Π is a long pyramid and by Lemma 5.4 P is a crosspath of Π. But then $G[\Pi \cup P]$ is a P-graph with special clique of size greater than 1, contradicting our choice of B (since $k = 1$). Therefore, $k > 1$. Let Q_1 and Q_2 be paths obtained when Lemma 4.7 is applied to S and K_1. Then $G[Q_1 \cup Q_2 \cup S \cup P]$ induces a theta or a wheel, a contradiction. So, by Lemma 5.3, $N_B(v)$ is a clique of size 2.

If $N_B(v) \cap K = \emptyset$, then, by (1) of Lemma 5.6, we have a contradiction to the maximality of B. So, $N_B(v) = \{v', v''\}$, where $v'' \in K$. If $v'' \neq w$, then $G[P_1 \cup P_2 \cup P_S \cup P]$ induces a wheel, a contradiction. So, $v'' = w$. If $k = 1$, then by Lemma 4.8 there exists a pyramid Π, contained in B, such that S and z are in different paths of Π, where z is a node of K_1 that belongs to an internal segment of B (it exists by the same argument as in the previous paragraph). Note that w is the center of the claw of Π. But then $G[\Pi \cup P]$ is a P-graph whose special clique is of size 3, contradicting our choice of B. So $k > 1$. Let Q_1 and Q_2 be paths obtained when Lemma 4.7 is applied to S and a node $z \in K_1$ that is on an internal segment of B. Then $G[Q_1 \cup Q_2 \cup S \cup P]$ induces a wheel, a contradiction.

Case 6: For some distinct segments S and S' of B, $N_B(u) \subseteq S$ and $N_B(v) \subseteq S'$.

Let K_1 and K_2 (resp. K_3 and K_4) be the end cliques of S (resp. S'). We divide this case in several subcases.

Case 6.1: $K \notin \{K_1, K_2, K_3, K_4\}$.

We may assume that $K_3 \notin \{K_1, K_2\}$. Let P_1, P_2 and P_3 be the 3 paths obtained by applying Lemma 4.4 to K_1, K_2 and K_3. Suppose that $N_B(u)$ is a single vertex u'. Since (2') does not hold, v has a neighbor in $S' \setminus (K_1 \cup K_2)$. But then $G[P_1 \cup P_2 \cup P_3 \cup P \cup S \cup (S' \setminus (K_1 \cup K_2))]$ contains a theta. So, by Lemma 5.3, $N_B(u)$ is a clique of size 2 in S, and similarly $N_B(v)$ is a clique of size 2 in S'. By (1) of Lemma 5.6, this contradicts the maximality of B.

Case 6.2: $K_4 = K$ and $K \notin \{K_1, K_2\}$.

Case 6.2.1: $k = 1$.

By Lemma 4.8, B contains a pyramid $\Pi = 3PC(x_1x_2x_3, v_1)$ such that S and S' are contained in different paths of Π. By part (1) of Lemma 5.6, P cannot be a loose crossing of Π. So by Lemma 5.4, P is a crosspath of Π. But this contradicts our choice of B since $k = 1$.

Case 6.2.2: $k \geq 2$.

For $i \in \{1, 2\}$, let x_i be the endnode of S that is in K_i, and let v_i and $v_{S'}$ be the endnodes of S'. First suppose that $K_2 = K_3$. Let $P_1 = w \dots w_1, P_2 = w \dots w_2$ and $P_{S'} = w \dots v_{S'}$ be the three paths obtained by applying Lemma 4.5 to S' and K_1 (where for $i \in \{1, 2\}$, $P_i \cap K_i = \{w_i\}$). Then $G[P_1 \cup P_2 \cup P_{S'} \cup S]$ is a pyramid $\Pi = 3PC(x_2w_2v_{S'}, w)$, and S and S' belong to different paths of Π. Suppose $v_i = w$ and $N_B(v) = v_i$. If u has a unique neighbor u' in S, then $G[\Pi \cup P]$ contains a $3PC(u', w)$, and otherwise by part (3) of Lemma 5.6 our choice of B is contradicted. So either $v_i \neq w$ or $N_B(v) \neq \{v_i\}$. But then by Lemma 5.4, P is a crosspath or a loose crossing of Π, and therefore by Lemma 5.6 our choice of B is contradicted.

So by symmetry, $K_3 \notin \{K_1, K_2\}$. Let $P_1 = w \dots w_1, P_2 = w \dots w_2$ and $P_3 = w \dots w_3$ be the three paths obtained by applying Lemma 4.4 to K_1, K_2 and K_3 (so $w \in K$ and for $i \in \{1, 2, 3\}$, $P_i \cap K_i = \{w_i\}$). Let Q be a direct connection from K_3 to $P_1 \cup P_2$ in $B \setminus K$ and H a hole in $G[P_1 \cup P_2 \cup P_3 \cup S \cup S' \cup Q]$ that contains S and S'. Suppose $N_B(v) = \{w\}$. If $N_B(u) = \{u'\}$, then $G[H \cup P]$ contains a $3PC(u', w)$. So $N_B(u)$ is a clique of size 2 in S, and hence by Lemma 5.6 our choice of B is contradicted. So $N_B(v) \neq \{w\}$. Now, let us assume that $v_i \neq w$ and that one of the paths P_1 and P_2 contains a vertex from $K \setminus \{w, v_i\}$. Note that then $P_3 \cap S' = \emptyset$. Let Π' be a pyramid contained in $G[P_1 \cup P_2 \cup P_3 \cup S \cup Q]$ (this pyramid contains S and its claw has center w). Then $G[P \cup P_3 \cup Q \cup S']$ contains a crossing of Π' with an endnode in u, and hence u has two neighbors in S (since u is not adjacent to w). If $N_B(v) = \{v'\} \neq \{v_i\}$, then $G[P_1 \cup P_2 \cup P_3 \cup S \cup S' \cup P]$ contains a $3PC(u', w)$, and if $N_B(v) = \{v_i\}$, then our choice of B is contradicted by Lemma 5.6. So, we may assume that $N_B(v) = \{v', v_i\}$, since otherwise our choice of B is contradicted by Lemma 5.6. But then $G[P_1 \cup P_2 \cup S \cup S' \cup P]$ contains a wheel with center v_i, a contradiction.

So $v_i = w$ or $P_1 \cap K, P_2 \cap K \in \{\{w\}, \{w, v_i\}\}$. Then $G[P_1 \cup P_2 \cup P_3 \cup S \cup S' \cup Q]$ contains a pyramid Π (whose claw has center w or v_i), such that S and S' belong to different paths of Π. By our choice of B and Lemma 5.6, P cannot be a loose crossing of Π. So, by Lemma 5.4, P is a crosspath of Π. If the center of the claw of Π is v_i and $v_i \neq w$, then $G[P_1 \cup P_2 \cup P_3 \cup S \cup S' \cup P]$ contains a theta or a wheel, a contradiction. So, the center of the claw of Π is w. Also $w = v_i$, since otherwise our choice of B is contradicted by Lemma 5.6. This implies that S' is a claw segment of B. Let Q_1 and Q_2 be the paths obtained when Lemma 4.7 is applied to K_1 and S' (we assume that $Q_1 \cap K_1 \neq \emptyset$). Furthermore, if Q_1 does not contain S, then we can extend Q_1 such that it contains one neighbor of u and such that we do not introduce edges between this new path and Q_2. But then, $G[Q_1 \cup Q_2 \cup S' \cup P]$ contains a wheel or a theta, a contradiction.

Case 6.3: $K_2 = K_4 = K$ and $K_1 \neq K_3$.

Let v_i (resp. v_j) be the endnode of S (resp. S') in K, and let x_S (resp. $x_{S'}$) be the other endnode of S (resp. S').

Case 6.3.1: $v_i = v_j$.

First, let $k = 1$. By Lemma 4.9 B contains a pyramid $\Pi = 3PC(x_1x_2x_3, v_i)$ such that S and S' are contained in different paths of Π. Since P does not satisfy (1) and does not satisfy (2') w.r.t. S nor w.r.t. S', P is a crossing of Π. By the choice of B and since $k = 1$, P cannot be a crosspath of Π. So by Lemma 5.4, P is a loose crossing of Π. But then by part (1) of Lemma 5.6, our choice of B is contradicted.

So, let $k \geq 2$. Let P_1 and P_2 be paths obtained when Lemma 4.6 is applied to S and S'. Since P does not satisfy (2'), node u (resp. v) has a neighbor in $S \setminus \{v_i\}$ (resp. $S' \setminus \{v_i\}$). If u or v is adjacent to v_i, then $G[S \cup S' \cup P \cup P_1 \cup P_2]$ contains a wheel with center v_i. Therefore, neither u nor v is adjacent to v_i. Suppose that u has the unique neighbor u' in S. If $u'v_i$ is not an edge, then $G[S \cup S' \cup P \cup P_1]$ contains a $3PC(u', v_i)$. If $u'v_i$ is an edge, then $G[S \cup S' \cup P \cup P_1 \cup P_2]$ contains a wheel with center v_i or a theta. So by Lemma 5.3, $N_B(u)$ is a clique of size 2 that belongs to $S \setminus v_i$, and by symmetry $N_B(v)$ is a clique of size 2 that belongs to $S' \setminus v_i$. By (1) of Lemma 5.6, this contradicts the maximality of B.

Case 6.3.2: $v_i \neq v_j$.

In particular, $k \geq 2$. First suppose that S and S' are both clique segments of B. Let $P_1 = w \dots w_1$, $P_2 = w \dots w_2$ and $P_{S'} = w \dots w_{S'}$ be the three paths obtained by applying Lemma 4.5 to S' and K_1. So $w \notin \{v_i, v_j\}$. Since (2') does not hold u (resp. v) has a neighbor in $S \setminus \{v_i\}$ (resp. $S' \setminus \{v_j\}$). Let u' be a neighbor of u in $S \setminus \{v_i\}$. If either $N_B(u) = \{u'\}$ or u is adjacent to v_i, then $G[P_1 \cup P_2 \cup S \cup (S' \setminus \{v_j\})]$ contains a wheel with center v_i or a $3PC(u', w)$. So by Lemma 5.3, $N_B(u)$ is a clique of size 2 in $S \setminus K$, and by symmetry $N_B(v)$ is a clique of size 2 in $S' \setminus K$. But then by (1) of Lemma 5.6 our choice of B is contradicted.

So w.l.o.g. we may assume that S is a claw segment. Let Q be a direct connection from K_1 to K_3 in $B \setminus K$. Let S_1 be a segment of B distinct from S that has endnode v_i. Let Q_1 be a direct connection from S_1 to Q in $B \setminus K$. Then $G[S \cup S' \cup S_1 \cup Q \cup Q_1]$ is a pyramid $\Pi = 3PC(x_1x_2x_3, v_i)$, in which S and S' are contained in different paths of Π. P cannot be a loose crossing of Π, since otherwise by (1) of Lemma 5.6 our choice of B is contradicted. Therefore by Lemma 5.4, P is a v_i'-crosspath of Π, where v_i' is the neighbor of v_i in S (since v has a neighbor in $S' \setminus \{v_j\}$). In particular, uv_i' is an edge and $N_B(u) \subseteq \{v_i', v_i\}$. By (xi) there exists a leaf segment S_2 of B with endnode v_k such that either $v_k \in K \setminus \{v_i, v_j\}$, or $v_k = v_j$ and $S_2 \neq S'$. Let Q_2 be a direct connection from S_2 to Π. Then $G[\Pi \cup S_2 \cup Q_2]$ contains a $3PC(v_i', v_j)$ (if $N_B(u) = \{v_i'\}$ and $j \neq k$) or a wheel with center v_i (otherwise).

Case 6.4: $K_2 = K_4 = K$ and $K_1 = K_3$.

Let v_i (resp. v_j) be the endnode of S (resp. S') in K, and let x_S (resp. $x_{S'}$) be the other endnode of S (resp. S').

Case 6.4.1: $k = 1$.

Then by (ix) R has no branches. By (i) B contains a pyramid $\Pi = 3PC(x_S x_{S'} x, v_1)$ where S and S' are paths of Π. Since P does not satisfy (1) and does not satisfy (2') w.r.t. S nor w.r.t. S', P is a crossing of Π. By the choice of B and since $k = 1$, P cannot be a crosspath of Π. So by Lemma 5.4, P is a loose crossing of Π. But then by part (1) of Lemma 5.6, our choice of B is contradicted.

Case 6.4.2: $k \geq 2$.

Then by (ix), $v_i \neq v_j$ and w.l.o.g. v_i is an endnode of a leaf segment $S_1 \neq S$. Let Q be a direct connection from S_1 to K_1 in $B \setminus K$. Then S, S', S_1 and

Q induce a pyramid $\Pi = 3PC(x_S x_{S'} x, v_i)$ (where x is an endnode of Q) such that S and $S'v_i$ are paths of Π. Since P does not satisfy (1) and it does not satisfy (2') w.r.t. S nor w.r.t. S', P is a crossing of Π. By (1) of Lemma 5.6 and our choice of B, P cannot be a loose crossing of Π. So by Lemma 5.4, P is a crosspath of Π. If P is a v_j-crosspath of Π then by part (3) of Lemma 5.6, our choice of B is contradicted. So for the neighbor v_i' of v_i in S, P is a v_i'-crosspath of Π. In particular, u is adjacent to v_i' and $N_B(u) \subseteq \{v_i, v_i'\}$, and $N_B(v)$ is a clique of size 2 of $S' \setminus \{v_j\}$. By (xi) there exists a leaf segment S_2 with endnode $v_k \in K \setminus \{v_i\}$ such that $S_2 \neq S'$. Let Q_2 be a direct connection from S_2 to $\Pi \setminus (S \cup S' \cup K)$. But then $G[(\Pi \setminus \{x_{S'}\}) \cup S_2 \cup Q_2]$ either contains a $3PC(v_i', v_j)$ (if $k = j$ and uv_i is not an edge) or a wheel with center v_i (otherwise). $\square$

Let $\Pi = 3PC(x_1x_2x_3, y)$ be a pyramid contained in a graph G. A *hat* of Π is a chordless path $P = p_1 \dots p_k$ in $G \setminus \Pi$ such that p_1 and p_k both have a single neighbor in Π and they are adjacent to different nodes of $\{x_1, x_2, x_3\}$, and no interior node of P has a neighbor in Π.

Lemma 5.8 *Let G be a (theta, wheel)-free graph. If G contains a pyramid with a hat, then G has a clique cutset.*

PROOF — Let $P = p_1 \dots p_k$ be a hat of $\Pi = 3PC(x_1x_2x_3, y)$ contained in G, with w.l.o.g. $N_\Pi(p_1) = \{x_1\}$ and $N_\Pi(p_k) = \{x_2\}$. Assume that G does not have a clique cutset. Then by Lemma 2.2, G is diamond-free. Let S be the set comprised of $\{x_1, x_2, x_3\}$ and all nodes $u \in G \setminus \Pi$ such that $N_\Pi(u) = \{x_1, x_2, x_3\}$. Since G is diamond-free, S is a clique. Let $Q = q_1 \dots q_l$ be a direct connection from P to $\Pi \setminus \{x_1, x_2, x_3\}$ in $G \setminus S$. We may assume w.l.o.g. that a hat P and direct connection Q are chosen so that $|V(P) \cup V(Q)|$ is minimized.

By Lemma 5.3, q_l either has a single neighbor in Π or $N_\Pi(q_l)$ are two adjacent nodes of a path of Π. If a node q_i, $i < l$, is adjacent to a node of $\{x_1, x_2, x_3\}$, then by definition of Q, q_i has a single neighbor in Π. If at least two nodes of $\{x_1, x_2, x_3\}$ have a neighbor in $Q \setminus q_l$, then a subpath of $Q \setminus q_l$ is a hat of Π, contradicting the minimality of $P \cup Q$. So at most one node of $\{x_1, x_2, x_3\}$ has a neighbor in $Q \setminus q_l$. Suppose x_i, for some $i \in \{1, 2, 3\}$, has a neighbor in $Q \setminus q_l$, and let q_t be such a neighbor with highest index. Then $N_\Pi(q_l) \subseteq S_i$, since otherwise $q_t \dots q_l$ is a crossing of Π that contradicts Lemma 5.4. If $i = 3$ then a subpath of $(P \setminus p_k) \cup Q$ or $(P \setminus p_1) \cup Q$ is a hat of Π, contradicting the minimality of $P \cup Q$. So w.l.o.g. $i = 1$. But then $G[(\Pi \setminus y_2) \cup P \cup Q]$ contains a wheel with center x_1. Therefore, no node of $\{x_1, x_2, x_3\}$ has a neighbor in $Q \setminus q_l$.

Without loss of generality we may assume that $P \cup \{q_1\}$ contains a chordless path P' from p_1 to q_1 that does not contain p_k. Then $N_\Pi(q_l) \subseteq S_1$, since otherwise the path induced by $P' \cup Q$ is a crossing of Π that contradicts Lemma 5.4. If $N_\Pi(q_l) = \{y\}$ then $P' \cup Q \cup S_1 \cup S_3$ induces a $3PC(x_1, y)$. So q_l has a neighbor in $S_1 \setminus \{x_1, y\}$. If p_1 is the unique neighbor of q_1 in P, then $G[P \cup Q \cup (\Pi \setminus y_2)]$ contains a wheel with center x_1. So $P \cup \{q_1\}$ must contain a chordless path P'' from p_k to q_1 that does not contain p_1. But then the path induced by $P'' \cup Q$ is a crossing of Π that contradicts Lemma 5.4. □

6 Proof of Theorem 1.2

A *strip* is a triple (H, A, A') that satisfies the following:

(i) H is a graph and A and A' are disjoint non-empty cliques of H;

(ii) every vertex of H is contained in a chordless path of H whose one endnode is in A, the other is in A', and no interior node is in $A \cup A'$ (such a path is called an AA'*-rung*).

Let B be a P-graph with special clique K, and let V_0 be the set of all vertices of B that are the unique vertex of some segment of length zero. A *strip system* $\mathcal{S}$ is any graph obtained from B as follows:

- for every segment $S = u \dots v$ of B of length at least 1, let $(H_S, Q_{u,S}, Q_{v,S})$ be a strip, such that $Q_{u,S} \cap S = \{u\}$ and $Q_{v,S} \cap S = \{v\}$;
- $V(\mathcal{S})$ is the union of vertices of H_S, for all segments S of B of length at least 1, and V_0;
- if $S = u \dots v$, $u \in K$, is a claw segment of B, then $Q_{u,S} = \{u\}$;
- for segments S and S' of length at least 1, if $S \cap S' = \emptyset$, then $V(H_S) \cap V(H_{S'}) = \emptyset$;
- a clique $Q_{x,S}$ is complete to a clique $Q_{x',S'}$ whenever x and x' are in the same clique of $\mathcal{K}$;
- a clique $Q_{x,S}$ is complete to x' whenever x and x' are in the same clique of $\mathcal{K}$ and $x' \in V_0$;

- these are the only edges of the strip system.

Furthermore, for a clique $K_1 \in \mathcal{K}$, we denote $Q_{K_1} = \bigcup_{u \in K_1} Q_{u,S} \cup K_1$ (where S is a segment of length at least 1 that contains u).

Note that any P-graph can be seen as a strip system, where every segment of length at least 1 is replaced by a strip equal to the segment. So, strip system can be seen as a way to thicken a P-graph. In the other direction, consider a graph T induced by V_0 and vertices of one rung from every strip of a strip system $\mathcal{S}$. We say that T is a *template* of $\mathcal{S}$. Note that in particular B is a strip system with unique template, namely B.

Lemma 6.1 *Let G be a (theta, wheel)-free graph. Then every template of a strip system of G is a P-graph.*

PROOF — We claim that given a P-graph B and a strip system obtained from B (that is contained in G), replacing one segment $S = u \dots v$ of B by a corresponding rung $S' = u' \dots v'$ yields another P-graph B'. The lemma then follows from this claim by induction on the number of segments. So let us prove the claim.

Let K be the special clique of B and R its skeleton. If u or v, say u, is in K, then let $K' = \{u'\} \cup K \setminus \{u\}$; otherwise let $K' = K$.

By [9] a graph is (claw,diamond)-free if and only if it is the line graph of a triangle-free graph. So, $B \setminus K$ is (claw,diamond)-free, and hence the same holds for $B' \setminus K'$, i.e. $B' \setminus K'$ is the line graph of a triangle-free graph R'. Observe that R' can be obtained from R by changing the length of a single branch or limb. Furthermore, in this way no branch of length 1 is obtained since the two cliques of any strip are disjoint. Therefore, R' satisfies all conditions of the definition of a skeleton, except possibly the ones that are concerned with the lengths of the limbs. So, we only need to check that R' satisfies (viii), which is true by Lemma 4.1. □

We are ready to prove Theorem 1.2.

Proof of Theorem 1.2: Let G be a (theta, wheel)-free graph, and assume that G does not have a clique cutset and that it is not a line graph of triangle-free chordless graph. By Lemma 2.2, G is diamond-free and by Theorem 1.1, G contains a pyramid, and hence a long pyramid (since G is wheel-free). So, by Lemma 2.3, G contains a P-graph. Let B be a P-graph contained in G with maximum size of the special clique K, say $|K| = k$, and such that out of all P-graphs with special clique of size k it has the maximum number of segments. Let $\mathcal{K}$ be the set that includes all big cliques of B and K, and let

R be the skeleton of B. Furthermore, let $\mathcal{S}$ be a maximal (w.r.t. inclusion) strip system obtained from B.

Claim 1. For every $w \in G \setminus \mathcal{S}$ either for some clique $K_1 \in \mathcal{K}$, $N_{\mathcal{S}}(w) = Q_{K_1}$, or for some segment S of B of length at least 1, $N_{\mathcal{S}}(w) \subseteq H_S$.

Proof of Claim 1. Suppose not. Observe that if for some $K_1 \in \mathcal{K}$, w has two distinct neighbors in Q_{K_1}, then since G is diamond-free, w is complete to Q_{K_1}.

First suppose that w is adjacent to a vertex $v \in V_0$. By Lemma 2.1, $X = (N_G(v) \setminus (\{w\} \cup \mathcal{S})) \cup \{v\}$ is not a star cutset of G, so there exists a chordless path $P = w \dots w'$ in $G \setminus (\mathcal{S} \cup X)$ such that w' has a neighbor u' in $\mathcal{S} \setminus \{v\}$ and no interior node of P has a neighbor in $\mathcal{S}$. By definition of a strip and $\mathcal{S}$, there is a template of $\mathcal{S}$ that contains u' and v. By Lemma 6.1 we may assume w.l.o.g. that B contains u' and v. By Lemma 5.7 applied to P and B, and since w' is not adjacent to v, $N_B(P) \subseteq K' \in \mathcal{K}$. In particular, $K' \in \mathcal{K} \setminus \{K\}$, $u', v \in K'$ and $N_B(w') = \{u'\}$. By Lemma 4.10, B contains a pyramid $\Pi = 3PC(u'vx, y)$, with $x \in K'$ and $y \in K$. If $N_\Pi(w) = \{v\}$ then P is a hat of Π, contradicting Lemma 5.8. So there exists $v' \in N_\Pi(w) \setminus \{v\}$. By Lemma 5.3, $N_\Pi(w)$ is a maximal clique of Π. If $N_\Pi(w) \neq \{u', v, x\}$ then $G[\Pi \cup P]$ contains a wheel with center v. So $N_\Pi(w) = \{u', v, x\}$, and hence w is complete to $Q_{K'}$. It follows that w has a neighbor u'' in $\mathcal{S} \setminus Q_{K'}$. Let B' be a template of $\mathcal{S}$ that contains v and u''. By Lemma 6.1, B' is a P-graph. By Lemma 5.3, $N_{B'}(w)$ is a maximal clique of B', and in particular vu'' is an edge. It follows that for some $K'' \in \mathcal{K} \setminus \{K', K\}$, w is complete to $Q_{K''}$. By Lemma 4.3 applied to B and v, there exists a hole H in B that contains v, and hence it contains a vertex of $K' \setminus \{v\}$ and a vertex of $K'' \setminus \{v\}$. But then (H, w) is a wheel, a contradiction.

Therefore, w is not adjacent to a vertex of V_0. It follows that there exist distinct segments S and S' of B, both of length at least 1, such that w has a neighbor u in H_S, a neighbor $v \in H_{S'}$, and there is no clique $K_1 \in \mathcal{K}$ such that u and v are both in Q_{K_1}. Let B' be a template of $\mathcal{S}$ that contains u and v (it exists by definition of a strip and $\mathcal{S}$). But then by Lemma 6.1, B' and w contradict Lemma 5.3. This completes the proof of Claim 1.

Claim 2. Let S be a segment of B of length at least 1 with endnodes $u \in K_1$ and $v \in K_2$, where K_1 and K_2 are distinct cliques of $\mathcal{K}$, $K_1 \neq K$, and let $(H_S, Q_{u,S}, Q_{v,S})$ be the corresponding strip of $\mathcal{S}$. Then $G \setminus \mathcal{S}$ cannot contain a chordless path $P = w_1 \dots w_2$ such that the following hold:

- $N_{\mathcal{S}}(w_1) = Q_{K_1}$,

- $N_{\mathcal{S}}(w_2) = Q_{K_2}$, or $N_{\mathcal{S}}(w_2) \subseteq H_S$ and w_2 has a neighbor in $H_S \setminus Q_{K_1}$, and
- no interior node of P has a neighbor in $\mathcal{S} \setminus Q_{u,S}$.

Proof of Claim 2. Assume such a path exists. Let $H'_S = H_S \cup P$ and $Q'_{u,S} = Q_{u,S} \cup \{w_1\}$. If $N_{\mathcal{S}}(w_2) = Q_{K_2}$ and either $K_2 \neq K$ or $k > 1$, then let $Q'_{v,S} = Q_{v,S} \cup \{w_2\}$, and otherwise let $Q'_{v,S} = Q_{v,S}$. Since w_2 has a neighbor in $H_S \setminus Q_{K_1}$, H'_S contains a rung with endnode w_1 that contains P, so $(H'_S, Q'_{u,S}, Q'_{v,S})$ is a strip. Since, by maximality of $\mathcal{S}$, $\mathcal{S}' = \mathcal{S} \cup P$ cannot be a strip system, it follows that S is a claw segment (so $K_2 = K$) and $N_{\mathcal{S}}(w_2) = K$ and $k > 1$. Since S is a claw segment of B, $Q_{v,S} = \{v\}$, and there exists another leaf segment S' of B with endnode v. Suppose that a node u_1 of $Q_{u,S}$ has a neighbor in interior of P. Let S_1 be a rung of H_S that contains u_1. By Lemma 6.1, $B' = (B \setminus S) \cup S_1$ is a P-graph where S_1 is a claw segment, so by (viii) of the definition of skeleton, u_1v is not an edge. Let H' be a hole of B' that contains S_1 and S'. But then $G[H' \cup (P \setminus w_1)]$ contains a $3PC(u_1, v)$, a contradiction. Therefore, no node of $\mathcal{S}$ has a neighbor in interior of P. But then by (2) of Lemma 5.6, the choice of B is contradicted. This completes the proof of Claim 2.

Claim 3. For a clique $K_1 \in \mathcal{K} \setminus \{K\}$, there cannot exist a vertex w of $G \setminus \mathcal{S}$ such that $N_{\mathcal{S}}(w) = Q_{K_1}$.

Proof of Claim 3. Suppose such a vertex exists. Let K' be a maximal clique of $G \setminus \{w\}$ that contains Q_{K_1}. Note that since G is diamond-free, no node of $G \setminus (K' \cup \{w\})$ is complete to Q_{K_1}. Since K' cannot be a clique cutset of G, there exists a chordless path $P = w \dots w'$ in $G \setminus (\mathcal{S} \cup K')$ such that w' has a neighbor u' in $\mathcal{S} \setminus Q_{K_1}$, no node of $P \setminus \{w\}$ is complete to Q_{K_1}, and no interior node of P has a neighbor in $\mathcal{S} \setminus Q_{K_1}$. By Claim 1 one of the following two cases hold.

Case 1: For some segment S of B of length at least 1, $N_{\mathcal{S}}(w') \subseteq H_S$.

First suppose that S has an endnode $u \in K_1$ and an endnode $v \in K_2$, for $K_2 \in \mathcal{K} \setminus \{K_1\}$. By Claim 2, a node of $Q_{K_1} \setminus Q_{u,S}$ must have a neighbor in $P \setminus w$. Let w'' be a node of $P \setminus \{w\}$ closest to w' that has a neighbor in $Q_{K_1} \setminus Q_{u,S}$. So, since G is diamond-free and $|K_1| \geq 3$, $N_{\mathcal{S}}(w'') = \{u''\}$, where $u'' \in Q_{K_1} \setminus Q_{u,S}$. Let B' be a template of $\mathcal{S}$ that contains u' and u''. By Lemma 6.1 B' is a P-graph, and so B' and the $w''w'$-subpath of P contradict Lemma 5.7.

So S does not have an endnode in K_1. Let w'' be a node of P closest to w' that has a neighbor in Q_{K_1}. Let u'' be a neighbor of w'' in Q_{K_1}, and let B' be a a template of $\mathcal{S}$ that contains u' and u''. By Lemma 6.1 B' is a P-graph, and so B' and the $w''w'$-subpath of P contradict Lemma 5.7.

Case 2: For some clique $K_2 \in \mathcal{K} \setminus \{K_1\}$, $N_{\mathcal{S}}(w') = Q_{K_2}$.

First suppose that there exists a segment S of B of length at least 1 with endnode $u \in K_1$ and an endnode $v \in K_2$. Then by Claim 2, a node of $Q_{K_1} \setminus Q_{u,S}$ has a neighbor in P. Let w'' be the node of P closest to w' that has a neighbor in $Q_{K_1} \setminus Q_{u,S}$. Then $N_{\mathcal{S}}(w'') = \{u''\}$. Let B' be a template of $\mathcal{S}$ that contains S and u''. By Lemma 6.1 B' is a P-graph, and so B' and the $w''w'$-subpath of P contradict Lemma 5.7.

So no segment of B of length at least 1 has an endnode in K_1 and an endnode in K_2. Let w'' be the node of P closest to w' that has a neighbor $u_1 \in Q_{K_1} \setminus Q_{K_2}$. Let B' be a template of $\mathcal{S}$ that contains u_1. Then by Lemma 6.1, B' and the $w''w'$-subpath of P contradict Lemma 5.7.

This completes the proof of Claim 3.

Claim 4. Let S be a clique segment of B with endnode $v \in K$. Then $G \setminus \mathcal{S}$ cannot contain a chordless path $P = w_1 \dots w_2$ such that the following hold:

- w_1 has a neighbor in $H_S \setminus Q_K$,
- $N_{\mathcal{S}}(w_2) = Q_K$, and
- no interior node of P has a neighbor in $\mathcal{S} \setminus Q_{v,S}$.

Proof of Claim 4. Assume such a path exists. By Lemma 6.1, w.l.o.g. we may assume that w_1 has a neighbor in $S \setminus K$. Let u be an endnode of S different from v, and let $K_1 \in \mathcal{K} \setminus \{K\}$ such that $u \in K_1$. By Claim 3, $N_{\mathcal{S}}(w_1) \neq Q_{K_1}$, and so by Claim 1, $N_{\mathcal{S}}(w_1) \subseteq H_S$. Let $H'_S = H_S \cup P$ and $Q'_{v,S} = Q_{v,S} \cup \{w_2\}$. Then $(H'_S, Q'_{v,S}, Q_{u,S})$ is a strip and $\mathcal{S}' = \mathcal{S} \cup P$ is a strip system that contradicts our choice of $\mathcal{S}$. This completes the proof of Claim 4.

Claim 5. For every connected component C of $G \setminus \mathcal{S}$, there exists a segment S of B of length at least 1 such that $N_{\mathcal{S}}(C) \subseteq H_S$.

Proof of Claim 5. Suppose that a connected component C of $G \setminus \mathcal{S}$ does not satisfy the stated property. Since Q_K is not a clique cutset, some node of C has a neighbor in $\mathcal{S} \setminus Q_K$. So by Claims 2 and 3 some node w_1 of C has a neighbor in $H_S \setminus Q_K$ for some segment S of B of length at least 1. So there exists a chordless path $P = w_1 \dots w_2$ in C such that w_2 has a neighbor in $\mathcal{S} \setminus H_S$. We choose P to be a minimal such path.

First suppose that S is an interior segment of B, and let $u \in K_1$ and $v \in K_2$ be endnodes of S, where $K_1, K_2 \in \mathcal{K} \setminus \{K\}$. By Lemma 6.1 w.l.o.g. we may assume that w_1 has a neighbor in S and w_2 has a neighbor in $B \setminus S$. By the choice of P, no interior node of P has a neighbor in B. But then by Lemma 5.7, w_2 is complete to K_1 or K_2, say K_1. By Claim 1 $N_{\mathcal{S}}(w_2) = Q_{K_1}$, contradicting Claim 3. Therefore S is a leaf segment of B.

Let $u \in K_1$ and $v \in K$ be the endnodes of S. By the choice of P, no interior node of P has a neighbor in $\mathcal{S} \setminus Q_{v,S}$. Suppose w_2 has a neighbor in $\mathcal{S} \setminus Q_K$. Then by Lemma 6.1, w.l.o.g. we may assume that w_1 has a neighbor in $S \setminus \{v\}$ and w_2 has a neighbor in $B \setminus (K \cup S)$. By Lemma 5.7 and Claims 1 and 3, an interior node of P is adjacent to v. Let w_1' be the interior node of P closest to w_2 that is adjacent to v. By Lemma 5.7 applied to $w_1'w_2$-subpath of P, for some leaf segment S' of B with endnode v, w_2 has a neighbor in $S' \setminus v$. Let K_2 be the clique in $\mathcal{K} \setminus \{K\}$ that contains a node of S'. Recall that interior nodes of P do not have neighbors in $B \setminus v$. Also by Claims 1 and 3 and by Lemma 5.3, $N_B(w_1)$ (resp. $N_B(w_2)$) is either a single node or an edge of S (resp. S'). Suppose $k \geq 2$. Then by (ix) of the definition of skeleton, $K_1 \neq K_2$. Let P_1 and P_2 be the paths obtained by applying Lemma 4.6 to S and S'. Then $G[S \cup S' \cup P_1 \cup P_2 \cup P]$ contains a theta or a wheel with center v. So $k = 1$.

By Lemma 4.9 let Π be a pyramid contained in B such that S and S' are contained in different paths of Π. If w_1 is adjacent to v then $G[\Pi \cup P]$ contains a wheel with center v. So w_1 is not adjacent to v and by symmetry neither is w_2. If both w_1 and w_2 have unique neighbors in Π, then $G[\Pi \cup P]$ contains a wheel with center v or a theta. So w.l.o.g. $N_B(w_1) = \{w_1', w_1''\}$ where $w_1'w_1''$ is an edge of S. Then $G[\Pi \cup P]$ contains a pyramid $\Pi' = 3PC(w_1w_1'w_1'', v)$. But then, by Lemma 5.4, $P \setminus \Pi'$ is a crosspath of Π' contradicting our choice of B (since $k = 1$). Therefore, $N_{\mathcal{S}}(w_2) \subseteq Q_K$.

Since w_2 has a neighbor outside S, $k > 1$. Let v_2 be a neighbor of w_2 in $K \setminus \{v\}$. Let w be a node of $B \setminus K$ adjacent to v_2, and let Q be a direct connection in $B \setminus K$ from w to K_1. By Lemma 6.1 w.l.o.g. w_1 has a neighbor in $S \setminus K$. Note that by Claims 1 and 3, $N_B(w_1) \subseteq S$. By Lemma 5.3, $N_B(w_1)$ is a clique of size 1 or 2 in S. If v has a neighbor in P, then $G[S \cup P \cup Q]$ contains a theta or a wheel. So v has no neighbor in P. Then by Lemma 5.7, w_2 is complete to K, and hence by Claim 1, $N_{\mathcal{S}}(w_2) = Q_K$. By Claim 4, S is a claw segment of B. So there is a node w' of $B \setminus (K \cup S)$ adjacent to v. Let Q' be a direct connection in $B \setminus K$ from w' to K_1, and let w_1' be the neighbor of w_1 in $S \setminus K$ which is the closest to K_1. If w_1' is adjacent to v, then $G[S \cup P \cup Q]$ is a wheel with center v. So, w_1' is not adjacent to v, and hence by Lemma 5.6, w_1' is the unique neighbor of w_1 in S. But then $G[S \cup P \cup Q']$ is a theta. This completes the proof of Claim 5.

Suppose $G \neq B$. Then by Claim 5, there exists a segment S of B of length at least 1 such that either $H_S \neq S$ or a node of $G \setminus \mathcal{S}$ has a neighbor in H_S. Let $\mathcal{C}$ be the union of all connected components C of $G \setminus \mathcal{S}$ that have a node with a neighbor in H_S. By Claim 5, $N_{\mathcal{S}}(\mathcal{C}) \subseteq H_S$. If S is not a claw segment of B, then $(H_S \cup \mathcal{C}, G \setminus (H_S \cup \mathcal{C}))$ is a 2-join of G. So we may assume that S is a claw segment of B with an endnode $u \in K$. Then, by Claim 5, $((H_S \setminus \{u\}) \cup \mathcal{C}, (G \setminus (H_S) \cup \mathcal{C}) \cup \{u\})$ is a 2-join of G (note that $Q_{u,S} = \{u\}$ and by (viii) of the definition of skeleton, every rung of H_S is of length at least 2). □

7 Recognition algorithm

In this section we give a recognition algorithm and a structure theorem for the class of (theta,wheel)-free graph. For this, most of the necessary work is already done in [7] (see Sections 6 and 7, where all important steps in the proof are given for (theta,wheel)-graphs).

To obtain a recognition algorithm for (theta,wheel)-free graphs we modify the algorithm given in Theorem 7.6 of [7] for only-pyramid graphs. In fact, the only modification that should be made is the change of the subroutine that checks whether a graph is basic. A recognition algorithm for basic (theta,wheel)-free graphs is given in the following lemma.

Lemma 7.1 *There is an $O(n^2m)$-time algorithm that decides whether an input graph is the line graph of a triangle-free chordless graph or a P-graph.*

PROOF — By Lemma 7.4 from [7], there is an $O(n^2m)$-time algorithm that decides whether an input graph is the line graph of a triangle-free chordless graph. So, it is enough to give an $O(n^2m)$-time algorithm that decides whether an input graph is a P-graph.

First, in time $O(n^2m)$ we can find the set S of all centers of claws in G. If $S = \emptyset$, or S does not induce a clique, then G is not a P-graph. So, assume that S induces a non-empty clique. Next, let K be a maximal clique of G that contains S, unless $|S| = 1$, in which case take $K = S$ if the vertex of S is not contained in a clique of size 3, or K is a maximal clique of size at least 3 that contains S otherwise. Now, let G' be the graph obtained from G by removing all vertices of K (and edges incident with them). Using Lemma 7.4 from [7] we decide (in time $O(n^2m)$) whether G' is a line graph of a triangle-free chordless graph, and if it is find R such that $G' = L(R)$ (if G' is not a line graph of a triangle-free chordless graph, then G is not a P-graph). Now, we check whether R is a k-skeleton, where $k = |K|$. To do this, first we find all pendant edges of R. We name vertices of K with numbers 1 to k, and give labels to the pendant edges of R according to their neighbor in K. We test whether (i), (vi), (vii), (viii) and (xi) in the definition of a k-skeleton are satisfied. Next, we check if (iii) is satisfied (in time $O(n(n+m))$) and then if (iv) is satisfied (in time $O(n^2(n+m))$). To check (v), note that an edge e is contained in a cycle of R if and only if $R \setminus e$ is connected, that is (v) can be check in time $O(m(n+m)) = O(n^2m)$. Branches and limbs of R can be found in time $O(n+m)$ and the number of them is $O(m)$. Hence, (ii) and (ix) can be checked in time $O(n+m+m^2) = O(n^2m)$. Finally, for an attaching vertex x of R all x-petals can be found in time $O(n+m)$, and hence (x) can be be checked in time $O(n(n+m))$. □

By the previous lemma, recognition of basic (theta,wheel)-free graphs can be done in the same running time as the recognition of basic only-pyramid graphs (used in [7]). Hence, the recognition algorithm for (theta,wheel)-free graphs, that was explained above, has the same running time as the algorithm given in Theorem 7.6 of [7]. This proves Theorem 1.3.

As in [7], our decomposition theorem for (theta,wheel)-free graphs can be turned into a structure theorem as follows.

Let G_1 be a graph that contains a clique K and G_2 a graph that contains the same clique K, and is node disjoint from G_1 apart from the nodes of K. The graph $G_1 \cup G_2$ is the graph obtained from G_1 and G_2 by *gluing along a clique*.

Let G_1 be a graph that contains a path $a_2c_2b_2$ such that c_2 has degree 2, and such that $(V(G_1) \setminus \{a_2, c_2, b_2\}, \{a_2, c_2, b_2\})$ is a consistent almost 2-join of G_1 (consistent almost 2-join is a special type of almost 2-joins – for the definition see [7]). Let G_2, a_1, c_1, b_1 be defined similarly. Let G be the graph built on $(V(G_1) \setminus \{a_2, c_2, b_2\}) \cup (V(G_2) \setminus \{a_1, c_1, b_1\})$ by keeping all edges inherited from G_1 and G_2, and by adding all edges between $N_{G_1}(a_2)$ and $N_{G_2}(a_1)$, and all edges between $N_{G_1}(b_2)$ and $N_{G_2}(b_1)$. Graph G is said to be *obtained from G_1 and G_2 by consistent 2-join composition*. Observe that $(V(G_1) \setminus \{a_2, c_2, b_2\}, V(G_2) \setminus \{a_1, c_1, b_1\})$ is a 2-join of G and that G_1 and G_1 are the blocks of decomposition of G with respect to this 2-join.

Using the results from [7], it is straightforward to check the following structure theorem. Every (theta,wheel)-free graph can be constructed as follows:

- Start with line graphs of triangle-free chordless graphs and P-graphs.
- Repeatedly use consistent 2-join compositions from previously constructed graphs.
- Gluing along a clique previously constructed graphs.

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